

# Neutrinos via Charm Decays in Astrophysical Sources

by

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# Abstract

## Acknowledgements

# Abbreviations

<b>AGN</b>	Active Galactic Nucleus
<b>CMB</b>	Cosmic Microwave Background
<b>DSA</b>	Diffusive Shock Acceleration
<b>EW</b>	Electroweak Theory
<b>FF</b>	Fragmentation Function
<b>FFE</b>	Force Free Electrodynamics
<b>GR</b>	General Relativity
<b>GZK</b>	Greisen-Zatsepin-Kuzmin
<b>MHD</b>	Magnetohydrodynamics
<b>QCD</b>	Quantum Chromodynamics
<b>QED</b>	Quantum Electrodynamics
<b>QFT</b>	Quantum Field Theory
<b>QM</b>	Quantum Mechanics
<b>SM</b>	Standard Model
<b>SMBH</b>	Super-Massive Black Hole
<b>SR</b>	Special Relativity
<b>UHECR</b>	Ultra-High-Energy Cosmic Ray

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# 1 Introduction

In their quest for a deeper understanding of astrophysical phenomena, researchers have moved to investigating a combination of different signal types. This has given rise to the independent field of multimessenger astronomy that studies electromagnetic radiation, cosmic rays, neutrinos and gravitational waves to infer which processes drive their sources [1]. While such environments are likely opaque to photons, charged particles may escape after being accelerated, predicting a non-thermal *Ultra-High-Energy Cosmic Ray* (UHECR) population. Experiments indeed support this expectation by regular measurements of energies from  $10^9$  GeV up to  $10^{11}$  GeV in accordance with the *Greisen-Zatsepin-Kuzmin* (GZK) cutoff [2]. The question then becomes whether it is possible to reconstruct any source region properties from these signals.

One basic constraint derived from observations of extreme energies is a relationship between the sizes and field strenghts of possible sources [3]. From this follows that plausible candidates for UHECR origins consist mostly of stellar remnants like white dwarfs or neutron stars as well as *Active Galactic Nuclei* (AGN) that are powered by *Super-Massive Black Hole* (SMBH) engines at their center [4–6]. Matching of cosmic ray signals to specific sources, however, proves to be an extremely difficult task because large scale magnetic fields isotropize the directions of charged messenger particles.

This leaves neutrinos and gravitational waves as multimessenger candidates, both of which are the subject of multiple promising discoveries at high-energy detection facilities [7–9]. Although gravitational signals provide valuable insights regarding the interaction and structure of massive objects, they do not carry information on particle acceleration. Neutrinos, on the other hand, are produced in hadronic reactions with proton or photon fields close to the accelerating regions. Their weakly interacting nature allows them to escape and propagate almost freely, making them reliable pointers to their sources [10].

Due to their strong physical connection, the origins of highly energetic neutrinos are thought to be closely related to those of hadronic cosmic rays. Recent combined analyses indicating simultaneous neutrino and gamma emissions from a flaring blazar seem to be in agreement with this assumption for the case of an AGN scenario [11, 12]. More compact sources such as strongly magnetized pulsars are also an area that is being actively researched [13]. The standard approach to modeling these sites starts with injecting protons accelerated by some astrophysical process into a photonic or baryonic target surrounding the potential source. In the resulting interactions, energetic pions and kaons are produced, which then decay to neutrinos. This method is generally self-consistent and capable of producing plausible results, but neglects certain effects that might reveal more detailed information about the respective source.

## 1 Introduction

When working towards describing sources of neutrinos in the upper UHECR energy ranges, intermediate hadrons must exhibit highly relativistic velocities. Accordingly, these particles experience extreme time dilation, leading to lifetimes appearing several orders of magnitude longer than in their rest frames when viewed from the surrounding fields. If sufficiently high matter densities are present, the typically investigated pions and kaons are therefore scattered to lower energies before decaying to neutrinos. Due to their much shorter lifetimes, more exotic particles, many of which contain at least one charm quark, then become significant contributors to the neutrino flux that would otherwise remain suppressed [14].

The present work estimates the relative neutrino contribution from decays of such charmed hadrons. For this purpose, results of the magnetar case presented in [15] are reproduced before applying the same procedure to a simplified AGN accretion disc model, assessing the feasibility of charm decay dominance in this latter scenario.

Following this introduction, chapter 2 summarizes the core physical concepts relevant to this thesis, including particle physics, multimessenger astronomy, and their connection to the treated astrophysical sources. Building on this background, chapter 3 introduces semianalytical methods involving empirical parametrizations, which are then employed for all further computations. Lastly, chapter 4 presents the results, with chapter 5 providing a concluding discussion as well as potential future prospects based on these findings.

On the topic of unit conventions, this work uses varying styles for different contexts. Physical equations are usually given in Gaussian units, while masses might have natural units of energy when concerning particle physics or be written as multiples of the solar mass in astrophysical settings. To avoid ambiguities, units will be clarified in the text whenever necessary.



## 2 Background

On their journey through space, messenger particles are subject to various influences, both in terms of type and scale, before reaching our planet. Understanding this propagation requires a thorough comprehension of physical processes at all scales of application, from production in astrophysical sources, through interactions with the vast radiation and matter fields that fill the cosmos, to entering the solar system and ultimately the terrestrial atmosphere. The following sections provide an incomplete overview of aspects relevant to the treatment of this complex topic, as well as references for further study on the various related subjects.

### 2.1 Particle Physics

In our modern view, interactions and categories of elementary particles are most accurately described by a construct called the *Standard Model* (SM) of particle physics. Underlying this formalism is the mathematical framework known as *Quantum Field Theory* (QFT) that combines and generalizes properties of *Quantum Mechanics* (QM) and *Special Relativity* (SR) to produce a consistent description of microscopic reality. Excitations in the associated fields are referred to as quanta and manifest themselves as observable particles. Since this work is mostly concerned with the statistical behaviour of large quantities instead of individual particle probabilistics, a more detailed treatment is omitted, with the exception of certain phenomena important to the justification of some later assumptions.

#### 2.1.1 Fundamental Interactions

With the theory of *Quantum Electrodynamics* (QED) representing a quantized description of classical electromagnetism, the first successful QFT formulation had been realized. It governs all interactions between electrical charges that involve photons and predicts observations with extremely high precision. Each subsequent QFT is built on a foundation of QED achievements, such as *Quantum Chromodynamics* (QCD) for strong force interactions or the unified framework of *Electroweak Theory* (EWT) that includes both weak force and electromagnetic phenomena.

According to the SM formalism, these interactions are mediated by bosonic elementary particles, namely massless photons  $\gamma$  for electrodynamics or gluons  $g$  in case of the strong force sector, as well as massive  $Z$  and  $W^\pm$  bosons that carry the weak force. The only remaining fundamental interaction not contained in the SM is gravity, the effects of which are currently best described in terms of *General Relativity* (GR) as the continuous macroscopic curvature of spacetime due to the presence of large masses. Because elementary particles exclusively inhabit microscopic scales, their behavior is usually indistinguishable from that described by SR in flat space.

## 2.1.2 Hadrons &amp; Leptons

In addition to the previously listed gauge bosons, the SM further includes elementary fermions as the fundamental constituents of matter. These sort into the categories of quarks and leptons, both of which are further grouped by generations of quark or lepton pairs. Ordered according to their generation, there are flavors of up and down, charm and strange, as well as top and bottom for quarks, which are denoted by their initial letters. For charged leptons, there exist electrons  $e$  as well as the heavier muon  $\mu$  and tau  $\tau$  that each have an associated neutrino  $\nu$  with no electrical charge. This picture is completed by the corresponding antiquarks and antileptons for all particles mentioned above.

Apart from an electrical charge, quarks are also carriers of so-called color charges that couple to gluons, which, in contrast to the electrically neutral photons, carry colors themselves. Gluons are therefore capable of self-coupling, leading to higher binding energies with increasing distances. If removed far enough, this energy is eventually released by producing a quark pair, ultimately leading to the creation of bound color-neutral states.

These resulting composite particles are called hadrons, with the entire process being referred to as hadronization. This naturally leads to the concept of confinement, which states that quarks cannot exist as free particles, with the notable exception of very high energies. In these regimes, strong force coupling becomes small and allows quarks to behave almost freely, defining the term asymptotic freedom. A major consequence of this is that perturbative QCD calculations become possible, which would otherwise require complex approaches such as discrete QCD on a lattice or some type of effective field theory instead.

The quark model enables classification of hadrons based on their constituent numbers. Baryons are comprised of three quarks, all with different colors, while mesons contain two quarks with opposite color and anticolor charges. From spin addition, it is clear that baryons obey fermionic statistics, while mesons act as bosons. Exemplary quark contents are given for baryons such as protons  $p \equiv uud$  and neutrons  $n \equiv udd$  as well as pions  $\pi^- \equiv \bar{u}d$  and kaons  $K^- \equiv \bar{u}s$  in the case of mesons. The charmed hadrons considered in this work are the same mesons  $D^0 \equiv c\bar{u}$  as well as  $D^- \equiv \bar{c}d$  and  $D_s^- \equiv \bar{c}s$  with the baryon  $\Lambda_c^+ \equiv udc$  as in [15]. Specific decay channels will be listed directly in the relevant sections with [16] providing a broad overview.

Contrary to hadrons, leptons are fundamental particles. Due to being colorless, they are not affected by the strong force, meaning that all leptonic interactions obey EWT instead. Noting that photons only couple to electrical charges, this leaves neutrinos as the only particles which interact exclusively via the weak force. Here, one distinctive feature of the weak interaction should be mentioned as well, namely that of parity violation. This manifests by only coupling to left-handed components of particles and right-handed components of antiparticles. Handedness in this context refers to chirality, which for massless particles is equivalent to helicity or the sign of the spin projection onto the momentum vector. Because the SM assigns zero mass to neutrinos, this EWT property suppresses certain decay modes and enforces correlated polarizations for the involved leptons. Not included in the SM is the observation of neutrino oscillations, which are, however, not relevant for this work, since flavor differences are neglected.

## 2.1.3 Reference Frames

Depending on the application, energies in particle physics are either given as viewed from a suitable rest frame or independent from the choice of coordinates altogether. One particularly convenient formulation uses incoming and outgoing momenta  $p_1, p_2$  and  $p_3, p_4$  to define

$$s = (p_1 + p_2)^2 \quad t = (p_1 - p_3)^2 \quad u = (p_1 - p_4)^2$$

as the Mandelstam variables, which assign different channels to scattering processes based on the squared momentum carried by an exchanged mediating particle. Implied in this context is a Minkowski inner product, making the above quantities manifestly Lorentz invariant. When working with parametrizations defined for use in different subdisciplines, it regularly becomes necessary to convert from center of mass energies  $\sqrt{s}$  to the energy  $E$  of a projectile as viewed in the system of a stationary target, where vectors  $P = (E, \mathbf{P}c)$  and  $p = (mc^2, 0)$  represent the respective particle momenta. With  $\mathbf{P}$  denoting a classical projectile momentum and  $c$  the speed of light, rest masses  $M$  and  $m$  for projectile and target lead to a relation  $E^2 = \mathbf{P}^2 c^2 + M^2 c^4$  and

$$s = (P + p)^2 = (E + mc^2)^2 - \mathbf{P}^2 c^2 = 2Emc^2 + m^2 c^4 + M^2 c^4$$

for the invariant mass. The trailing terms in this equation become negligible at high energies, allowing for an approximation  $s = 2Emc^2$  in the target rest frame.

## 2.1.4 Particle Collisions

Another important aspect of particle physics is the description of collision processes. For this purpose, the concept of cross sections  $\sigma$  should be understood. In classical mechanics, these quantities are solely related to geometric properties of the objects involved, assuming they only interact upon impact. If longer range forces are included, the cross section represents a larger effective area that measures how the trajectory of an incoming projectile is influenced by the target. Additionally, differential cross sections with respect to some independent variable such as solid angle  $d\sigma/d\Omega$  or kinetic energy  $d\sigma/dE$  can be defined. Measurements of this quantity often reveal more information about the inner structure of targets. To distinguish these cases, one might refer to  $\sigma$  as the integrated cross section. Since QM is at the core of particle physics, cross sections in this context are instead interpreted as probability measures of fundamentally stochastic collision processes, such as the production of specific particles. It can also be useful to separate cross sections into elastic and inelastic components, especially for collisions in which particles lose energy, as is the case for cooling processes via scattering.

## 2.1.5 Cooling &amp; Decay

Consider an infinitesimally thin slice of a target medium with particle number density  $n$  and volume  $V = Sdx$  where  $S$  and  $dx$  measure surface area and thickness, respectively. Accordingly, there exist  $\tilde{N} = nV$  targets that each have effective interaction cross sections  $\tilde{\sigma}$  with a total coverage of  $\tilde{S} = \tilde{\sigma}\tilde{N}$  as viewed by an incident projectile.

The probability for dissipating these beam constituent then corresponds to the ratio  $\mathcal{P} = \tilde{S}/S$  of both areas or explicitly  $\mathcal{P} = \tilde{\sigma} n dx$  in case of  $dx$  as the covered distance. Expressing  $\tilde{\sigma} = \kappa \sigma$  in terms of the inelastic scattering cross section  $\sigma$  and a dimensionless factor  $\kappa$  called inelasticity, one can identify a length scale  $\lambda = (\kappa \sigma n)^{-1}$  as the mean free path between collisions. Multiplying with  $\kappa$  includes the ratio of remaining to initial energy, which is taken to be constant. From this follows a reduction in beam particles  $dN = -N \lambda^{-1} dx$  proportional to  $N$  as the total projectile count and  $\mathcal{P} = \lambda^{-1} dx$  for the reformulated probability which represents an ordinary differential equation of first order. The solution is found to follow an exponential law

$$N(x) = N_0 \exp\left(-\frac{x}{\lambda}\right)$$

where  $N$  particles remain over some distance  $x$  with  $N_0$  as the initial amount. Furthermore,

$$P(x) = 1 - \exp\left(-\frac{x}{\lambda}\right)$$

gives the probability of a particle having been scattered after travelling  $x$  length units. Similar steps for time instead of distance lead to the well known equation

$$N(t) = N_0 \exp\left(-\frac{t}{\tau}\right)$$

describing exponential decay. It commonly appears in the context of radioactive materials but also applies to hadrons and leptons or more generally any quantity which decreases at a rate proportional to itself. Analogous to the previous case, a particle decays with probability

$$P(t) = 1 - \exp\left(-\frac{t}{\tau}\right)$$

before a time  $t$  has passed and for  $\tau$  as the mean lifetime. Translating this from rest frame to laboratory coordinates defines the decay timescale  $t_{\text{dec}} = \tau \Gamma$  with a Lorentz factor  $\Gamma = E/m$  via projectile energy and invariant mass. This is equivalent to a characteristic decay length given by  $\lambda_{\text{dec}} = v t_{\text{dec}}$  where the velocity  $v = c$  can be set for highly relativistic particles. Additionally, mean free path and cooling distance  $\lambda_{\text{cool}} = (\kappa \sigma n)^{-1}$  refer to exchangeable concepts. Particles lose energy in every collision, which is the same as reducing temperature from a thermodynamics perspective. Dividing by the speed of light  $c$  translates this expression to  $t_{\text{cool}} = (\kappa \sigma n c)^{-1}$  as a cooling timescale. Analogously, the distance  $\lambda_{\text{dec}} = c \tau E/m$  can be rewritten as  $t_{\text{dec}} = \tau E/m$  in units of time. Substituting into the decay formula yields a cooling factor

$$C = 1 - \exp\left(-\frac{t_{\text{cool}}}{t_{\text{dec}}}\right) \quad (1)$$

that rescales spectra from direct production to account for decay processes taking place after collisional energy losses have occurred. This is an essential mechanism for the hypothesis that neutrinos from charm dominate pion and kaon contributions at high energies. While longer lived particles experience significant cooling thanks to time dilation, charmed hadrons decay promptly in comparison, resulting in a neutrino flux that directly traces the underlying hadronic spectrum. One requirement for the validity of (1) is that  $\lambda_{\text{dec}} \ll d$  holds, with  $d$  measuring the target field size to ensure decays occur exclusive inside this region.

## 2.2 Multimessenger Astronomy

An extension of traditional astronomy, the field of multimessenger astrophysics continues to thrive, providing valuable insights via the combination of complementary information carried by multiple messenger species. This section begins by expanding on physical concepts alluded to in chapter 1 before continuing to describe the available multimessenger candidates in some detail. For a more extensive overview, reference [1] should be consulted.

### 2.2.1 Magnetic Field Scales

Particles carrying electric charges  $Q = Ze$  and moving at velocity  $v$  orthogonal to a homogenous magnetic field  $B$  are acted on by the Lorentz force  $F = QvB$  that produces a gyrating motion. With the elementary charge  $e$  and  $\Gamma$  as a Lorentz factor, this must balance a relativistic centripetal force  $F = m\Gamma v^2/R$  on the resulting circular path with  $R$  giving the Larmor radius. Rearranging and identifying  $p = m\Gamma v$  leads to the solution

$$R = \frac{p}{QB}$$

on which the magnetic field extent  $D$  imposes  $R < D$  as a condition. In case of highly relativistic energies, one can substitute  $E = pc$  for the momentum to obtain an inequality  $E < QcBD$  from the above considerations. Realistic astrophysical magnetic fields are not ordered, instead following turbulences that travel through the plasma. To incorporate effects of moving scattering centers, a factor  $\beta$  proportional to the Alfvén velocity is included, giving

$$E < Ze\beta cBD \quad (2)$$

as the Hillas criterion. It is named after its description in [3] and connects source region sizes to the strength of prevailing magnetic fields, with observed cosmic ray energies as a constraint.

### 2.2.2 High Energy Cutoff

At very high energies, protons and nuclei interact with cosmic photons, which can be blueshifted up to extreme gamma regimes due to the Doppler effect. In these processes, the photon spin is absorbed to produce a delta resonance  $\Delta^+$  representing an excited proton state. This decays almost immediately to pairs of nucleons and pions, leading to  $p\gamma \rightarrow n\pi^+$  or  $p\gamma \rightarrow p\pi^0$  as probable reaction channels and resulting in a change of momentum for the produced particles. For the case of *Cosmic Microwave Background* (CMB) radiation, a close to perfect black body spectrum

$$\frac{dn}{d\epsilon}(\epsilon) = \frac{\epsilon^2}{\pi^2 \hbar^3 c^3 (e^{\epsilon/k_B T} - 1)}$$

is given by [17] as the number density. From a CMB temperature of  $T = 2.725$  K follows that significant photon numbers exist with energies around  $\epsilon = 10^{-12}$  GeV in the CMB rest frame.

To find a threshold in center of mass coordinates, the scalar product is used for

$$s = (p_p + p_\gamma)^2 = p_p^2 + p_\gamma^2 + 2p_p p_\gamma = m_p^2 c^4 + 2E\epsilon = M^2 c^4$$

where  $M = m_p + m_\pi$  and  $m_\gamma = 0$  as well as head on collisions have been assumed. The solution

$$E = \left( (m_p + m_\pi)^2 c^4 - m_p^2 c^4 \right) / (2\epsilon)$$

corresponds to roughly  $E = 10^{11}$  GeV for the so-called GZK cutoff. Energies that exceed this value when viewed from CMB coordinates lead to significant losses, making the universe opaque to such protons. If pair production  $\gamma \rightarrow e^+ e^-$  from proton interactions is considered, one finds

$$E = \left( (m_p + 2m_e)^2 c^4 - m_p^2 c^4 \right) / (2\epsilon)$$

or approximately  $E = 10^9$  GeV by setting  $M = m_p + 2m_e$  instead. For alpha particles and heavier nuclei, similar steps apply, while electrons are limited mainly through Compton downscattering. Additionally, there exist infrared and radio backgrounds as potentially relevant radiation fields. From the multiple competing mechanisms for energy losses and momentum isotropization, it is as a consequence extremely challenging to reliably interpret cosmic ray signals and practically impossible to reconstruct any information about specific sources.

### 2.2.3 Stochastic Acceleration

From the introduction of theoretical limits in sections 2.2.1 and 2.2.2 arises the question of how particles reach these high energy regimes in the first place. Due to its relatively simple derivation and surprising accuracy in the description of measured spectra, probabilistic collisions are often viewed as one of the most plausible mechanisms responsible for cosmic ray acceleration. The general case is described in [18] and supposes that for each collision, particles gain some energy proportional to a constant factor  $\eta$  and remain in the accelerating region with probability  $\varsigma$  on average. With initial conditions  $N_0$  for the number of particles and  $E_0$  as the mean energy, this results in  $N = N_0 \varsigma^k$  and  $E = E_0 \eta^k$  after  $k$  collisions. Using  $\ln x^k = k \ln x$  in

$$\frac{\ln(N/N_0)}{\ln(E/E_0)} = \frac{\ln(\varsigma)}{\ln(\eta)}$$

eliminates the exponent and by rearranging gives the relation

$$N = N_0 \left( \frac{E}{E_0} \right)^{\ln(\varsigma)/\ln(\eta)}$$

as a connection between energy and particle number. This incidentally follows a power law, which is an almost ubiquitous feature observed in cosmic ray physics.

One can interpret the above expression as an integrated spectrum, leading to

$$\frac{dN}{dE} = \frac{N_0}{E_0} \left( \frac{E}{E_0} \right)^\alpha$$

for the differential case with a characteristic spectral index

$$\alpha = \frac{\ln(\zeta)}{\ln(\eta)} - 1$$

constrained by  $\ln(\zeta)/\ln(\eta) < 0$  due to  $\zeta < 1$  and  $\eta > 1$  as implied per the definitions.

The basic case of *Diffusive Shock Acceleration* (DSA) considers shock fronts that propagate with velocity  $\beta = v/c$  in a fully ionized gas. Requiring momentum isotropization without significant energy losses on each side of the discontinuity results in  $\ln(\zeta)/\ln(\eta) = -1$  and a corresponding spectral dependence  $dN/dE \propto E^{-2}$  that is discussed by [18] as well. A slightly steeper index around  $\alpha = -2.5$  can be derived when nonlinear effects are accounted for.

Energy gain increasing linearly with  $\beta$  leads this mechanism to be categorized as Fermi type acceleration of first order, whereas the originally proposed formulation scales like  $\beta^2$  or as second order. Though shocks exceed the local speed of sound in the astrophysical medium, relativistic velocities are typically not achieved. Consequently, ratios of  $\beta \ll 1$  mean that lower order processes are much more efficient in reaching high particle energies.

#### 2.2.4 Cosmic Rays

Consisting of charged particles with strictly non-thermal origins, cosmic rays were first detected at high altitudes over a century ago. Since then, they have become a foundational piece in our understanding of astrophysical sources. Although their flux is dominated by protons, other nuclei as heavy as lead can contribute significant portions at high energies as well. Besides this baryonic component, there are also charged leptons adding to the total population. The exact ratios of these constituents contain information about source compositions, though any reconstruction of individual sources is likely impossible. This is a result of cosmic magnetic fields that scramble the directions of charged particles, leading to a largely isotropic distribution. At the same time, it is possible to constrain the size and field parameters of sources based on the energy of cosmic rays. For this purpose, section 2.2.1 provides the so-called Hillas condition. In combination with the characteristic broken power law spectrum, one concludes that most cosmic rays are of galactic origin, except for the highest energies, which require extragalactic sources [4, 6]. Beyond these ranges, there appears to be a sharp drop that agrees with 2.2.2 in case of a GZK cutoff. Both the spectral shape and the extreme energies are also consistent with a slightly modified DSA scenario as described in section 2.2.3 for cosmic rays [10].

### 2.2.5 Neutrinos

Their weakly interacting nature introduced by section 2.1.2 is precisely what makes neutrinos so interesting. After being produced, they penetrate most obstacles that would be completely opaque to any other messenger particle. This comes at the cost of being notoriously difficult to detect, of course, but opens up the opportunity of measuring otherwise inaccessible processes occurring deep inside astrophysical sources. Several different populations are either known or hypothesized, for example those of solar origin, neutrinos from core-collapse supernovae, stellar remnants or active galaxies that are treated in this work, as well as neutrinos from pion decay implied in section 2.2.2 after GZK photodissociation. Additionally, a primordial neutrino background is expected to exist with a history analogous to that of CMB formation, in which photons decoupled from matter during the recombination epoch. All of these astrophysical and cosmological prospects as well as the previously mentioned flavor oscillations make neutrino astronomy a promising field of research.

### 2.2.6 Photons

With visual observations being conducted for millenia, photonic signals undoubtedly form the most ancient form of astronomy. Advances in optical instrumentation have driven revolutionary progress over the last centuries, culminating in modern detectors unlocking wavelengths across the entire electromagnetic spectrum. Especially radio and gamma frequencies have the potential of contributing to multimessenger astrophysics, with radio waves having the ability to resolve regions obscured by dust, and gamma photons being closely associated with hadronic processes in a similar manner as neutrinos, for instance.

### 2.2.7 Gravitational Waves

Setting aside the possibility of gravitons as hypothetical mediating particles, gravitation seems to be a fundamentally continuous phenomenon, in the same way as classical theories like that of electromagnetism by Maxwell. According to Einstein, masses lead to distortions of spacetime that propagate at the speed of light. From this perspective, it readily follows that especially massive systems such as black hole or neutron star binaries emit gravitational waves through their movement. These excitations have an advantage compared to other messengers in that they experience essentially no absorption or scattering by matter. Alongside unimpeded insights into highly energetic merger events, there should also exist a gravitational background radiation containing information about the very early universe.



## 2.3 Astrophysical Sources

### 2.3.1 Magnetars

### 2.3.2 Pulsar Spindown

To explain observations of rapidly spinning neutron stars or pulsars, there has to exist some mechanism by which rotational energy is lost. Reference [19] gives a brief overview of possible radiation candidates such as gravitational quadrupolar or higher order electromagnetic moments. Because this work is limited in its scope and concerns itself with the acceleration of electric charges, a pure magnetic dipole approach will be adopted. For more compact and convenient notation, Gaussian units are used.

In an idealized view like [20] of stars as sharply bound and uniformly magnetized spheres, one finds  $\mu = R^3 B/2$  for the external magnetic moment. The parameters  $R$  and  $B$  measure stellar radius and polar magnetic flux density, respectively. In case of a rotating dipole in vacuo,

$$L = \frac{2\mu^2 \omega^4 \sin^2 \chi}{3c^3} \quad (3)$$

is the exact expression for radiant power derived in [21] as a standard reference, where a static angle  $\chi$  between the dipole field and rotational axis with angular frequency  $\omega$  is assumed. If instead a FFE limit is applied, variational calculations [22] and MHD simulations [23] indicate

$$L = \frac{\mu^2 \omega^4 (1 + \sin^2 \chi)}{c^3} \quad (4)$$

as an appropriate expression of the luminosity. This is likely somewhat more accurate than the previous result, as it has been shown by [24] that the surroundings of a neutron star cannot support a vacuum but must instead be filled with a plasma of charge carriers originating from instabilities at the surface. However, one immediately encounters the problem that  $\mathbf{E} \cdot \mathbf{B} = 0$  as a condition of force free magnetospheres prevents particle acceleration. As [25] and [26] discuss, this can be overcome by introducing deviations from such a global solution on local scales.

Spindown is described by an exponential energy decay  $E = E_0 \exp(-t/t_{\text{sd}})$  with a characteristic timescale  $t_{\text{sd}}$  and  $\dot{E} = -E/t_{\text{sd}}$  for the time derivative. Assuming all energy is stored in rotational form, the expressions  $E = I\omega^2/2$  and  $\dot{E} = I\omega\dot{\omega}$  are also valid with  $I$  as the neutron star moment of inertia. Equations (3) and (4) can be generalized as  $L = K\omega^4$  by using a coefficient  $K$  containing all other information except the dipolar  $\omega^4$  dependence. Identifying this with the energy loss via  $\dot{E} = -L$  and evaluating at  $t = 0$  yields  $L_0 = K\omega_0^4$  as well as  $\dot{E}_0 = -I\omega_0^2/2t_{\text{sd}}$  for an expression of  $t_{\text{sd}} = I/2K\omega_0^2$  as the spindown time. It further follows for any  $t$  that  $I\dot{\omega} = -K\omega^3$  must hold. This is a special case of the power law differential equation  $\dot{\omega} \propto \omega^n$  with  $n = 3$  being the braking index characteristic of a rotating magnetic dipole. Separating variables and integrating leads to finding  $\omega = \omega_0(1 + t/t_{\text{sd}})^{-1/2}$  as the time dependent frequency solution.

### 2.3.3 Active Galactic Nuclei

[\[27\]](#) [\[28\]](#) [\[29\]](#) [\[30\]](#)

### 2.3.4 Accretion Disks

Diffusely distributed material in orbit around a central massive object naturally produces disk like structures. This is a consequence of gravitational forces being compensated in the radial plane by rotational effects while matter is relatively free to collapse in the axial direction.

Compression and friction during inward spiraling heat the disk, thereby emitting intense thermal radiation. Settings where these phenomena likely occur are protoplanetary disks surrounding newly formed stars or accretion flows on a SMBH inside the galactic core region, among others. The following describes some related effects that are for example discussed by [18] as well.

### Hydrostatic Equilibrium

In Newtonian physics, the gravitational force at distance  $R$  from a mass  $M$  is given as

$$F = \frac{GMm}{R^2} \quad (5)$$

for particles with masses  $m$  orbiting the central object. The resulting acceleration is therefore of magnitude  $g = GM/R^2$  with vertical component  $\ddot{z} = g \sin(\vartheta)$  and  $\vartheta$  as the angular elevation. In case of  $\vartheta \ll \pi/2$  one can approximate  $\sin(\vartheta) = z/R$  to obtain

$$\ddot{z} = \frac{GMz}{R^3}$$

which is related to pressure  $p$  and density  $\rho$  via  $dp/dz = -\rho\ddot{z}$  for a stable equilibrium to exist. Using the speed of sound  $u^2 = dp/d\rho$  allows reformulation to

$$\frac{d\rho}{dz} = -\rho \frac{GMz}{u^2 R^3}$$

and  $\rho = \rho_0 \exp(-z^2/2h^2)$  after integration. According to this result, the disk density follows a centered Gaussian distribution with height parameter  $h$  defined via

$$h^2 = \frac{u^2 R^3}{GM}$$

as the standard deviation. Assuming  $M \gg m$  and circular Keplerian orbits, expression (5) equals the acting centripetal force, leading to  $v^2 = GM/R$  for the orbital velocity. One is then able to write  $h^2 = u^2 R^2/v^2$  and from the necessary condition  $h \ll R$  for the small angle approximation follows  $u \ll v$  or that orbital velocities must greatly exceed the speed of sound specific to the medium. This special case is called a thin disk and can further be extended to slim or even thick types of accretion structures.

### Luminosity Limit

Any luminous object radiating with spherical symmetry exerts a pressure

$$P = \frac{L}{4\pi R^2 c}$$

as a function of luminosity  $L$  at distance  $R$  from the source. Suppose a gaseous cloud containing particles with masses  $m$  falling towards the same bright central mass  $M$  due to gravitational attraction. During this process, each particle experiences  $F = \kappa m P$  as an opposing radiative force, where the opacity  $\kappa$  measures cross section per unit mass.

Balancing of (5) and rearranging leads to the Eddington limit

$$L = \frac{4\pi GMc}{\kappa} \quad (6)$$

on the luminosity beyond which additional matter is immediately blown away from the central object. By assuming that infalling material consists exclusively of ionized hydrogen, one can approximate  $\kappa = m/\sigma$  with proton mass  $m$  and Thomson cross section  $\sigma$  from electron scattering. Though originally applied in the context of stellar structures, this approach can also be used to describe accretion disks. If a compact object increases its mass with rate  $\dot{M}$  due to accreting matter, some of the corresponding gravitational potential may be converted to radiation. In terms of rest energy, a luminosity  $L = \eta \dot{M} c^2$  is obtained where  $\eta$  denotes the efficiency of this mechanism. With (6) one finds an analogous steady state limit

$$\dot{M} = \frac{4\pi GM}{\eta c \kappa}$$

for the accretion rate, enforced by continuous balancing of radiation pressure and gravitational forces as a natural feedback process.

## 3 Methods

### 3.1 Cross Sections

By defining an effective area perpendicular to the velocity vectors of projectiles and targets, cross sections measure the probability of collision processes in particle physics. Due to depending on the strength of an interaction, these quantities generally scale with energy. Distinct from the integrated case, differential cross sections are usually given with respect to some independent variable such as angle or momentum of the particle.

#### 3.1.1 Scattering

To model total cross sections in hadron proton scattering, this work uses the formula

$$\sigma_{hp} = H_h \ln^2(s/s_h) + P_h + R_h^1(s_h/s)^{\eta_1} + R_h^2(s_h/s)^{\eta_2} \quad (7)$$

as given in [31] for a universal analytic parametrization of the corresponding amplitudes.

All adjustable parameters are listed in table 1 together with relevant meson lifetimes for cooling. In this approach, the variable  $M$  relates to  $H = \pi(\hbar c/M)^2$  and  $s_h = (m_h + m_p + M)^2$  as an effective mass. Coefficients in (7) denote Heisenberg, Pomeranchuk and Regge terms which have some qualitative motivation, though the formula itself is primarily a quantitative result.

TABLE 1: Fits to the total inclusive scattering cross sections in hadron proton collisions. Parameters are taken from [31] with  $M = 2.121$  GeV for  $H = 0.272$  mb as the rate of growth. Both  $\eta_1 = 0.447$  and  $\eta_2 = 0.5486$  are dimensionless exponents. Decay times  $\tau_h$  and rest masses  $m_h$  can be found in the particle listings [16] where the latter are given in natural units.

$h$	$P_h / \text{mb}$	$R_h^1 / \text{mb}$	$R_h^2 / \text{mb}$	$\tau_h / \text{ns}$	$m_h / \text{GeV}$	$s_h / \text{GeV}^2$
$p$	34.41	13.07	7.39		0.938	15.98
$\pi$	18.75	9.56	1.767	26.03	0.140	10.23
$K$	16.36	4.29	3.408	12.38	0.494	12.62

Assuming a quasi universal ratio  $\mathcal{R}$  between elastic and total hadron cross sections, one obtains the inelastic cross section  $\sigma_{\text{inel}} = (1 - \mathcal{R})\sigma_{\text{tot}}$  from  $\sigma_{\text{el}} = \mathcal{R}\sigma_{\text{tot}}$  and  $\sigma_{\text{el}} + \sigma_{\text{inel}} = \sigma_{\text{tot}}$  as a unitarity condition. Provided in [32] is the model independent parametrization

$$\mathcal{R}(s) = \frac{\sigma_{\text{el}}(s)}{\sigma_{\text{tot}}(s)} = \mathcal{A} \tanh(\gamma_1 - \gamma_2 \ln(s) + \gamma_3 \ln^2(s)) \quad (8)$$

with a constant asymptote  $\mathcal{A}$  at very high energies. Coefficients are given in table 2 for different physical settings. Both equations (7) and (8) use units of  $\text{GeV}^2$  for the  $s$  variables.

TABLE 2: Almost model independent ratio of hadronic elastic and total scattering cross sections. Factors  $\gamma$  are taken from [32] for varying  $\mathcal{A}$  asymptotes.

$\mathcal{A}$	$\gamma_1$	$\gamma_2$	$\gamma_3$
1/2	0.466	0.0259	0.00177
1	0.2204	0.0111	0.00076

Reference [33] tests the asymptotic rise  $\sigma(s) \propto \ln^2(s)$  derived in [34] as a theoretical upper bound and concludes that it is somewhat exceeded. Additionally, a ratio  $\mathcal{A} = 1/3$  due to diffraction as opposed to the black disc limit  $\mathcal{A} = 1/2$  from optical theorem predictions is suggested. Because parameters are only available in the latter case, all calculations of  $\mathcal{R}$  use function (8) as defined by an asymptote  $\mathcal{A} = 1/2$  for this work. Data matching  $\mathcal{A} = 1/3$  then implies underestimated values of  $\sigma_{\text{inel}}(s)$  which should however not significantly influence the overall results.

### 3.1.2 Production

For charm quark production in proton air collisions, reference [35] gives

$$x_F \frac{d\sigma}{dx_F}(x_F, E_p) = ax_F^b(1 - x_F^m)^n$$

as the parametrized differential cross section with components

$$a = a_1 \ln(E_p) - a_2 \quad b = b_1 - b_2 \ln(E_p) \quad n = n_1 - n_2 \ln(E_p)$$

for which table 3 lists all necessary constants. Here proton energies  $E_p$  are defined as viewed by air nuclei at rest, while the Feynman scaling variable  $x_F = p_c/p_s$  specifies magnitude ratios of produced charm quark longitudinal momentum to all available momentum in center mass coordinates of the colliding particles. Application of appendix ?? shows that this approximately fulfills  $x_F = x_c$  where  $x_c = E_c/E_p$  in the relevant energy ranges.

TABLE 3: Parametrization of the inclusive charm quark production differential cross section. Coefficients are calculated from [35] to write  $E_p$  in units of GeV without needing redundant conversion steps. The number  $m = 1.2$  is a constant at all energies. For the application at hand, energy ranges beyond the given validity intervals might be used when mentioned in the text.

$E_p / \text{GeV}$	$a_1 / \mu\text{b}$	$a_2 / \mu\text{b}$	$b_1$	$b_2$	$n_1$	$n_2$
$10^4 - 10^8$	826	8411	0.197	0.016	8.486	0.107
$10^8 - 10^{11}$	403	2002	0.237	0.023	7.639	0.102

As in [36] it is assumed that the cross section scales linearly with nucleon number, yielding

$$\frac{d\sigma}{dx_c}(x_c, E_p) = A^{-1} \frac{d\sigma}{dx_F}(x_c, E_p)$$

for inclusive charm production in proton proton collisions. Approximating air as a gas mixture of roughly 75 % nitrogen and 25 % oxygen, one finds  $A = 14.5$  for this scaling. Translation of charm quarks to charmed hadrons is achieved with a folding integral

$$\frac{d\sigma}{dx_h}(x_h, E_p) = \int_{x_h}^1 dz z^{-1} \frac{d\sigma}{dx_c}(x_c, E_p) D_c^h(z) \quad (9)$$

where  $z = E_h/E_c$  and  $x_h = E_h/E_p$  as well as  $x_c = x_h/z$  are fractional energies. Limits for the integration follow from a basic inequality  $E_h \leq E_c \leq E_p$  to incorporate kinematic constraints. Furthermore, the probability of observing any final state  $h$  originating from a  $c$  quark is encoded in a *Fragmentation Function* (FF)  $D_c^h(z)$  dependent on the fraction of hadron to charm energy. Reference [37] addresses the connection between this concept and that of a *Parton Distribution Function* (PDF) among other things.

Where a PDF represents the probability density of finding a parton with given momentum in a color neutral particle, probabilities for color neutral states existing inside individual partons are given by the appropriate FF instead. The partons described here are either quarks or gluons, which can be free only asymptotically at high energies due to carrying color charges. In this limit, the running coupling of QCD is small enough for a power series expansion to be a sensible approach, leading to the definition of terms such as LO and NLO in reference to exponent order. There exist different factorization methods to separate these parts from the nonperturbative contributions contained in any PDF and FF for the confined constituents of hadrons. By fitting to existing data or perturbative results, models can extrapolate to low momentum fractions that have not yet been probed experimentally. A similar procedure has lead [38] to obtain

$$D_c^h(z) = \frac{N_h z(1-z)^2}{((1-z)^2 + \epsilon_h z)^2} \quad (10)$$

with parameters from  $e^+e^-$  data in table 4 as the charm hadron FF used throughout this work. It is important to note that such functions are invariant under charge conjugation, so that there is no differentiation between quark to particle or antiquark to antiparticle processes.

## 3.2 Spectral Distributions

Constructing spectra  $dN_h/dE_h$  from proton injection requires folding of  $F_{p \rightarrow h}$  as the hadronic distribution for a single  $pp$  interaction with the number of protons per energy interval given by the function  $dN_p/dE_p$  obtained from source specific modelling. An analogous approach can be applied to compute neutrino spectra  $dN_\nu/dE_\nu$  from  $dN_h/dE_h$  via distributions  $F_{h \rightarrow \nu}$  formulated according to the involved decay modes.

### 3.2.1 Charm

Spectral distributions for charmed hadron production are calculated according to [15] through

$$F_{p \rightarrow h}(E_h, E_p) = E_p^{-1} \sigma_{pp}^{-1}(E_p) \frac{d\sigma}{dx_h}(x_h, E_p)$$

with  $E_h = x_h E_p$  translating between variables. This formula can be understood as a normalization of (9) against proton energies and the inelastic  $pp$  cross section from appendix 3.1 to yield hadron numbers per unit energy.

To find neutrino spectra from charmed hadrons, the same approach as in [15] is used, which assumes an effective energy distribution approximated by three body decays for the semileptonic channel to a less massive pseudoscalar meson. By neglecting lepton masses, one obtains

$$\tilde{F}_{h \rightarrow \nu}(y) = D_h^{-1} \left( 6b_h a_h^2 - 4a_h^3 - 12\lambda_h^3 a_h + 12\lambda_h^2 y - 6b_h y^2 + 4y^3 + 12\lambda_h^2 \ln((1-y)/\lambda_h) \right)$$

as a distribution with  $y = E_\nu/E_h$  and  $F_{h \rightarrow \nu}(E_\nu, E_h) = \mathcal{F}_h \tilde{F}_{h \rightarrow \nu}(y)/E_h$  for conversion.

Hadron specific coefficients for this equation are defined with the parameter  $\lambda_h = \tilde{s}_h/m_h^2$  as

$$a_h = 1 - \lambda_h \quad b_h = 1 - 2\lambda_h \quad D_h = 1 - 8\lambda_h - 12\lambda_h^2 \ln(\lambda_h) + 8\lambda_h^3 - \lambda_h^4$$

where both  $s_h$  and  $m_h$  are listed in table 4 for all included charmed hadrons. The assumption of three body decays like  $D^+ \rightarrow \bar{K}^0 e^+ \nu_e$  can be justified by consulting [16] for information on the relevant particles and comparing branching ratios, which indicate that purely leptonic modes are strongly suppressed. Hadronic channels such as  $D^+ \rightarrow \pi^+ \pi^0$  or  $D^+ \rightarrow K^- \pi^+ \pi^+$  are either very improbable as well or occur at significant rates but do not contribute many high energy neutrinos due to pions and kaons being subject to further cooling before decaying to leptons. By the same logic, secondary muon decay is neglected when determining the neutrino spectrum.



TABLE 4: Coefficients for charm hadron production, cooling and decay to neutrinos.

All parameters  $\epsilon_h$  are taken from fits to LO QCD results via the FF as defined and described in [38] with normalizations  $N_h$  given by [15] to rescale the integration of (10) over  $[0,1]$  to approximately match the fractions  $f_h$  provided in [39] from measurements. Effective masses  $\sqrt{s_h}$  and branching fractions  $\mathcal{F}_h$  are determined by [40] and [41] from fitting decay rates. Mean lifetimes  $\tau_h$  and masses  $m_h$  are adopted from [16] in the particle listings. Mass type quantities use natural units.

$h$	$N_h$	$\epsilon_h$	$\tau_h / \text{fs}$	$\mathcal{F}_h$	$\sqrt{s_h} / \text{GeV}$	$m_h / \text{GeV}$
$D^0$	0.577	0.101	410	0.067	0.67	1.86
$D^+$	0.238	0.104	1033	0.176	0.63	1.87
$D_s^+$	0.0327	0.0322	501	0.065	0.84	1.97
$\Lambda_c^+$	0.0067	0.00418	203	0.045	1.27	2.29

### 3.2.2 Pions & Kaons

By parametrizing event generator results, a neutral pion production spectrum of the form

$$\tilde{F}_\pi(x_\pi, E_p) = 4\alpha B x_\pi^{\alpha-1} \left( \frac{1 - x_\pi^\alpha}{1 + r x_\pi^\alpha (1 - x_\pi^\alpha)} \right)^4 \left( (1 - x_\pi^\alpha)^{-1} + \frac{r(1 - 2x_\pi^\alpha)}{1 + r x_\pi^\alpha (1 - x_\pi^\alpha)} \right) \left( 1 - \frac{m_\pi}{x_\pi E_p} \right)^{1/2}$$

is found in [42] with  $m_\pi = 0.135 \text{ GeV}$  [16] translated to natural units and parameters

$$B = \tilde{B} + C \quad \alpha = \frac{\tilde{\alpha}}{\sqrt{C}} \quad r = \frac{\tilde{r}}{\sqrt{C}}$$

where a low energy cutoff is enforced via  $E_\pi = x_\pi E_p$  in the mass term. From

$$C = c_1 - c_2 \ln(E_p) + c_3 \ln^2(E_p)$$

results a dependence on projectile energy for the shape of this distribution.

Coefficients are specified in table 5 and recalculated for  $E_p$  in GeV instead of TeV units. Under the assumption of a  $\pi^0$  cross section approximately equal to the  $\pi^\pm$  average and with identical spectra for charged pions, it follows that  $F_\pi = \tilde{F}_\pi/E_p$  should describe pion production regardless of charge reasonably well for the purpose of this work.

TABLE 5: Parametrized spectral distribution for neutral pion production. Factors are taken from [42] and converted to write  $E_p$  in units of GeV for  $c_k$  coefficients.

$\tilde{B}$	$\tilde{\alpha}$	$\tilde{r}$	$c_1$	$c_2$	$c_3$
0.25	0.98	2.6	1.515	0.206	0.075

For a convenient formulation of kaon production, references [43] and [44] indicate a constant ratio  $\pi/K$  at moderately high energies. Similar fractions are retrieved from multiplicities given in [45] and lead to  $F_K/F_\pi = 0.12$  as a simplifying assumption, the validity of which cannot be guaranteed for the application at hand. Calculations of kaon spectra still employ this approach but are subject to considerable reservations as a result.

Decays of pions and kaons to neutrinos are approximated via the  $h \rightarrow \mu^+ \nu_\mu$  two body channel with branching fractions of  $\mathcal{F}_\pi = 99.99\%$  and  $\mathcal{F}_K = 63.56\%$  given in the [16] particle listings. By decaying, muons produced in these processes can significantly impact the neutrino spectrum. Results from [15] suggest that this is particularly relevant for pions. Muonic three body decays of type  $\mu^- \rightarrow e^- \bar{\nu}_e \nu_\mu$  as well as cooling factors depend on the polarization of participating leptons due to the nature of weak force coupling. This complicates computations and is thus omitted in service of restricting the present work to a manageable scope, though it should be remembered as an important caveat for the final results.

The remaining two body decays of ultrarelativistic hadrons  $h$  to leptons  $l$  obey a distribution

$$F_{h \rightarrow l}(E_l, E_h) = \mathcal{F}_h E_h^{-1} (1 - \lambda_h)^{-1}$$

with  $m_\nu = 0$  and  $m_\mu = 0.106$  GeV [16] as well as  $\lambda_h = m_\mu^2/m_h^2$  as a parameter. This formula is the same whether  $l = \nu$  or  $l = \mu$  because there is one muon for each neutrino. In addition, kinematic considerations lead to  $E_\mu/E_h > \lambda_h$  and  $E_\nu/E_h < 1 - \lambda_h$  for integral bounds  $E_\mu < E_h < E_\mu/\lambda_h$  in the case of muons or  $E_\nu/(1 - \lambda_h) < E_h < E_p$  when considering neutrinos.

### 3.3 Implementation

In order to calculate neutrino spectra from hadronic distributions, several integrals have to be computed. Discretizing this task allows the general case

$$F(x, y) = \int_{z_-}^{z_+} dz G(x, z) H(z, y)$$

to be rewritten as a Riemann sum. Assuming  $G$  and  $H$  are integrable over a given interval,

$$F_{ij} = \sum_k D_{kk} G_{ik} H_{kj}$$

converges to the exact solution for sufficiently small steps. Transforming variables

$$x \rightarrow x_i \quad y \rightarrow y_j \quad z \rightarrow z_k$$

and defining  $D_{kk} = z_{k+1} - z_k$  leads to the above notation. It is easily shown how this expression in terms of indices translates to the product of corresponding matrices

$$\mathbf{F} = \mathbf{G} \mathbf{D} \mathbf{H}$$

as an equivalent formulation. Here the output  $\mathbf{F} \in \mathbb{R}^{m \times n}$  is obtained from the inputs  $\mathbf{G} \in \mathbb{R}^{m \times l}$  and  $\mathbf{H} \in \mathbb{R}^{l \times n}$  as well as the square matrix  $\mathbf{D} \in \mathbb{R}^{l \times l}$  that encodes all step sizes on its diagonal. These results enable a quick and efficient implementation of the required calculations as program code, where array arithmetic operations can greatly increase execution speed.<sup>1</sup>

<sup>1</sup> In service of reproducibility, all implementations can be viewed in [this](#) repository.

## 4 Results

## 5 Conclusion & Outlook

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