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The multidimensional Mundlak estimator

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ABSTRACT

Mundlak (1978) shows that the fixed effects estimator is equivalent to the random effects estimator in the one-way error component model once the random individual effects are modeled as a linear function of all the averaged regressors over time. In the spirit of Mundlak, this paper shows that this result also holds for the multidimensional error component model. This is a generalization of Baltagi (2023) from the two-way Mundlak model to higher order multidimensional error components model, see Balazsi et al. (2024a) for the multidimensional fixed effects model and Balazsi et al. (2024b) for the multidimensional random effects model. The F test suggested by Mundlak (1978) to test for this correlation between the random effects and the regressors generate Hausman (1978) type tests that are easily generalizable to the multi-dimensional Mundlak regression.

1. Introduction

Mundlak (1978) showed that for the one-way error component regression, correlated random individual effects can be modeled as a linear function of *all* the regressors averaged over time plus a stochastic random individual (time invariant) disturbance. Substituting these correlated random effects in the original regression yields an augmented regression with averaged regressors and random individual uncorrelated effects. Mundlak (1978) then applied the random effects estimator (using GLS) on this augmented regression and showed that this yields the fixed effects estimator. This allows a Hausman (1978) type test for the correlated random effects that can be implemented in this augmented regression by testing the significance of the averaged regressors. Not rejecting the null, yields to non-rejection of the random effects model, while rejecting the null means random effects is not consistent and the Mundlak estimator reduces to the fixed effects estimator. These important Mundlak results were extended to the two-way error component model with time and individual effects by Yang (2022) and Baltagi (2023). This paper shows that these results can be easily extended to higher order error components models. This is useful for multidimensional panel data, see Balazsi et al. (2024a) for the multidimensional fixed effects model, and Balazsi et al. (2024b) for the multidimensional random effects model. These two chapters cite several multidimensional panel applications in the literature especially in trade and housing, too numerous to cite here.

Empirical applications of the Mundlak correlated random effects model use OLS rather than GLS on this augmented regression. Baltagi (2023) showed that OLS is equivalent to GLS for the two-way error component model and both yield the two-way fixed effects estimator. However, the standard errors are different for OLS and GLS as they assume different variance–covariance structures. This also affects test of hypotheses and in particular the test for correlated random effects. In fact, the GLS regression yields the Hausman (1978) test for correlated effects (based on fixed versus between estimators) suggested by Mundlak (1978). While, the OLS estimator yields a different statistic for the significance of these additional averaged regressors. Not rejecting the null in the OLS augmented regression yields to non-rejection of the pooled OLS model, while non-rejection of the additional averaged regressors using GLS yield a non-correlated random effects model as in Mundlak (1978). For the higher dimensional error component model, this paper also shows that OLS on the augmented Mundlak regression still yields the same multidimensional fixed effects estimator as GLS, but once again the standard errors are different and test of hypotheses yield different statistics especially for the test for correlated random effects.

Section 2 presents the multidimensional Mundlak model extending Mundlak's (1978) idea from the one-way error component to higher order error components models. This is derived for the three-way error component model to keep the presentation simple and show that it can be easily extended to higher order error components depending on the application. The paper also shows that GLS or OLS on this augmented higher order Mundlak regression yield the same (multidimensional fixed effects) estimator. However, we emphasize that the standard

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errors and test of hypotheses are different for these two estimators of the augmented regressions.

2. The Mundlak multi-dimensional error component regression

Consider the three-dimensional error component regression model with say firm, country and time effects

$$y_{ijt} = X'_{ijt}\beta + u_{ijt} \tag{1}$$

with

$$u_{ijt} = \mu_i + \eta_j + \lambda_t + \nu_{ijt} \tag{2}$$

where $i = 1,\ldots,N$ may denote firms; $j = 1,\ldots,M$ may denote countries; and $t = 1,\ldots,T$ denotes time, see Ghosh (1976) for an early use of this model citing its usefulness for international or interregional studies. These four *independent* error components are assumed to be random with $\mu_i \sim \mathrm{IID}(0,\sigma_\mu^2)$, $\eta_j \sim \mathrm{IID}(0,\sigma_\eta^2)$, $\lambda_t \sim \mathrm{IID}(0,\sigma_\lambda^2)$ and $\nu_{ijt} \sim \mathrm{IID}(0,\sigma_\nu^2)$. Order the observations such that the slowest index is i, then j and the faster index is t, so that

$$\begin{aligned} u' &= (u_{111},\ldots,u_{11T},u_{121},\ldots,u_{12T},\ldots,u_{1M1},\ldots,\\ &\quad u_{1MT},\ldots,u_{NM1},\ldots,u_{NMT}) \end{aligned}$$

In vector form

$$u = Z_\mu\mu + Z_\eta\eta + Z_\lambda\lambda + \nu \tag{3}$$

where $Z_\mu = (I_N \otimes \iota_M \otimes \iota_T)$ is the selector matrix that picks up the proper μ_i from $\mu' = (\mu_1,\mu_2,\ldots,\mu_N)$; $Z_\eta = (\iota_N \otimes I_M \otimes \iota_T)$ is the selector matrix that picks up the proper η_j from $\eta' = (\eta_1,\eta_2,\ldots,\eta_M)$; $Z_\lambda = (\iota_N \otimes \iota_M \otimes I_T)$ is the selector matrix that picks up the proper λ_t from $\lambda' = (\lambda_1,\lambda_2,\ldots,\lambda_T)$. I_N is an identity matrix of dimension N, and ι_T is a vector of ones of dimension T.

The three-way error components model can be written in matrix form as:

$$y = X\beta + Z_\mu\mu + Z_\eta\eta + Z_\lambda\lambda + \nu \tag{4}$$

Following Mundlak (1978), one can model the correlated firm effects as a linear function of *all* the explanatory variables averaged across j and t

$$\mu_i = \bar{X}'_{i..}\pi_\mu + \epsilon_i \tag{5}$$

where $\epsilon_i \sim \mathrm{IIN}(0,\sigma_\epsilon^2)$ and $\bar{X}'_{i..}$ is a $1\times K$ vector of observations on the explanatory variables averaged over j and t. The dot subscript indicates summation over that subscript and the bar indicates averaging. These firm effects are uncorrelated with the explanatory variables if and only if $\pi_\mu = 0$, otherwise this is a correlated random effects regression. Mundlak (1978) assumed, without loss of generality, that the X's are deviations from their sample mean. In vector form (5) can be written as follows:

$$\mu = Z'_\mu X\pi_\mu/MT + \epsilon \tag{6}$$

where $\epsilon' = (\epsilon_1,\ldots,\epsilon_N)$. Similarly, the correlated country effects are modeled as a linear function of *all* the explanatory variables averaged across i and t

$$\eta_j = \bar{X}'_{.j.}\pi_\eta + \zeta_j \tag{7}$$

where $\zeta_j \sim \mathrm{IIN}(0,\sigma_\epsilon^2)$ and $\bar{X}'_{.j.}$ is a $1\times K$ vector of observations on the explanatory variables averaged over i and t. These country effects are uncorrelated with the explanatory variables if and only if $\pi_\eta = 0$. In vector form this can be written as follows:

$$\eta = Z'_\eta X\pi_\eta/NT + \zeta \tag{8}$$

where $\varepsilon' = (\zeta_1,\ldots,\zeta_M)$. The correlated time effects are modeled as a linear function of *all* the explanatory variables averaged across i and j

$$\lambda_t = \bar{X}'_{..t}\pi_\lambda + \varepsilon_t \tag{9}$$

where $\varepsilon_t \sim \mathrm{IIN}(0,\sigma_\varepsilon^2)$ and $\bar{X}'_{..t}$ is a $1\times K$ vector of observations on the explanatory variables averaged over i and j. In other words, this assumes that the time effects are a linear function of *all* the explanatory variables averaged across i and j. These time effects are uncorrelated with the explanatory variables if and only if $\pi_\lambda = 0$. In vector form this can be written as follows:

$$\lambda = Z'_\lambda X\pi_\lambda/NM + \varepsilon \tag{10}$$

where $\varepsilon' = (\varepsilon_1,\ldots,\varepsilon_T)$.

Substituting the correlated effects for μ, η and λ, into the regression defined in (4), one gets the Mundlak 3-way model

$$y = X\beta + P_\mu X\pi_\mu + P_\eta X\pi_\eta + P_\lambda X\pi_\lambda + (Z_\mu\epsilon + Z_\eta\zeta + Z_\lambda\varepsilon + \nu) \tag{11}$$

Denote by $J_s = \iota_s\iota'_s$ a matrix of ones of dimension s which could be N, M or T. This is the summing matrix. Also denote by $\bar{J}_s = J_s/s$ as the averaging matrix for $s = N, M$ or T. In this case, $P_\mu = Z_\mu Z'_\mu/MT = (I_N \otimes J_M \otimes J_T)/MT = (I_N \otimes \bar{J}_M \otimes \bar{J}_T)$ is the averaging matrix across j and t. $P_\eta = Z_\eta Z'_\eta/NT = (J_N \otimes I_M \otimes J_T)/NT = (\bar{J}_N \otimes I_M \otimes \bar{J}_T)$ is the averaging matrix over i and t. $P_\lambda = Z_\lambda Z'_\lambda/NM = (J_N \otimes J_M \otimes I_T)/NM = (\bar{J}_N \otimes \bar{J}_M \otimes I_T)$ is the averaging matrix over i and j.

Using the fact that the ϵ, ζ, ε and υ are uncorrelated, the three-way error components disturbances in parentheses have zero mean and variance–covariance matrix

$$\begin{aligned} \Omega &= E(Z_\mu\epsilon + Z_\eta\zeta + Z_\lambda\varepsilon + \nu)(Z_\mu\epsilon + Z_\eta\zeta + Z_\lambda\varepsilon + \nu)' \\ &= \sigma_\epsilon^2(I_N \otimes J_M \otimes J_T) + \sigma_\zeta^2(J_N \otimes I_M \otimes J_T) \\ &\quad + \sigma_\varepsilon^2(J_N \otimes J_M \otimes I_T) + \sigma_\nu^2(I_N \otimes I_M \otimes I_T) \end{aligned} \tag{12}$$

We apply the Wansbeek and Kapteyn (1982) trick by replacing J_s by its idempotent counterpart $s\bar{J}_s$, for $s = N, M$ or T. Also replace I_s by its components $(E_s + \bar{J}_s)$, where $E_s = I_s - \bar{J}_s$ is the deviations from mean matrix for $s = N, M$ or T. Next, we collect like terms with the same matrices. This gives

$$\Omega = \sum_{l=1}^{5} w_l Q_l \tag{13}$$

where $w_1 = \sigma_\nu^2, w_2 = NM\sigma_\varepsilon^2 + \sigma_\nu^2, w_3 = TM\sigma_\epsilon^2 + \sigma_\nu^2, w_4 = NT\sigma_\zeta^2 + \sigma_\nu^2$ and $w_5 = NM\sigma_\varepsilon^2 + TM\sigma_\epsilon^2 + NT\sigma_\zeta^2 + \sigma_\nu^2$. Also

$$\begin{aligned} Q_1 &= (E_N \otimes E_M \otimes E_T) + (E_N \otimes \bar{J}_M \otimes E_T) + (\bar{J}_N \otimes E_M \otimes E_T) \\ &\quad + (E_N \otimes E_M \otimes \bar{J}_T) \\ &= I_N \otimes I_M \otimes I_T - \bar{J}_N \otimes \bar{J}_M \otimes I_T - I_N \otimes \bar{J}_M \otimes \bar{J}_T \\ &\quad - \bar{J}_N \otimes I_M \otimes \bar{J}_T + 2(\bar{J}_N \otimes \bar{J}_M \otimes \bar{J}_T) \\ Q_2 &= \bar{J}_N \otimes \bar{J}_M \otimes E_T = P_\lambda - \bar{J}_N \otimes \bar{J}_M \otimes \bar{J}_T \\ Q_3 &= E_N \otimes \bar{J}_M \otimes \bar{J}_T = P_\mu - \bar{J}_N \otimes \bar{J}_M \otimes \bar{J}_T \\ Q_4 &= \bar{J}_N \otimes E_M \otimes \bar{J}_T = P_\eta - \bar{J}_N \otimes \bar{J}_M \otimes \bar{J}_T \\ \text{and}\quad Q_5 &= \bar{J}_N \otimes \bar{J}_M \otimes \bar{J}_T \end{aligned}$$

The w_l are the distinct characteristic roots of Ω for $l = 1,2,\ldots,5$. The Q_l are the corresponding matrices of eigenvectors. w_1 is of multiplicity $(NMT-T-N-M+2)$, w_2 is of multiplicity $(T-1)$, w_3 is of multiplicity $(N-1)$, w_4 is of multiplicity $(M-1)$, and w_5 is of multiplicity 1. Each Q_l is symmetric and idempotent with its rank equal to its trace. Moreover, the Q_l are pairwise orthogonal and sum to the identity matrix, see Baltagi (1987). The advantages of this spectral decomposition of Ω are that

$$\Omega^r = \sum_{l=1}^{5} w_l^r Q_l \tag{14}$$

where r is an arbitrary scalar. This obtains the inverse Ω^{-1} for $r = -1$, and $\Omega^{-1/2}$ for $r = -1/2$. Rewrite X as $(\sum_{l=1}^5 Q_l)X$, since $\sum_{l=1}^5 Q_l = I_{NT}$. Substitute this X in (11) and collect like terms, one gets

$$y = Q_1X\beta + P_\mu X(\pi_\mu+\beta) + P_\eta X(\pi_\eta+\beta) + P_\lambda X(\pi_\lambda+\beta) + (Z_\mu\epsilon + Z_\eta\zeta + Z_\lambda\varepsilon + \nu) \tag{15}$$

where we used the fact that the $X's$ are in deviations from their sample mean, so that $(\bar{J}_N \otimes \bar{J}_M \otimes \bar{J}_T)X = Q_5X = 0$. Note that $Q_1X, P_\mu X, P_\eta X$ and $P_\lambda X$ are orthogonal to each other, since $E_s\bar{J}_s = 0$ and $P_\mu P_\lambda X = P_\mu P_\eta X = P_\lambda P_\eta X = (\bar{J}_N \otimes \bar{J}_M \otimes \bar{J}_T)X = 0$. Shortly, we will prove that OLS on this regression (15) is equivalent to GLS. In fact, OLS on (15) yields

$$\hat{\beta}_{OLS} = (X'Q_1X)^{-1}X'Q_1y = \tilde{\beta}_w \tag{16}$$

$$(\widehat{\pi_\mu + \beta})_{OLS} = (X'P_\mu X)^{-1}X'P_\mu y = \hat{\beta}_{between1} \tag{17}$$

$$(\widehat{\pi_\eta + \beta})_{OLS} = (X'P_\eta X)^{-1}X'P_\eta y = \hat{\beta}_{between2} \tag{18}$$

$$(\widehat{\pi_\lambda + \beta})_{OLS} = (X'P_\lambda X)^{-1}X'P_\lambda y = \hat{\beta}_{between3} \tag{19}$$

where $\tilde{\beta}_w$ is the three-way Within estimator which runs $\tilde{y} = Q_1y$ with typical element $\tilde{y}_{ijt} = (y_{ijt} - \bar{y}_{i..} - \bar{y}_{.j.} - \bar{y}_{..t} + 2\bar{y}_{...})$ on $\tilde{X} = Q_1X$ with typical element $\tilde{x}_{ijt} = (x_{ijt} - \bar{x}_{i..} - \bar{x}_{.j.} - \bar{x}_{..t} + 2\bar{x}_{...})$ for every regressor in X, see Balazsi et al. (2024a) for a similar transformation for the three-way fixed effects model. $\hat{\beta}_{between1}$ is the between estimator running OLS on the averaged equation over countries and time, $\hat{\beta}_{between2}$ is the between estimator running OLS on the averaged equation over firms and time. $\hat{\beta}_{between3}$ is the between estimator running OLS on the averaged equation over firms and countries. This means that $\hat{\pi}_\mu = \hat{\beta}_{between1} - \tilde{\beta}_w$. Also, $\hat{\pi}_\eta = \hat{\beta}_{between2} - \tilde{\beta}_w$, and $\hat{\pi}_\lambda = \hat{\beta}_{between3} - \tilde{\beta}_w$. A joint test for $H_0^a; \pi_\mu = \pi_\eta = \pi_\lambda = 0$ would test jointly whether there is correlation between the averaged regressors with the firm, country and time effects. This can be done with an F test from the OLS regression of y on $\tilde{X}$, $\bar{X}_{i..}$, $\bar{X}_{..t}$, $\bar{X}_{.j.}$, see Baltagi (2023) for the case of the two-way Mundlak regression. For the three-way Mundlak model, the test for $H_0^a; \pi_\mu = \pi_\eta = \pi_\lambda = 0$ is a joint test of three between estimators (each averaging across two indices), being equal to a three-way Within estimator. Note that one can also test $H_0^b; \pi_\mu = 0$; as well as $H_0^c; \pi_\eta = 0$; and $H_0^d; \pi_\lambda = 0$ or any pair of them not being significantly different from zero using the F tests. Having rejected H_0^a, this may shed some light on whether this is due to correlated firm, country or time effects, any pair, or all. This also gives a Hausman type test interpretation for these F-tests when they are based on contrasts between any pair of estimators, with one being efficient and the other being consistent under the null. In fact, $H_0^b; \pi_\mu = 0$ can be formulated as a Hausman test for $E(\mu_i \mid X_{ijt}) = 0$ based on $\hat{\beta}_{between1} - \tilde{\beta}_w$. Similarly, $H_0^c; \pi_\eta = 0$ can be formulated as a Hausman test for $E(\eta_j \mid X_{ijt}) = 0$ based on $\hat{\beta}_{between2} - \tilde{\beta}_w$, and $H_0^d; \pi_\lambda = 0$ can be formulated as a Hausman test for $E(\lambda_t \mid X_{ijt}) = 0$ based on $\hat{\beta}_{between3} - \tilde{\beta}_w$.[1]

Theorem. *The OLS estimator is equivalent to the GLS estimator for the three-way Mundlak regression in* (15).

Proof. Using Zyskind's (1967) necessary and sufficient condition for the equivalence of OLS and GLS, one has to show that $P_Z\Omega = \Omega P_Z$ where $Z = [Q_1X, P_\mu X, P_\eta X, P_\lambda X]$ is the matrix of regressors in (15) and $P_Z = Z(Z'Z)^{-1}Z'$ with $\Omega = \sum_{l=1}^{5} w_lQ_l$ from (13). The regressors are orthogonal, so

$P_Z = P_{Q_1X} + P_{P_\mu X} + P_{P_\eta X} + P_{P_\lambda X}$
and
$P_Z\Omega = w_1P_{Q_1X} + w_2P_{P_\lambda X} + w_3P_{P_\mu X} + w_4P_{P_\lambda X} = \Omega P_Z$.

This uses the properties of the $Q_l's$. For example, $P_{Q_1X}\Omega = w_1P_{Q_1X}$ since $Q_l\Omega = w_lQ_l$ for $l = 1, 2, 3, 4, 5$; Also $X'P_\mu\Omega = w_3X'P_\mu$, $X'P_\eta\Omega = w_4X'P_\eta$ and $X'P_\lambda\Omega = w_2X'P_\lambda$.

Zyskind's (1967) necessary and sufficient condition is satisfied and OLS is equivalent to GLS on (15). OLS yields the estimators in (16) to (19). This extends Baltagi's (2023) results from the two-way to the 3-way Mundlak augmented regression. This also shows that the 3-way Mundlak estimator can be computed using OLS rather than GLS as Mundlak proposed. Having said that, it is important to emphasize that Mundlak (1978) applied random effects GLS to his augmented regression because the disturbances have a one-way random error components variance–covariance structure even after including the time averaged regressors. Yet most applications of the Mundlak (1978) estimator apply least squares. This is because OLS on the Mundlak model is equivalent to GLS and both yield the fixed effects estimator. However, the test for the significance of the averaged regressors based on OLS is different from that based on GLS. Only the latter yields the Hausman type test reported by Mundlak (1978) based on the contrast between the fixed and between estimators. This is important because not rejecting the null, Mundlak (1978) does not reject a random effects model (where the individual effects are uncorrelated with the averaged regressors). In contrast, not rejecting the null based on OLS reverts to pooled least squares and not a random effects estimator. Also, the standard errors of the fixed effects estimates obtained from least squares on the Mundlak (1978) augmented regression do not produce the standard errors obtained from the application of the fixed effects regression. In contrast, the standard errors of the fixed effects estimator obtained from GLS (or random effects) on the augmented Mundlak regression yield exactly the same standard errors reported by the within regression. The same argument applies to the higher order Mundlak error components models derived in this paper. In sum, while OLS is equivalent to GLS for the augmented 3-way Mundlak model, the standard errors are different, the tests for the significance of the correlated effects are different. Also, not rejecting the null of no correlated effects yields to not rejecting pooled OLS in one scenario and random effects in another. Mundlak (1978)'s idea is to test for random versus fixed effects and his omitted variables F-test in his augmented regression yields a Hausman test, not a test for whether pooled OLS is not rejected.

In conclusion, the results in this paper can be easily extended to higher order error components models, as the same proof applies but with a little more matrix Algebra and notation. We demonstrated it for the three-way Mundlak model, but it applies also to a four-way or higher order Mundlak model.

Data availability

No data was used for the research described in the article.

References

Balazsi, L., Baltagi, B.H., Matyas, L., Pus, D., 2024a. Random effects models. In: Matyas, Laszlo (Ed.), Chapter 3 in the Econometrics of Multi-dimensional Panels, 2nd edition In: Advanced Studies in Theoretical and Applied Econometrics, Springer, pp. 61–98.

Balazsi, L., Matyas, L., Wansbeek, T., 2024b. Fixed effects models. In: Matyas, Laszlo (Ed.), Chapter 1 in the Econometrics of Multi-Dimensional Panels, second ed. In: Advanced Studies in Theoretical and Applied Econometrics, Springer, pp. 1–37.

Baltagi, B.H., 1987. On estimating from a more general time-series cum cross-section data structure. Am. Econ. 31, 69–71.

Baltagi, B.H., 2023. The two-way Mundlak estimator. Econometric Rev. 42 (2), 240–246.

Ghosh, S.K., 1976. Estimating from a more general time-series cum cross-section data structure. Am. Econ. 20, 15–21.

Hausman, J.A., 1978. Specification tests in econometrics. Econometrica 46, 1251–1271.

Kang, S., 1985. A note on the equivalence of specification tests in the two-factor multivariate variance components model. J. Econometrics 28, 193–203.

Mundlak, Y., 1978. On the pooling of time series and cross-section data. Econometrica 46, 69–85.

Wansbeek, T.J., Kapteyn, A., 1982. A simple way to obtain the spectral decomposition of variance components models for balanced data. Commun. Stat. A 11, 2105–2112.

Yang, Y., 2022. A correlated random effects approach to the estimation of models with multiple fixed effects. Econ. Lett. (ISSN: 0165-1765) 213, 110408. http://dx.doi.org/10.1016/j.econlet.2022.110408.

Zyskind, G., 1967. On canonical forms, non-negative covariance matrices and best and simple least squares linear estimators in linear models. Ann. Math. Stat. 36, 1092–1109.

[1] See Kang (1985) for the Hausman tests in the two-way error component model, and Baltagi (2023) for the relationship of these Hausman tests with the corresponding F tests for the two-way Mundlak error component model.