习题7.1

1. (2) $\sin x\sin 3x=\frac{1}{2}[\cos(x-3x)-\cos(x+3x)]$

(3) $\cos^2 x=\frac{1}{2}(1+\cos 2x)$

(4) $x^4+3=x^2(x^2+1)-(x^2+1)+4$

(5) $\cot^2 x=\csc^2 x-1$

(6) $1=\sec^2 x-\tan^2 x$, $\cos^2 x=1/\sec^2 x=\frac{1}{2}(1+\cos 2x)$

(8) $\dfrac{1-x+x^2}{x+x^3}=\dfrac{(1+x^2)-x}{x(1+x^2)}$

(9) $\sqrt{x\sqrt{x}}=(x\cdot x^{\frac{1}{2}})^{\frac{1}{2}}=x^{\frac{3}{4}}$

(10) $5^x e^x=(5e)^x$.

(11) $(2^x+3^{-x})^2=(2^x)^2+2\cdot 2^x\cdot 3^{-x}+(3^{-x})^2=4^x+2\cdot(\frac{2}{3})^x+(\frac{1}{9})^x$

(12) $(\sin x-\cos x)^2=1-2\sin x\cos x(=1-\sin 2x)$

2. 即 $f'(x)=e^{3x}$ $\therefore f(x)=\int e^{3x}dx=\frac{1}{3}e^{3x}+C$

代入 $x=0$, $f(0)=1\Rightarrow C=\frac{2}{3}$,

3. $\int f(x)dx=\begin{cases}\int e^x dx=e^x+C_1, & x<0\\ \int(2x+1)dx=x^2+x+C_2, & x>0\end{cases}$

由于 $(\int f(x)dx)'=f(x)$, 故 $\int f(x)dx$ 在 $x=0$ 处连续,

故 $1+C_1=C_2$, 从而可取

$\int f(x)dx=\begin{cases}e^x, & x<0\\ x^2+x+1, & x>0.\end{cases}$

习题 7.2

1. (1) $\tan x = \sin x / \cos x$，$\sin x dx = -d\cos x$.

(3) $\cot x = \cos x / \sin x$，$\cos x dx = d\sin x$

(2) 法一类似P7例6；(4) 类似.

法二：$\int \frac{1}{\sin x}dx = \int \frac{\sin^2\frac{x}{2}+\cos^2\frac{x}{2}}{2\sin\frac{x}{2}\cos\frac{x}{2}}dx = \int \frac{\cos\frac{x}{2}dx}{2\sin\frac{x}{2}} + \int \frac{\sin\frac{x}{2}}{2\cos\frac{x}{2}}dx$

$= +\ln|\sin\frac{x}{2}| - \ln|\cos\frac{x}{2}| + C = \ln|\tan\frac{x}{2}| + C$

(5) $\sin 2x = 2\sin x\cos x$.　　(6) $dx = \frac{1}{4}d(4x+5)$

(7) $\sin^2 x = \frac{1}{2}(1-\cos 2x)$；　(8) $\frac{1}{x}dx = d(\ln x + 3)$

(9)(11) $x^2 dx = \frac{1}{3}d(x^3+C)$；　(12) $x dx = \frac{1}{2}dx^2$

(10) 注意到 $1-\cos x = 2\sin^2\frac{x}{2}$，$\csc^2\frac{x}{2} = \frac{1}{\sin^2\frac{x}{2}}$.

(13) $\sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha+\beta)+\sin(\alpha-\beta)]$.

(14) 利用 $1 = \sin^2 x + \cos^2 x$.

(15) $\frac{1}{\sqrt{x}}dx = 2d(\sqrt{x}+1)$；　(16) $I = \int \frac{e^{-x}dx}{e^{-x}\sqrt{e^{2x}-1}} = \int \frac{-de^{-x}}{\sqrt{1-(e^{-x})^2}}$

2. (1) 令 $t^6 = x-1$，则 $I = \int \frac{t^3}{1+t^2}\cdot 6t^5 dt$

再利用 $t^8 = t^6(1+t^2) - t^2(1+t^2) + (t^2+1) - 1$

(2) 令 $x = a\tan t$，则 $I = \int \frac{1}{|a^3\sec^3 t|}\cdot a\sec^2 t\, dt = \cdots$

$(1+\tan^2 t = \sec^2 t)$

(3) $I = \int \frac{d(x+1)}{\sqrt{(x+1)^2-(\sqrt{2})^2}}$　　(6) $I = \int \sqrt{2^2-(x+1)^2}\,dx$ (公式法或三角代换)

习题 7.2

2.(4) 原式 $= \int \frac{dx}{|a|\sqrt{1-(\frac{x}{|a|})^2}} = \int \frac{d(\frac{x}{|a|})}{\sqrt{1-(\frac{x}{|a|})^2}} = \arcsin\frac{x}{|a|} + C$

(5) 原式 $\overset{\sqrt{x}=t}{=\!=} \int \frac{\cos t}{t}\cdot 2t\,dt = 2\int \cos t\,dt = 2\sin\sqrt{x} + C$

(7) 原式 $\overset{t=x-1}{=\!=} \int \frac{(t+1)^3+5}{t^2}dt = \int \frac{t^3+3t^2+3t+6}{t^2}dt$

$= \int (t+3+3t^{-1}+6t^{-2})dt = \cdots = \frac{1}{2}(x-1)^2 + \cdots$

(8) 原式 $\overset{t=\sqrt{x-1}}{=\!=} \int \frac{t+1}{t-1}\cdot 2t\,dt = \int \frac{2t(t-1)+4(t-1)+4}{t-1}dt$

$= t^2 + 4t + 4\ln|t-1| + C = \cdots$

3.(1) 原式 $= \int x\,d(-\cos x) = -x\cos x + \int \cos x\,dx = \sin x - x\cos x + C$

(2) 原式 $= \frac{1}{3}\int \arctan x\,dx^3 = \frac{1}{3}x^3\arctan x - \frac{1}{3}\int \frac{x^3}{1+x^2}dx$

$= \frac{1}{3}x^3\arctan x - \frac{1}{3}\int \frac{x(x^2+1)-x}{1+x^2}dx$

$= \frac{1}{3}x^3\arctan x - \frac{1}{3}\int x\,dx - \frac{1}{6}\int \frac{d(1+x^2)}{1+x^2} = \cdots$

(3) 原式 $= \frac{1}{2}\int \ln^2 x\,dx^2 = \frac{1}{2}x^2\ln^2 x - \frac{1}{2}\int x^2\cdot 2\ln x\cdot \frac{1}{x}dx$

$= \frac{1}{2}x^2\ln^2 x - \frac{1}{2}\int \ln x\,dx^2 = \cdots$

(4) 原式 $= 2\int \arcsin x\,d\sqrt{x+1} = 2\sqrt{x+1}\arcsin x - 2\int \frac{\sqrt{x+1}}{\sqrt{1-x^2}}dx$

$= 2\sqrt{x+1}\arcsin x - 2\int \frac{1}{\sqrt{1-x}}dx$

$= 2\sqrt{x+1}\arcsin x + 4\sqrt{1-x} + C$

习题7.2

3.(5) $I=\frac{1}{3}\int e^{2x}d\sin 3x=\frac{1}{3}e^{2x}\sin 3x-\frac{2}{3}\int e^{2x}\sin 3x\,dx$

$=\frac{1}{3}e^{2x}\sin 3x+\frac{2}{9}\int e^{2x}d\cos 3x=\frac{1}{3}e^{2x}(\sin 3x+\frac{2}{3}\cos 3x)+I$

$\Rightarrow I=\frac{1}{13}e^{2x}(2\cos 3x+3\sin 3x)+C$

(6) 原式 $=I=x\sin(\ln x)-\int x\cdot\cos(\ln x)\cdot\frac{1}{x}dx$

$=x\sin(\ln x)-\int\cos(\ln x)dx=x[\sin(\ln x)-\cos(\ln x)]-I$

$\therefore I=\frac{x}{2}[\sin(\ln x)-\cos(\ln x)]+C$

(7) 原式 $=\frac{1}{2}\int x(1+\cos 2x)dx=\cdots=\frac{1}{4}[x^2+x\sin 2x+\frac{1}{2}\cos 2x]+C$

(8) 原式 $\xlongequal{t=\sqrt{x}}\int t\cdot(\ln t^2)^2\cdot 2t\,dt=8\int t^2\ln^2 t\,dt$

$=\frac{8}{3}\int\ln^2 t\,d(t^3)=\cdots$

(9) 原式 $\xlongequal{t=\sqrt{3-2x}}\int e^t\cdot(-t)dt=\cdots$

(10) 原式 $\xlongequal{t=\sqrt[3]{x}}\int\cos t\cdot 3t^2dt=\cdots$

4. 类似例16，分部积分可得

5. 代入4中的公式可得.

习题 7.3

1. (1) 利用 $x^5 = x^4(1+x) - x^3(1+x) + x^2(1+x) - x(1+x) + (1+x) - 1$

(2) 法一：令 $R(x) = \frac{A}{x-1} + \frac{Bx+C}{x^2+1} \Rightarrow A(x^2+1)+(Bx+C)(x-1)=1$

法二：$1=(1-x)+x$（用处不大）

(3) 法一：$R(x)=A/(x+2)+B/(x+2)^2+C/(x+2)^3+D/(x+2)^4$.(×)

法二：平移代换，令 $t=x+2$.(√)

(4) 注意到分母的分解. 令 $1+x^4=(x^2+ax+1)(x^2+bx+1)$

$\Rightarrow a=\sqrt{2}, b=-\sqrt{2}$. 此时, $x^2+\sqrt{2}x+1>0$, $x^2-\sqrt{2}x+1>0$.

$\frac{1+x}{1+x^4} = \frac{1-x}{(x^2+\sqrt{2}x+1)(x^2-\sqrt{2}x+1)}$, 作分解后, 出现形如

$$I_1=\int\frac{b\,dx}{x^2+ax+1}=\int\frac{\frac{b}{2}d(x+\frac{a}{2})}{(x+\frac{a}{2})^2+(1-\frac{a^2}{4})}$$ （对比 $\int\frac{x\,dx}{x^2+c}$）

$$I_2=\int\frac{x\,dx}{x^2+ax+1}=\int\frac{\frac{1}{2}d(x^2+ax+1)-\frac{a}{2}dx}{x^2+ax+1}=\cdots$$

(5) $$I_1=\int\frac{x\,dx}{(1+x^2)^2}=\frac{1}{2}\int\frac{d(1+x^2)}{(1+x^2)^2}$$

$$I_2=\int\frac{1}{(1+x^2)^2}dx=\int\frac{(1+x^2)-x^2}{(1+x^2)^2}dx=\int\frac{1}{1+x^2}dx-\int\frac{x^2}{(1+x^2)^2}dx$$

$$I_3=\int\frac{x^2}{(1+x^2)^2}dx=\frac{1}{2}\int x\,d\frac{1}{1+x^2}=\frac{1}{2}\frac{x}{1+x^2}-\frac{1}{2}\int\frac{1}{1+x^2}dx$$

（对 I_3，也可用三角代换，即令 $x=\tan t$）

(6) $1-x^3=-(x-1)(x^2+x+1)$; (7) $R(x)=\frac{1}{x^3(1+x^2)}=\frac{(1+x^2)-x^2}{x^3(1+x^2)}$ $=\frac{1}{x^3}-\frac{1}{x(1+x^2)}$ 再分解.

(8) $\frac{x^2}{1-x^4}=\frac{x^2}{(1-x^2)(1+x^2)}=\frac{1}{2}\left(\frac{1}{1-x^2}-\frac{1}{1+x^2}\right)$

$=\cdots$

习题 7.3

2.(1) 令 $t=\tan\frac{x}{2}$，则 $I=\int \frac{1}{3+\frac{10t}{1+t^2}}\cdot\frac{2t}{1+t^2}dt=\frac{1}{4}\int\left(\frac{3}{3t+1}-\frac{1}{t+3}\right)dt=\cdots$

(2) $I=\int\frac{1}{\frac{1}{\cos^2x}+1}\,\frac{1}{\cos^2x}dx=\int\frac{\sec^2x\,dx}{\sec^2x+1}=\int\frac{d\tan x}{\tan^2x+2}=\frac{1}{\sqrt2}\arctan\frac{\tan x}{\sqrt2}+C$

(3) 注意到 $I=\int\frac{\cos x}{\cos x-\sin x}dx$，且 $\cos x=\frac{\cos x-\sin x}{2}+\frac{\cos x+\sin x}{2}$.

$I=\int\frac{1}{2}dx-\frac{1}{2}\int\frac{d(\cos x-\sin x)}{\cos x-\sin x}=\frac{1}{2}x-\ln|\cos x-\sin x|+C$

(法二：令 $t=\tan\frac{x}{2}$，…

(4) 注意到 $\sin x\cos x=\frac{1}{2}[(\sin x+\cos x)^2-1]$，故

$I=\frac{1}{2}\int(\sin x+\cos x)dx-\frac{1}{2}\int\frac{1}{\sin x+\cos x}dx$ 利用 $t=\tan\frac{x}{2}$ 处理.

3.(1) 令 $\sqrt{x^2+x+2}=xt+\sqrt2$，则 $x=\frac{2\sqrt2 t}{1-t^2}$，$dx=\cdots$

法二：$I=\int\frac{1}{x^2\sqrt{1+\frac{1}{x}+(\frac{1}{x})^2\cdot 2}}=-\frac{1}{\sqrt2}\int\frac{d(\frac{1}{x}+\frac{1}{4})}{\sqrt{(\frac{1}{x}+\frac{1}{4})^2+(\frac{\sqrt7}{4})^2}}=\cdots$

(2) $I=\int\frac{1}{\sqrt{(x-\frac{1}{2})^2-(\frac{1}{2})^2}}d(x-\frac{1}{2})=\cdots$

代入附录中对应公式.

(3) $I=\int\frac{1+t^3}{1-t^3}\cdot t\cdot d\,\frac{1-t^3}{1+t^3}=\cdots$，其中 $t=\sqrt[3]{\frac{1-x}{1+x}}$

(4) $I=\int\sqrt{(x-\frac{1}{2})^2+(\frac{\sqrt3}{2})^2}\,d(x-\frac{1}{2})=\cdots$

复习题七

1. $\int x f'(x)\,dx = xf(x) - \int f(x)\,dx = xf(x) + (1+\sin x)\ln x + C$

2. $xf'(x^2)=1 \Leftrightarrow [f(x^2)]' = \frac{1}{2} \Leftrightarrow \int [f(x^2)]'\,dx = \int \frac{1}{2}\,dx \Leftrightarrow f(x^2) = \frac{1}{2}x + C$

从而 $f(x) = \frac{1}{2}\sqrt{x} + C$，又 $\because f(1)=1 \Rightarrow C=\frac{1}{2}$，故 $f(x) = \frac{\sqrt{x}}{2} + \frac{1}{2}$

3. $\because [f(x)e^{-x}+C]' = [f'(x)-f(x)]e^{-x}$，$\therefore \int [f'(x)-f(x)]e^{-x}\,dx = f(x)e^{-x}+C$.

4. (1) $\because \int g(x)\,dx = \int x f(x)\,dx = \int x F'(x)\,dx = xF(x) - \int F(x)\,dx$

注意到 $F'(x)$ 存在，故 $F(x)$ 连续，从而上式第二个积分存在，即证。

(2) $\int h(x)\,dx = \int \frac{F'(x)}{F(x)}\,dx = \ln|F(x)| + C$.

5. (1) 利用 $\cos^4 x = \left(\frac{1+\cos 2t}{2}\right)^2 = \frac{1}{4} + \cos 2t + \frac{1}{4}\cos^2 2t = \frac{1}{4} + \cos x + \frac{1+\cos 4t}{8}$

(2) 令 $x=t^6$，则 $I = \int \frac{t^6+t^3-2}{t^2-1}\cdot 6t^5\,dt \cdots$

(3) $I = \int \frac{\frac{1}{\cos^2 x}dx}{\sqrt{\sin x\frac{\cos^3 x}{\cos^4 x}}} = \int \frac{1}{\sqrt{\tan x}}\,d\tan x = 2\sqrt{\tan x} + C$

(4) $I = \int \ln\cos x \cdot \underline{\csc^2 x\,dx} = -\cot x \ln\cos x + \int \cot x \cdot \frac{\sin x}{\cos x}\,dx = \cdots$

(5) 令 $t=\sqrt{x}$，则 $x=t^2$，$I = \int 2t\sin t\,dt = \cdots$

(6) 令 $t = 1-\sqrt[3]{x}$，则 $x=(t+1)^3$，$I = \int \frac{1}{t}\cdot 3(t+1)^2\,dt = \cdots$

(7) $\frac{1}{\sin^4 x} = \csc^4 x = \csc^2 x\cdot\csc^2 x = (1+\cot^2 x)\cdot\csc^2 x$，

而 $\csc^2 x\,dx = -d\cot x$.

(8) $I = \int e^{-x}\frac{(1+x^2)+2x}{(1+x^2)^2}\,dx = \underbrace{\int \frac{e^{-x}}{1+x^2}\,dx}_{\text{(保留)}} + \int e^{-x}\cdot\left(-d\frac{1}{1+x^2}\right) = -e^{-x}\frac{1}{1+x^2} + C$

复习题七

5.(9) $x^2+2x=(x+1)^2-1$

(10) $x^5=x^3(1+x^2)-x(1+x^2)+x$

(11) $t=x+1$ (平移代换.

(12) 易知 $x^3-3x^2+4=(x+1)(x^2+ax+4)$

$\Rightarrow a=-4$. 故

$x^3-3x^2+4=(x+1)(x-2)^2$. …

(13) 令 $t=\sqrt{x}$, 则

$I=\int 2t\arctan t\,dt=\cdots$

(14) $I=\int \arccos x\, d\frac{1}{x}=\frac{1}{x}\arccos x+\int\frac{1}{x}\frac{1}{\sqrt{1-x^2}}dx$

$=\frac{1}{x}\arccos x+\int\frac{-d(\frac{1}{x})}{\sqrt{(\frac{1}{x})^2-1}}=\cdots$

(15) $[x\ln\ln x]'=\ln\ln x+\frac{1}{\ln x}$.

(16) $I=\int\sec x\, d\tan x=\sec x\tan x-\int\tan^2 x\sec x\, dx$

$=\sec x\tan x-\int[\sec^2 x-1]\sec x\, dx=\sec x\tan x+\int\sec x\, dx-I$

(17) $I=\int\frac{1}{2\sin x\cos^2 x}dx=\int\frac{\sin^2 x+\cos^2 x}{2\sin x\cos^2 x}dx$

$=\int\frac{\sin x\, dx}{2\cos^2 x}+\int\frac{dx}{2\sin x}$ (用代"1"$=\cos^2\frac{x}{2}+\sin^2\frac{x}{2}$)

(18) $1=(1+x^{10})-x^{10}$

(19) $I=\int\frac{(1+x)e^x dx}{xe^x(1+xe^x)}=\int\frac{d(xe^x)}{xe^x(1+xe^x)}\overset{u=xe^x}{=\!=}\int(\frac{1}{u}-\frac{1}{1+u})du=\cdots$

(20) $I=\int\ln(x+\sqrt{x^2+1})\, d\sqrt{x^2+1}$

(21) $\tan^3 x/\sqrt{\cos x}=\sin^3 x/\cos^3 x\sqrt{\cos x}=\frac{(1-\cos^2 x)\sin x}{\cos^{7/2} x}$ (凑微分)

(22) $I_n=\int\tan^{n-2}x(\sec^2 x-1)dx=\frac{1}{n-1}\tan^{n-1}x-I_{n-2}$ ($n\geq 2$)

6. $0=f'(1)=a+b+c$; $0=f''(-1)=-2a+b$.

$f(x)=\int f'(x)dx=\frac{1}{3}ax^3+\frac{1}{2}bx^2+cx+C_1$, $f(1)=16$

且 $f(-1)=0$