Appendix

Proof of Theorem 1: We first show that the random effects estimator of β in equation (9) is identical to the fixed effects estimator. The random effects estimator can be characterized as the OLS estimator based on the transformed variables. Let $\tilde{\mathrm{x}}_{ij} = \mathrm{x}_{ij} - \psi_1 \bar{\mathrm{x}}_{i\bullet} - \psi_2 \bar{\mathrm{x}}_{\bullet j}$ denote the transpose of the jth row in the ith block of the transformed matrix $\Omega_1^{-1/2}\mathrm{X}$. $\bar{\tilde{\mathrm{x}}}_{i\bullet}$ and $\bar{\tilde{\mathrm{x}}}_{\bullet j}$ are similarly defined. Let $\tilde{\mathrm{x}}_{ij}^m$ denote the mth covariate in $\tilde{\mathrm{x}}_{ij}$. Consider the regression of $\tilde{\mathrm{x}}_{ij}^m$ on $\bar{\tilde{\mathrm{x}}}_{i\bullet}$ and $\bar{\tilde{\mathrm{x}}}_{\bullet j}$. The least squares estimation is as the following

$$\min \sum_{i=1}^{N} \sum_{j=1}^{L} (\tilde{\mathrm{x}}_{ij}^m - \bar{\tilde{\mathrm{x}}}_{i\bullet}' \varphi_1^m - \bar{\tilde{\mathrm{x}}}_{\bullet j}' \varphi_2^m)^2 \tag{A.1}$$

The first order conditions are obtained as

$$\begin{pmatrix} \hat{\varphi}_1^m \\ \hat{\varphi}_2^m \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^N \sum_{j=1}^L \bar{\tilde{x}}_{i\bullet} \bar{\tilde{\mathrm{x}}}_{i\bullet}' & \sum_{i=1}^N \sum_{j=1}^L \bar{\tilde{\mathrm{x}}}_{i\bullet} \bar{\tilde{\mathrm{x}}}_{\bullet j}' \\ \sum_{i=1}^N \sum_{j=1}^L \bar{\tilde{\mathrm{x}}}_{\bullet j} \bar{\tilde{\mathrm{x}}}_{i\bullet}' & \sum_{i=1}^N \sum_{j=1}^L \bar{\tilde{\mathrm{x}}}_{\bullet j} \bar{\tilde{\mathrm{x}}}_{\bullet j}' \end{pmatrix}^{-1} \begin{pmatrix} \sum_{i=1}^N \sum_{j=1}^L \bar{\tilde{\mathrm{x}}}_{i\bullet} \tilde{\mathrm{x}}_{ij}^m \\ \sum_{i=1}^N \sum_{j=1}^L \bar{\tilde{\mathrm{x}}}_{\bullet j} \tilde{\mathrm{x}}_{ij}^m \end{pmatrix} \tag{A.2}$$

Recall that we assume that all variables are demeaned by their sample averages so $\bar{\mathrm{x}}_{\bullet\bullet} = 0$ which implies that $\sum_{i=1}^N \sum_{j=1}^L \bar{\tilde{\mathrm{x}}}_{i\bullet} \bar{\tilde{\mathrm{x}}}_{\bullet j}' = 0$. It follows that $\hat{\varphi}_1^m$ and $\hat{\varphi}_2^m$ are identical to a vector of zeros except for the mth element being one. The residual obtained by regressing $\tilde{\mathrm{x}}_{ij}^m$ on $\bar{\tilde{\mathrm{x}}}_{i\bullet}$ and $\bar{\tilde{\mathrm{x}}}_{\bullet j}$ is $\tilde{\mathrm{x}}_{ij}^m - \bar{\tilde{\mathrm{x}}}_{i\bullet}^m - \bar{\tilde{\mathrm{x}}}_{\bullet j}^m = \mathrm{x}_{ij}^m - \bar{\mathrm{x}}_{i\bullet}^m - \bar{\mathrm{x}}_{\bullet j}^m$. The partition theorem leads to the equivalence result that $\hat{\beta}_{RE}^A = \hat{\beta}_{FE}$, where $\hat{\beta}_{RE}^A$ denotes the pooled OLS estimator of β in the augmented regression in equation (9). The same equivalence holds for the pooled OLS estimator by setting $\psi_1 = \psi_2 = 0$.

Next we prove the second part of the theorem that the Hausman specification test is identical to a variable addition test. Here we only consider the random effects estimator of δ_1 and δ_2. Standard results of the least squares estimation imply that

$$\begin{pmatrix} \hat{\delta}_{1RE} \\ \hat{\delta}_{2RE} \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^N \sum_{j=1}^L \bar{\tilde{\mathrm{x}}}_{i\bullet} \bar{\tilde{\mathrm{x}}}_{i\bullet}' & \sum_{i=1}^N \sum_{j=1}^L \bar{\tilde{\mathrm{x}}}_{i\bullet} \bar{\tilde{\mathrm{x}}}_{\bullet j}' \\ \sum_{i=1}^N \sum_{j=1}^L \bar{\tilde{\mathrm{x}}}_{\bullet j} \bar{\tilde{\mathrm{x}}}_{i\bullet}' & \sum_{i=1}^N \sum_{j=1}^L \bar{\tilde{\mathrm{x}}}_{\bullet j} \bar{\tilde{\mathrm{x}}}_{\bullet j}' \end{pmatrix}^{-1} \begin{pmatrix} \sum_{i=1}^N \sum_{j=1}^L \bar{\tilde{\mathrm{x}}}_{i\bullet} (\tilde{\mathrm{y}}_{ij} - \tilde{\mathrm{x}}_{ij}' \hat{\beta}_{FE}) \\ \sum_{i=1}^N \sum_{j=1}^L \bar{\tilde{\mathrm{x}}}_{\bullet j} (\tilde{\mathrm{y}}_{ij} - \tilde{\mathrm{x}}_{ij}' \hat{\beta}_{FE}) \end{pmatrix} \tag{A.3}$$

Again using the fact that $\sum_{i=1}^N \sum_{j=1}^L \bar{\tilde{\mathrm{x}}}_{i\bullet} \bar{\tilde{\mathrm{x}}}_{\bullet j}' = NL \bar{\bar{\tilde{\mathrm{x}}}}_{\bullet\bullet} \bar{\bar{\tilde{\mathrm{x}}}}_{\bullet\bullet}' = 0$, $\bar{\tilde{\mathrm{x}}}_{i\bullet} = (1-\psi_1)\bar{\mathrm{x}}_{i\bullet}$ and $\bar{\tilde{\mathrm{x}}}_{\bullet j} = (1-\psi_2)\bar{\mathrm{x}}_{\bullet j}$, it can be shown that

$$\begin{pmatrix} \hat{\delta}_{1RE} \\ \hat{\delta}_{2RE} \end{pmatrix} = \begin{pmatrix} (\sum_{i=1}^{N} \bar{\mathrm{x}}_{i\bullet}\bar{\mathrm{x}}_{i\bullet}')^{-1}(\sum_{i=1}^{N} \bar{\mathrm{x}}_{i\bullet}\bar{\mathrm{y}}_{i\bullet}) - \hat{\beta}_{FE} \\ (\sum_{j=1}^{L} \bar{\mathrm{x}}_{\bullet j}\bar{\mathrm{x}}_{\bullet j}')^{-1}(\sum_{j=1}^{L} \bar{\mathrm{x}}_{\bullet j}\bar{\mathrm{y}}_{\bullet j}) - \hat{\beta}_{FE} \end{pmatrix} \equiv \begin{pmatrix} \hat{\beta}_{o\bullet} - \hat{\beta}_{FE} \\ \hat{\beta}_{\bullet o} - \hat{\beta}_{FE} \end{pmatrix} \tag{A.4}$$

where $\hat{\beta}_{o\bullet}$ and $\hat{\beta}_{\bullet o}$ are the between-group estimators.

With some algebra, the random effects estimator can be decomposed as a weighted average of these two between estimators and the fixed effects estimator.

$$\begin{aligned} \hat{\beta}_{RE} &= \left[\mathrm{X}'(\mathrm{M}_1 + \mathrm{d}_1(\mathrm{I_N} \otimes \mathrm{P_L} - \mathrm{P_N} \otimes \mathrm{P_L}) + \mathrm{d}_2(\mathrm{P_N} \otimes \mathrm{I_L} - \mathrm{P_N} \otimes \mathrm{P_L}))\mathrm{X}\right]^{-1} \\ &\quad \left[\mathrm{X}'(\mathrm{M}_1 + \mathrm{d}_1(\mathrm{I_N} \otimes \mathrm{P_L} - \mathrm{P_N} \otimes \mathrm{P_L}) + \mathrm{d}_2(\mathrm{P_N} \otimes \mathrm{I_L} - \mathrm{P_N} \otimes \mathrm{P_L}))\mathrm{Y}\right] \end{aligned} \tag{A.5}$$

where d_1 and d_2 are some constants determined by the variances of the error components. Now we can rewrite the random effects estimator as the following

$$\hat{\beta}_{RE} = \Delta_1\hat{\beta}_{o\bullet} + \Delta_2\hat{\beta}_{\bullet o} + (\mathrm{I_k} - \Delta_1 - \Delta_2)\hat{\beta}_{FE} \tag{A.6}$$

where $\Delta_1 = \left[\mathrm{X}'(\mathrm{M}_1 + \mathrm{d}_1(\mathrm{I_N} \otimes \mathrm{P_L} - \mathrm{P_N} \otimes \mathrm{P_L}) + \mathrm{d}_2(\mathrm{P_N} \otimes \mathrm{I_L} - \mathrm{P_N} \otimes \mathrm{P_L}))\mathrm{X}\right]^{-1}\mathrm{d}_1\mathrm{X}'(\mathrm{I_N} \otimes \mathrm{P_L} - \mathrm{P_N} \otimes \mathrm{P_L})\mathrm{X}$ and $\Delta_2 = \left[\mathrm{X}'(\mathrm{M}_1 + \mathrm{d}_1(\mathrm{I_N} \otimes \mathrm{P_L} - \mathrm{P_N} \otimes \mathrm{P_L}) + \mathrm{d}_2(\mathrm{P_N} \otimes \mathrm{I_L} - \mathrm{P_N} \otimes \mathrm{P_L}))\mathrm{X}\right]^{-1}\mathrm{d}_2\mathrm{X}'$ $(\mathrm{P_N} \otimes \mathrm{I_L} - \mathrm{P_N} \otimes \mathrm{P_L})\mathrm{X}$. Simple rearrangement gives

$$\hat{\beta}_{RE} - \hat{\beta}_{FE} = \Delta_1(\hat{\beta}_{o\bullet} - \hat{\beta}_{FE}) + \Delta_2(\hat{\beta}_{\bullet o} - \hat{\beta}_{FE}) = \Delta_1\hat{\delta}_{1RE} + \Delta_2\hat{\delta}_{2RE} \tag{A.7}$$

This shows that the Hausman specification test based on $\hat{\beta}_{RE} - \hat{\beta}_{FE}$ is identical to the modified variable addition test of $\Delta_1\delta_1 + \Delta_2\delta_2 = 0$, which completes the proof. ***Q.E.D***

Proof of Theorem 2: We first show that the random effects estimator of β in the augmented regression (17) is identical to the fixed effects estimator. As shown in Matyas (2017, Chapter 2), $\Omega_2^{-1/2}Y$ transforms a typical element y_{ijt} into $\tilde{\mathrm{y}}_{ijt} = \mathrm{y}_{ijt} - \zeta_1\bar{\mathrm{y}}_{i\bullet\bullet} - \zeta_2\bar{\mathrm{y}}_{\bullet j\bullet} - \zeta_3\bar{\mathrm{y}}_{\bullet\bullet t}$. It thus suffices to prove that the residuals obtained from regressing $\tilde{\mathrm{x}}_{ijt}$ on $\tilde{\bar{\mathrm{x}}}_{i\bullet\bullet}$, $\tilde{\bar{\mathrm{x}}}_{\bullet j\bullet}$ and $\tilde{\bar{\mathrm{x}}}_{\bullet\bullet t}$ are the same as the fixed effects residuals $\mathrm{M}_2\mathrm{X}$. Consider the regression of $\tilde{\mathrm{x}}_{ijt}^m$, the mth covariate in $\tilde{\mathrm{x}}_{ijt}$, on $\tilde{\bar{\mathrm{x}}}_{i\bullet\bullet}$, $\tilde{\bar{\mathrm{x}}}_{\bullet j\bullet}$ and $\tilde{\bar{\mathrm{x}}}_{\bullet\bullet t}$

$$\min \sum_{i=1}^{N}\sum_{j=1}^{L}\sum_{t=1}^{T}(\tilde{\mathrm{x}}_{ijt}^m - \tilde{\bar{\mathrm{x}}}_{i\bullet\bullet}'\varphi_1^m - \tilde{\bar{\mathrm{x}}}_{\bullet j\bullet}'\varphi_2^m - \tilde{\bar{\mathrm{x}}}_{\bullet\bullet t}'\varphi_3^m)^2 \tag{A.8}$$

Using the fact that $\tilde{\bar{\mathrm{x}}}_{i\bullet\bullet} = (1-\zeta_1)\bar{\mathrm{x}}_{i\bullet\bullet}$, $\tilde{\bar{\mathrm{x}}}_{\bullet j\bullet} = (1-\zeta_2)\bar{\mathrm{x}}_{\bullet j\bullet}$ and $\tilde{\bar{\mathrm{x}}}_{\bullet\bullet t} = (1-\zeta_3)\bar{x}_{\bullet\bullet t}$, it is easy to show that $\hat{\varphi}_1^m$, $\hat{\varphi}_2^m$ and $\hat{\varphi}_3^m$ are identical to a vector of zeros except for the mth element being one. It follows that the residuals obtained from regressing $\tilde{\mathrm{x}}_{ijt}$ on $\tilde{\bar{\mathrm{x}}}_{i\bullet\bullet}$, $\tilde{\bar{\mathrm{x}}}_{\bullet j\bullet}$ and $\tilde{\bar{\mathrm{x}}}_{\bullet\bullet t}$ are exactly the same as the fixed effects residuals $\mathrm{M}_2\mathrm{X}$. Thus the random effects estimator of β in the augmented regression (17) is identical to the fixed effects estimator. The equivalence between the pooled OLS estimator and the fixed effects estimator follows as a special case by setting $\zeta_1 = \zeta_2 = \zeta_3 = 0$

Next we show that the random effects estimators and the pooled OLS estimators of the δs are identical to the differences between the between-group estimators and the fixed effects estimator.

$$\begin{pmatrix} \hat{\delta}_{1RE} \\ \hat{\delta}_{2RE} \\ \hat{\delta}_{3RE} \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^N \sum_{j=1}^L \sum_{t=1}^T \tilde{\bar{\mathrm{x}}}_{i\bullet\bullet}\tilde{\bar{\mathrm{x}}}'_{i\bullet\bullet} & \sum_{i=1}^N \sum_{j=1}^L \sum_{t=1}^T \tilde{\bar{\mathrm{x}}}_{i\bullet\bullet}\tilde{\bar{\mathrm{x}}}'_{\bullet j\bullet} & \sum_{i=1}^N \sum_{j=1}^L \sum_{t=1}^T \tilde{\bar{\mathrm{x}}}_{i\bullet\bullet}\tilde{\bar{\mathrm{x}}}'_{\bullet\bullet t} \\ \sum_{i=1}^N \sum_{j=1}^L \sum_{t=1}^T \tilde{\bar{\mathrm{x}}}_{\bullet\bullet j}\tilde{\bar{\mathrm{x}}}'_{i\bullet\bullet} & \sum_{i=1}^N \sum_{j=1}^L \sum_{t=1}^T \tilde{\bar{\mathrm{x}}}_{\bullet\bullet j}\tilde{\bar{\mathrm{x}}}'_{\bullet j\bullet} & \sum_{i=1}^N \sum_{j=1}^L \sum_{t=1}^T \tilde{\bar{\mathrm{x}}}_{\bullet j\bullet}\tilde{\bar{\mathrm{x}}}'_{\bullet\bullet t} \\ \sum_{i=1}^N \sum_{j=1}^L \sum_{t=1}^T \tilde{\bar{\mathrm{x}}}_{\bullet\bullet t}\tilde{\bar{\mathrm{x}}}'_{i\bullet\bullet} & \sum_{i=1}^N \sum_{j=1}^L \sum_{t=1}^T \tilde{\bar{\mathrm{x}}}_{\bullet\bullet t}\tilde{\bar{\mathrm{x}}}'_{\bullet j\bullet} & \sum_{i=1}^N \sum_{j=1}^L \sum_{t=1}^T \tilde{\bar{\mathrm{x}}}_{\bullet\bullet t}\tilde{\bar{\mathrm{x}}}'_{\bullet\bullet t} \end{pmatrix}^{-1}$$
$$\begin{pmatrix} \sum_{i=1}^N \sum_{j=1}^L \sum_{t=1}^T \tilde{\bar{\mathrm{x}}}_{i\bullet\bullet}(\tilde{\mathrm{y}}_{ijt} - \tilde{\mathrm{x}}'_{ijt}\hat{\beta}_{FE}) \\ \sum_{i=1}^N \sum_{j=1}^L \sum_{t=1}^T \tilde{\bar{\mathrm{x}}}_{\bullet j\bullet}(\tilde{\mathrm{y}}_{ijt} - \tilde{\mathrm{x}}'_{ijt}\hat{\beta}_{FE}) \\ \sum_{i=1}^N \sum_{j=1}^L \sum_{t=1}^T \tilde{\bar{\mathrm{x}}}_{\bullet\bullet t}(\tilde{\mathrm{y}}_{ijt} - \tilde{\mathrm{x}}'_{ijt}\hat{\beta}_{FE}) \end{pmatrix} \tag{A.9}$$

Using the fact that $\bar{\mathrm{x}}_{\bullet\bullet\bullet} = 0$ we can show that all the off diagonal blocks of the inverse matrix are zeros. It then follows that

$$\begin{pmatrix} \hat{\delta}_{1RE} \\ \hat{\delta}_{2RE} \\ \hat{\delta}_{3RE} \end{pmatrix} = \begin{pmatrix} \hat{\beta}_{o\bullet\bullet} - \hat{\beta}_{FE} \\ \hat{\beta}_{\bullet o\bullet} - \hat{\beta}_{FE} \\ \hat{\beta}_{\bullet\bullet o} - \hat{\beta}_{FE} \end{pmatrix} \tag{A.10}$$

With some algebra we can rewrite the random effects estimator in equation (12) as

$$\begin{aligned} \hat{\beta}_{RE} &= \left[\mathrm{X}'(\mathrm{M}_2 + \check{\mathrm{d}}_1(\mathrm{Q_N} \otimes \mathrm{P_L} \otimes \mathrm{P_T}) + \check{\mathrm{d}}_2(\mathrm{P_N} \otimes \mathrm{Q_L} \otimes \mathrm{P_T}) + \check{\mathrm{d}}_3(\mathrm{P_N} \otimes \mathrm{P_L} \otimes \mathrm{Q_T}))\mathrm{X}\right]^{-1} \\ &\quad \left[\mathrm{X}'(\mathrm{M}_2 + \check{\mathrm{d}}_1(\mathrm{Q_N} \otimes \mathrm{P_L} \otimes \mathrm{P_T}) + \check{\mathrm{d}}_2(\mathrm{P_N} \otimes \mathrm{Q_L} \otimes \mathrm{P_T}) + \check{\mathrm{d}}_3(\mathrm{P_N} \otimes \mathrm{P_L} \otimes \mathrm{Q_T}))\mathrm{Y}\right] \end{aligned} \tag{A.11}$$

where $\check{d}_1$, $\check{d}_2$ and $\check{d}_3$ are some constants associated with the σ^2s. Now we can show that the random effects estimator can be expressed as a matrix average of the between estimators and the fixed effects estimator.

$$
\begin{aligned}
\hat{\beta}_{RE} = \tilde{\Delta}_1\hat{\beta}_{o\bullet\bullet} \quad &+ \quad \tilde{\Delta}_2\hat{\beta}_{\bullet o\bullet} + \tilde{\Delta}_3\hat{\beta}_{\bullet\bullet o} + (\mathrm{I} - \tilde{\Delta}_1 - \tilde{\Delta}_2 - \tilde{\Delta}_3)\hat{\beta}_{FE} \\
\tilde{\Delta}_1 \quad &= \quad \tilde{\mathrm{M}}\check{\mathrm{d}}_1\mathrm{X}'(\mathrm{Q_N} \otimes \mathrm{P_L} \otimes \mathrm{P_T})\mathrm{X} \\
\tilde{\Delta}_2 \quad &= \quad \tilde{\mathrm{M}}\check{\mathrm{d}}_2\mathrm{X}'(\mathrm{P_N} \otimes \mathrm{Q_L} \otimes \mathrm{P_T})\mathrm{X} \\
\tilde{\Delta}_3 \quad &= \quad \tilde{\mathrm{M}}\check{\mathrm{d}}_3\mathrm{X}'(\mathrm{P_N} \otimes \mathrm{P_L} \otimes \mathrm{Q_T})\mathrm{X}
\end{aligned}
\tag{A.12}
$$

where $\tilde{\mathrm{M}} = \left[\mathrm{X}'(\mathrm{M}_2 + \check{\mathrm{d}}_1(\mathrm{Q_N} \otimes \mathrm{P_L} \otimes \mathrm{P_T}) + \check{\mathrm{d}}_2(\mathrm{P_N} \otimes \mathrm{Q_L} \otimes \mathrm{P_T}) + \check{\mathrm{d}}_3(\mathrm{P_N} \otimes \mathrm{P_L} \otimes \mathrm{Q_T}))\mathrm{X}\right]^{-1}$. It is now straightforward to show that

$$
\hat{\beta}_{RE} - \hat{\beta}_{FE} = \tilde{\Delta}_1(\hat{\beta}_{o\bullet\bullet} - \hat{\beta}_{FE}) + \tilde{\Delta}_2(\hat{\beta}_{\bullet o\bullet} - -\hat{\beta}_{FE}) + \tilde{\Delta}_3(\hat{\beta}_{\bullet\bullet o} - \hat{\beta}_{FE}) \tag{A.13}
$$

which demonstrates that the Hausman specification test based on $\hat{\beta}_{RE} - \hat{\beta}_{FE}$ is identical to the modified variable addition test of $\tilde{\Delta}_1\delta_1 + \tilde{\Delta}_2\delta_2 + \tilde{\Delta}_3\delta_3 = 0$ ***Q.E.D***