数学分析 3 习题参考

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目录

第 13 章 多元函数及其微分学

习题 13.1 平面中的点集

1. 由 Cauchy-Schwarz 不定式 $\boldsymbol{r}_1 \cdot \boldsymbol{r}_2 \leqslant |\boldsymbol{r}_1 \cdot \boldsymbol{r}_2| \leqslant \|\boldsymbol{r}_1\| \cdot \|\boldsymbol{r}_2\|$, 得

$$2\boldsymbol{r}_1 \cdot \boldsymbol{r}_2 \leqslant 2\|\boldsymbol{r}_1\| \cdot \|\boldsymbol{r}_2\|$$
$$\Rightarrow \boldsymbol{r}_1 \cdot \boldsymbol{r}_1 + 2\boldsymbol{r}_1 \cdot \boldsymbol{r}_2 + \boldsymbol{r}_2 \cdot \boldsymbol{r}_2 \leqslant \|\boldsymbol{r}_1\|^2 + 2\|\boldsymbol{r}_1\| \cdot \|\boldsymbol{r}_2\| + \|\boldsymbol{r}_2\|^2$$
$$\Rightarrow (\boldsymbol{r}_1 + \boldsymbol{r}_2) \cdot (\boldsymbol{r}_1 + \boldsymbol{r}_2) \leqslant (\|\boldsymbol{r}_1\| + \|\boldsymbol{r}_2\|)^2$$
$$\Rightarrow (\|\boldsymbol{r}_1 + \boldsymbol{r}_2\|)^2 \leqslant (\|\boldsymbol{r}_1\| + \|\boldsymbol{r}_2\|)^2,$$

两端开方即得 $\|\boldsymbol{r}_1 + \boldsymbol{r}_2\| \leqslant \|\boldsymbol{r}_1\| + \|\boldsymbol{r}_2\|$. 记 $P_1 - P_3 = \boldsymbol{r}_1, P_3 - P_2 = \boldsymbol{r}_2$, 则 $P_1 - P_2 = \boldsymbol{r}_1 + \boldsymbol{r}_2$, 代入上式即得 $\|P_1 - P_2\| \leqslant \|P_1 - P_3\| + \|P_3 - P_2\|$. □

2. ⇒: E 是有界集, 则存在 $R > 0$, 使得 $E \subset U(O; R)$, 因此 $d(E) < 2R < +\infty$;
⇐: 设 $d(E) < +\infty$, 若 $E = \varnothing$, 显然 E 有界; 若 $E \neq \varnothing$, 任取 $P_0 \in E$, 则 $E \subset U(P_0; d(E))$, 即 E 有界. □

4. 不妨设 $U(P_0) = B_\varepsilon(P_0)$ 是圆邻域. $\forall P \in B_\varepsilon(P_0)$, 取 $\delta = \frac{\varepsilon - \|P - P_0\|}{2}$, 则 $B_\delta(P) \subset B_\varepsilon(P_0)$, 从而 P 是 $U(P_0)$ 的内点, 即 $U(P_0)$ 是开集. □

5. 见课后习题解答.

6. (1) 仅证明开集的余集是闭集. 设 A 是开集, $B = \mathbb{R}^2 \backslash A$ 是 A 的余集. 要证 B 是闭集, 只需证明 B 的余集不含 B 的聚点, 即要证明: $\forall P \in A$, P 不是 B 的聚点. 事实上, P 是 A 的内点, 所以 $\exists U(P) \subset A$, 从而 $U(P) \cap B = \varnothing$, 因而 P 不是 B 的聚点.

(2) 仅证明有限个开集的交是开集. 设 $A_i, i = 1, 2, ..., k$ 是有限的 k 个开集, $A = \bigcap_{i=1}^k A_i$. 若 $A = \varnothing$, 则 A 是开集. 设 $A \neq \varnothing$, 对 $\forall P \in A$, 有 $P \in A_i, i = 1, 2, ..., k$. 因为 $A_i, i = 1, 2, ..., k$ 是开集, $\exists \delta_i > 0$, 使得 $U(P; \delta_i) \subset A_i, i = 1, 2, ..., k$. 取 $\delta = \min\{\delta_i\}$, 则 $U(P; \delta) \subset A_i, i = 1, 2, ..., k$, 即 $U(P; \delta) \subset \bigcap_{i=1}^k A_i$, 亦即 P 是 A 的内点, 从而 A 是开集. □

7. 对 $\forall(x, y) \in A \times B$, 有 $x \in A$ 且 $y \in B$. 因为 A, B 都是 $\mathbb{R}$ 中的开集, 所以存在 $\varepsilon > 0$ 和 $\delta > 0$, 使得 $U(x; \varepsilon) \subset A$ 且 $U(y; \delta) \subset B$, 因而 $U(x; \varepsilon) \times U(y; \delta) \subset A \times B$. 取 $\eta = \min\{\varepsilon, \delta\}$, 则 $U((x, y); \eta) \subset A \times B$, 从而 (x, y) 是 $A \times B$ 的内点. 由 (x, y) 的任意性知 $A \times B$ 是开集. □

8. 设 $U((x_0, y_0); \varepsilon) = U(x_0; \varepsilon) \times U(y_0; \varepsilon)$ 是 (x_0, y_0) 的任一方邻域, 由题设, x_0 的邻域 $U(x_0; \varepsilon)$ 含有 A 中的无穷多点, y_0 的邻域 $U(y_0; \varepsilon)$ 含有 B 中的无穷多点, 从而 $U((x_0, y_0); \varepsilon)$ 含有 $A \times B$ 中的无穷多点, 因而 (x_0, y_0) 是 $A \times B$ 的聚点. □

9. 对 $\forall P \notin E \cup \partial E$, P 是 E 的外点, 即 $\exists \delta > 0$, 使得 $U(P; \delta) \cap E = \varnothing$. 下面使用反证法证明 $U(P; \delta) \cap \partial E = \varnothing$, 假设 $U(P; \delta) \cap \partial E \neq \varnothing$, 则 $\exists P_0 \in U(P; \delta) \cap \partial E$ 即 $p_0 \in \partial E$ 且 $P_0 \in U(P; \delta)$. 取 $\varepsilon = \frac{1}{2}(\delta - \|P - P_0\|)$, 由 $P_0 \in \partial E$ 知 $U(P_0; \varepsilon) \cap E \neq \varnothing$; 由 $P_0 \in U(P; \delta)$ 知 $U(P_0; \varepsilon) \subset U(P; \delta)$, 而 $U(P; \delta) \cap E = \varnothing$, 所以 $U(P_0; \varepsilon) \cap E = \varnothing$, 矛盾, 因而 $U(P; \delta) \cap \partial E = \varnothing$. 所以 $U(P; \delta) \cap (E \cup \partial E) = \varnothing$, 即 P 不是 $E \cup \partial E$ 的聚点, $E \cup \partial E$ 的聚点都在 $E \cup \partial E$ 之内, 即 $E \cup \partial E$ 是闭集. □

习题 13.2 $\mathbb{R}^2$ 的完备性

1. 若 $\{P_n\}$ 是有限点集, 则 $\{P_n\}$ 有一个常驻子列自然收敛.

若 $\{P_n\}$ 是无限点集, 因为 $\{P_n\}$ 有界, 由聚点定理, $\{P_n\}$ 在 $\mathbb{R}^2$ 中至少有一个聚点, 不妨设 P_0 是 $\{P_n\}$ 在 $\mathbb{R}^2$ 中的一个聚点. 分别取 $\delta_k = 1/k, k = 1, 2, ...$, 则邻域 $U(P_0; \delta_k), k = 1, 2, ...$ 都含有 $\{P_n\}$ 的无穷多点. 分别取 $P_{n_k} \in U(P_0; \delta_k) \cap \{P_n\}, k = 1, 2, ...$, 则 $\{P_{n_k}\}$ 是 $\{P_n\}$ 的一个子列, 下面证明 $\{P_{n_k}\}$ 收敛. 事实上, 由于 $P_{n_k} \in U(P_0; \delta_k)$, 因而 $\|P_{n_k} - P_0\| < \delta_k = 1/k \to 0, k \to \infty$, 即 $\lim_{k \to \infty} P_{n_k} = P_0$. $\qquad\square$

2. 设 $\{P_n\}$ 是基本点列, 且收敛于 P_0, 则对于取定的 $\varepsilon = 1, \exists N \in \mathbb{N}_+$, 当 $n > N$ 时总有 $P_n \in U(P_0; 1)$. 取 $R = \max\{1, \|P_n - P_0\|, n = 1, 2, ..., N\}$, 就有 $P_n \in B_R(P_0), n = 1, 2, ...$, 即 $\{P_n\}$ 有界. $\qquad\square$

3. $\Leftarrow$: 设 $\lim_{n \to \infty} P_n = P_0$ 且 $\{P_n\} \subset E$ 各项互异, 则 $\forall \varepsilon > 0, \exists N \in \mathbb{N}_+$, 当 $n > N$ 时总有 $P_n \in U(P_0; \varepsilon)$, 由 ε 的任意性和 $\{P_n\}$ 的互异性知 P_0 是 E 的聚点.

$\Rightarrow$: 设 P_0 是 E 的聚点, 分别取 $\delta_n = 1/n, n = 1, 2, ...$, 则邻域 $U(P_0; \delta_n)$ 均含有 E 的无穷多点, 依次取 $P_n \in U(P_0; \delta_n) \cap E \backslash \{P_k\}_{k=0}^{n-1}, n = 1, 2, ...$, 则 $\{P_n\} \subset E$ 且各项互异, 进一步因为 $P_n \in U(P_0; \delta_n)$, 有 $\|P_n - P_0\| < \delta_n = 1/n \to 0, n \to \infty$, 即 $\lim_{n \to \infty} P_n = P_0$. $\qquad\square$

4. 设 E 是 $\mathbb{R}^2$ 中的有界无限点集, 需证 E 在 $\mathbb{R}^2$ 中至少有一个聚点. 由于 E 有界, 可设 E 包含在一个闭正方形 D_1 中, 将 D_1 等分为 4 个小的闭正方形, 其中必有一个含有 E 中的无限个点, 记为 $D_2, ...$, 以此类推, 得到一列闭正方形 $\{D_n\}_{n=1}^{\infty}$, 满足 $D_{n+1} \subset D_n$, 且每个 $D_n \cap E$ 均是无限点集. 注意到 $d(D_{n+1}) = d(D_n)/2 = ... = d(D_1)/2^n \to 0, n \to \infty, \{D_n \cap E\}$ 满足闭集套定理的条件, 故 $\exists! P_0 \in D_n \cap E, n = 1, 2, ...$, 下面证明 P_0 是 E 的聚点. 事实上, 对 $\forall \varepsilon > 0, \exists N = \max\{[\log_2(d(D_1)/\varepsilon)] + 1, 1\} \in \mathbb{N}_+$, 当 $n > N$ 时, $d(D_n) = d(D_1)/2^{n-1} < 2\varepsilon$, 由 $P_0 \in D_n$ 有 $D_n \subset U(P_0; \varepsilon)$, 从而 $D_n \cap E \subset U(P_0; \varepsilon)$, 而每个 $D_n \cap E$ 都是无限点集, 从而 P_0 是 E 的聚点. $\qquad\square$

5. 设 $\{P_n\}$ 是 Cauchy 点列, 需证 $\{P_n\}$ 收敛. 由第 2 题, $\{P_n\}$ 有界, 再由致密性定理知, $\{P_n\}$ 有收敛子列, 设为 $\{P_{n_k}\}$, 并且假设 $\{P_{n_k}\}$ 收敛到 P_0, 即有, 对 $\forall \varepsilon > 0, \exists K \in \mathbb{N}_+$, 当 $k > K$ 时,

$$\|P_{n_k} - P_0\| < \varepsilon/2. \tag{1}$$

另一方面, $\{P_n\}$ 是 Cauchy 点列, 从而对上述的 $\varepsilon > 0, \exists N_0 \in \mathbb{N}_+$, 当 $n, m > N_0$ 时,

$$\|P_n - P_m\| < \varepsilon/2. \tag{2}$$

由于 $k > K$ 时 $n_k > K$ 也成立, 取 $N = \max\{K, N_0\}$, 则当 $k, n > N$ 时, 由 (1) 和 (2) 得

$$\|P_n - P_0\| \leqslant \|P_n - P_{n_k}\| + \|P_{n_k} - P_0\| < \varepsilon,$$

即 $\{P_n\}$ 收敛. $\qquad\square$

习题 13.3 二元函数的极限和连续性

1. 首先限制 (x,y) 处于方邻域 $U((2,-1);1)$ 之内, 则 $|x+2-4y| < 13$ 且 $|3y+5| < 5$. 对 $\forall \varepsilon > 0, \exists \delta = \min\{\varepsilon/18, 1\} > 0$, 对方邻域 $U^\circ((2,-1);\delta)$ 内的任意点 (x,y), 有

$$
\begin{aligned}
|x^2 - 4xy - 3y^2 - 9| &= |(x-2)(x+2-4y) - (y+1)(3y+5)| \\
&\leqslant |x+2-4y| \cdot |x-2| + |3y+5| \cdot |y+1| \\
&\leqslant 13\delta + 5\delta < \varepsilon,
\end{aligned}
$$

由重极限的定义知 $\lim_{(x,y)\to(2,-1)}(x^2 - 4xy - 3y^2) = 9$. □

    2. (1) 令 $x^2 + y^2 = r^2$, 则

$$
\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{1+x^2+y^2} = \lim_{r\to 0} \frac{r^2}{1+r^2} = 0.
$$

(2) 令 $x = r\cos\theta, y = r\sin\theta, r > 0$, 则

$$
\lim_{(x,y)\to(0,0)} \frac{x^2 y^2}{x^2+y^2} = \lim_{r\to 0^+} r^2 \cos^2\theta \sin^2\theta = 0.
$$

(3) 令 $x^2 + y^2 = r^2$, 则
$$
\lim_{(x,y)\to(0,0)} \frac{\sin(x^2+y^2)}{x^2+y^2} = \lim_{r\to 0} \frac{\sin r^2}{r^2} = 1.
$$

(4)

$$
\begin{aligned}
\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{1-\sqrt{(1+x^2)(1+y^2)}} &= -\lim_{(x,y)\to(0,0)} \frac{(x^2+y^2)(1+\sqrt{(1+x^2)(1+y^2)})}{x^2+y^2+x^2 y^2} \\
&= -\lim_{(x,y)\to(0,0)} \frac{1+\sqrt{(1+x^2)(1+y^2)}}{1+\frac{x^2 y^2}{x^2+y^2}},
\end{aligned}
$$

由 (2) 知 $\lim_{(x,y)\to(0,0)} \frac{x^2 y^2}{x^2+y^2} = 0$, 所以 $\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{1-\sqrt{(1+x^2)(1+y^2)}} = -2$. □

    3. (1) 因为

$$
\lim_{y\to 0}\lim_{x\to 0} \frac{xy}{x^2+y^2} = \lim_{y\to 0} 0 = 0, \quad \lim_{x\to 0}\lim_{y\to 0} \frac{xy}{x^2+y^2} = \lim_{x\to 0} 0 = 0,
$$

所以 $f(x,y)$ 在 $(0,0)$ 的两个累次极限都是 0. 又因为

$$
\lim_{\substack{(x,y)\to(0,0) \\ y=kx}} \frac{xy}{x^2+y^2} = \frac{k}{1+k^2},
$$

上式对不同的 k 得到的极限也不同, 因而 $f(x,y)$ 在 $(0,0)$ 的重极限不存在.

(2) 因为 $\lim_{x\to 0}\sin\frac{1}{x}$ 不存在, 所以 $f(x,y)$ 在 $(0,0)$ 的两个累次极限都不存在. 又因为无穷小与有界量的乘积也是无穷小, 因而 $f(x,y)$ 在 $(0,0)$ 的重极限为 0.

(3) 因为

$$\lim_{y\to 0}\lim_{x\to 0}\frac{x^2y^2}{x^2y^2+(x-y)^2}=\lim_{y\to 0}0=0,\ \ \lim_{x\to 0}\lim_{y\to 0}\frac{x^2y^2}{x^2y^2+(x-y)^2}=\lim_{x\to 0}0=0,$$

所以 $f(x,y)$ 在 $(0,0)$ 的两个累次极限都是 0. 又因为

$$\lim_{\substack{(x,y)\to(0,0)\\y=x}}\frac{x^2y^2}{x^2y^2+(x-y)^2}=1,\ \ \ \lim_{\substack{(x,y)\to(0,0)\\y=-x}}\frac{x^2y^2}{x^2y^2+(x-y)^2}=\lim_{x\to 0}\frac{x^2}{x^2+4}=0,$$

上式两个极限不同, 因而 $f(x,y)$ 在 $(0,0)$ 的重极限不存在.

(4) 因为

$$\lim_{y\to 0}\lim_{x\to 0}\frac{x^2y^2}{x^3+y^3}=\lim_{y\to 0}0=0,\ \ \lim_{x\to 0}\lim_{y\to 0}\frac{x^2y^2}{x^3+y^3}=\lim_{x\to 0}0=0,$$

所以 $f(x,y)$ 在 $(0,0)$ 的两个累次极限都是 0. 又因为

$$\lim_{\substack{(x,y)\to(0,0)\\y=x}}\frac{x^2y^2}{x^3+y^3}=\lim_{x\to 0}\frac{x}{2}=0,\ \ \ \lim_{\substack{(x,y)\to(0,0)\\y=x(x-1)}}\frac{x^2y^2}{x^3+y^3}=\lim_{x\to 0}\frac{x^2-2x+1}{x^2-3x+3}=\frac{1}{3},$$

上式两个极限不同, 因而 $f(x,y)$ 在 $(0,0)$ 的重极限不存在. $\qquad\qquad\square$

5. (1) 对 $\forall(x_0,y_0)\in\mathbb{R}^2$, 如果 $x_0+y_0\neq 0$, $\lim_{(x,y)\to(x_0,y_0)}f(x,y)=f(x_0,y_0)$, 从而 (x_0,y_0) 是 $f(x,y)$ 的连续点. 假设 $x_0+y_0=0$, 则 $f(x_0,y_0)=0$, 而 $\lim_{(x,y)\to(x_0,y_0)}f(x,y)=\infty$, 因而 (x_0,y_0) 是 $f(x,y)$ 的间断点, 满足 $x+y=0$ 的所有点都是 $f(x,y)$ 的间断点.

(2) 对 $\forall(x_0,y_0)\in\mathbb{R}^2$, 如果 $(x_0,y_0)\neq(0,0)$, $\lim_{(x,y)\to(x_0,y_0)}f(x,y)=f(x_0,y_0)$, 从而 (x_0,y_0) 是 $f(x,y)$ 的连续点. 对 $(0,0)$ 点, $f(0,0)=0$, 而 $\lim_{(x,y)\to(0,0)}f(x,y)$ 不存在 (因为 $f(x,y)$ 在 $(0,0)$ 点的两个累次极限都存在但是不相等), 因而 $(0,0)$ 是 $f(x,y)$ 的间断点. $\qquad\square$

6. 任取 $E\subset D,E\neq\varnothing$, 对 $\forall P_0\in E$, 有 $P_0\in D$. 因为 f 在 D 连续, 所以 f 在 P_0 连续, 即, 对 $\varepsilon>0,\exists\delta>0$, 对 $\forall P\in U(P_0;\delta)\cap D$, 有

$$|f(P)-f(P_0)|<\varepsilon. \tag{3}$$

因此, 对 $\forall P\in U(P_0;\delta)\cap E$, 显然有 $P\in U(P_0;\delta)\cap D$, (3) 式自然也成立, 从而 f 在 E 也连续. $\qquad\qquad\square$

7. 对 $\forall(x_0,y_0)\in D$, 需要证明 $f(x,y)$ 在 (x_0,y_0) 连续. 首先, 因为 f 对 x 连续, 从而一元函数 $f(x,y_0)$ 在 x_0 连续, 即, 对 $\forall\varepsilon>0,\exists\delta_1>0$, 当 $|x-x_0|<\delta_1$ 时,

$$|f(x,y_0)-f(x_0,y_0)|<\varepsilon/2. \tag{4}$$

其次, 因为 f 对 y (关于 x) 一致连续, 所以对上述的 $\varepsilon, \exists \delta_2 > 0$, 当 $|y - y_0| < \delta_2$ 且 $(x, y), (x, y_0) \in D$ 时,

$$|f(x, y) - f(x, y_0)| < \varepsilon/2. \tag{5}$$

取 $\delta = \min\{\delta_1, \delta_2\}$, 则当 $|x - x_0| < \delta, |y - y_0| < \delta$ 时, 由 (4) 式和 (5) 式有

$$|f(x, y) - f(x_0, y_0)| \leqslant |f(x, y) - f(x, y_0)| + |f(x, y_0) - f(x_0, y_0)| < \varepsilon,$$

即 $f(x, y)$ 在 (x_0, y_0) 连续. $\qquad\square$

8. (参考 3.4 节例 8 的证明) 由于 $\lim_{\|P\| \to +\infty} f(P)$ 存在且有限, 由 Cauchy 收敛准则, 对 $\forall \varepsilon > 0, \exists R > 1 > 0$, 只要 $\|P_1\|, \|P_2\| \geqslant R$, 就有

$$|f(P_1) - f(P_2)| < \varepsilon, \tag{6}$$

即, f 在 $\mathbb{R}^2 \backslash B_R(O)$ 内一致连续. 同时, 由于 f 在有界闭集 $\bar{B}_{R+1}(O)$ 内连续, 由一致连续性定理 (定理 13.3.6), f 在 $\bar{B}_{R+1}(O)$ 内一致连续, 即, 对上述的 $\varepsilon, \exists \delta_1 > 0$, 当 $\|P_1 - P_2\| < \delta_1$ 且 $P_1, P_2 \in \bar{B}_{R+1}(O)$ 时, (6) 式成立. 取 $\delta = \min\{1, \delta_1\}$, 则当 $P_1, P_2 \in \mathbb{R}^2$ 且 $\|P_1 - P_2\| < \delta$ 时, P_1, P_2 同时处于 $\bar{B}_{R+1}(O)$ 或 $\mathbb{R}^2 \backslash B_R(O)$ 内, 因此 (6) 式总成立. 由定义, f 在 $\mathbb{R}^2$ 一致连续. $\qquad\square$

9. 令 $\bar{D} = D \cup D'$, 则 $\bar{D}$ 是有界闭集. 构造函数

$$\bar{f}(P) = \begin{cases} f(P), & P \in D, \\ \lim_{\substack{Q \to P \\ Q \in D}} f(Q), & P \in D' \backslash D, \end{cases}$$

则 $\bar{f}(P)$ 在 $\bar{D}$ 上连续, 从而在 $\bar{D}$ 上一致连续, 因此, $\bar{f}(P)$ 在 D 上, 即 $f(P)$ 在 D 上, 也一致连续. $\qquad\square$

10. 仅证最小值情形. 因为 f 在 $\mathbb{R}^2$ 连续, 所以 $f(O)$ 存在, 其中 O 是坐标原点. 由 $\lim_{\|P\| \to +\infty} f(P) = +\infty$, 对 $M = |f(O)| + 1 > 0, \exists R > 0$, 当 $P \in \mathbb{R}^2 \backslash \bar{B}_R(O)$, 即 $\|P\| > R$ 时, $|f(P)| > M$. 而在 $\bar{B}_R(O)$ 中, f 有最小值 m, 显然 $m \leqslant f(O)$. 对 $\forall P \in \mathbb{R}^2 \backslash \bar{B}_R(O)$, $|f(P)| > M = |f(O)| + 1 > f(O) \geqslant m$, 所以 m 是 f 在 $\mathbb{R}^2$ 中的最小值. $\qquad\square$

习题 13.4 多元函数的偏导数和全微分

6. 全增量

$$\Delta z = \Delta x \Delta y \sin \frac{1}{(\Delta x)^2 + (\Delta y)^2} = 0 \cdot \Delta x + 0 \cdot \Delta y + \Delta y \sin \frac{1}{(\Delta x)^2 + (\Delta y)^2} \cdot \Delta x + 0 \cdot \Delta y,$$

取 $A = B = 0, \alpha = \Delta y \sin \frac{1}{(\Delta x)^2 + (\Delta y)^2}, \beta = 0$, 显然 α, β 是 $(\Delta x, \Delta y) \to (0, 0)$ 时的无穷小, 因而 $f(x, y)$ 在 $(0, 0)$ 可微, 并且 $\mathrm{d}f|_{(0,0)} = 0$. $\qquad\square$

7. 令 $x = r\cos\theta, y = r\sin\theta, r > 0$, 则

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{r\to 0^+} r^2 \sin\frac{1}{r^2} = 0 = f(0,0),$$

即, $f(x,y)$ 在 $(0,0)$ 连续. 易求得偏导数

$$f_x(x,y) = \begin{cases} 2x\sin\frac{1}{x^2+y^2} - \frac{2x}{(x^2+y^2)}\cos\frac{1}{x^2+y^2}, & (x,y) \neq (0,0), \\ \lim_{\Delta x\to 0}\frac{f(\Delta x,0)-f(0,0)}{\Delta x} = 0, & (x,y) = (0,0), \end{cases}$$

$$f_y(x,y) = \begin{cases} 2y\sin\frac{1}{x^2+y^2} - \frac{2y}{(x^2+y^2)}\cos\frac{1}{x^2+y^2}, & (x,y) \neq (0,0), \\ \lim_{\Delta y\to 0}\frac{f(0,\Delta y)-f(0,0)}{\Delta y} = 0, & (x,y) = (0,0), \end{cases}$$

因为

$$\lim_{(x,y)\to(0,0)} \frac{2x}{(x^2+y^2)}\cos\frac{1}{x^2+y^2} \text{ 和 } \lim_{(x,y)\to(0,0)} \frac{2y}{(x^2+y^2)}\cos\frac{1}{x^2+y^2}$$

均不存在, 所以

$$\lim_{(x,y)\to(0,0)} f_x(x,y) \text{ 和 } \lim_{(x,y)\to(0,0)} f_y(x,y)$$

不存在, 即, 偏导数在 $(0,0)$ 不连续. 而全增量

$$\begin{aligned} \Delta z =& [(\Delta x)^2 + (\Delta y)^2]\sin\frac{1}{(\Delta x)^2+(\Delta y)^2} \\ =& 0\cdot\Delta x + 0\cdot\Delta y + \Delta x\sin\frac{1}{(\Delta x)^2+(\Delta y)^2}\cdot\Delta x + \Delta y\sin\frac{1}{(\Delta x)^2+(\Delta y)^2}\cdot\Delta y, \end{aligned}$$

取 $A = B = 0, \alpha = \Delta x\sin\frac{1}{(\Delta x)^2+(\Delta y)^2}, \beta = \Delta y\sin\frac{1}{(\Delta x)^2+(\Delta y)^2}$, 显然 α, β 是 $(\Delta x, \Delta y)\to(0,0)$ 时的无穷小, 因而 $f(x,y)$ 在 $(0,0)$ 可微, 并且 $\mathrm{d}f|_{(0,0)} = 0$. $\square$

8. 由 $z = f(x,y)$ 在 (x_0,y_0) 可微, 有 $\Delta z = A\Delta x + B\Delta y + \alpha\Delta x + \beta\Delta y$, 其中 A, B 是常数, α, β 是当 $(\Delta x, \Delta y)\to(0,0)$ 时的无穷小. 因此 $\lim_{(\Delta x,\Delta y)\to(0,0)}\Delta z = 0$, 即, $\lim_{(\Delta x,\Delta y)\to(0,0)} f(x_0 + \Delta x, y_0 + \Delta y) = f(x_0, y_0)$, 此即表明 $z = f(x,y)$ 在 (x_0, y_0) 连续. $\square$

9. 设 $z = f(x,y)$ 在邻域 $U((x_0,y_0))$ 中偏导数存在且有界, 对 $\forall (x,y) \in U((x_0,y_0))$, 需证 $z = f(x,y)$ 在 (x,y) 连续. 取 $(x + \Delta x, y + \Delta y) \in U((x_0,y_0))$, 由 Lagrange 中值定理, $\exists \theta_1, \theta_2 \in (0,1)$, 使得

$$\begin{aligned} f(x+\Delta x, y+\Delta y) - f(x,y) =& f(x+\Delta x, y+\Delta y) - f(x+\Delta x, y) + f(x+\Delta x, y) - f(x,y) \\ =& f_y(x+\Delta x, y+\theta_1\Delta y)\Delta y + f_x(x+\theta_2\Delta x, y)\Delta x, \end{aligned}$$

因为偏导数存在且有界, 因而有 $\lim_{(\Delta x,\Delta y)\to(0,0)}[f(x+\Delta x, y+\Delta y) - f(x,y)] = 0$, 此即表明 $z = f(x,y)$ 在 (x,y) 连续. $\square$

10. 令 $x = r\cos\theta, y = r\sin\theta, r > 0$, 则

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{r\to 0^+} r\cos^2\theta\sin\theta = 0 = f(0,0),$$

即, $f(x,y)$ 在 $(0,0)$ 连续. 偏导数

$$f_x(x,y) = \begin{cases} \frac{2xy^3}{(x^2+y^2)^2}, & (x,y)\neq(0,0), \\ \lim_{\Delta x\to 0}\frac{f(\Delta x,0)-f(0,0)}{\Delta x}=0, & (x,y)=(0,0), \end{cases}$$

$$f_y(x,y) = \begin{cases} \frac{x^2(x^2-y^2)}{(x^2+y^2)^2}, & (x,y)\neq(0,0), \\ \lim_{\Delta y\to 0}\frac{f(0,\Delta y)-f(0,0)}{\Delta y}=0, & (x,y)=(0,0) \end{cases}$$

均存在. 而全增量 $\Delta z = \frac{(\Delta x)^2\Delta y}{(\Delta x)^2+(\Delta y)^2}$, 假设 $f(x,y)$ 在 $(0,0)$ 可微, 则 $\mathrm{d}z|_{(0,0)} = 0$, 所以 $\Delta z - \mathrm{d}z|_{(0,0)} = \frac{(\Delta x)^2\Delta y}{(\Delta x)^2+(\Delta y)^2} = o(\rho)$, 其中 $\rho = \sqrt{(\Delta x)^2+(\Delta y)^2}$, 即, $\frac{(\Delta x)^2\Delta y}{(\Delta x)^2+(\Delta y)^2}\cdot\frac{1}{\rho} = o(1)$. 然而

$$\lim_{\substack{(\Delta x,\Delta y)\to(0,0)\\ \Delta y=k\Delta x}} \frac{(\Delta x)^2\Delta y}{(\Delta x)^2+(\Delta y)^2}\cdot\frac{1}{\rho} = \pm\frac{k}{(1+k^2)^{\frac{3}{2}}},$$

由于上式对不同的 k 其取值也不同, 从而 $\frac{(\Delta x)^2\Delta y}{(\Delta x)^2+(\Delta y)^2} \neq o(\rho)$, 导致矛盾, 因而 $f(x,y)$ 在 $(0,0)$ 不可微. $\qquad\square$

习题 13.5 复合函数的微分法

1. (1) $\frac{\mathrm{d}z}{\mathrm{d}x} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial z}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}x} = ye^{xy} + xe^{xy}\frac{1}{1+x^2} = e^{x\arctan x}\left(\arctan x + \frac{x}{1+x^2}\right)$.

(2) 令 $u = \frac{1}{x} + \frac{1}{y}$, 则 $z = e^u\sin u$, 因而

$$\frac{\partial z}{\partial x} = \frac{\mathrm{d}z}{\mathrm{d}u}\frac{\partial u}{\partial x} = (e^u\sin u + e^u\cos u)\left(-\frac{1}{x^2}\right) = -\frac{1}{x^2}e^{\frac{1}{x}+\frac{1}{y}}\left[\sin\left(\frac{1}{x}+\frac{1}{y}\right) + \cos\left(\frac{1}{x}+\frac{1}{y}\right)\right],$$
$$\frac{\partial z}{\partial y} = \frac{\mathrm{d}z}{\mathrm{d}u}\frac{\partial u}{\partial y} = (e^u\sin u + e^u\cos u)\left(-\frac{1}{y^2}\right) = -\frac{1}{y^2}e^{\frac{1}{x}+\frac{1}{y}}\left[\sin\left(\frac{1}{x}+\frac{1}{y}\right) + \cos\left(\frac{1}{x}+\frac{1}{y}\right)\right].$$

(3) $\frac{\mathrm{d}z}{\mathrm{d}t} = \frac{\partial z}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial z}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} = (e^x + \cos x\cos y)\cdot 2t + (-\sin x\sin y + e^y) = 2t[e^{t^2} + \cos(t^2)\cos(1+t)] - \sin(t^2)\sin(1+t) + e^{1+t}$.

(4) 由链式法则,

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} = \frac{1}{x}\ln y + \frac{1}{y}\ln x = \frac{\ln(u-v)}{u+v} + \frac{\ln(u+v)}{u-v},$$
$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial v}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v} = \frac{1}{x}\ln y - \frac{1}{y}\ln x = \frac{\ln(u-v)}{u+v} - \frac{\ln(u+v)}{u-v}.$$

$\qquad\square$

2. 易得 $\frac{\partial z}{\partial x} = -\frac{y}{x^2+y^2}, \frac{\partial z}{\partial y} = \frac{x}{x^2+y^2}$, 因而 $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 0$. $\qquad\square$

3. 易得 $\frac{\partial z}{\partial x} = 2nx(x^2+y^2)^{n-1}, \frac{\partial z}{\partial y} = 2ny(x^2+y^2)^{n-1}$, 因而 $\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = 4n^2(x^2+y^2)^{2n-1} =$

$4n^2 z^{\frac{2n-1}{n}}.$ □

4. 易得 $\frac{\partial z}{\partial x} = y + e^{\frac{y}{x}} - \frac{y}{x} e^{\frac{y}{x}}, \frac{\partial z}{\partial y} = x + e^{\frac{y}{x}}$, 因而 $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = xy + xe^{\frac{y}{x}} + xy = xy + z.$ □

5. (1) 令 $u = x + y, v = xy$, 则 $z = f(x, u, v)$,

$$
\begin{aligned}
z_x &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x} = f_1' + f_2' + yf_3', \\
z_y &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial y} + \frac{\partial f}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial y} = f_2' + xf_3'.
\end{aligned}
$$

(2) 令 $x = r\cos\theta, y = r\sin\theta$, 则 $z = f(x, y)$,

$$
\begin{aligned}
z_r &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r} = f_1'\cos\theta + f_2'\sin\theta, \\
z_\theta &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \theta} = -rf_1'\sin\theta + rf_2'\cos\theta.
\end{aligned}
$$

(3) 令 $s = \frac{x}{y} + \frac{y}{z} + \frac{z}{x}, t = xyz$, 则 $u = f(s, t)$,

$$
\begin{aligned}
u_x &= \frac{\partial f}{\partial s}\frac{\partial s}{\partial x} + \frac{\partial f}{\partial t}\frac{\partial t}{\partial x} = f_1'\left(\frac{1}{y} - \frac{z}{x^2}\right) + yzf_2', \\
u_y &= \frac{\partial f}{\partial s}\frac{\partial s}{\partial y} + \frac{\partial f}{\partial t}\frac{\partial t}{\partial y} = f_1'\left(\frac{1}{z} - \frac{x}{y^2}\right) + xzf_2', \\
u_z &= \frac{\partial f}{\partial s}\frac{\partial s}{\partial z} + \frac{\partial f}{\partial t}\frac{\partial t}{\partial z} = f_1'\left(\frac{1}{x} - \frac{y}{z^2}\right) + xyf_2'.
\end{aligned}
$$

(4) 令 $w = \sqrt[3]{x^2 + y^2 + z^2}$, 则 $u = f(w)$,

$$
\begin{aligned}
u_x &= \frac{\mathrm{d}f}{\mathrm{d}w}\frac{\partial w}{\partial x} = \frac{2}{3}x(x^2 + y^2 + z^2)^{-\frac{2}{3}}f', \\
u_y &= \frac{\mathrm{d}f}{\mathrm{d}w}\frac{\partial w}{\partial y} = \frac{2}{3}y(x^2 + y^2 + z^2)^{-\frac{2}{3}}f', \\
u_z &= \frac{\mathrm{d}f}{\mathrm{d}w}\frac{\partial w}{\partial z} = \frac{2}{3}z(x^2 + y^2 + z^2)^{-\frac{2}{3}}f'.
\end{aligned}
$$

□

6. (1) 令 $u = x + y, v = xy, w = \frac{x}{y}$, 则 $z = f(u, v, w)$,

$$
\begin{aligned}
z_x &= \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial f}{\partial w}\frac{\partial w}{\partial x} = f_1' + yf_2' + \frac{1}{y}f_3', \\
z_y &= \frac{\partial f}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial y} + \frac{\partial f}{\partial w}\frac{\partial w}{\partial y} = f_1' + xf_2' - \frac{x}{y^2}f_3',
\end{aligned}
$$

因此, $\mathrm{d}z = \left(f_1' + yf_2' + \frac{1}{y}f_3'\right)\mathrm{d}x + \left(f_1' + xf_2' - \frac{x}{y^2}f_3'\right)\mathrm{d}y.$

(2) 令 $w = \frac{x}{y} + \frac{y}{z} + \frac{z}{x}$, 则 $u = f(w)$,

$$u_x = \frac{\mathrm{d}f}{\mathrm{d}w}\frac{\partial w}{\partial x} = \left(\frac{1}{y} - \frac{z}{x^2}\right)f', u_y = \frac{\mathrm{d}f}{\mathrm{d}w}\frac{\partial w}{\partial y} = \left(\frac{1}{z} - \frac{x}{y^2}\right)f', u_z = \frac{\mathrm{d}f}{\mathrm{d}w}\frac{\partial w}{\partial z} = \left(\frac{1}{x} - \frac{y}{z^2}\right)f',$$

因此, $\mathrm{d}u = f'\left[\left(\frac{1}{y} - \frac{z}{x^2}\right)\mathrm{d}x + \left(\frac{1}{z} - \frac{x}{y^2}\right)\mathrm{d}y + \left(\frac{1}{x} - \frac{y}{z^2}\right)\mathrm{d}z\right]$. $\qquad\square$

7. (1) 令 $\tau = x^2 + y^2$, 则 $z = F(\tau)$ 可微. $\frac{\partial z}{\partial x} = \frac{\mathrm{d}F}{\mathrm{d}\tau}\frac{\partial \tau}{\partial x} = 2xF', \frac{\partial z}{\partial y} = \frac{\mathrm{d}F}{\mathrm{d}\tau}\frac{\partial \tau}{\partial y} = 2yF'$, 因此 $y\frac{\partial z}{\partial x} - x\frac{\partial z}{\partial y} = 0$;

(2) 令 $\tau = x^2 + y^2 + z^2$, 则 $u = F(\tau)$ 可微. $\frac{\partial u}{\partial x} = \frac{\mathrm{d}F}{\mathrm{d}\tau}\frac{\partial \tau}{\partial x} = 2xF', \frac{\partial u}{\partial y} = \frac{\mathrm{d}F}{\mathrm{d}\tau}\frac{\partial \tau}{\partial y} = 2yF', \frac{\partial u}{\partial z} = \frac{\mathrm{d}F}{\mathrm{d}\tau}\frac{\partial \tau}{\partial z} = 2zF'$, 因此 $\left(1 - \frac{y}{x}\right)\frac{\partial u}{\partial x} + \left(1 - \frac{z}{y}\right)\frac{\partial u}{\partial y} + \left(1 - \frac{x}{z}\right)\frac{\partial u}{\partial z} = 0$. $\qquad\square$

8. (Euler 定理) 因为 $f(tx, ty, tz) = t^n f(x, y, z)$, 左边对 t 求导得到

$$
\begin{aligned}
\frac{\mathrm{d}f(tx, ty, tz)}{\mathrm{d}t} =& \frac{\partial f(tx, ty, tz)}{\partial(tx)}\frac{\partial(tx)}{\partial t} + \frac{\partial f(tx, ty, tz)}{\partial(ty)}\frac{\partial(ty)}{\partial t} + \frac{\partial f(tx, ty, tz)}{\partial(tz)}\frac{\partial(tz)}{\partial t} \\
=& x\frac{\partial f(tx, ty, tz)}{\partial(tx)} + y\frac{\partial f(tx, ty, tz)}{\partial(ty)} + z\frac{\partial f(tx, ty, tz)}{\partial(tz)},
\end{aligned}
\tag{7}
$$

右边对 t 求导得到

$$\frac{\mathrm{d}t^n f(x, y, z)}{\mathrm{d}t} = nt^{n-1}u, \tag{8}$$

取 $t = 1$ 并由 (7) 和 (8) 得证. $\qquad\square$

10. 函数都是各阶连续可微的.

(1) 令 $u = x + y, v = xy$, 则 $z = f(u, v)$,

$$
\begin{aligned}
z_x =& \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x} = f_1' + yf_2', \ z_y = \frac{\partial f}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial y} = f_1' + xf_2', \\
z_{xx} =& (f_1' + yf_2')_x = \frac{\partial(f_1')}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial(f_1')}{\partial v}\frac{\partial v}{\partial x} + y\left[\frac{\partial(f_2')}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial(f_2')}{\partial v}\frac{\partial v}{\partial x}\right] \\
=& f_{11}'' + yf_{12}'' + y(f_{21}'' + yf_{22}'') = f_{11}'' + 2yf_{12}'' + y^2 f_{22}'', \\
z_{xy} =& (f_1' + yf_2')_y = \frac{\partial(f_1')}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial(f_1')}{\partial v}\frac{\partial v}{\partial y} + f_2' + y\left[\frac{\partial(f_2')}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial(f_2')}{\partial v}\frac{\partial v}{\partial y}\right] \\
=& f_{11}'' + xf_{12}'' + f_2' + y(f_{21}'' + xf_{22}'') = f_{11}'' + (x + y)f_{12}'' + xyf_{22}'' + f_2', \\
z_{yy} =& (f_1' + xf_2')_y = \frac{\partial(f_1')}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial(f_1')}{\partial v}\frac{\partial v}{\partial y} + x\left[\frac{\partial(f_2')}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial(f_2')}{\partial v}\frac{\partial v}{\partial y}\right] \\
=& f_{11}'' + xf_{12}'' + x(f_{21}'' + xf_{22}'') = f_{11}'' + 2xf_{12}'' + x^2 f_{22}''.
\end{aligned}
$$

(2) 令 $w = \frac{x}{y} + \frac{y}{z} + \frac{z}{x}$, 则 $u = f(w)$,

$$u_x = \frac{\mathrm{d}f}{\mathrm{d}w}\frac{\partial w}{\partial x} = \left(\frac{1}{y} - \frac{z}{x^2}\right)f', \ u_y = \frac{\mathrm{d}f}{\mathrm{d}w}\frac{\partial w}{\partial y} = \left(\frac{1}{z} - \frac{x}{y^2}\right)f', \ u_z = \frac{\mathrm{d}f}{\mathrm{d}w}\frac{\partial w}{\partial z} = \left(\frac{1}{x} - \frac{y}{z^2}\right)f',$$

$$u_{xx} = \left[\left(\frac{1}{y} - \frac{z}{x^2}\right)f'\right]_x = \frac{2z}{x^3}f' + \left(\frac{1}{y} - \frac{z}{x^2}\right)\frac{\mathrm{d}(f')}{\mathrm{d}w}\frac{\partial w}{\partial x} = \frac{2z}{x^3}f' + \left(\frac{1}{y} - \frac{z}{x^2}\right)^2 f'',$$

$$u_{xy} = \left[\left(\frac{1}{y} - \frac{z}{x^2}\right)f'\right]_y = -\frac{1}{y^2}f' + \left(\frac{1}{y} - \frac{z}{x^2}\right)\frac{\mathrm{d}(f')}{\mathrm{d}w}\frac{\partial w}{\partial y} = -\frac{1}{y^2}f' + \left(\frac{1}{y} - \frac{z}{x^2}\right)\left(\frac{1}{z} - \frac{x}{y^2}\right)f'',$$

$$u_{yy} = \left[\left(\frac{1}{z} - \frac{x}{y^2}\right)f'\right]_y = \frac{2x}{y^3}f' + \left(\frac{1}{z} - \frac{x}{y^2}\right)\frac{\mathrm{d}(f')}{\mathrm{d}w}\frac{\partial w}{\partial y} = \frac{2x}{y^3}f' + \left(\frac{1}{z} - \frac{x}{y^2}\right)^2 f''.$$

(3) 因为 $\frac{\mathrm{d}z}{\mathrm{d}x} = \frac{\partial f}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}x} + \frac{\partial f}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}x} = f_1' + f_2'\varphi'$, 所以

$$\begin{aligned}
\frac{\mathrm{d}^2 z}{\mathrm{d}x^2} &= \frac{\mathrm{d}}{\mathrm{d}x}(f_1' + f_2'\varphi') = \frac{\partial(f_1')}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}x} + \frac{\partial(f_1')}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}x} + \left[\frac{\partial(f_2')}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}x} + \frac{\partial(f_2')}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}x}\right]\varphi' + f_2'\varphi'' \\
&= f_{11}'' + f_{12}''\varphi' + (f_{21}'' + f_{22}''\varphi')\varphi' + f_2'\varphi'' \\
&= f_{11}'' + 2f_{12}''\varphi' + f_{22}''\varphi'^2 + f_2'\varphi''.
\end{aligned}$$

(4) 令 $u = x^2, v = x^3$, 则 $z = f(x, u, v)$,

$$\begin{aligned}
\frac{\mathrm{d}z}{\mathrm{d}x} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x} = f_1' + 2xf_2' + 3x^2 f_3', \\
\frac{\mathrm{d}^2 z}{\mathrm{d}x^2} &= \frac{\mathrm{d}}{\mathrm{d}x}(f_1' + 2xf_2' + 3x^2 f_3') \\
&= \left[\frac{\partial(f_1')}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial(f_1')}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial(f_1')}{\partial v}\frac{\partial v}{\partial x}\right] + 2f_2' + 2x\left[\frac{\partial(f_2')}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial(f_2')}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial(f_2')}{\partial v}\frac{\partial v}{\partial x}\right] \\
&\quad + 6xf_3' + 3x^2\left[\frac{\partial(f_3')}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial(f_3')}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial(f_3')}{\partial v}\frac{\partial v}{\partial x}\right] \\
&= (f_{11}'' + 2xf_{12}'' + 3x^2 f_{13}'') + 2f_2' + 2x(f_{21}'' + 2xf_{22}'' + 3x^2 f_{23}'') + 6xf_3' + 3x^2(f_{31}'' + 2xf_{32}'' + 3x^2 f_{33}'') \\
&= f_{11}'' + 4xf_{12}'' + 6x^2 f_{13}'' + 4x^2 f_{22}'' + 12x^3 f_{23}'' + 9x^4 f_{33}'' + 2f_2' + 6xf_3'.
\end{aligned}$$

$\square$

11. 计算得

$$\begin{aligned}
\frac{\partial z}{\partial x} &= \alpha x^{\alpha-1}y^\beta, \ \frac{\partial z}{\partial y} = \beta x^\alpha y^{\beta-1}, \ \frac{\partial^2 z}{\partial x \partial y} = (\alpha x^{\alpha-1}y^\beta)_y = \alpha\beta x^{\alpha-1}y^{\beta-1}, \\
\frac{\partial^2 z}{\partial x^2} &= (\alpha x^{\alpha-1}y^\beta)_x = \alpha(\alpha-1)x^{\alpha-2}y^\beta, \ \frac{\partial^2 z}{\partial y^2} = (\beta x^\alpha y^{\beta-1})_y = \beta(\beta-1)x^\alpha y^{\beta-2},
\end{aligned}$$

由 $(\alpha-1)(\beta-1) = \alpha\beta - (\alpha+\beta) + 1 = \alpha\beta$, 直接验证可知

$$\frac{\partial^2 z}{\partial x^2}\frac{\partial^2 z}{\partial y^2} = \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2.$$

$\square$

12. $z = \frac{1}{2}\ln((x-a)^2 + (y-b)^2)$, 计算得

$$\frac{\partial z}{\partial x} = \frac{x-a}{(x-a)^2 + (y-b)^2}, \ \frac{\partial z}{\partial y} = \frac{y-b}{(x-a)^2 + (y-b)^2},$$

所以有

$$\frac{\partial^2 z}{\partial x^2} = \left[\frac{x-a}{(x-a)^2 + (y-b)^2}\right]_x = \frac{(y-b)^2 - (x-a)^2}{[(x-a)^2 + (y-b)^2]^2},$$

$$\frac{\partial^2 z}{\partial y^2} = \left[\frac{y-b}{(x-a)^2 + (y-b)^2}\right]_y = \frac{(x-a)^2 - (y-b)^2}{[(x-a)^2 + (y-b)^2]^2},$$

显然满足 Laplace 方程

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0.$$

$\square$

13. 因为 $z = f(u, v)$ 在 (u, v) 可微, 有

$$\Delta z = f(u + \Delta u, v + \Delta v) - f(u, v) = f_u(u, v)\Delta u + f_v(u, v)\Delta v + \alpha_1 \Delta u + \beta_1 \Delta v, \qquad (9)$$

其中 α_1, β_1 满足

$$\lim_{(\Delta u, \Delta v) \to (0,0)} \alpha_1 = \lim_{(\Delta u, \Delta v) \to (0,0)} \beta_1 = 0. \qquad (10)$$

又因为 $u = \varphi(x, y)$ 和 $v = \psi(x, y)$ 在 (x, y) 均可微, 有

$$\begin{aligned}\Delta u =& \varphi(x + \Delta x, y + \Delta y) - \varphi(x, y) = \varphi_x(x, y)\Delta x + \varphi_y(x, y)\Delta y + \alpha_2 \Delta x + \beta_2 \Delta y, \\ \Delta v =& \psi(x + \Delta x, y + \Delta y) - \psi(x, y) = \psi_x(x, y)\Delta x + \psi_y(x, y)\Delta y + \alpha_3 \Delta x + \beta_3 \Delta y,\end{aligned} \qquad (11)$$

其中 α_i, β_i 满足

$$\lim_{(\Delta x, \Delta y) \to (0,0)} \alpha_i = \lim_{(\Delta x, \Delta y) \to (0,0)} \beta_i = 0, i = 2, 3. \qquad (12)$$

由 (11) 式,

$$\lim_{(\Delta x, \Delta y) \to (0,0)} \Delta u = \lim_{(\Delta x, \Delta y) \to (0,0)} \Delta v = 0, \qquad (13)$$

因此, (10) 式可写为

$$\lim_{(\Delta x, \Delta y) \to (0,0)} \alpha_1 = \lim_{(\Delta x, \Delta y) \to (0,0)} \beta_1 = 0. \qquad (14)$$

对复合函数 $z = f(\varphi(x,y), \psi(x,y))$, 由 (9), (11) 式,

$$\begin{aligned} \Delta z = & f_u \Delta u + f_v \Delta v + \alpha_1 \Delta u + \beta_1 \Delta v \\ = & f_u(\varphi_x \Delta x + \varphi_y \Delta y + \alpha_2 \Delta x + \beta_2 \Delta y) + f_v(\psi_x \Delta x + \psi_y \Delta y + \alpha_3 \Delta x + \beta_3 \Delta y) \\ & + \alpha_1(\varphi_x \Delta x + \varphi_y \Delta y + \alpha_2 \Delta x + \beta_2 \Delta y) + \beta_1(\psi_x \Delta x + \psi_y \Delta y + \alpha_3 \Delta x + \beta_3 \Delta y) \\ = & (f_u \varphi_x + f_v \psi_x) \Delta x + (f_u \varphi_y + f_v \psi_y) \Delta y + \alpha \Delta x + \beta \Delta y, \end{aligned}$$

其中 $\alpha = f_u \alpha_2 + f_v \alpha_3 + \varphi_x \alpha_1 + \psi_x \beta_1 + \alpha_1 \alpha_2 + \alpha_3 \beta_1$, $\beta = f_u \beta_2 + f_v \beta_3 + \varphi_y \alpha_1 + \psi_y \beta_1 + \alpha_1 \beta_2 + \beta_1 \beta_3$. 由 (12), (14) 式, $\lim_{(\Delta x, \Delta y) \to (0,0)} \alpha = \lim_{(\Delta x, \Delta y) \to (0,0)} \beta = 0$, 从而 $z = f(\varphi(x,y), \psi(x,y))$ 在 (x,y) 可微, 并且

$$\mathrm{d}z = (f_u \varphi_x + f_v \psi_x)\mathrm{d}x + (f_u \varphi_y + f_v \psi_y)\mathrm{d}y.$$

$\square$

14. $z = f(x,y), x = r\cos\theta, y = r\sin\theta$, 则

(1) 因为
$$r\frac{\partial z}{\partial r} = r\left(\frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r}\right) = r(f_x \cos\theta + f_y \sin\theta) = xf_x + yf_y = 0,$$

即, $\frac{\partial z}{\partial r} = 0$, 所以 $f(r\cos\theta, r\sin\theta) = F(\theta)$;

(2) 因为

$$\frac{\partial z}{\partial \theta} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \theta} = f_x \cdot (-r\sin\theta) + f_y \cdot r\cos\theta = -yf_x + xf_y = 0,$$

所以 $f(r\cos\theta, r\sin\theta) = G(r)$.

$\square$

复习题

1. (1) 令 $x = r\cos\theta, y = r\sin\theta, r > 0$, 则

$$\lim_{(x,y) \to (0,0)} \frac{x^3 + y^3}{x^2 + y^2} = \lim_{r \to 0^+} r(\cos^3\theta + \sin^3\theta) = 0.$$

(2) 令 $x = r\cos\theta, y = r\sin\theta, r > 0$, 由 L'Hospital 法则,

$$\lim_{(x,y) \to (0,0)} (x^2 + y^2)^{xy} = \lim_{r \to 0^+} r^{r^2 \sin 2\theta} = \lim_{r \to 0^+} e^{r^2 \sin 2\theta \ln r} = e^{\sin 2\theta \lim_{r \to 0^+} \frac{\ln r}{r^{-2}}} = e^{\sin 2\theta \lim_{r \to 0^+} (-r^2/2)} = 1.$$

$\square$

2. 见课后习题解答. $\square$

3. 任取 $(x_0, y_0) \in D$, 需证 $f(x,y)$ 在 (x_0, y_0) 连续. 因为 $f(x,y)$ 在 D 对 x 连续, 对

$\forall \varepsilon > 0, \exists \delta_1 > 0$, 对 $\forall x : |x - x_0| < \delta_1$, 有

$$|f(x, y_0) - f(x_0, y_0)| < \frac{\varepsilon}{2}. \tag{15}$$

又因为 $f_y(x, y)$ 有界, 可设 $|f_y(x, y)| \leqslant M, M > 0$. 取 $\delta_2 = \frac{\varepsilon}{2M}$, 对 $\forall y : |y - y_0| < \delta_2$, 由 Lagrange 中值定理, 有

$$|f(x, y) - f(x, y_0)| = |f_y(x, y_0 + \theta(y - y_0))||y - y_0| < \frac{\varepsilon}{2}. \tag{16}$$

取 $\delta = \min\{\delta_1, \delta_2\}$, 则对 $\forall (x, y) \in U((x_0, y_0); \delta) \cap D$, 由 (15) 和 (16) 式有

$$|f(x, y) - f(x_0, y_0)| \leqslant |f(x, y) - f(x, y_0) + f(x, y_0) - f(x_0, y_0)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

即, $f(x, y)$ 在 (x_0, y_0) 连续. $\qquad \square$

- 注: 对 $\forall \varepsilon > 0, \exists \delta_1 > 0$, 对 $\forall x : |x - x_0| < \delta_1$, 有 $|f(x, y) - f(x_0, y)| < \frac{\varepsilon}{2}$. 这样做是错误的, 因为 $f(x, y)$ 只是对 x 连续, 而不是一致连续, 这样的 δ_1 可能不存在.

4. 任取 $(x_0, y_0) \in D$, 需证 $f(x, y)$ 在 (x_0, y_0) 连续. 因为 $f(x, y)$ 在 D 对 x 连续, 对 $\forall \varepsilon > 0, \exists \delta_1 > 0$, 对 $\forall x : |x - x_0| < \delta_1$, 有

$$|f(x, y_0) - f(x_0, y_0)| < \frac{\varepsilon}{2} \text{ 或 } f(x_0, y_0) - \frac{\varepsilon}{2} < f(x, y_0) < f(x_0, y_0) + \frac{\varepsilon}{2}. \tag{17}$$

又因为 $f(x, y)$ 在 D 对 y 连续, $\exists \delta_2 > 0$, 对 $\forall y : |y - y_0| < \delta_2$, 有

$$\left| f\left(x_0 - \frac{\delta_1}{2}, y\right) - f\left(x_0 - \frac{\delta_1}{2}, y_0\right) \right| < \frac{\varepsilon}{2}, \ \left| f\left(x_0 + \frac{\delta_1}{2}, y\right) - f\left(x_0 + \frac{\delta_1}{2}, y_0\right) \right| < \frac{\varepsilon}{2}. \tag{18}$$

因此, 取 $\delta = \min\left\{\frac{\delta_1}{2}, \delta_2\right\} > 0$, 对 $\forall (x, y) \in U((x_0, y_0); \delta)$, (17) 和 (18) 式成立. 不妨设 $f(x, y)$ 关于 x 单调递增, 由 (18) 式得

$$f\left(x_0 - \frac{\delta_1}{2}, y_0\right) - \frac{\varepsilon}{2} < f\left(x_0 - \frac{\delta_1}{2}, y\right) < f(x, y) < f\left(x_0 + \frac{\delta_1}{2}, y\right) < f\left(x_0 + \frac{\delta_1}{2}, y_0\right) + \frac{\varepsilon}{2}. \tag{19}$$

对 $\forall (x, y) \in U((x_0, y_0); \delta)$, 由 (17) 式得

$$f\left(x_0 + \frac{\delta_1}{2}, y_0\right) + \frac{\varepsilon}{2} < f(x_0, y_0) + \varepsilon, \ f\left(x_0 - \frac{\delta_1}{2}, y_0\right) - \frac{\varepsilon}{2} > f(x_0, y_0) - \varepsilon. \tag{20}$$

由 (19) 和 (20) 式得

$$f(x_0, y_0) - \varepsilon < f(x, y) < f(x_0, y_0) + \varepsilon,$$

即对 $\forall \varepsilon > 0, \exists \delta > 0$, 对 $\forall (x, y) \in U((x_0, y_0); \delta)$, 有 $|f(x, y) - f(x_0, y_0)| < \varepsilon$, 即 $f(x, y)$ 在

(x_0, y_0) 连续, 由 (x_0, y_0) 的任意性, $f(x, y)$ 在 D 连续. □

- 注: 单调性条件很重要, 以下证明是错误的.

- 证明: 因为 $f(x, y)$ 对 x 连续, 对 $\forall \varepsilon > 0, \exists \delta_1 > 0$, 对 $\forall x : |x - x_0| < \delta_1$, 有 $|f(x, y_0) - f(x_0, y_0)| < \frac{\varepsilon}{2}$; 又因为 $f(x, y)$ 对 y 连续, $\exists \delta_2 > 0$, 对 $\forall y : |y - y_0| < \delta_2$, 有 $|f(x, y) - f(x, y_0)| < \frac{\varepsilon}{2}$. 因此, $\exists \delta = \min\{\delta_1, \delta_2\}$, 对 $\forall (x, y) \in U((x_0, y_0); \delta)$, 有 $|f(x, y) - f(x_0, y_0)| \leqslant |f(x, y) - f(x, y_0)| + |f(x, y_0) - f(x_0, y_0)| < \varepsilon$, 即连续. □

- 因为 δ_2 可能不存在, 除非 $f(x, y)$ 对 y 一致连续.

- $f(x, y) = \begin{cases} \frac{xy}{x^2 + y^2}, & (x, y) \neq (0, 0), \\ 0, & (x, y) = (0, 0), \end{cases}$ 对 x, y 均连续, 但是 $\lim_{(x,y) \to (0,0)} f(x, y)$ 不存在, 因而 $f(x, y)$ 在 $(0, 0)$ 不连续.

5. 见课后习题解答. □

6. 任取 $(x_0, y_0) \in D$, 需证 $f(x, y)$ 在 (x_0, y_0) 连续. 因为 $f(x, y)$ 在 D 关于 x 连续, 对 $\forall \varepsilon > 0, \exists \delta_1 > 0$, 对 $\forall x : |x - x_0| < \delta_1$, 有

$$|f(x, y_0) - f(x_0, y_0)| < \frac{\varepsilon}{2}, \tag{21}$$

取 $\delta_2 = \frac{\varepsilon}{2L}$, 对 $\forall y : |y - y_0| < \delta_2$, 由 Lipschitz 条件, 有

$$|f(x, y) - f(x, y_0)| \leqslant L|y - y_0| < \frac{\varepsilon}{2}. \tag{22}$$

取 $\delta = \min\{\delta_1, \delta_2\}$, 则对 $\forall (x, y) \in U((x_0, y_0); \delta) \cap D$, 由 (21) 和 (22) 式有

$$|f(x, y) - f(x_0, y_0)| \leqslant |f(x, y) - f(x, y_0)| + |f(x, y_0) - f(x_0, y_0)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

即, $f(x, y)$ 在 (x_0, y_0) 连续. □

- 注: 对 $\forall \varepsilon > 0, \exists \delta_1 > 0$, 对 $\forall x : |x - x_0| < \delta_1$, 有 $|f(x, y) - f(x_0, y)| < \frac{\varepsilon}{2}$. 这样做是错误的, 因为 $f(x, y)$ 只是对 x 连续, 而不是一致连续, 这样的 δ_1 可能不存在.

7. (1) 令 $x = r \cos\theta, y = r \sin\theta, r > 0$, 则

$$\lim_{(x,y) \to (0,0)} f(x, y) = \lim_{r \to 0^+} r^2 \sin\frac{1}{r} = 0 = f(0, 0),$$

即, $f(x, y)$ 在 $(0, 0)$ 连续.

(2) 易求得偏导数

$$
f_x(x,y) = \begin{cases} 2x \sin \frac{1}{\sqrt{x^2+y^2}} - \frac{x}{\sqrt{x^2+y^2}} \cos \frac{1}{\sqrt{x^2+y^2}}, & (x,y) \neq (0,0), \\ \lim_{\Delta x \to 0} \frac{f(\Delta x, 0) - f(0,0)}{\Delta x} = 0, & (x,y) = (0,0), \end{cases}
$$

$$
f_y(x,y) = \begin{cases} 2y \sin \frac{1}{\sqrt{x^2+y^2}} - \frac{y}{\sqrt{x^2+y^2}} \cos \frac{1}{\sqrt{x^2+y^2}}, & (x,y) \neq (0,0), \\ \lim_{\Delta y \to 0} \frac{f(0, \Delta y) - f(0,0)}{\Delta y} = 0, & (x,y) = (0,0), \end{cases}
$$

因为

$$
\lim_{(x,y) \to (0,0)} \frac{x}{\sqrt{x^2+y^2}} \cos \frac{1}{\sqrt{x^2+y^2}} \text{ 和 } \lim_{(x,y) \to (0,0)} \frac{y}{\sqrt{x^2+y^2}} \cos \frac{1}{\sqrt{x^2+y^2}}
$$

均不存在, 所以

$$
\lim_{(x,y) \to (0,0)} f_x(x,y) \text{ 和 } \lim_{(x,y) \to (0,0)} f_y(x,y)
$$

不存在, 即, 偏导数在 $(0,0)$ 不连续.

(3) 全增量

$$
\begin{aligned}
\Delta z =& [(\Delta x)^2 + (\Delta y)^2] \sin \frac{1}{\sqrt{(\Delta x)^2 + (\Delta y)^2}} \\
=& 0 \cdot \Delta x + 0 \cdot \Delta y + \Delta x \sin \frac{1}{\sqrt{(\Delta x)^2 + (\Delta y)^2}} \cdot \Delta x + \Delta y \sin \frac{1}{\sqrt{(\Delta x)^2 + (\Delta y)^2}} \cdot \Delta y,
\end{aligned}
$$

取 $A = B = 0$, $\alpha = \Delta x \sin \frac{1}{\sqrt{(\Delta x)^2 + (\Delta y)^2}}, \beta = \Delta y \sin \frac{1}{\sqrt{(\Delta x)^2 + (\Delta y)^2}}$, 显然 α, β 是 $(\Delta x, \Delta y) \to (0,0)$ 时的无穷小, 因而 $f(x,y)$ 在 $(0,0)$ 可微, 并且 $\mathrm{d}f|_{(0,0)} = 0$. $\qquad \square$

8. 记 $\Delta z = f(x,y) - f(x_0, y_0)$. 因为 $f_x(x,y)$ 在 (x_0, y_0) 存在, 有 $\lim_{\Delta x \to 0} \frac{f(x,y_0) - f(x_0,y_0)}{\Delta x} = f_x(x_0, y_0)$, 即

$$
f(x, y_0) - f(x_0, y_0) = (f_x(x_0, y_0) + \alpha) \Delta x, \tag{23}
$$

其中 α 满足 $\lim_{\Delta x \to 0} \alpha = 0$. 又因为 $f_y(x,y)$ 在 (x_0, y_0) 连续, 由 Lagrange 中值定理, 有

$$
f(x, y) - f(x, y_0) = f_y(x_0 + \Delta x, y_0 + \theta \Delta y) \Delta y = (f_y(x_0, y_0) + \beta) \Delta y, \tag{24}
$$

其中, $\theta \in (0,1), \beta$ 满足 $\lim_{(\Delta x, \Delta y) \to (0,0)} \beta = 0$. 由 (23) 和 (24) 式, 有

$$
\Delta z = f(x,y) - f(x, y_0) + f(x, y_0) - f(x_0, y_0) = (f_x(x_0, y_0) + \alpha) \Delta x + (f_y(x_0, y_0) + \beta) \Delta y,
$$

即, $f(x,y)$ 在 (x_0, y_0) 可微. $\qquad \square$

9. $z_x = f'(x+y) + f'(x-y), z_y = f'(x+y) - f'(x-y), z_{xy} = f''(x+y) - f''(x-y)$. $\qquad \square$

10. 直接计算可得. $\qquad \square$

11. 直接计算可得

$$\frac{\partial^m z}{\partial x^k \partial y^{m-k}} = \begin{cases} \frac{i!}{(i-k)!} \frac{j!}{[j-(m-k)]!}(x-x_0)^{i-k}(y-y_0)^{j-(m-k)}, & k \leqslant i \text{ and } m-k \leqslant j, \\ 0, & k > i \text{ or } m-k > j, \end{cases}$$

因此,

$$\left. \frac{\partial^m z}{\partial x^k \partial y^{m-k}} \right|_{(x_0,y_0)} = \begin{cases} i!j!, & k=i \text{ and } m-k=j, \\ 0, & \text{otherwise.} \end{cases}$$

$\square$

12. 形式符号

$$\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^m f(a+th, b+tk) = \sum_{i=0}^{m} C_m^i \frac{\partial^m f(a+th, b+tk)}{\partial x^i \partial y^{m-i}} h^i k^{m-i}, m=1,2,...,n.$$

当 $m=1$ 时, $F'(t) = \frac{\partial f}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial f}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} = \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right) f(a+th, b+tk)$. 假设 $F^{(m)}(t) = \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^m f(a+th, b+tk)$, 则

$$\begin{aligned} F^{(m+1)}(t) &= \left[\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^m f(a+th, b+tk)\right]_t \\ &= \left[\sum_{i=0}^{m} C_m^i \frac{\partial^m f(a+th, b+tk)}{\partial x^i \partial y^{m-i}} h^i k^{m-i}\right]_t \\ &= \sum_{i=0}^{m} C_m^i \left[\frac{\partial^m f(a+th, b+tk)}{\partial x^i \partial y^{m-i}}\right]_t h^i k^{m-i} \\ &= \sum_{i=0}^{m} C_m^i \left[h\frac{\partial^{m+1} f(a+th, b+tk)}{\partial x^{i+1} \partial y^{m-i}} + k\frac{\partial^{m+1} f(a+th, b+tk)}{\partial x^i \partial y^{m+1-i}}\right] h^i k^{m-i} \\ &= \sum_{i=0}^{m} C_m^i \frac{\partial^{m+1} f(a+th, b+tk)}{\partial x^{i+1} \partial y^{m-i}} h^{i+1} k^{m-i} + \sum_{i=0}^{m} C_m^i \frac{\partial^{m+1} f(a+th, b+tk)}{\partial x^i \partial y^{m+1-i}} h^i k^{m+1-i} \\ &= \sum_{i=1}^{m+1} C_m^{i-1} \frac{\partial^{m+1} f(a+th, b+tk)}{\partial x^i \partial y^{m+1-i}} h^i k^{m+1-i} + \sum_{i=0}^{m+1} C_m^i \frac{\partial^{m+1} f(a+th, b+tk)}{\partial x^i \partial y^{m+1-i}} h^i k^{m+1-i} \\ &= \sum_{i=0}^{m+1} (C_m^{i-1} + C_m^i) \frac{\partial^{m+1} f(a+th, b+tk)}{\partial x^i \partial y^{m+1-i}} h^i k^{m+1-i} \\ &= \sum_{i=0}^{m+1} C_{m+1}^i \frac{\partial^{m+1} f(a+th, b+tk)}{\partial x^i \partial y^{m+1-i}} h^i k^{m+1-i} \\ &= \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^{m+1} f(a+th, b+tk). \end{aligned}$$

由数学归纳法, 得证. $\square$

13. (Euler 定理) 记 $v(t) = f(tx, ty, tz)$, 由条件有

$$
\begin{aligned}
t\frac{\mathrm{d}v(t)}{\mathrm{d}t} =& t\left[\frac{\partial f}{\partial(tx)}\frac{\partial(tx)}{\partial t} + \frac{\partial f}{\partial(ty)}\frac{\partial(ty)}{\partial t} + \frac{\partial f}{\partial(tz)}\frac{\partial(tz)}{\partial t}\right] \\
=& tx\frac{\partial f(tx, ty, tz)}{\partial(tx)} + ty\frac{\partial f(tx, ty, tz)}{\partial(ty)} + tz\frac{\partial f(tx, ty, tz)}{\partial(tz)} \\
=& nf(tx, ty, tz) = nv(t),
\end{aligned}
$$

即有 $\frac{\mathrm{d}v(t)}{v(t)} = n\frac{\mathrm{d}t}{t}$, 解此微分方程得 $v(t) = Ct^n$, C 为常数. 易知, $C = v(1) = f(x, y, z)$, 因此, $v(t) = t^n v(1)$, 即, $f(tx, ty, tz) = t^n f(x, y, z)$, $u = f(x, y, z)$ 是 n 次齐次函数. $\qquad\square$

第 14 章 多元函数微分法的应用

习题 14.1 方向导数

1. 设单位方向向量 $\boldsymbol{l}_0 = (\cos\alpha, \cos\beta)$, 其中 $\cos\alpha, \cos\beta$ 是方向余弦, 梯度 $\nabla z(1,1) = (z_x(1,1), z_y(1,1)) = (2,-2)$, 则方向导数 $\left.\frac{\partial z}{\partial \boldsymbol{l}_0}\right|_{(1,1)} = \nabla z(1,1) \cdot \boldsymbol{l}_0 = 2\cos\alpha - 2\cos\beta$. 当 $\boldsymbol{l}_0$ 与 $\nabla z(1,1)$ 同向, 即 $\boldsymbol{l}_0 = (1/\sqrt{2}, -1/\sqrt{2})$ 时方向导数最大, 最大值为 $2\sqrt{2}$; 当 $\boldsymbol{l}_0$ 与 $\nabla z(1,1)$ 反向, 即 $\boldsymbol{l}_0 = (-1/\sqrt{2}, 1/\sqrt{2})$ 时方向导数最小, 最小值为 $-2\sqrt{2}$. $\qquad\square$

2. $w = xy^2 + yz^2 + zx^2$, 单位方向向量 $\boldsymbol{l}_0 = (\cos\alpha, \cos\beta, \cos\gamma) = (2,1,-1)/\sqrt{6}$, 梯度 $\nabla w(2,1,-1) = (w_x(2,1,-1), w_y(2,1,-1), w_z(2,1,-1)) = (-3,5,2)$, 则方向导数 $\left.\frac{\partial w}{\partial \boldsymbol{l}_0}\right|_{(2,1,-1)} = \nabla w(2,1,-1) \cdot \boldsymbol{l}_0 = -\sqrt{6}/2$. $\qquad\square$

3. 单位方向向量 $\boldsymbol{l}_0 = (\cos\alpha, \cos\beta, \cos\gamma) = (1/2, 1/\sqrt{2}, 1/2)$, 梯度 $\nabla u(1,1,1) = (1,1,1)$, 则方向导数 $\left.\frac{\partial u}{\partial \boldsymbol{l}_0}\right|_{(1,1,1)} = \nabla u(1,1,1) \cdot \boldsymbol{l}_0 = 1 + \sqrt{2}/2$. $\qquad\square$

4. (1) $\nabla z = (z_x, z_y) = \left(\frac{2x}{x^2+y^2}, \frac{2y}{x^2+y^2}\right), \|\nabla z\| = \frac{2}{\sqrt{x^2+y^2}}$;

(2) $\nabla u = (u_x, u_y, u_z) = \left(\frac{x}{\sqrt{x^2+y^2+z^2}}, \frac{y}{\sqrt{x^2+y^2+z^2}}, \frac{z}{\sqrt{x^2+y^2+z^2}}\right), \|\nabla u\| = 1$;

(3) $\nabla u = (u_x, u_y, u_z) = \left(\frac{-x}{\sqrt{(x^2+y^2+z^2)^3}}, \frac{-y}{\sqrt{(x^2+y^2+z^2)^3}}, \frac{-z}{\sqrt{(x^2+y^2+z^2)^3}}\right), \|\nabla u\| = \frac{1}{x^2+y^2+z^2}$. $\qquad\square$

5. $\mathbf{grad}u = (u_x, u_y, u_z) = (2x - 3yz, 2y - 3xz, 2z - 3xy)$, 欲使 $\mathbf{grad}u$ 垂直于 x 轴, 只需 $\mathbf{grad}u \cdot \boldsymbol{i} = 0$, 即有 $2x - 3yz = 0$, 即, 曲面 $2x = 3yz$ 上的点可使 $\mathbf{grad}u$ 垂直于 x 轴. $\qquad\square$

6. 仅证 (3). 设 $u = u(x,y)$, 则

$$\mathbf{grad}f(u) = \left(\frac{\mathrm{d}f}{\mathrm{d}u}\frac{\partial u}{\partial x}, \frac{\mathrm{d}f}{\mathrm{d}u}\frac{\partial u}{\partial y}\right) = f'(u)(u_x, u_y) = f'(u)\mathbf{grad}u.$$

$\qquad\square$

习题 14.2 多元函数 Taylor 公式

1. 由于 $f(x,y) = 2x^2 - xy - 3y^2 - 7x + y + 1$, 直接计算得

$$
\begin{aligned}
&f(1,-2) = -16, & &f_x(x,y) = 4x - y - 7, & &f_x(1,-2) = -1, \\
&f_y(x,y) = -x - 6y + 1, & &f_y(1,-2) = 12, & &f_{xx}(x,y) = 4, & &f_{xx}(1,-2) = 4, \\
&f_{xy}(x,y) = -1, & &f_{xy}(1,-2) = -1, & &f_{yy}(x,y) = -6, & &f_{yy}(1,-2) = -6,
\end{aligned}
$$

而三阶及以上的偏导数均为 0. 因此,

$$f(x,y) = f(1,-2) + (x-1)f_x(1,-2) + (y+2)f_y(1,-2)$$
$$+ \frac{1}{2}[(x-1)^2 f_{xx}(1,-2) + 2(x-1)(y+2)f_{xy}(1,-2) + (y+2)^2 f_{yy}(1,-2)]$$
$$= 2(x-1)^2 - (x-1)(y+2) - 3(y+2)^2 - (x-1) + 12(y+2) - 16.$$

$\square$

3. 因为 $f_x(x,y), f_y(x,y)$ 有界, 所以存在 $M_1, M_2 > 0$, 使得 $|f_x(x,y)| \leqslant M_1, |f_y(x,y)| \leqslant M_2$. 对 $\forall \varepsilon > 0, \exists \delta = \frac{\varepsilon}{M_1 + M_2}$, 对 $\forall (x_1,y_1),(x_2,y_2) \in D$ 只要 $|x_2 - x_1| < \delta, |y_2 - y_1| < \delta$, 就有

$$|f(x_1,y_1) - f(x_2,y_2)| = |f_x(x_1 + \theta(x_2 - x_1), y_1 + \theta(y_2 - y_1))(x_2 - x_1)$$
$$+ f_y(x_1 + \theta(x_2 - x_1), y_1 + \theta(y_2 - y_1))(y_2 - y_1)|$$
$$< M_1\delta + M_2\delta = \varepsilon,$$

即 $f(x,y)$ 在 D 一致连续. $\square$

4. 设 $\boldsymbol{p} = (a_1,b_1), \boldsymbol{q} = (a_2,b_2)$, 则方向余弦 $\cos\alpha_1 = \frac{a_1}{\|\boldsymbol{p}\|}, \cos\beta_1 = \frac{b_1}{\|\boldsymbol{p}\|}, \cos\alpha_2 = \frac{a_2}{\|\boldsymbol{q}\|}, \cos\beta_2 = \frac{b_2}{\|\boldsymbol{q}\|}$, 从而有

$$f_x \cos\alpha_1 + f_y \cos\beta_1 = f_{\boldsymbol{p}} = 0,$$
$$f_x \cos\alpha_2 + f_y \cos\beta_2 = f_{\boldsymbol{q}} = 0.$$

因为 $\boldsymbol{p}, \boldsymbol{q}$ 线性无关, 上述关于 f_x, f_y 的齐次线性方程组的系数行列式不为 0, 因而只有零解, 即 $f_x = f_y = 0$, 从而 f 是常值函数. $\square$

习题 14.3 多元函数的极值

1. (1) 由 $\begin{cases} \frac{\partial f}{\partial x} = 2x + y - 4 = 0, \\ \frac{\partial f}{\partial y} = 2y + x - 2 = 0 \end{cases}$ 知 $(2,0)$ 是 $f(x,y)$ 的唯一稳定点且 $f(2,0) = 0$, 同时 $\det \boldsymbol{H}_f(2,0) = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 3 > 0$ 且 $f_{xx}(2,0) = 2 > 0$, 所以 Hesse 矩阵 $\boldsymbol{H}_f(2,0)$ 正定, $(2,0)$ 是 $f(x,y)$ 的极小值点且极小值 $f(2,0) = 0$.

(2) 由 $\begin{cases} \frac{\partial f}{\partial x} = 3x^2 - 3y = 0, \\ \frac{\partial f}{\partial y} = 3y^2 - 3x = 0 \end{cases}$ 知 $(1,1),(0,0)$ 是 $f(x,y)$ 的稳定点, 同时 $\det \boldsymbol{H}_f(1,1) = \begin{vmatrix} 6 & -3 \\ -3 & 6 \end{vmatrix} = 27 > 0$ 且 $f_{xx}(1,1) = 6 > 0$, 所以 Hesse 矩阵 $\boldsymbol{H}_f(1,1)$ 正定, $(1,1)$ 是 $f(x,y)$ 的极小值点且极小值 $f(1,1) = -1$. 对稳定点 $(0,0)$, 易知 $\det \boldsymbol{H}_f(0,0) = \begin{vmatrix} 0 & -3 \\ -3 & 0 \end{vmatrix} = -9 < 0$, 因此 $(0,0)$ 点不是极值点.

(3) 由 $\begin{cases} \frac{\partial f}{\partial x} = 3x^2y + y^3 - y = 0, \\ \frac{\partial f}{\partial y} = 3xy^2 + x^3 - x = 0 \end{cases}$ 知 $P_0(0,0), P_{1,2}(\pm 1, 0), P_{3,4}(0, \pm 1), P_{5,6}(\pm\frac{1}{2}, \pm\frac{1}{2}), P_{7,8}(\pm\frac{1}{2}, \mp\frac{1}{2})$
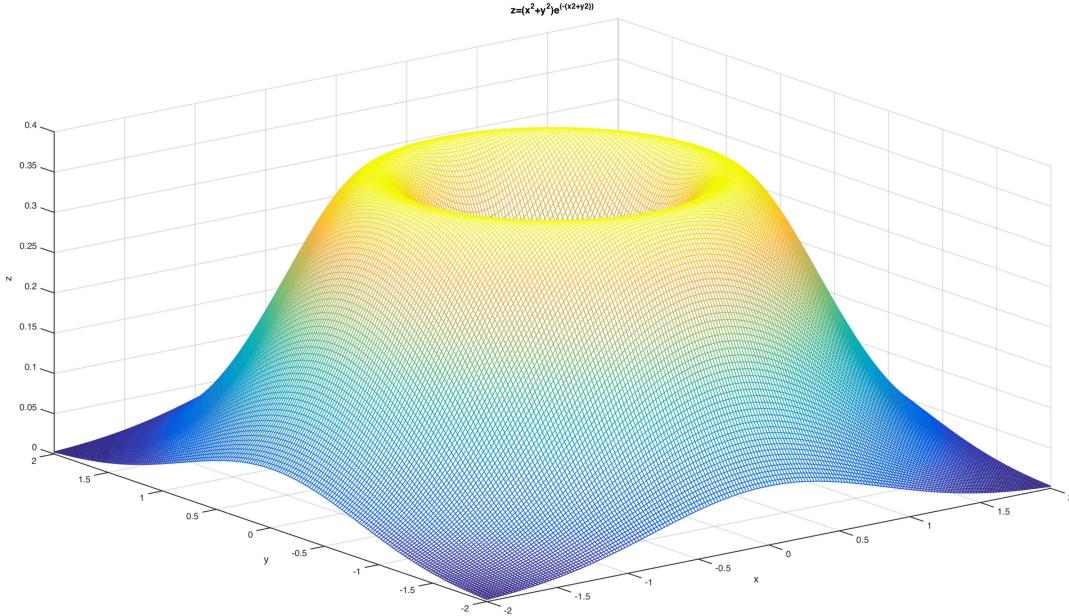
图 1: $f(x,y) = (x^2+y^2)e^{-(x^2+y^2)}$

是 $f(x,y)$ 的稳定点. 同时 $\det \boldsymbol{H}_f(x,y) = \begin{vmatrix} 6xy & 3(x^2+y^2)-1 \\ 3(x^2+y^2)-1 & 6xy \end{vmatrix}$. 依次考虑各点.

$\det \boldsymbol{H}_f(P_0) = -1 < 0$, P_0 不是极值点; $\det \boldsymbol{H}_f(P_{1,2,3,4}) = -4 < 0$, $P_{1,2,3,4}$ 不是极值点; $\det \boldsymbol{H}_f(P_{5,6}) = 2 > 0$ 且 $6xy|_{P_{5,6}} = \frac{3}{2} > 0$, 所以 $\boldsymbol{H}_f(P_{5,6})$ 正定, $P_{5,6}$ 是极小值点, 极小值是 $f(P_{5,6}) = -\frac{1}{8}$; $\det \boldsymbol{H}_f(P_{7,8}) = 2 > 0$ 且 $6xy|_{P_{7,8}} = -\frac{3}{2} < 0$, 所以 $\boldsymbol{H}_f(P_{7,8})$ 负定, $P_{7,8}$ 是极大值点, 极大值是 $f(P_{7,8}) = \frac{1}{8}$.

(4) 令 $x = r\cos\theta, y = r\sin\theta$, 则 $z = r^2 e^{-r^2}$. 由 $z' = 2r(1-r^2)e^{-r^2} = 0$ 知 $r_0 = 0, r_1 = 1$ 是 z 的稳定点, 同时 $z''(0) = 2(1-3r^2+2r^4)e^{-r^2}|_0 = 2 > 0$, 因此, $r = 0$ 即 $(x,y) = (0,0)$ 是极小值点, 极小值是 $z(0) = 0$. 对 $r = 1$, 计算知 $z''(1) = 0$, 但是易知, $0 < r < 1$ 时 $z' > 0$, $r > 1$ 时 $z' < 0$, 从而知 $r_1 = 1$ 即 $(x,y) \in \{(x,y)|x^2+y^2=1\}$ 是 z 的极大值点, 极大值是 $z(x,y) = e^{-1}$. (原函数 $f(x,y) = (x^2+y^2)e^{-(x^2+y^2)}$ 的图像见图 1)

$\square$

2. (1) 由 $\begin{cases} \frac{\partial f}{\partial x} = -8xy + 12x^3 = 0, \\ \frac{\partial f}{\partial y} = 2y - 4x^2 = 0 \end{cases}$ 知 $(0,0)$ 是 $f(x,y)$ 的稳定点且 $f(0,0) = 0$, 然而

$\det \boldsymbol{H}_f(0,0) = \begin{vmatrix} 0 & 0 \\ 0 & 2 \end{vmatrix} = 0$. 沿曲线 $y = 2x^2$, $f(x,y) = -x^4 \leqslant 0 = f(0,0), x \in U^{\circ}(0)$, 沿曲线 $y = 4x^2$, $f(x,y) = 3x^4 \geqslant 0 = f(0,0), x \in U^{\circ}(0)$, 即, 对 $\forall (x,y) \in U^{\circ}(0,0)$, 可能有 $f(x,y) < f(0,0)$, 也可能有 $f(x,y) > f(0,0)$, 从而 $f(x,y)$ 在原点取不到极值. (见图 2)

(2) 沿直线 $x = 0$, $f(0,y) = y^2$ 在原点取到极小值; 沿直线 $y = 0$, $f(x,0) = 3x^4$ 在原点取到极小值; 沿直线 $y = kx, k \neq 0$, $g(x) \triangleq f(x,kx) = k^2x^2 - 4kx^3 + 3x^4$, 易知,
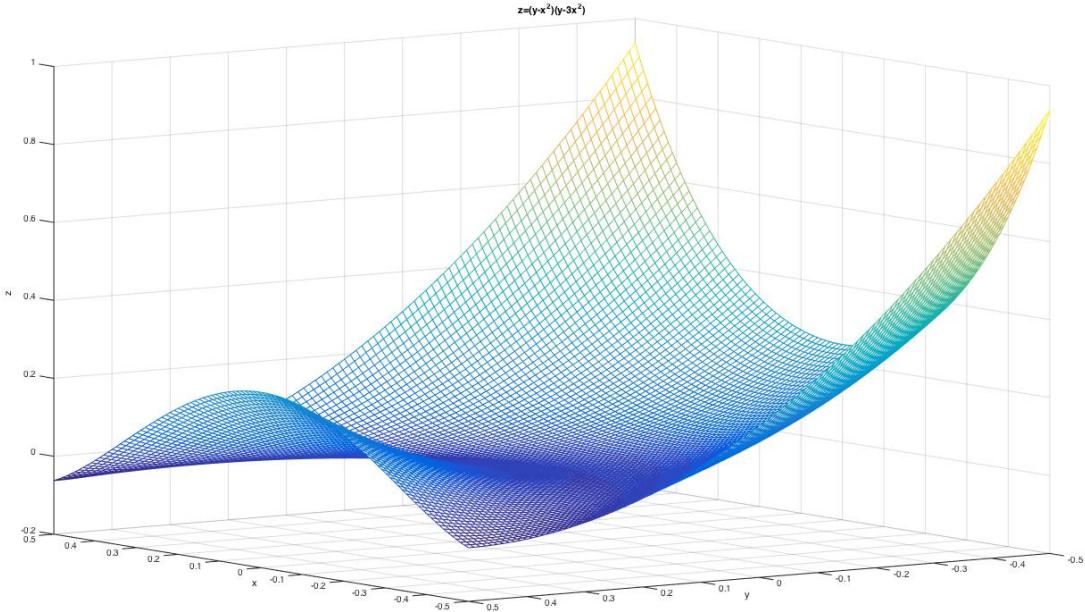
图 2: $f(x, y) = (y - x^2)(y - 3x^2)$

$g'(0) = 0, g''(0) = 2k^2 > 0$, 因而 $g(x)$ 在原点取到极小值. 综上, 沿过原点的任意直线, $f(x, y)$ 均在原点取到极小值. $\qquad\square$

3. (1) 首先考虑内部点的极值. 由 $\begin{cases} \frac{\partial z}{\partial x} = y = 0, \\ \frac{\partial z}{\partial y} = x = 0 \end{cases}$ 得稳定点 $P_1(0, 0)$, 并且 $z(P_1) = 0$. 再考虑边界 $x^2 + y^2 = 4$ 上的最值. 对边界曲线 $y = \sqrt{4 - x^2}, -2 \leqslant x \leqslant 2$, $z = x\sqrt{4 - x^2}$ 是一元函数, 由 $z' = \frac{2(2 - x^2)}{\sqrt{4 - x^2}} = 0$ 得稳定点 $x_{2,3} = \pm\sqrt{2}$, 对应的 $y_{2,3} = \sqrt{2}$. 又 $z'' = -\frac{4x}{\sqrt{4 - x^2}} - \frac{2 - x^2}{(4 - x^2)\sqrt{4 - x^2}}$, 所以 $z''(x_{2,3}) = \mp 4$, $x_2 = \sqrt{2}$ 是极大值点, $z(P_2) = z(x_2, y_2) = z(\sqrt{2}, \sqrt{2}) = 2$; $x_3 = -\sqrt{2}$ 是极小值点, $z(P_3) = z(x_3, y_3) = z(-\sqrt{2}, \sqrt{2}) = -2$. 记 $x_{4,5} = \pm 2$, 对应的 $y_{4,5} = 0$, $z(P_4) = z(x_4, y_4) = z(2, 0) = 0, z(P_5) = z(x_5, y_5) = z(-2, 0) = 0$. 对边界曲线 $y = -\sqrt{4 - x^2}, -2 \leqslant x \leqslant 2$, $z = -x\sqrt{4 - x^2}$ 是一元函数, 由 $z' = -\frac{2(2 - x^2)}{\sqrt{4 - x^2}} = 0$ 得稳定点 $x_{6,7} = \pm\sqrt{2}$, 对应的 $y_{6,7} = -\sqrt{2}$. 又 $z'' = \frac{4x}{\sqrt{4 - x^2}} + \frac{2 - x^2}{(4 - x^2)\sqrt{4 - x^2}}$, 所以 $z''(x_{6,7}) = \pm 4$, $x_6 = \sqrt{2}$ 是极小值点, $z(P_6) = z(x_6, y_6) = z(\sqrt{2}, -\sqrt{2}) = -2$; $x_7 = -\sqrt{2}$ 是极大值点, $z(P_7) = z(x_7, y_7) = z(-\sqrt{2}, -\sqrt{2}) = 2$. 比较 $z(P_i), i = 1, 2, ..., 7$ 的值可知, $P_2(\sqrt{2}, \sqrt{2}), P_7(-\sqrt{2}, -\sqrt{2})$ 是最大值点, 最大值 $z_{\max} = 2$; $P_3(-\sqrt{2}, \sqrt{2}), P_6(\sqrt{2}, -\sqrt{2})$ 是最小值点, 最小值 $z_{\min} = -2$.

(2) 首先考虑内部点的极值. 由 $\begin{cases} \frac{\partial z}{\partial x} = 2x + y = 0, \\ \frac{\partial z}{\partial y} = x - 2y = 0 \end{cases}$ 得稳定点 $P_1(0, 0)$, 并且 $z(P_1) = 0$. 再考虑边界 $|x| + |y| = 1$ 上的最值. 对边界曲线 $y = -x + 1, 0 \leqslant x \leqslant 1$, $z = x^2 + x(-x + 1) - (-x + 1)^2 = -x^2 + 3x - 1$ 是一元函数, 由 $z' = -2x + 3 = 0$ 得零点 $\frac{3}{2} \notin [0, 1]$, 因此无稳定点. 对边界曲线 $y = x + 1, -1 \leqslant x \leqslant 0$, $z = x^2 + x(x + 1) - (x + 1)^2 = x^2 - x - 1$ 是一元函数, 由 $z' = 2x - 1 = 0$ 得零点 $\frac{1}{2} \notin [-1, 0]$, 因此无稳定点. 对边界曲线 $y = -x - 1, -1 \leqslant x \leqslant 0$,

$z = x^2 + x(-x-1) - (-x-1)^2 = -x^2 - 3x - 1$ 是一元函数, 由 $z' = -2x - 3 = 0$ 得零点 $-\frac{3}{2} \notin [-1, 0]$, 因此无稳定点. 对边界曲线 $y = x - 1, 0 \leqslant x \leqslant 1, z = x^2 + x(x-1) - (x-1)^2 = x^2 + x - 1$ 是一元函数, 由 $z' = 2x + 1 = 0$ 得零点 $-\frac{1}{2} \notin [0, 1]$, 因此无稳定点. 同时, $z(P_{2,3}) = z(\pm 1, 0) = 1, z(P_{4,5}) = z(0, \pm 1) = -1$. 比较 $z(P_i), i = 1, 2, ..., 5$ 的值可知, $P_{2,3}(\pm 1, 0)$ 是最大值点, 最大值 $z_{\max} = 1$; $P_{4,5}(0, \pm 1)$ 是最小值点, 最小值 $z_{\min} = -1$. $\qquad \square$

4. 设三角形三个角分别是 A, B, C, 则 $C = \pi - (A+B)$. 令 $x = \cos A, y = \cos B, z = \cos C$, 要求 $u = \sin A \sin B \sin C$ 的最大值. 因为

$$u = \sin A \sin B \sin C = \sin A \sin B \sin(A + B) = \sin A \sin B (\sin A \cos B + \cos A \sin B)$$
$$= \cos B (1 - \cos^2 A) \sqrt{1 - \cos^2 B} + \cos A (1 - \cos^2 B) \sqrt{1 - \cos^2 A}$$
$$= y(1 - x^2) \sqrt{1 - y^2} + x(1 - y^2) \sqrt{1 - x^2},$$

由

$$\frac{\partial u}{\partial x} = \frac{\sqrt{1 - y^2}}{\sqrt{1 - x^2}} [-2xy\sqrt{1 - x^2} + (1 - 2x^2)\sqrt{1 - y^2}] = 0,$$
$$\frac{\partial u}{\partial y} = \frac{\sqrt{1 - x^2}}{\sqrt{1 - y^2}} [-2xy\sqrt{1 - y^2} + (1 - 2y^2)\sqrt{1 - x^2}] = 0,$$

得稳定点 $(1/2, 1/2) \in (-1, 1)^2$, 即, 当 $A = B = C = \frac{\pi}{3}$ 时, u 取得最大值, $u_{\max} = \frac{3}{8}\sqrt{3}$. $\qquad \square$

5. 设三角形三边长分别为 a, b, c, 面积为 S, 则 $a + b + c = 2p$, 由海伦公式,

$$S^2 = p(p-a)(p-b)(p-c) = p(p-a)(p-b)(a+b-p).$$

由

$$\frac{\partial S^2}{\partial a} = p(p-b)(2p - 2a - b) = 0,$$
$$\frac{\partial S^2}{\partial b} = p(p-a)(2p - a - 2b) = 0,$$

得稳定点 $(2p/3, 2p/3) \in (0, 2p)^2$, 即, 当 $a = b = c = 2p/3$ 时, S 取到最大, $S_{\max} = \frac{p^2}{9}\sqrt{3}$. $\qquad \square$

6. 设圆 O 半径 r 已知, 三角形 ABC 外切于圆 O, 三角形三个顶点到切点的距离分别是 x, y, z, 则 $p = x + y + z$ 恰是半周长. 由海伦公式, $S^2 = p(p-x-y)(p-y-z)(p-x-z) = pxyz$, 同时, $S = pr$, 因此, $p^2 r^2 = pxyz$, 从而

$$z = \frac{(x+y)r^2}{xy - r^2}.$$

目标函数

$$S = pr = \left[x + y + \frac{(x+y)r^2}{xy - r^2}\right] r = \frac{xy(x+y)}{xy - r^2} r.$$

由

$$\frac{\partial S}{\partial x} = \frac{x^2 y - (2x+y)r^2}{(xy-r^2)^2} yr = 0,$$

$$\frac{\partial S}{\partial y} = \frac{xy^2 - (x+2y)r^2}{(xy-r^2)^2} xr = 0,$$

得稳定点 $(\sqrt{3}r, \sqrt{3}r)$, 此时 $z = \sqrt{3}r$. 因此当 $x = y = z = \sqrt{3}r$, 即三角形为正三角形时面积最小, $S_{\min} = 3\sqrt{3}r^2$. $\hfill\square$

习题 14.4 隐函数

1. 令 $F(x,y) = \cos x + \sin y - \ln(e+xy)$, 则 $F(0,0) = 0$, $F(x,y)$ 在 $U((0,0))$ 连续, 且 $F_y(0,0) = \cos y - \frac{x}{e+xy}\Big|_{(0,0)} = 1 \neq 0$, 由定理 14.4.2, $F(x,y) = 0$ 在 $U((0,0))$ 确定了隐函数 $y = f(x)$. 但是 $F(x,y) = 0$ 在 $U((0,0))$ 不能确定隐函数 $x = g(y)$. 事实上, 对于 $F_x(x,y) = -\sin x - \frac{y}{e+xy}$, 如果有 $\sin x_0 < 0, \frac{y_0}{e+x_0 y_0} < 0$, 从而 $F_x(x_0, y_0) < 0$, 则必有 $F_x(-x_0, -y_0) > 0$, 即, F_x 在 $U((0,0))$ 的符号不定, 因而 F 关于 x 不具有单调性, 从而 $F(x,y) = 0$ 在 $U((0,0))$ 不能确定隐函数 $x = g(y)$. $\hfill\square$

2. 令 $F(x,y,z) = x + y + z + xyz - 3e^{xz} + e^{yz}$, 则 $F(0,2,0) = 0$, $F(x,y,z)$ 在 $U((0,2,0))$ 连续, 且 $F_x(0,2,0) = F_y(0,2,0) = 1 \neq 0, F_z(0,2,0) = 3 \neq 0$, 由定理 14.4.3, $F(x,y,z) = 0$ 在 $U((0,2,0))$ 确定了隐函数 $x = x(y,z), y = y(x,z), z = z(x,y)$. $\hfill\square$

4. 把 y 看作 x 的函数, $e^{x+y} = xy$ 两端对 x 求导两次得

$$e^{x+y}(1+y') = y + xy',$$

$$e^{x+y}(1+y')^2 + e^{x+y}y'' = 2y' + xy'',$$

从而有

$$y'' = \frac{e^{x+y}(1+y')^2 - 2y'}{x - e^{x+y}},$$

其中

$$y' = \frac{e^{x+y} - y}{x - e^{x+y}}.$$

$\hfill\square$

6. 把 z 看作 x, y 的函数, 对

$$\cos^2 x + \cos^2 y + \cos^2 z = 1$$

式两端求偏导, 有

$$-\sin 2x - z_x \sin 2z = 0,$$

$$-\sin 2y - z_y \sin 2z = 0,$$

继续求偏导有

$$-2\cos 2x - z_{xx}\sin 2z - 2(z_x)^2\cos 2z = 0,$$

$$-2\cos 2y - z_{yy}\sin 2z - 2(z_y)^2\cos 2z = 0,$$

$$-z_{xy}\sin 2z - 2z_x z_y\cos 2z = 0,$$

从而得

$$z_{xx} = -2\frac{\cos 2x}{\sin 2z}(1 + (z_x)^2),$$

$$z_{yy} = -2\frac{\cos 2y}{\sin 2z}(1 + (z_y)^2),$$

$$z_{xy} = -2\frac{\cos 2z}{\sin 2z}z_x z_y,$$

其中,

$$z_x = -\frac{\sin 2x}{\sin 2z}, z_y = -\frac{\sin 2y}{\sin 2z}.$$

$\square$

7. 令 $F(x,y,z) = x - az - \varphi(y - bz)$, 则 $F_x = 1, F_y = -\varphi', F_z = -a + b\varphi'$, 因此,

$$a\frac{\partial z}{\partial x} + b\frac{\partial z}{\partial y} = -a\frac{F_x}{F_z} - b\frac{F_y}{F_z} = -a\left(\frac{1}{-a + b\varphi'}\right) - b\left(\frac{-\varphi'}{-a + b\varphi'}\right) = 1.$$

$\square$

8. 易知, $F_x = F_1' - \frac{z}{x^2}F_2', F_y = -\frac{z}{y^2}F_1' + F_2', F_z = \frac{1}{y}F_1' + \frac{1}{x}F_2'$, 因此

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = -x\frac{F_x}{F_z} - y\frac{F_y}{F_z} = z - xy.$$

$\square$

10. 对 $\forall(x_0, y_0) \in D$, 易知, $\nabla f(x_0, y_0) = (z_x(x_0,y_0), z_y(x_0,y_0))$. f 的等高线由 $f(x,y) = c$ 确定, 令 $F(x,y) = f(x,y) - c$, 则 $F(x,y) = 0$ 确定的隐函数即是等高线, 在 (x_0, y_0) 点的切线斜率为 $\frac{\mathrm{d}y}{\mathrm{d}x}\Big|_{(x_0,y_0)} = -\frac{F_x(x_0,y_0)}{F_y(x_0,y_0)} = -\frac{z_x(x_0,y_0)}{z_y(x_0,y_0)}$, 故过 (x_0, y_0) 点的切线是 $y - y_0 = -\frac{z_x(x_0,y_0)}{z_y(x_0,y_0)}(x - x_0)$, 即

$$\frac{x - x_0}{-\frac{z_y(x_0,y_0)}{z_x(x_0,y_0)}} = \frac{y - y_0}{1},$$

其方向是 $\boldsymbol{l} = \left(-\frac{z_y(x_0,y_0)}{z_x(x_0,y_0)}, 1\right)$. 显然, $\boldsymbol{l} \cdot \nabla f = 0$, 得证. $\square$

习题 14.5 隐函数组

1. 在哪些点, 由隐函数组定理可以保证方程组 $\begin{cases} x^2 + y^2 + z^2 = 1, \\ x + y + z = 0 \end{cases}$ 关于变量 z 可以参数化?

并求 $\frac{dx}{dz}, \frac{dy}{dz}$.

解. 令 $\begin{cases} F(x,y,z) = x^2 + y^2 + z^2 - 1, \\ G(x,y,z) = x + y + z, \end{cases}$ 由 $\frac{\partial(F,G)}{\partial(x,y)} = \begin{vmatrix} 2x & 2y \\ 1 & 1 \end{vmatrix} = 2(x-y) \neq 0$ 知, 方程

组 $\begin{cases} F(x,y,z) = 0, \\ G(x,y,z) = 0 \end{cases}$ 在 $x \neq y$ 的点附近可以关于变量 z 可以参数化. 因为 x, y 都是

z 的函数, 方程组 $\begin{cases} x^2 + y^2 + z^2 = 1, \\ x + y + z = 0 \end{cases}$ 两端对 z 求导, 得到 $\begin{cases} 2x\frac{dx}{dz} + 2y\frac{dy}{dz} + 2z = 0, \\ \frac{dx}{dz} + \frac{dy}{dz} + 1 = 0, \end{cases}$ 即

$\begin{cases} 2x\frac{dx}{dz} + 2y\frac{dy}{dz} = -2z, \\ \frac{dx}{dz} + \frac{dy}{dz} = -1. \end{cases}$ 当 $x \neq y$ 时, $\frac{\partial(F,G)}{\partial(x,y)} = \begin{vmatrix} 2x & 2y \\ 1 & 1 \end{vmatrix} \neq 0$, 由 Cramer 法则,

$$\frac{dx}{dz} = \frac{\begin{vmatrix} -2z & 2y \\ -1 & 1 \end{vmatrix}}{\begin{vmatrix} 2x & 2y \\ 1 & 1 \end{vmatrix}} = \frac{y-z}{x-y}, \quad \frac{dy}{dz} = \frac{\begin{vmatrix} 2x & -2z \\ 1 & -1 \end{vmatrix}}{\begin{vmatrix} 2x & 2y \\ 1 & 1 \end{vmatrix}} = \frac{z-x}{x-y}.$$

$\square$

2. 把 y, z 看作 x 的函数, 方程组 $\begin{cases} x^2 + y^2 + z^2 = r^2, \\ x^2 + y^2 = rx \end{cases}$ 两端对 x 求导, 得到关于 $\frac{dy}{dx}, \frac{dz}{dx}$

的方程组 $\begin{cases} 2x + 2y\frac{dy}{dx} + 2z\frac{dz}{dx} = 0, \\ 2x + 2y\frac{dy}{dx} = r, \end{cases}$ 即 $\begin{cases} 2y\frac{dy}{dx} + 2z\frac{dz}{dx} = -2x, \\ 2y\frac{dy}{dx} = r - 2x. \end{cases}$ 当 $\begin{vmatrix} 2y & 2z \\ 2y & 0 \end{vmatrix} = -4yz \neq 0$ 时,

由 Cramer 法则,

$$\frac{dy}{dx} = \frac{\begin{vmatrix} -2x & 2z \\ r-2x & 0 \end{vmatrix}}{\begin{vmatrix} 2y & 2z \\ 2y & 0 \end{vmatrix}} = \frac{r-2x}{2y}, \quad \frac{dz}{dx} = \frac{\begin{vmatrix} 2y & -2x \\ 2y & r-2x \end{vmatrix}}{\begin{vmatrix} 2y & 2z \\ 2y & 0 \end{vmatrix}} = -\frac{r}{2z}.$$

把 x, z 看作 y 的函数, 方程组 $\begin{cases} x^2 + y^2 + z^2 = r^2, \\ x^2 + y^2 = rx \end{cases}$ 两端对 y 求导, 得到关于 $\frac{dx}{dy}, \frac{dz}{dy}$ 的方程

组 $\begin{cases} 2x\frac{dx}{dy} + 2y + 2z\frac{dz}{dy} = 0, \\ 2x\frac{dx}{dy} + 2y = r\frac{dx}{dy}, \end{cases}$ 即 $\begin{cases} 2x\frac{dx}{dy} + 2z\frac{dz}{dy} = -2y, \\ (2x-r)\frac{dx}{dy} = -2y. \end{cases}$ 当 $\begin{vmatrix} 2x & 2z \\ 2x-r & 0 \end{vmatrix} = -2z(2x-r) \neq 0$

时, 由 Cramer 法则,

$$\frac{dx}{dy} = \frac{\begin{vmatrix} -2y & 2z \\ -2y & 0 \end{vmatrix}}{\begin{vmatrix} 2x & 2z \\ 2x-r & 0 \end{vmatrix}} = \frac{2y}{r-2x}, \quad \frac{dz}{dy} = \frac{\begin{vmatrix} 2x & -2y \\ 2x-r & -2y \end{vmatrix}}{\begin{vmatrix} 2x & 2z \\ 2x-r & 0 \end{vmatrix}} = \frac{yr}{z(2x-r)}.$$

$\square$

3. 方程组分别对 x, y 求偏导, 注意到 $u = u(x, y), v = v(x, y)$, 得

$$\begin{cases} u + xu_x - yv_x = 0, \\ yu_x + v + xv_x = 0, \end{cases} \quad \begin{cases} xu_y - v - yv_y = 0, \\ u + yu_y + xv_y = 0, \end{cases}$$

当 $\begin{vmatrix} x & -y \\ y & x \end{vmatrix} = x^2 + y^2 \neq 0$ 即 $(x, y) \neq (0, 0)$ 时, 由 Cramer 法则,

$$u_x = -\frac{xu + yv}{x^2 + y^2}, \ v_x = -\frac{yu - xv}{x^2 + y^2}, \ u_y = \frac{xv - yu}{x^2 + y^2}, \ v_y = -\frac{xu + yv}{x^2 + y^2}.$$

□

4. 方程组 $\begin{cases} x - u^2 - yv = 0, \\ y - v^2 - xu = 0 \end{cases}$ 确定 u, v 是 x, y 的函数, 求 u_x, v_x, u_y, v_y.

解. 方程组分别对 x, y 求偏导, 注意到 $u = u(x, y), v = v(x, y)$, 得

$$\begin{cases} 1 - 2uu_x - yv_x = 0, \\ -2vv_x - u - xu_x = 0, \end{cases} \quad \begin{cases} -2uu_y - v - yv_y = 0, \\ 1 - 2vv_y - xu_y = 0, \end{cases}$$

当 $\begin{vmatrix} -2u & -y \\ -x & -2v \end{vmatrix} = 4uv - xy \neq 0$ 时, 由 Cramer 法则,

$$u_x = \frac{2v + yu}{4uv - xy}, \ v_x = -\frac{x + 2u^2}{4uv - xy}, \ u_y = -\frac{y + 2v^2}{4uv - xy}, \ v_y = \frac{2u + xv}{4uv - xy}.$$

□

5. 方程组分别对 x, y 求偏导, 注意到 $u = u(x, y), v = v(x, y)$, 得

$$\begin{cases} u_x = f_1'(u + xu_x) + f_2'v_x, \\ v_x = g_1'(u_x - 1) + g_2' \cdot 2yvv_x, \end{cases} \quad \begin{cases} u_y = f_1' \cdot xu_y + f_2'(1 + v_y), \\ v_y = g_1' \cdot u_y + g_2'(v^2 + 2yvv_y), \end{cases}$$

即

$$\begin{cases} (1 - xf_1')u_x - f_2'v_x = uf_1', \\ g_1'u_x + (1 - 2yvg_2')v_x = -g_1', \end{cases} \quad \begin{cases} (1 - xf_1')u_y - f_2'v_y = f_2', \\ g_1'u_y + (1 - 2yvg_2')v_y = v^2g_2', \end{cases}$$

当 $\begin{vmatrix} 1 - xf_1' & -f_2' \\ g_1' & 1 - 2yvg_2' \end{vmatrix} \neq 0$ 时, 由 Cramer 法则,

$$u_x = \frac{\begin{vmatrix} uf_1' & -f_2' \\ -g_1' & 1 - 2yvg_2' \end{vmatrix}}{\begin{vmatrix} 1 - xf_1' & -f_2' \\ g_1' & 1 - 2yvg_2' \end{vmatrix}}, \quad v_x = \frac{\begin{vmatrix} 1 - xf_1' & uf_1' \\ g_1' & 1 - g_1' \end{vmatrix}}{\begin{vmatrix} 1 - xf_1' & -f_2' \\ g_1' & 1 - 2yvg_2' \end{vmatrix}},$$

$$u_y = \frac{\begin{vmatrix} f_2' & -f_2' \\ v^2g_2' & 1 - 2yvg_2' \end{vmatrix}}{\begin{vmatrix} 1 - xf_1' & -f_2' \\ g_1' & 1 - 2yvg_2' \end{vmatrix}}, \quad v_y = \frac{\begin{vmatrix} 1 - xf_1' & f_2' \\ g_1' & v^2g_2' \end{vmatrix}}{\begin{vmatrix} 1 - xf_1' & -f_2' \\ g_1' & 1 - 2yvg_2' \end{vmatrix}}.$$

$\square$

6. 方程组 $\begin{cases} x = u + v, \\ y = u^2 + v^2 \end{cases}$ 确定了反函数组, 求 u_x, v_x, u_y, v_y.

解. 方程组分别对 x, y 求偏导, 注意到 $u = u(x, y), v = v(x, y)$, 得

$$\begin{cases} 1 = u_x + v_x, \\ 0 = 2uu_x + 2vv_x, \end{cases} \quad \begin{cases} 0 = u_y + v_y, \\ 1 = 2uu_y + 2vv_y, \end{cases}$$

当 $\begin{vmatrix} 1 & 1 \\ 2u & 2v \end{vmatrix} = 2(v - u) \neq 0$ 即 $v \neq u$ 时, 由 Cramer 法则,

$$u_x = \frac{v}{v - u}, \; v_x = \frac{u}{u - v}, \; u_y = \frac{1}{2(u - v)}, \; v_y = \frac{1}{2(v - u)}.$$

$\square$

习题 14.6 几何应用

1. $t = 3$ 时, $x_0 = \frac{9}{4}, y_0 = \frac{4}{9}, z_0 = 9$, 并且 $\frac{dx}{dt}\big|_3 = \frac{15}{16}, \frac{dy}{dt}\big|_3 = -\frac{5}{27}, \frac{dz}{dt}\big|_3 = 6$, 所以切线方程为

$$\begin{cases} x = \frac{9}{4} + \frac{15}{16}t, \\ y = \frac{4}{9} - \frac{5}{27}t, \\ z = 9 + 6t, \end{cases}$$

法平面方程为 $\frac{15}{16}\left(x - \frac{9}{4}\right) - \frac{5}{27}\left(y - \frac{4}{9}\right) + 6(z - 9) = 0$. $\square$

2. 依题意, 曲线 $x = t, y = -t^2, z = t^3$ 在 t 点的切线方向 $\boldsymbol{\tau} = (1, -2t, 3t^2)$ 与平面 $x + 2y + z = 1$ 的法方向 $\boldsymbol{n} = (1, 2, 1)$ 正交, 因此有 $\boldsymbol{\tau} \cdot \boldsymbol{n} = 0$, 得到 $t_1 = 1, t_2 = \frac{1}{3}$, 对应的切线

方程分别为

$$
\begin{cases}
x = 1 + t, \\
y = -1 - 2t, \\
z = 1 + 3t,
\end{cases}
\quad
\begin{cases}
x = \frac{1}{3} + t, \\
y = -\frac{1}{9} - \frac{2}{3}t, \\
z = \frac{1}{27} + \frac{1}{9}t.
\end{cases}
$$

$\square$

3. 记 $F(x, y, z) = x^2 + y^2 + z^2 - 4$, $G(x, y, z) = x^2 + y^2 - 2y$, 则

$$
\frac{\partial F}{\partial x} = 2x, \frac{\partial F}{\partial y} = 2y, \frac{\partial F}{\partial z} = 2z, \ \frac{\partial G}{\partial x} = 2x, \frac{\partial G}{\partial y} = 2y - 2, \frac{\partial G}{\partial z} = 0,
$$

因此,

$$
\frac{\partial(F, G)}{\partial(y, z)}\bigg|_{(1,1,\sqrt{2})} = \begin{vmatrix} 2 & 2\sqrt{2} \\ 0 & 0 \end{vmatrix} = 0,
$$

$$
\frac{\partial(F, G)}{\partial(z, x)}\bigg|_{(1,1,\sqrt{2})} = \begin{vmatrix} 2\sqrt{2} & 2 \\ 0 & 2 \end{vmatrix} = 4\sqrt{2},
$$

$$
\frac{\partial(F, G)}{\partial(x, y)}\bigg|_{(1,1,\sqrt{2})} = \begin{vmatrix} 2 & 2 \\ 2 & 0 \end{vmatrix} = -4,
$$

切线方程为

$$
\begin{cases}
x = 1, \\
y = 1 + 4\sqrt{2}t, \\
z = \sqrt{2} - 4t,
\end{cases}
$$

法平面方程为 $\sqrt{2}y - z = 0$. $\square$

4. 记 $F(x, y, z) = x^2 + y^2 + z^2 - 3xyz$, $P_0(1, 1, 1)$, 则

$$
F_x(P_0) = (2x - 3yz)|_{P_0} = -1, F_y(P_0) = (2y - 3xz)|_{P_0} = -1, F_z(P_0) = (2z - 3xy)|_{P_0} = -1.
$$

切平面方程为 $x + y + z = 3$, 法线方程为 $x = y = z$. $\square$

5. 所求切平面的法线方向为 $(1, -1, 2)$, 设其与已知曲面相切于 (x, y, z) 点, 则有 $(2x, 2y, 4z) // (1, -1, 2)$, 因此 $x = t, y = -t, z = t$, 代入已知曲面方程得到 $t = \pm\frac{\sqrt{22}}{2}$, 所求切平面为 $x - y + 2z = \pm 2\sqrt{22}$. $\square$

6. 设所求点为 $P_0(x_0, y_0, z_0)$, 已知曲面在该点处的切平面的法方向是 $\boldsymbol{\tau} = (6x_0, 2y_0, 4z_0)$. 依题意, $\boldsymbol{\tau}$ 与 $\boldsymbol{n}_1 = (1, 1, 1)$ 和 $\boldsymbol{n}_2 = (4, 4, 8)$ 都正交, 因此有

$$
\begin{cases}
\boldsymbol{\tau} \cdot \boldsymbol{n}_1 = 6x_0 + 2y_0 + 4z_0 = 0, \\
\boldsymbol{\tau} \cdot \boldsymbol{n}_2 = 4(6x_0 + 2y_0 + 8z_0) = 0,
\end{cases}
$$

又知 P_0 是已知曲面上的点, 有 $3x_0^2 + y_0^2 + 2z_0^2 = 16$, 从而得到 $P_0(\pm\frac{2}{3}\sqrt{3}, \mp 2\sqrt{3}, 0)$. $\square$

7. 先证必要性. 设三平面在 P 点的切平面共线于 l, 同时设 l 的方向是非零向量 (a, b, c), 则 l 与三个切平面的法向量 $(F_x(P), F_y(P), F_z(P)), (G_x(P), G_y(P), G_z(P)), (H_x(P), H_y(P), H_z(P))$ 均正交, 即, 关于 a, b, c 的线性方程组

$$\begin{cases} aF_x(P) + bF_y(P) + cF_z(P) = 0, \\ aG_x(P) + bG_y(P) + cG_z(P) = 0, \\ aH_x(P) + bH_y(P) + cH_z(P) = 0 \end{cases}$$

有非零解, 因而其系数行列式

$$\left. \frac{\partial(F, G, H)}{\partial(x, y, z)} \right|_P = 0.$$

以上过程回推同时注意到三个切平面过同一个点 P 即得充分性. □

习题 14.7 条件极值

1. (1) 构造 Lagrange 函数:

$$L(x, y, \lambda) = x^2 + y^2 + \lambda(x + y - 2),$$

由 $L_x = L_y = L_\lambda = 0$ 得稳定点 $P_0(1, 1)$, 下面说明 P_0 点也是目标函数在约束条件下的最小值点. 注意到约束条件 $x + y = 2$ 所确定的函数 $y = -x + 2$ 的定义域是 $(-\infty, +\infty)$, 显然当 $x \to \pm\infty$ 时, $f \to +\infty$, 因此 f 一定存在最小值, 唯一的稳定点 P_0 即是最小值点, 最小值为 $f(P_0) = 2$.

(2) 构造 Lagrange 函数:

$$L(x, y, \lambda) = x^2 - 2x + y^2 + 2y + 2 + \lambda(x^2 + y^2 - 4),$$

由 $L_x = L_y = L_\lambda = 0$ 得稳定点 $P_1(\sqrt{2}, -\sqrt{2}), P_2(-\sqrt{2}, \sqrt{2})$, 而 $f(P_1) = 6 - 4\sqrt{2}, f(P_2) = 6 + 4\sqrt{2}$, 因此, $P_1(\sqrt{2}, -\sqrt{2})$ 是最小值点, 最小值为 $f(P_1) = 6 - 4\sqrt{2}$, $P_2(-\sqrt{2}, \sqrt{2})$ 是最大值点, 最大值为 $f(P_2) = 6 + 4\sqrt{2}$.

(3) 构造 Lagrange 函数:

$$L(x, y, z, \lambda, \mu) = xyz + \lambda(x^2 + y^2 + z^2 - 1) + \mu(x + y + z),$$

由 $L_x = L_y = L_z = L_\lambda = L_\mu = 0$ 得稳定点 $P_1\left(\frac{\sqrt{6}}{6}, \frac{\sqrt{6}}{6}, -\frac{\sqrt{6}}{3}\right), P_2\left(\frac{\sqrt{6}}{6}, -\frac{\sqrt{6}}{3}, \frac{\sqrt{6}}{6}\right), P_3\left(-\frac{\sqrt{6}}{3}, \frac{\sqrt{6}}{6}, \frac{\sqrt{6}}{6}\right),$ $P_4\left(-\frac{\sqrt{6}}{6}, -\frac{\sqrt{6}}{6}, \frac{\sqrt{6}}{3}\right), P_5\left(-\frac{\sqrt{6}}{6}, \frac{\sqrt{6}}{3}, -\frac{\sqrt{6}}{6}\right), P_6\left(\frac{\sqrt{6}}{3}, -\frac{\sqrt{6}}{6}, -\frac{\sqrt{6}}{6}\right),$ 并且, $f(P_i) = -\frac{\sqrt{6}}{18}, i = 1, 2, 3, f(P_i) = \frac{\sqrt{6}}{18}, i = 4, 5, 6$. 故, $P_{1,2,3}$ 是最小值点, 最小值为 $-\frac{\sqrt{6}}{18}$, $P_{4,5,6}$ 是最大值点, 最大值为 $\frac{\sqrt{6}}{18}$. □

2. 设长方体三边长 a, b, c, 表面积 $S = 2(ab + bc + ac)$, 体积 $V = abc$.

(1) 构造 Lagrange 函数 $L(a,b,c,\lambda) = abc + \lambda(2(ab+bc+ac)-S)$, 由 $L_a = L_b = L_c = L_\lambda = 0$ 得稳定点 $\left(\sqrt{\frac{S}{6}}, \sqrt{\frac{S}{6}}, \sqrt{\frac{S}{6}}\right)$. 由于体积存在最大值, 故唯一的稳定点就是最大值点, 最大值为 $V_{\max} = \frac{S}{6}\sqrt{\frac{S}{6}}$.

(2) 构造 Lagrange 函数 $L(a,b,c,\lambda) = 2(ab+bc+ac) + \lambda(abc-V)$, 由 $L_a = L_b = L_c = L_\lambda = 0$ 得稳定点 $(\sqrt[3]{V}, \sqrt[3]{V}, \sqrt[3]{V})$. 由于表面积存在最小值, 故唯一的稳定点就是最小值点, 最小值为 $S_{\min} = 6V^{\frac{2}{3}}$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

3. 空间中一点 (X_0, Y_0, Z_0) 到平面 $Ax + By + Cz + D = 0$ 的距离是这个点到平面上点的距离的最小值, 因此, 目标函数可以取为 $d^2 = (x-X_0)^2 + (y-Y_0)^2 + (z-Z_0)^2$, 约束条件为 $Ax + By + Cz + D = 0$. 令 $\Delta x = x - X_0, \Delta y = y - Y_0, \Delta z = z - Z_0$, 目标函数和约束条件可以化为

$$\begin{cases} \min \quad d^2 = (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2, \\ \text{s.t.} \quad A\Delta x + B\Delta y + C\Delta z = -(AX_0 + BY_0 + CZ_0 + D). \end{cases}$$

构造 Lagrange 函数:

$$L(\Delta x, \Delta y, \Delta z, \lambda) = (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2 + \lambda(A\Delta x + B\Delta y + C\Delta z + AX_0 + BY_0 + CZ_0 + D),$$

由 $L_{\Delta x} = L_{\Delta y} = L_{\Delta z} = L_\lambda = 0$ 得稳定点

$$\left(-\frac{A(AX_0 + BY_0 + CZ_0 + D)}{A^2 + B^2 + C^2}, -\frac{B(AX_0 + BY_0 + CZ_0 + D)}{A^2 + B^2 + C^2}, -\frac{C(AX_0 + BY_0 + CZ_0 + D)}{A^2 + B^2 + C^2}\right).$$

又知最小距离存在, 这个唯一的稳定点就是最小值点, 最小值为

$$d_{\min} = \frac{|AX_0 + BY_0 + CZ_0 + D|}{\sqrt{A^2 + B^2 + C^2}}.$$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

4. 目标函数 $u = x^3 + y^3 + z^3$, 约束条件 $x^2 + y^2 + z^2 = 4$. 构造 Lagrange 函数:

$$L(x, y, z, \lambda) = x^3 + y^3 + z^3 + \lambda(x^2 + y^2 + z^2 - 4),$$

由 $L_x = L_y = L_z = L_\lambda = 0$ 得稳定点 $P_{1,2}(\pm 2/\sqrt{3}, \pm 2/\sqrt{3}, \pm 2/\sqrt{3})$, 并且 $u(P_{1,2}) = \pm 8/\sqrt{3}$. 约束条件确定的隐函数的定义域为 $D = \{(x,y)|x^2+y^2 \leqslant 4\}$, 其边界为 $\partial D = \{(x,y)|x^2+y^2 = 4\}$. 作极坐标换元, 令 $x = r\cos\theta, y = r\sin\theta, r \geqslant 0, 0 \leqslant \theta < 2\pi$, 则在边界 $\partial D = \{r|r = 2\}$ 上, 目标函数转化为 $u = r^3(\cos^3\theta + \sin^3\theta) = 8(\cos^3\theta + \sin^3\theta)$, 显然当 $\theta = 0, \frac{\pi}{2}$ 时, 即在点 $P_3(2,0,0), P_4(0,2,0)$ 处 u 在边界 ∂D 上取到最大值 8, 当 $\theta = \pi, \frac{3\pi}{2}$ 时, 即在点 $P_5(-2,0,0), P_6(0,-2,0)$ 处 u 在边界 ∂D 上取到最小值 -8. 与稳定点处的值做比较知, 目标函数在约束条件下的最大值点是 $P_{3,4}$, 最大值是 8, 最小值点是 $P_{5,6}$, 最小值是 -8. $\qquad\square$

5. 目标函数 $d^2 = x^2 + y^2 + z^2$, 约束条件 $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$. 构造 Lagrange 函数:

$$L(x, y, z, \lambda) = x^2 + y^2 + z^2 + \lambda \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 \right),$$

分别对 x, y, z, λ 求偏导并令其为 0, 得

$$\begin{aligned}
L_x &= 2x \left(1 + \frac{\lambda}{a^2} \right) = 0, \\
L_y &= 2y \left(1 + \frac{\lambda}{b^2} \right) = 0, \\
L_z &= 2z \left(1 + \frac{\lambda}{c^2} \right) = 0, \\
L_\lambda &= \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0.
\end{aligned} \tag{25}$$

由 (25) 之第一式知 $x = 0$ 或 $\lambda = -a^2$.

1) 若 $\lambda = -a^2$, 由 (25) 之第二式和第三式得 $y = z = 0$, 再由 (25) 之第四式得 $x = \pm a$, 得稳定点 $P_{1,2}(\pm a, 0, 0)$;

2) 若 $x = 0$, 由 (25) 之第二式得 $y = 0$ 或 $\lambda = -b^2$.

    2.1) 若 $\lambda = -b^2$, 由 (25) 之第三式得 $z = 0$, 又因为 $x = 0$, 再由 (25) 之第四式得 $y = \pm b$, 得稳定点 $P_{3,4}(0, \pm b, 0)$;

    2.2) 若 $y = 0$, 又因为 $x = 0$, 由 (25) 之第四式得 $z = \pm c$, 得稳定点 $P_{5,6}(0, 0, \pm c)$.

约束条件所确定的函数的定义域为 $D = \{(x, y) | \frac{x^2}{a^2} + \frac{y^2}{b^2} \leqslant 1\}$, 其边界为 $\partial D = \{(x, y) | \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\}$. 易知, 上述 $P_i, i = 1, 2, ..., 6$ 都在 ∂D 上, 也容易验证, d^2 在 ∂D 上的最大和最小值点正好分别是 $P_{1,2}$ 和 $P_{5,6}$, 因此, $P_{1,2}$ 到原点最远, 最远距离是 a; $P_{5,6}$ 到原点最近, 最近距离是 c. $\quad \square$

6. 在椭圆上任取一点 (x, y), 该点的切线方程斜率是 $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{4x}{9y}$, 切线方程是

$$Y - y = -\frac{4x}{9y}(X - x).$$

该切线与坐标轴所围面积为

$$S = \left| \left(y + \frac{4x^2}{9y} \right) \left(x + \frac{9y^2}{4x} \right) \right|,$$

目标函数为

$$S^2 = \frac{(4x^2 + 9y^2)^4}{36^2 x^2 y^2},$$

约束条件为

$$4x^2 + 9y^2 = 72,$$

从而新的目标函数可以取为

$$z = \frac{1}{x^2 y^2}.$$

构造 Lagrange 函数:

$$L(x, y, \lambda) = \frac{1}{x^2 y^2} + \lambda(4x^2 + 9y^2 - 72),$$

由 $L_x = L_y = L_\lambda = 0$ 得稳定点 $P_{1,2}(\pm 3, \pm 2), P_{3,4}(\pm 3, \mp 2)$, 并且 $z(P_{1,2,3,4}) = \frac{1}{36}$, 而最小值一定存在, 所以 $P_{1,2,3,4}$ 都是最小值点, 此时的切线方程是

$$y - (\pm 2) = -\frac{2}{3}(x - (\pm 3)), \ y - (\mp 2) = \frac{2}{3}(x - (\pm 3)).$$

$\square$

7. 设 $Q(x_0, y_0)$, $P(x, y)$, 且 $\varphi(x, y) = 0$. 目标函数

$$d^2 = (x - x_0)^2 + (y - y_0)^2,$$

约束条件

$$\varphi(x, y) = 0.$$

构造 Lagrange 函数:

$$L(x, y, \lambda) = (x - x_0)^2 + (y - y_0)^2 + \lambda \varphi(x, y),$$

由 $L_x = L_y = L_\lambda = 0$ 得

$$\begin{cases} 2(x - x_0) + \lambda \varphi_x = 0, \\ 2(y - y_0) + \lambda \varphi_y = 0, \\ \varphi(x, y) = 0. \end{cases} \tag{26}$$

由于最短距离一定存在, 满足 (26) 式的 (x, y) 就是最短距离点. 由 (26) 的前两式得 $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{\varphi_x}{\varphi_y} = -\frac{x - x_0}{y - y_0}$, 即为最短距离点 P 处的切线斜率, 因此法线斜率为 $\frac{y - y_0}{x - x_0}$, 恰好是 PQ 的斜率. $\square$

复习题

1. 设方向 $\boldsymbol{l} = (x, y)$, 由方向导数的定义,

$$z_{\boldsymbol{l}} = \lim_{(x,y)\to(0,0)} \frac{f(x,y) - f(0,0)}{\|(x,y) - (0,0)\|} = 1.$$

$\square$

2. 方向 $\boldsymbol{l} = (2,1,3)$, 方向余弦 $\cos\alpha = \frac{2}{\sqrt{14}}, \cos\beta = \frac{1}{\sqrt{14}}, \cos\gamma = \frac{3}{\sqrt{14}}$, 点 $P_0(2,1,3)$ 处沿 $\boldsymbol{l}$ 的方向导数

$$\begin{aligned}
u_{\boldsymbol{l}}(P_0) =& u_x(P_0)\cos\alpha + u_y(P_0)\cos\beta + u_z(P_0)\cos\gamma \\
=& 2(x+y+z)|_{P_0}(\cos\alpha + \cos\beta + \cos\gamma) = \frac{72}{\sqrt{14}}
\end{aligned}$$

$\square$

4. 计算各阶偏导数:

$$\begin{aligned}
z(1,1) =& 0, \ z_x(1,1) = \frac{1}{x}\ln y\Big|_{(1,1)} = 0, \ z_y(1,1) = \frac{1}{y}\ln x\Big|_{(1,1)} = 0, \\
z_{xx}(1,1) =& -\frac{1}{x^2}\ln y\Big|_{(1,1)} = 0, \ z_{xy}(1,1) = \frac{1}{xy}\Big|_{(1,1)} = 1, \ z_{yy}(1,1) = -\frac{1}{y^2}\ln x\Big|_{(1,1)} = 0, \\
z_{x^3}(1,1) =& \frac{2}{x^3}\ln y\Big|_{(1,1)} = 0, \ z_{y^3}(1,1) = \frac{2}{y^3}\ln x\Big|_{(1,1)} = 0, \\
& z_{x^2y}(1,1) = -\frac{1}{x^2y} = -1, z_{xy^2}(1,1) = -\frac{1}{xy^2} = -1, \\
z_{x^4}(x,y) =& -\frac{6}{x^4}\ln y, \ z_{y^4}(x,y) = -\frac{6}{y^4}\ln x, \\
& z_{x^3y}(x,y) = \frac{2}{x^3y}, z_{x^2y^2}(x,y) = \frac{1}{x^2y^2}, z_{xy^3}(x,y) = \frac{2}{xy^3}.
\end{aligned}$$

展开到三阶的含 Lagrange 型余项的 Taylor 公式:

$$\begin{aligned}
z(x,y) =& z(1+(x-1), 1+(y-1)) = z(1,1) + [z_x(1,1)(x-1) + z_y(1,1)(y-1)] \\
& + \frac{1}{2}[z_{xx}(1,1)(x-1)^2 + 2z_{xy}(1,1)(x-1)(y-1) + z_{yy}(1,1)(y-1)^2] \\
& + \frac{1}{6}[z_{x^3}(1,1)(x-1)^3 + 3z_{x^2y}(1,1)(x-1)^2(y-1) + 3z_{xy^2}(1,1)(x-1)(y-1)^2 \\
& + z_{y^3}(1,1)(y-1)^3] + R_3, \\
=& (x-1)(y-1) - \frac{1}{2}(x-1)^2(y-1) - \frac{1}{2}(x-1)(y-1)^2 + R_3,
\end{aligned}$$

其中,

$$R_3 = \frac{1}{24}\left((x-1)\frac{\partial}{\partial x} + (y-1)\frac{\partial}{\partial y}\right)^4 z(1+\theta(x-1), 1+\theta(y-1)).$$

展开到三阶的含 Peano 型余项的 Taylor 公式:

$$z(x,y) = (x-1)(y-1) - \frac{1}{2}(x-1)^2(y-1) - \frac{1}{2}(x-1)(y-1)^2 + o((\sqrt{(x-1)^2+(y-1)^2})^3).$$

$\square$

5. 设三角形三边长分别为 a, b, c, 面积为 S, 则 $a+b+c = 2p$, 由海伦公式,

$$S^2 = p(p-a)(p-b)(p-c) = p(p-a)(p-b)(a+b-p).$$

由

$$\frac{\partial S^2}{\partial a} = p(p-b)(2p-2a-b) = 0,$$
$$\frac{\partial S^2}{\partial b} = p(p-a)(2p-a-2b) = 0,$$

得稳定点 $(2p/3, 2p/3) \in (0, 2p)^2$, 即, 当 $a = b = c = 2p/3$ 时, S 取到最大, $S_{\max} = \frac{p^2}{9}\sqrt{3}$.　$\square$

6. 首先证明

$$\sum_{k=1}^{n}\cos\frac{2k\pi}{n} = \sum_{k=1}^{n}\sin\frac{2k\pi}{n} = 0. \tag{27}$$

事实上, 考虑 $x^n - 1 = 0$ 的 n 个根 $x_k = \cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n}, k = 1, 2, ..., n$, 由方程的根与系数间的关系即知 (27) 式成立. 设 $\boldsymbol{l}_1 = (\cos\alpha, \cos\beta)$, 则

$$\boldsymbol{l}_k = \left(\cos\left(\alpha + \frac{2\pi}{n}(k-1)\right), \cos\left(\beta - \frac{2\pi}{n}(k-1)\right)\right), k = 1, 2, ..., n,$$

有

$$\begin{aligned}
\sum_{k=1}^{n}\frac{\partial f}{\partial \boldsymbol{l}_k}\bigg|_{P_0} &= \sum_{k=1}^{n}\left[\frac{\partial f}{\partial x}(P_0)\cos\left(\alpha + \frac{2\pi}{n}(k-1)\right) + \frac{\partial f}{\partial y}(P_0)\cos\left(\beta - \frac{2\pi}{n}(k-1)\right)\right] \\
&= \frac{\partial f}{\partial x}(P_0)\sum_{k=1}^{n}\left[\cos\left(\alpha - \frac{2\pi}{n}\right)\cos\frac{2k\pi}{n} - \sin\left(\alpha - \frac{2\pi}{n}\right)\sin\frac{2k\pi}{n}\right] \\
&\quad + \frac{\partial f}{\partial y}(P_0)\sum_{k=1}^{n}\left[\cos\left(\beta + \frac{2\pi}{n}\right)\cos\frac{2k\pi}{n} + \sin\left(\beta + \frac{2\pi}{n}\right)\sin\frac{2k\pi}{n}\right] \\
&= \frac{\partial f}{\partial x}(P_0)\left[\cos\left(\alpha - \frac{2\pi}{n}\right)\sum_{k=1}^{n}\cos\frac{2k\pi}{n} - \sin\left(\alpha - \frac{2\pi}{n}\right)\sum_{k=1}^{n}\sin\frac{2k\pi}{n}\right] \\
&\quad + \frac{\partial f}{\partial y}(P_0)\left[\cos\left(\beta + \frac{2\pi}{n}\right)\sum_{k=1}^{n}\cos\frac{2k\pi}{n} + \sin\left(\beta + \frac{2\pi}{n}\right)\sum_{k=1}^{n}\sin\frac{2k\pi}{n}\right] \\
&= 0.
\end{aligned}$$

$\square$

7. 取平面上一点 (x, y), 目标函数为

$$S = \sum_{i=1}^{n} (x - a_i)^2 + (y - b_i)^2.$$

由 $S_x = S_y = 0$ 得

$$\begin{cases} \sum_{i=1}^{n}(x - a_i) = 0, \\ \sum_{i=1}^{n}(y - b_i) = 0, \end{cases}$$

解得稳定点 $(\bar{a}, \bar{b})$, 其中 $\bar{a} = \frac{1}{n}\sum_{i=1}^{n} a_i, \bar{b} = \frac{1}{n}\sum_{i=1}^{n} b_i$. 由于 S 的 Hesse 矩阵 $\boldsymbol{H}_S = \begin{pmatrix} n & 0 \\ 0 & n \end{pmatrix}$ 正定, 此稳定点就是最小值点, $S_{\min} = \sum_{i=1}^{n}(a_i - \bar{a})^2 + (b_i - \bar{b})^2$. □

8. f 分别对 x, y 求偏导并令其为 0 得,

$$\begin{cases} (\cos y - 1 - x)e^x, \\ -(1 + e^x)\sin y = 0, \end{cases}$$

稳定点为 $P_{2k}(0, 2k\pi), P_{2k+1}(-2, (2k+1)\pi), k \in \mathbb{N}_+$. Hesse 矩阵

$$\boldsymbol{H}_f(x, y) = \begin{pmatrix} (\cos y - 2 - x)e^x & -e^x \sin y \\ -e^x \sin y & -(1 + e^x)\cos y \end{pmatrix}.$$

所以,

$$\det \boldsymbol{H}_f(P_{2k}) = \begin{vmatrix} -1 & 0 \\ 0 & -2 \end{vmatrix} = 2 > 0,$$

而 $f_{xx}(P_{2k}) = -1 < 0$, 所以 $\boldsymbol{H}_f(P_{2k})$ 负定, P_{2k} 是极大值点, 有无穷多个, 极大值为 $f(P_{2k}) = 0, k \in \mathbb{N}_+$. 下面说明 $P_{2k+1}, k \in \mathbb{N}_+$ 不是极值点. 事实上, 沿着 $y = (2k+1)\pi, k \in \mathbb{N}_+$, $f(x, (2k+1)\pi) = -3 - (1+x)e^x$, 显然, $\lim_{x \to +\infty} f(x, (2k+1)\pi) = -\infty, \lim_{x \to -\infty} f(x, (2k+1)\pi) = 0$, 故 $f(x, (2k+1)\pi)$ 取不到极小值, 因而原二元函数 $f(x, y)$ 无极小值点.(见图 3) □

9. 设三角形三个定点 $A(2, 0), B(x, y), C(x, -y), x \neq 2, y > 0$, 则面积为 $S = (2 - x)|y|$, 因此, 目标函数可以取作

$$S^2 = (2 - x)^2 y^2,$$

约束条件为

$$3x^2 + y^2 = 12.$$

构造 Lagrange 函数:

$$L(x, y, \lambda) = (2 - x)^2 y^2 + \lambda(3x^2 + y^2 - 12),$$

由 $L_x = L_y = L_\lambda = 0$ 得 $(-1, 3)$. 由于最大面积一定存在, 此唯一的稳定点就是最大值点, 最大面积是 $S_{\max} = 9$. □
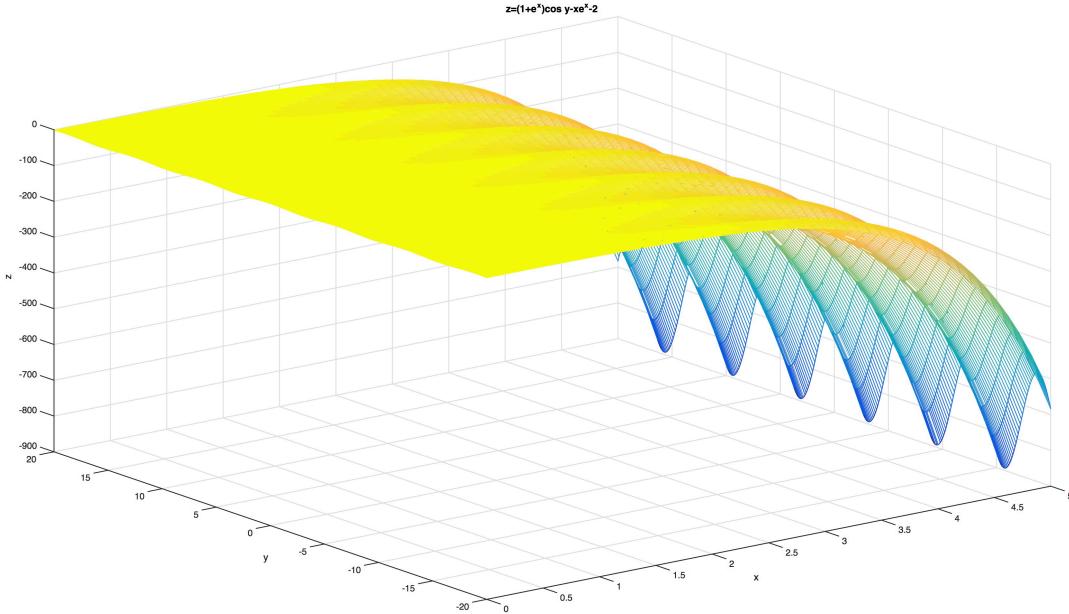
图 3: $f(x, y) = (1 + e^x) \cos y - xe^x - 2$

10. 设长方体与半球底面相交的长方形边长为 x, y, 另一棱长为 z. 目标函数 $V = xyz$, 约束条件 $x^2 + y^2 + 4z^2 = 4a^2$. 构造 Lagrange 函数 $L(x, y, z, \lambda) = xyz + \lambda(x^2 + y^2 + 4z^2 - 4a^2)$, 由 $L_x = L_y = L_z = L_\lambda = 0$ 得 $(2a/\sqrt{3}, 2a/\sqrt{3}, a/\sqrt{3})$. 由于最大值一定存在, 此唯一的稳定点就是最大值点, 最大体积是 $V_{\max} = 4a^3/(3\sqrt{3})$. □

11. 设长方体与圆锥底面相交的长方形边长为 x, y, 另一棱长为 z. 目标函数 $V = xyz$, 约束条件

$$\frac{z}{h} = \frac{a - \frac{\sqrt{x^2+y^2}}{2}}{a},$$

即 $4a^2(h-z)^2 = h^2(x^2+y^2)$. 构造 Lagrange 函数 $L(x, y, z, \lambda) = xyz + \lambda(4a^2(h-z)^2 - h^2(x^2+y^2))$, 由 $L_x = L_y = L_z = L_\lambda = 0$ 得 $(2\sqrt{2}a/3, 2\sqrt{2}a/3, h/3)$. 由于最大值一定存在, 此唯一的稳定点就是最大值点, 最大体积是 $V_{\max} = 8a^2h/27$. □

12. 设平面与三个坐标轴的交点分别为 $(a, 0, 0), (0, b, 0), (0, 0, c)$, 则平面方程可写为

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1.$$

目标函数 $V = \frac{1}{6}abc$, 约束条件 $\frac{1}{a} + \frac{1}{3b} + \frac{2}{c} = 1$. 构造 Lagrange 函数 $L(a, b, c, \lambda) = \frac{1}{6}abc + \lambda(\frac{1}{a} + \frac{1}{3b} + \frac{2}{c} - 1)$, 由 $L_a = L_b = L_c = L_\lambda = 0$ 得 $(3, 1, 6)$. 由于最小值一定存在, 此唯一的稳定点就是最小值点, 最小体积是 $V_{\min} = 3$, 此时的平面方程是

$$\frac{x}{3} + \frac{y}{1} + \frac{z}{6} = 1.$$

13. $F(x, y, z) = 1$ 在 P_0 处的切平面方程是 $F_x(P_0)(x-x_0)+F_y(P_0)(y-y_0)+F_z(P_0)(z-z_0) = 0$, 即 $F_x(P_0)x + F_y(P_0)y + F_z(P_0)z = x_0F_x(P_0) + y_0F_y(P_0) + z_0F_z(P_0)$. 由齐次函数的性质 (习题 13.5 第 8 题) 知 $x_0F_x(P_0) + y_0F_y(P_0) + z_0F_z(P_0) = nF(P_0)$, 而 $F(P_0) = 1$, 得证. □

14. 在曲面 $\sqrt{x} + \sqrt{y} + \sqrt{z} = \sqrt{a}$ 上任取一点 $P_0(x_0, y_0, z_0)$, 则 $\sqrt{x_0} + \sqrt{y_0} + \sqrt{z_0} = \sqrt{a}$. P_0 点处的切平面方程是 $\frac{x-x_0}{2\sqrt{x_0}} + \frac{y-y_0}{2\sqrt{y_0}} + \frac{z-z_0}{2\sqrt{z_0}} = 0$, 即

$$\frac{x}{\sqrt{x_0}} + \frac{y}{\sqrt{y_0}} + \frac{z}{\sqrt{z_0}} = \sqrt{a},$$

因此, 该切平面与三个坐标轴的交点分别是 $(\sqrt{ax_0}, 0, 0), (0, \sqrt{ay_0}, 0), (0, 0, \sqrt{az_0})$, 截距之和为 $\sqrt{ax_0} + \sqrt{ay_0} + \sqrt{az_0} = a$. □

第 15 章 含参变量积分

习题 15.1 含参变量正常积分及其分析性质

1. (1) 函数 $f(x,t) = \sqrt[2n]{x^{2n} + t^{2n}}$ 在矩形区域 $[-1, 1] \times [-1, 1]$ 连续, 因而

$$\lim_{t \to 0} \int_{-1}^1 \sqrt[2n]{x^{2n} + t^{2n}} \mathrm{d}x = \int_{-1}^1 (\lim_{t \to 0} \sqrt[2n]{x^{2n} + t^{2n}}) \mathrm{d}x = \int_{-1}^1 |x| \mathrm{d}x = 1.$$

(2) 函数 $f(x,t) = e^{x+t^2x^2}$ 在矩形区域 $[0, 1] \times [-1, 1]$ 连续, 因而

$$\lim_{t \to 0} \int_0^1 e^{x+t^2x^2} \mathrm{d}x = \int_0^1 (\lim_{t \to 0} e^{x+t^2x^2}) \mathrm{d}x = \int_0^1 e^x \mathrm{d}x = e - 1.$$

$\square$

2. 直接计算得

$$\begin{aligned}
I'(x) &= \int_{3x}^{x^3} \frac{\partial}{\partial x} \frac{\sin(xy)}{y} \mathrm{d}y + \frac{\sin(x^4)}{x^3} \cdot 3x^2 - \frac{\sin(3x^2)}{3x} \cdot 3 \\
&= \int_{3x}^{x^3} \cos(xy) \mathrm{d}y + \frac{3\sin(x^4) - \sin(3x^2)}{x} \\
&= \frac{\sin(xy)}{x} \Big|_{y=3x}^{y=x^3} + \frac{3\sin(x^4) - \sin(3x^2)}{x} \\
&= \frac{4\sin(x^4) - 2\sin(3x^2)}{x}.
\end{aligned}$$

$\square$

3. 易知 $\frac{x^b - x^a}{\ln x} = \frac{x^y}{\ln x} \Big|_{y=a}^{y=b} = \int_a^b x^y \mathrm{d}y$.

(1) 记

$$f(x,y) = \begin{cases} (\sin \ln x) x^y, & (x,y) \in (0,1] \times [a,b], \\ \lim_{x \to 0} (\sin \ln x) x^y = 0, & x = 0, y \in [a,b], \end{cases}$$

则 $f(x,y)$ 在矩形区域 $D = [0,1] \times [a,b]$ 上连续. 易得

$$\begin{aligned}
\int_0^1 \sin \ln x \frac{x^b - x^a}{\ln x} \mathrm{d}x &= \int_0^1 \sin \ln x \left(\int_a^b x^y \mathrm{d}y \right) \mathrm{d}x = \int_0^1 \mathrm{d}x \int_a^b (\sin \ln x) x^y \mathrm{d}y \\
&= \int_a^b \mathrm{d}y \int_0^1 (\sin \ln x) x^y \mathrm{d}x,
\end{aligned}$$

令 $I = \int_0^1 (\sin \ln x) x^y \mathrm{d}x$, 由分部积分,

$$I = \int_0^1 (\sin\ln x)x^y\mathrm{d}x = \frac{1}{y+1}\int_0^1 \sin\ln x\mathrm{d}(x^{y+1})$$
$$=\frac{1}{y+1}\left[\sin\ln x(x^{y+1})\Big|_{x=0}^{x=1} - \int_0^1 (\cos\ln x)x^y\mathrm{d}x\right] \tag{28}$$
$$=-\frac{1}{y+1}\int_0^1 (\cos\ln x)x^y\mathrm{d}x,$$

继续使用分部积分有

$$I = -\frac{1}{(y+1)^2}\int_0^1 \cos\ln x\mathrm{d}(x^{y+1})$$
$$=-\frac{1}{(y+1)^2}\left[\cos\ln x(x^{y+1})\Big|_{x=0}^{x=1} + \int_0^1 (\sin\ln x)x^y\mathrm{d}x\right]$$
$$=-\frac{1}{(y+1)^2}(1+I),$$

得 $I = -\frac{1}{1+(y+1)^2}$, 因此有

$$\int_0^1 \sin\ln x\frac{x^b - x^a}{\ln x}\mathrm{d}x = -\int_a^b \frac{1}{1+(y+1)^2}\mathrm{d}y = \arctan(a+1) - \arctan(b+1).$$

(2) 类似于 (1) 得

$$\int_0^1 \cos\ln x\frac{x^b - x^a}{\ln x}\mathrm{d}x = \int_a^b \mathrm{d}y\int_0^1 (\cos\ln x)x^y\mathrm{d}x,$$

由 (28) 式知

$$\int_0^1 (\cos\ln x)x^y\mathrm{d}x = -(y+1)\int_0^1 (\sin\ln x)x^y\mathrm{d}x = \frac{y+1}{1+(y+1)^2},$$

所以,

$$\int_0^1 \cos\ln x\frac{x^b - x^a}{\ln x}\mathrm{d}x = \int_a^b \frac{y+1}{1+(y+1)^2}\mathrm{d}y = \frac{1}{2}\ln\frac{1+(b+1)^2}{1+(a+1)^2}.$$

$\square$

4. 直接计算得

$$u_t = \frac{1}{2}[-af'(x-at) + af'(x+at)] + \frac{1}{2}[g(x+at) + g(x-at)],$$
$$u_{tt} = \frac{1}{2}[a^2 f''(x-at) + a^2 f''(x+at)] + \frac{1}{2}[ag'(x+at) - ag'(x-at)],$$
$$u_x = \frac{1}{2}[f'(x-at) + f'(x+at)] + \frac{1}{2a}[g(x+at) - g(x-at)],$$
$$u_{xx} = \frac{1}{2}[f''(x-at) + f''(x+at)] + \frac{1}{2a}[g'(x+at) - g'(x-at)],$$

直接验证即得. □

5. 直接计算得

$$u' = \int_0^\pi \frac{\partial}{\partial x} \cos(n\theta - x\sin\theta)\mathrm{d}\theta = \int_0^\pi \sin\theta \sin(n\theta - x\sin\theta)\mathrm{d}\theta,$$

$$u'' = \int_0^\pi \frac{\partial}{\partial x} \sin\theta \sin(n\theta - x\sin\theta)\mathrm{d}\theta = -\int_0^\pi \sin^2\theta \cos(n\theta - x\sin\theta)\mathrm{d}\theta,$$

代入 Bessel 方程得

$$\begin{aligned}
&x^2 u'' + xu' + (x^2 - n^2)u\\
&= \int_0^\pi (x^2\cos^2\theta - n^2)\cos(n\theta - x\sin\theta)\mathrm{d}\theta + \int_0^\pi x\sin\theta\sin(n\theta - x\sin\theta)\mathrm{d}\theta\\
&\triangleq I_1 + I_2,
\end{aligned}$$

而

$$\begin{aligned}
I_1 &= \int_0^\pi (x^2\cos^2\theta - n^2)\cos(n\theta - x\sin\theta)\mathrm{d}\theta\\
&= -\int_0^\pi (n + x\cos\theta)\mathrm{d}\sin(n\theta - x\sin\theta)\\
&= -\left[(n + x\cos\theta)\sin(n\theta - x\sin\theta)\Big|_{\theta=0}^{\theta=\pi} - \int_0^\pi \sin(n\theta - x\sin\theta)(-x\sin\theta)\mathrm{d}\theta\right]\\
&= -\int_0^\pi x\sin\theta\sin(n\theta - x\sin\theta)\mathrm{d}\theta = -I_2,
\end{aligned}$$

所以, $x^2 u'' + xu' + (x^2 - n^2)u = 0.$ □

6. (1) $a \neq 0$ 时, 因为 $a^2\sin^2 x + \cos^2 x > 0$, 故 $\ln(a^2\sin^2 x + \cos^2 x)$ 为连续函数且有连续的对 a 的导数, 从而可以在积分号下求导. 记 $I(a) = \int_0^{\frac{\pi}{2}} \ln(a^2\sin^2 x + \cos^2 x)\mathrm{d}x$, 则当 $a^2 = 1$ 时 $I(a) = 0$, 当 $a^2 \neq 1$ 时, 令 $t = \tan x$, 有

$$\begin{aligned}
I'(a) &= \int_0^{\frac{\pi}{2}} \frac{\partial}{\partial a} \ln(a^2\sin^2 x + \cos^2 x)\mathrm{d}x = 2a\int_0^{\frac{\pi}{2}} \frac{\sin^2 x}{a^2\sin^2 x + \cos^2 x}\mathrm{d}x\\
&= 2a\int_0^{\frac{\pi}{2}} \frac{1}{a^2 + \cot^2 x}\mathrm{d}x = 2a\int_0^{+\infty} \frac{1}{a^2 + 1/t^2}\frac{1}{t^2 + 1}\mathrm{d}t\\
&= \frac{2a}{a^2 - 1}\int_0^{+\infty} \left(\frac{1}{t^2 + 1} - \frac{1}{a^2 t^2 + 1}\right)\mathrm{d}t\\
&= \frac{2a}{a^2 - 1}\arctan t\Big|_0^{+\infty} - \frac{2}{a^2 - 1}\arctan(at)\Big|_0^{+\infty}\\
&= \begin{cases} \frac{\pi}{a+1}, & a > 0,\\ \frac{\pi}{a-1}, & a < 0 \end{cases}
\end{aligned}$$

上式两端从 1 到 a 积分, 并注意到 $I(1) = 0$, 得到 $I(a) = \pi\ln\frac{|a|+1}{2}$. 显然, 当 $a^2 = 1$ 时此式也成立. 因此,

$$I(a) = \pi \ln \frac{|a|+1}{2}, a \neq 0.$$

(2) 因为 $1 - 2t\cos\tau + t^2 \geqslant 1 - 2|t| + t^2 = (1-|t|)^2 > 0$, 故 $\ln(1 - 2t\cos\tau + t^2)$ 为连续函数且有连续的对 t 的导数, 从而可以在积分号下求导. 当 $t = 0$ 时 $I(0) = 0$; 当 $t \neq 0$ 时, 令 $x = \tan\frac{\tau}{2}$, 注意到 $|t| < 1$, 有

$$\begin{aligned}
I'(t) &= \int_0^\pi \frac{\partial}{\partial t} \ln(1 - 2t\cos\tau + t^2)\mathrm{d}\tau \\
&= 2\int_0^\pi \frac{t - \cos\tau}{1 - 2t\cos\tau + t^2}\mathrm{d}\tau \\
&= \frac{1}{t}\int_0^\pi \left(1 + \frac{t^2 - 1}{1 - 2t\cos\tau + t^2}\right)\mathrm{d}\tau \\
&= \frac{\pi}{t} + \frac{t^2 - 1}{t}\int_0^\pi \frac{1}{1 - 2t\cos\tau + t^2}\mathrm{d}\tau \\
&= \frac{\pi}{t} + \frac{t^2 - 1}{t}\int_0^{+\infty} \frac{1}{1 - 2t\frac{1-x^2}{1+x^2} + t^2}\frac{2}{1+x^2}\mathrm{d}x \\
&= \frac{\pi}{t} + \frac{t^2 - 1}{t}\int_0^{+\infty} \frac{2}{(1+t)^2 x^2 + (1-t)^2}\mathrm{d}x \\
&= \frac{\pi}{t} + \frac{2}{t}\arctan\frac{t+1}{t-1}x\Big|_{x=0}^{x=+\infty} \\
&= 0,
\end{aligned}$$

上式两端从 0 到 t 积分, 并注意到 $I(0) = 0$, 得到

$$I(t) = 0.$$

$\square$

习题 15.2 含参变量反常积分及一致收敛判别法

1. 直接计算得

$$\psi(t) = \begin{cases} 0, & t = 0, \\ -e^{-xt}\big|_{x=0}^{x=+\infty} = 1, & t > 0, \end{cases} \tag{29}$$

即, 对 $\forall t \in [0, +\infty), \psi(t)$ 均收敛. 下面证明 $\psi(t)$ 在 $[0, +\infty)$ 不一致收敛.

方法一. 反证法. 假设 $\psi(t)$ 一致收敛, 则因为被积函数 te^{-xt} 在 $[0, +\infty)^2$ 连续, 由定理 15.3.1, $\psi(t)$ 应该在 $[0, +\infty)$ 连续, 但由 (29) 式知 $\psi(t)$ 不连续, 因此 $\psi(t)$ 不一致收敛.

方法二. 使用不一致收敛的 Cauchy 准则. 取 $\varepsilon_0 = e^{-1/3} - e^{-1/2} > 0$, 对 $\forall G > 0$, 存在 $G_1 = 2G, G_2 = 3G, t_0 = \frac{1}{G}$, 使得

$$\left|\int_{G_1}^{G_2} t_0 e^{-xt_0}\mathrm{d}x\right| = \left|-e^{-xt_0}\big|_{x=G_1}^{x=G_2}\right| = \varepsilon_0,$$

因此 $\psi(t)$ 不一致收敛. $\square$

2. (1) 因为 $\left|\frac{\sin(tx)}{x^2}\right| \leqslant \frac{1}{x^2}$, 而 $\int_1^{+\infty} \frac{1}{x^2}\mathrm{d}x$ 收敛, 由 M 判别法, 含参变量无穷限积分 $\int_1^{+\infty} \frac{\sin(tx)}{x^2}\mathrm{d}x$ 在 $(-\infty, +\infty)$ 一致收敛.

(2) 因为 $\frac{1}{1+(x+t)^2} \leqslant \frac{1}{1+x^2}$, 而 $\int_1^{+\infty} \frac{1}{1+x^2}\mathrm{d}x$ 收敛, 由 M 判别法, 含参变量无穷限积分 $\int_1^{+\infty} \frac{1}{1+(x+t)^2}\mathrm{d}x$ 在 $[0, +\infty)$ 一致收敛.

(3) 因为 $\left|\frac{\sin(tx)}{1+x^p}\right| \leqslant \frac{1}{1+x^p}$, 而 $\int_1^{+\infty} \frac{1}{1+x^p}\mathrm{d}x$ 收敛, 由 M 判别法, 含参变量无穷限积分 $\int_1^{+\infty} \frac{\sin(tx)}{1+x^p}\mathrm{d}x$ 在 $(-\infty, +\infty)$ 一致收敛.

(4) 因为 $\left|\cos(tx)e^{-x(1+t^2)}\right| \leqslant e^{-x}$, 而 $\int_1^{+\infty} e^{-x}\mathrm{d}x$ 收敛, 由 M 判别法, 含参变量无穷限积分 $\int_1^{+\infty} \cos(tx)e^{-x(1+t^2)}\mathrm{d}x$ 在 $(-\infty, +\infty)$ 一致收敛. $\square$

3. 取 $f(x) = \sin\beta x, g(x) = \frac{1}{x}$, 则

$$F(u) = \int_0^u \sin\beta x\mathrm{d}x = \begin{cases} 0, & \beta = 0, \\ -\frac{1}{\beta}(\cos\beta u - 1), & \beta > 0 \end{cases}$$

在 $u \in [0, +\infty)$ 上有界, 并且 $g(x)$ 在 $(0, +\infty)$ 单调趋于 0, 由 Dirichlet 判别法 (定理 9.8.5), $\psi(\beta) = \int_0^{+\infty} \frac{\sin\beta x}{x}\mathrm{d}x$ 在 $[0, +\infty)$ 收敛. 下面使用不一致收敛的 Cauchy 准则证明 $\psi(\beta)$ 在 $[0, +\infty)$ 不一致收敛. 首先由积分中值定理, 对 $G_2 > G_1 > 0, \exists \xi \in (G_1, G_2)$, 使得

$$\left|\int_{G_1}^{G_2} \frac{\sin\beta_0 x}{x}\mathrm{d}x\right| = \frac{|\sin\beta_0\xi|}{\xi}(G_2 - G_1),$$

取 $\varepsilon_0 = 1 > 0$, 对 $\forall G > 0$, 存在 $G_1 = 2G, G_2 = 2G + \xi, \beta_0 = \frac{\pi}{2\xi}$, 使得

$$\left|\int_{G_1}^{G_2} \frac{\sin\beta_0 x}{x}\mathrm{d}x\right| = \frac{|\sin\beta_0\xi|}{\xi}(G_2 - G_1) = \varepsilon_0,$$

因此 $\psi(\beta)$ 在 $[0, +\infty)$ 不一致收敛. $\square$

4. 方法一. 令 $f(x, p) = \sin(px), g(x, p) = \frac{x}{1+x^2}$, 易知, 对 $\forall b > 1$ 和 $\forall p \geqslant a, \left|\int_1^b \sin(px)\mathrm{d}x\right| = \frac{1}{p}|\cos(pb) - \cos p| \leqslant \frac{2}{a}$; 当 $x \geqslant 1$ 时, $g(x, p)$ 关于 x 单调递减; 当 $x \to +\infty$ 时 $g(x, p)$ 一致趋于 0. 由 Dirichlet 判别法知 $I(p)$ 一致收敛.

方法二. 令 $f(x, p) = \frac{\sin(px)}{x}, g(x, p) = \frac{x^2}{1+x^2}$, 由 Dirichlet 判别法知 $\int_1^{+\infty} f(x, p)\mathrm{d}x$ 在 $p \in [a, +\infty)$ 一致收敛; 又当 $x \geqslant 1$ 时, $g(x, p)$ 关于 x 单调递增; $|g(x, p)| \leqslant 1$. 由 Abel 判别法知 $I(p)$ 一致收敛. $\square$

5. 令 $g(x, t) = \frac{1}{x^\lambda}$, 易知, $g(x, t)$ 关于 x 单调递减; 当 $x \to +\infty$ 时 $g(x, t)$ 一致趋于 0. 又因为 $\varphi(x, t)$ 有界, 由 Dirichlet 判别法知 $I(t)$ 一致收敛. $\square$

习题 15.3 含参变量反常积分的分析性质

1. (2) 不妨设 $b > a$.

方法一: 交换累次积分顺序. 易知, $\frac{\cos(bx) - \cos(ax)}{x} = -\int_a^b \sin(xy)\mathrm{d}y$. 令 $\varphi(y) = \int_0^{+\infty} \frac{\sin(xy)}{x}\mathrm{d}x$, 由

Dirichlet 判别法易知 $\varphi(y)$ 在 $[a, b]$ 一致收敛, 因此,

$$\int_0^{+\infty} \frac{\cos(bx) - \cos(ax)}{x^2} \mathrm{d}x = -\int_0^{+\infty} \mathrm{d}x \int_a^b \frac{\sin(xy)}{x} \mathrm{d}y = -\int_a^b \mathrm{d}y \int_0^{+\infty} \frac{\sin(xy)}{x} \mathrm{d}x$$
$$= -\frac{\pi}{2} \int_a^b \mathrm{sgn}(y) \mathrm{d}y.$$

若 $b > a > 0$, $\int_a^b \mathrm{sgn}(y) \mathrm{d}y = \int_a^b \mathrm{d}y = b - a$;

若 $a < b < 0$, $\int_a^b \mathrm{sgn}(y) \mathrm{d}y = -\int_a^b \mathrm{d}y = a - b$;

若 $a < 0 < b$, $\int_a^b \mathrm{sgn}(y) \mathrm{d}y = -\int_a^0 \mathrm{d}y + \int_0^b \mathrm{d}y = a + b$.

因此, $\int_0^{+\infty} \frac{\cos(bx) - \cos(ax)}{x^2} \mathrm{d}x = \frac{\pi}{2}(|a| - |b|)$.

方法二: 积分号下求导. 记

$$I(t) = \int_0^{+\infty} \frac{\cos(tx) - \cos(ax)}{x^2} \mathrm{d}x, t \in [a, b],$$

则 $I(b)$ 即为所求. 由反常积分收敛的 Dirichlet 判别法 (定理 9.8.5), $\int_0^{+\infty} \frac{\cos(tx)}{x^2} \mathrm{d}x$ 和 $\int_0^{+\infty} \frac{\cos(ax)}{x^2} \mathrm{d}x$ 均收敛, 因此 $I(t)$ 收敛. 又由含参变量反常积分一致收敛的 Dirichlet 判别法 (定理 15.2.3),

$$\int_0^{+\infty} \frac{\partial}{\partial t} \frac{\cos(tx) - \cos(ax)}{x^2} \mathrm{d}x = -\int_0^{+\infty} \frac{\sin(tx)}{x} \mathrm{d}x$$

在 $[a, b]$ 一致收敛. 因此,

$$I'(t) = \int_0^{+\infty} \frac{\partial}{\partial t} \frac{\cos(tx) - \cos(ax)}{x^2} \mathrm{d}x = -\int_0^{+\infty} \frac{\sin(tx)}{x} \mathrm{d}x = -\frac{\pi}{2} \mathrm{sgn}(t).$$

上式两端从 a 到 b 积分, 并注意到 $I(a) = 0$, 得到

$$I(b) = -\frac{\pi}{2} \int_a^b \mathrm{sgn}(t) \mathrm{d}t = \frac{\pi}{2}(|a| - |b|).$$

$\square$

2. 首先计算

$$\begin{aligned}
I(y) &= \int_0^{+\infty} e^{-xy} \cos(mx) \mathrm{d}x = \frac{1}{m} \int_0^{+\infty} e^{-xy} \mathrm{d}\sin(mx) \\
&= \frac{1}{m} \left(e^{-xy} \sin(mx)|_{x=0}^{x=+\infty} + y \int_0^{+\infty} e^{-xy} \sin(mx) \mathrm{d}x \right) \\
&= \frac{y}{m} \int_0^{+\infty} e^{-xy} \sin(mx) \mathrm{d}x = -\frac{y}{m^2} \int_0^{+\infty} e^{-xy} \mathrm{d}\cos(mx) \\
&= -\frac{y}{m^2} \left(e^{-xy} \cos(mx)|_{x=0}^{x=+\infty} + y \int_0^{+\infty} e^{-xy} \cos(mx) \mathrm{d}x \right) \\
&= \frac{y}{m^2} - \frac{y^2}{m^2} I(y),
\end{aligned}$$

于是, $I(y) = \frac{y}{m^2 + y^2}$.

方法一: 交换累次积分顺序. 易知, $\frac{e^{-ax}-e^{-bx}}{x} = \int_a^b e^{-xy}\mathrm{d}y$. 由含参变量反常积分一致收敛的 Dirichlet 判别法 (定理 15.2.3), $I(y) = \int_0^{+\infty} e^{-xy}\cos(mx)\mathrm{d}x$ 在 $[a,b]$ 一致收敛. 因此,

$$\int_0^{+\infty} \frac{e^{-ax}-e^{-bx}}{x}\cos(mx)\mathrm{d}x = \int_0^{+\infty}\mathrm{d}x\int_a^b e^{-xy}\cos(mx)\mathrm{d}y$$
$$= \int_a^b \mathrm{d}y\int_0^{+\infty} e^{-xy}\cos(mx)\mathrm{d}x = \int_a^b \frac{y}{m^2+y^2}\mathrm{d}y$$
$$= \frac{1}{2}\ln(m^2+y^2)|_a^b = \frac{1}{2}\ln\frac{m^2+b^2}{m^2+a^2}.$$

方法二: 积分号下求导. 记

$$J(t) = \int_0^{+\infty} \frac{e^{-tx}-e^{-bx}}{x}\cos(mx)\mathrm{d}x, t\in[a,b],$$

则 $J(a)$ 即为所求. 由反常积分收敛的 Dirichlet 判别法 (定理 9.8.5), $\int_0^{+\infty}\frac{e^{-tx}}{x}\cos(mx)\mathrm{d}x$ 和 $\int_0^{+\infty}\frac{e^{-bx}}{x}\cos(mx)\mathrm{d}x$ 均收敛, 因此 $J(t)$ 收敛. 又由含参变量反常积分一致收敛的 Dirichlet 判别法 (定理 15.2.3),

$$\int_0^{+\infty}\frac{\partial}{\partial t}\frac{e^{-tx}-e^{-bx}}{x}\cos(mx)\mathrm{d}x = -\int_0^{+\infty} e^{-tx}\cos(mx)\mathrm{d}x$$

在 $[a,b]$ 一致收敛. 因此,

$$J'(t) = \int_0^{+\infty}\frac{\partial}{\partial t}\frac{e^{-tx}-e^{-bx}}{x}\cos(mx)\mathrm{d}x = -\int_0^{+\infty} e^{-tx}\cos(mx)\mathrm{d}x = -\frac{t}{m^2+t^2},$$

上式两端从 a 到 b 积分, 并注意到 $J(b) = 0$, 得到

$$J(a) = \int_a^b \frac{t}{m^2+t^2}\mathrm{d}t = \frac{1}{2}\ln(m^2+t^2)|_a^b = \frac{1}{2}\ln\frac{m^2+b^2}{m^2+a^2}.$$

$\square$

3. 令

$$I(y) = \int_0^{+\infty}\frac{\ln(1+x^2y^2)}{1+x^2}\mathrm{d}x,$$
$$J(y) = \int_0^{+\infty}\frac{\partial}{\partial y}\frac{\ln(1+x^2y^2)}{1+x^2}\mathrm{d}x.$$

(1) 首先证明: $I(y)$ 在 $(-\infty,+\infty)$ 连续. 事实上, $I(y)$ 在 $\{y||y|\leqslant M\}$ 一致收敛. 因为

$$\left|\frac{\ln(1+x^2y^2)}{1+x^2}\right| \leqslant \frac{\ln(1+x^2M^2)}{1+x^2},$$

而

$$\lim_{x\to+\infty} x^{\frac{3}{2}} \frac{\ln(1+x^2M^2)}{1+x^2} = \lim_{x\to+\infty} \frac{\ln(1+x^2M^2)}{x^{\frac{1}{2}}} = \lim_{x\to+\infty} \frac{4M^2 x^{\frac{3}{2}}}{1+x^2M^2} = \lim_{x\to+\infty} \frac{3}{\sqrt{x}} = 0,$$

由 Cauchy 判别法 (推论 9.8.3) 知 $\int_0^{+\infty} \frac{\ln(1+x^2M^2)}{1+x^2}\mathrm{d}x$ 收敛, 再由 M 判别法知 $I(y)$ 在 $\{y||y| \leqslant M\}$ 一致收敛. 由 M 的任意性知 $I(y)$ 在 $(-\infty,+\infty)$ 一致收敛, 因此, $I(y)$ 在 $(-\infty,+\infty)$ 连续.

(2) 接着证明: $J(y)$ 在 $\{y|0 < m \leqslant |y| \leqslant M\}$ 一致收敛. 事实上,

$$\left| \frac{\partial}{\partial y} \frac{\ln(1+x^2y^2)}{1+x^2} \right| = \left| \frac{2x^2y}{(1+x^2)(1+x^2y^2)} \right| \leqslant \frac{2x^2M}{(1+x^2)(1+x^2m^2)},$$

而 $\lim_{x\to+\infty} x^2 \frac{2x^2M}{(1+x^2)(1+x^2m^2)} = \frac{2M}{m^2}$, 同样由 Cauchy 判别法 (推论 9.8.3) 知 $\int_0^{+\infty} \frac{2x^2M}{(1+x^2)(1+x^2m^2)}\mathrm{d}x$ 收敛, 再由 M 判别法知 $J(y)$ 在 $\{y|0 < m \leqslant |y| \leqslant M\}$ 一致收敛.

(3) 最后计算 $I(y)$.

$$\begin{aligned} I'(y) =& J(y) = \int_0^{+\infty} \frac{2x^2y}{(1+x^2)(1+x^2y^2)}\mathrm{d}x = \frac{2y}{y^2-1} \int_0^{+\infty} \left(\frac{1}{1+x^2} - \frac{1}{1+x^2y^2} \right) \mathrm{d}x \\ =& \frac{2y}{y^2-1} \cdot \frac{\pi}{2} \left(1 - \frac{1}{|y|} \right) = \begin{cases} \frac{\pi}{y+1}, & y > 0, \\ \frac{\pi}{y-1}, & y < 0. \end{cases} \end{aligned}$$

由 m, M 的任意性知上式对 $(-\infty,0) \cup (0,+\infty)$ 均成立. 两端求不定积分, 得到

$$I(y) = \pi \ln(|y| + 1) + C.$$

因为 $I(y)$ 在 $(-\infty,+\infty)$ 连续, 因此,

$$0 = I(0) = \lim_{y\to 0} I(y) = \lim_{y\to 0}[\pi \ln(|y| + 1) + C] = C,$$

从而, $I(y) = \pi \ln(|y| + 1)$. □

复习题

1. 对 $\forall y_0 \in \mathbb{R}\backslash 0$, 取 $J = [y_0 - \delta, y_0 + \delta] \in \mathbb{R}\backslash 0$, 则 $\frac{yf(x)}{x^2+y^2}$ 在 $[0,1] \times J$ 连续, 因而 $F(y)$ 在 J 连续, 从而在 y_0 连续. 由 y_0 的任意性, $F(y)$ 在 $\mathbb{R}\backslash 0$ 连续. □

2. 直接计算,

$$\frac{\partial F}{\partial x} = \int_{x/y}^{xy} \frac{\partial}{\partial x}(x - yt)f(t)\mathrm{d}t + (x - xy^2)f(xy)y - (x - x)f(x/y)(1/y)$$

$$= \int_{x/y}^{xy} f(t)\mathrm{d}t + xy(1 - y^2)f(xy),$$

$$\frac{\partial^2 F}{\partial x \partial y} = \frac{\partial F}{\partial y}\left[\int_{x/y}^{xy} f(t)\mathrm{d}t + xy(1 - y^2)f(xy)\right]$$

$$= [f(xy)x - f(x/y)(-x/y^2)]$$

$$\quad + [x(1 - y^2)f(xy) + xy(-2y)f(xy) + xy(1 - y^2)f'(xy)x]$$

$$= x(2 - 3y^2)f(xy) + \frac{x}{y^2}f(x/y) + x^2 y(1 - y^2)f'(xy).$$

$\square$

3. 直接计算得

$$f'(u) = \int_0^u \frac{\partial}{\partial u}g(x + u, x - u)\mathrm{d}x + g(2u, 0)$$

$$= \int_0^u [g'(x + u) - g'(x - u)]\mathrm{d}x + g(2u, 0).$$

$\square$

4. 记

$$f(t, x) = \frac{\sin(tx)}{t}\int_0^1 e^{(txy)^2}\mathrm{d}y, \ 0 \leqslant x \leqslant 1, t \neq 0.$$

补充定义

$$f(0, x) = \lim_{t \to 0}\frac{\sin(tx)}{t}\int_0^1 e^{(txy)^2}\mathrm{d}y = x, \ x \in [0, 1],$$

则 $f(t, x)$ 在 $[-1, 1] \times [0, 1]$ 连续, 因此,

$$\lim_{t \to 0}\int_0^1 f(t, x)\mathrm{d}x = \int_0^1 \lim_{t \to 0} f(t, x)\mathrm{d}x = \lim_{t \to 0}\int f(0, x)\mathrm{d}x = \lim_{t \to 0}\int x\mathrm{d}x = \frac{1}{2}.$$

$\square$

5. 记

$$f(x, t) = \begin{cases} \lim_{x \to 0+0}\frac{\arctan(t\tan x)}{\tan x} = t, & x = 0, \\ \frac{\arctan(t\tan x)}{\tan x}, & x \in (0, \frac{\pi}{2}), \\ \lim_{x \to \frac{\pi}{2}-0}\frac{\arctan(t\tan x)}{\tan x} = 0, & x = \frac{\pi}{2}, \end{cases}$$

则 $f(x, t)$ 在 $[0, \frac{\pi}{2}] \times J$ 上连续, 其中 J 是以 0 和 a 为端点的闭区间. 记

$$I(t) = \int_0^{\frac{\pi}{2}} \frac{\arctan(t\tan x)}{\tan x}\mathrm{d}x, t \in J,$$

则 $I(t)$ 在 J 连续, 并且 $I(a)$ 即为所求. 直接计算得

$$
\begin{aligned}
I'(t) &= \int_0^{\frac{\pi}{2}} \frac{\partial}{\partial t} \frac{\arctan(t \tan x)}{\tan x} \mathrm{d}x = \int_0^{\frac{\pi}{2}} \frac{1}{1 + t^2 \tan^2 x} \mathrm{d}x \\
&= \int_0^{+\infty} \frac{1}{(1 + t^2 u^2)(1 + u^2)} \mathrm{d}u = \frac{1}{t^2 - 1} \int_0^{+\infty} \left(\frac{t^2}{1 + t^2 u^2} - \frac{1}{1 + u^2} \right) \mathrm{d}u \\
&= \left\{ \begin{array}{ll} \frac{\pi}{2(t+1)}, & t > 0, \\ \frac{\pi}{2(1-t)}, & t < 0 \end{array} \right. = \frac{\pi}{2(1 + |t|)},
\end{aligned}
$$

上式从 0 到 a 积分并注意到 $I(0) = 0$, 有

$$
I(a) = \left\{ \begin{array}{ll} \frac{\pi}{2} \ln(1 + a), & a > 0, \\ -\frac{\pi}{2} \ln(1 - a), & a < 0 \end{array} \right. = \frac{\pi}{2} \mathrm{sgn}(a) \ln(1 + |a|),
$$

即为所求. $\qquad\qquad\square$

6. 记

$$
f(x, t) = \left\{ \begin{array}{ll} \frac{1}{\cos x} \ln \frac{1 + t \cos x}{1 - t \cos x}, & x \in [0, \frac{\pi}{2}), \\ \lim_{x \to \frac{\pi}{2} - 0} \frac{1}{\cos x} \ln \frac{1 + t \cos x}{1 - t \cos x} = 2t, & x = \frac{\pi}{2}, \end{array} \right.
$$

则 $f(x, t)$ 在 $[0, \frac{\pi}{2}] \times [-|a|, |a|]$ 连续. 记

$$
I(t) = \int_0^{\frac{\pi}{2}} \ln \frac{1 + t \cos x}{1 - t \cos x} \frac{\mathrm{d}x}{\cos x}, \ t \in [-|a|, |a|],
$$

则 $I(t)$ 在 $[-|a|, |a|]$ 连续, 并且 $I(a)$ 即为所求. 直接计算得

$$
I'(t) = \int_0^{\frac{\pi}{2}} \frac{\partial}{\partial t} \ln \frac{1 + t \cos x}{1 - t \cos x} \frac{1}{\cos x} \mathrm{d}x = \int_0^{\frac{\pi}{2}} \left[\frac{1}{1 + t \cos x} - \frac{1}{1 - t \cos x} \right] \mathrm{d}x = \int_0^{\frac{\pi}{2}} \frac{4 \mathrm{d}x}{2 - t^2 - t^2 \cos(2x)},
$$

令 $u = \tan x$, 得

$$
I'(t) = 2 \int_0^{+\infty} \frac{1}{(1 - t^2) + u^2} \mathrm{d}u = \frac{\pi}{\sqrt{1 - t^2}}.
$$

上式从 0 到 a 积分并注意到 $I(0) = 0$, 有

$$
I(a) = \pi \arcsin a,
$$

即为所求. $\qquad\qquad\square$

7. 函数 $f(x, y) = \frac{2a}{1 - a^2 y^2 \cos^2 x}$ 在 $[0, \frac{\pi}{2}] \times [0, 1]$ 连续, 注意到 $\frac{1}{\cos x} \ln \frac{1 + t \cos x}{1 - t \cos x} = 2a \int_0^1 \frac{\mathrm{d}y}{1 - a^2 y^2 \cos^2 x}$,

有

$$I = \int_0^{\frac{\pi}{2}} \ln \frac{1 + t\cos x}{1 - t\cos x} \frac{\mathrm{d}x}{\cos x} = 2a \int_0^{\frac{\pi}{2}} \mathrm{d}x \int_0^1 \frac{1}{1 - a^2 y^2 \cos^2 x} \mathrm{d}y$$

$$= 2a \int_0^1 \mathrm{d}y \int_0^{\frac{\pi}{2}} \frac{1}{1 - a^2 y^2 \cos^2 x} \mathrm{d}x = 4a \int_0^1 \left(\int_0^{\frac{\pi}{2}} \frac{1}{2 - a^2 y^2 - a^2 y^2 \cos(2x)} \mathrm{d}x \right) \mathrm{d}y$$

$$\overset{u = \tan x}{=\!=\!=} 2a \int_0^1 \mathrm{d}y \int_0^{+\infty} \frac{1}{1 - a^2 y^2 + u^2} \mathrm{d}u = \int_0^1 \frac{\pi a}{\sqrt{1 - a^2 y^2}} \mathrm{d}y$$

$$= \pi \arcsin(ay)|_0^1 = \pi \arcsin a.$$

$\square$

8. (1) $I(1)$ 即为所求. 函数 $f(x, t) = \frac{\ln(1 + tx)}{1 + x^2}$ 在 $[0, 1]^2$ 连续, 所以,

$$I'(t) = \int_0^1 \frac{\partial}{\partial t} \frac{\ln(1 + tx)}{1 + x^2} \mathrm{d}x = \int_0^1 \frac{x}{(1 + x^2)(1 + tx)} \mathrm{d}x$$

$$= \frac{1}{1 + t^2} \int_0^1 \left(\frac{x + t}{1 + x^2} - \frac{t}{1 + tx} \right) \mathrm{d}x$$

$$= \frac{1}{1 + t^2} \left[\frac{1}{2} \ln(1 + x^2) + t \arctan x - \ln(1 + tx) \right] \Big|_{x=0}^{x=1}$$

$$= \frac{1}{1 + t^2} \left[\frac{1}{2} \ln 2 + \frac{\pi}{4} t - \ln(1 + t) \right],$$

上式两端从 0 到 1 积分并注意到 $I(0) = 0$ 得

$$I(1) = \frac{\ln 2}{2} \arctan t|_0^1 + \frac{\pi}{8} \ln(1 + t^2)|_0^1 - I(1),$$

所以 $I(1) = \frac{\pi}{8} \ln 2$, 即为所求.

(2) 令 $x = \tan \alpha$, 则

$$J = \int_0^1 \frac{\ln(1 + x)}{1 + x^2} \mathrm{d}x = \int_0^{\frac{\pi}{4}} \ln(1 + \tan \alpha) \mathrm{d}\alpha = \int_0^{\frac{\pi}{4}} [\ln(\sin \alpha + \cos \alpha) - \ln \cos \alpha] \mathrm{d}\alpha$$

$$= \int_0^{\frac{\pi}{4}} \left[\ln \sqrt{2} + \ln \cos \left(\frac{\pi}{4} - \alpha \right) - \ln \cos \alpha \right] \mathrm{d}\alpha.$$

令 $\beta = \frac{\pi}{4} - \alpha$, 则

$$\int_0^{\frac{\pi}{4}} \ln \cos \left(\frac{\pi}{4} - \alpha \right) \mathrm{d}\alpha = \int_0^{\frac{\pi}{4}} \ln \cos \beta \mathrm{d}\beta = \int_0^{\frac{\pi}{4}} \ln \cos \alpha \mathrm{d}\alpha,$$

因此, $J = \int_0^{\frac{\pi}{4}} \ln \sqrt{2} \mathrm{d}\alpha = \frac{\pi}{8} \ln 2$. $\square$

9. 因为 $|f(x)\cos(xy)| \leqslant |f(x)|$, 而 $\int_{-\infty}^{+\infty} |f(x)| \mathrm{d}x$ 收敛, 由 M 判别法, $I(y)$ 在 $(-\infty, +\infty)$ 一致收敛. 下面证明 $I(y)$ 在 $(-\infty, +\infty)$ 一致连续. 首先由 $I(y)$ 在 $(-\infty, +\infty)$ 一致收敛, 所以

$I(y)$ 在 $(-\infty, +\infty)$ 连续, 并且, 对 $\forall \varepsilon > 0, \exists G > 0, \forall G' > G, \forall y \in (-\infty, +\infty)$, 有

$$\left| \int_{G'}^{+\infty} f(x)\cos(xy)\mathrm{d}x \right| < \varepsilon, \quad \left| \int_{-\infty}^{-G'} f(x)\cos(xy)\mathrm{d}x \right| < \varepsilon.$$

对上述的 $\varepsilon > 0$, $\exists \delta = \varepsilon > 0$, 对 $\forall y_1, y_2 : |y_1 - y_2| < \delta$, 有

$$
\begin{aligned}
|I(y_1) - I(y_2)| &= \left| \int_{-\infty}^{+\infty} f(x)[\cos(xy_1) - \cos(xy_2)]\mathrm{d}x \right| \\
&= \left| \int_{-\infty}^{-G'} f(x)[\cos(xy_1) - \cos(xy_2)]\mathrm{d}x + \int_{-G'}^{G'} f(x)[\cos(xy_1) - \cos(xy_2)]\mathrm{d}x \right.\\
&\quad \left. + \int_{G'}^{+\infty} f(x)[\cos(xy_1) - \cos(xy_2)]\mathrm{d}x \right| \\
&\leqslant \left| \int_{-\infty}^{-G'} f(x)\cos(xy_1)\mathrm{d}x \right| + \left| \int_{-\infty}^{-G'} f(x)\cos(xy_2)\mathrm{d}x \right| \\
&\quad + \left| \int_{-G'}^{G'} f(x)[\cos(xy_1) - \cos(xy_2)]\mathrm{d}x \right| \\
&\quad + \left| \int_{G'}^{+\infty} f(x)\cos(xy_1)\mathrm{d}x \right| + \left| \int_{G'}^{+\infty} f(x)\cos(xy_2)\mathrm{d}x \right| \\
&< 4\varepsilon + \left| \int_{-G'}^{G'} f(x)[\cos(xy_1) - \cos(xy_2)]\mathrm{d}x \right| \\
&= 4\varepsilon + \left| \int_{-G'}^{G'} \left[\int_{y_1}^{y_2} \frac{f(x)}{x} \sin(xy)\mathrm{d}y \right] \mathrm{d}x \right|
\end{aligned}
$$

令 $g(x,y) = \frac{f(x)}{x}\sin(xy)$, 补充定义 $g(0,y) = \lim_{x\to 0} \frac{f(x)}{x}\sin(xy) = yf(0)$, 则 $g(x,y)$ 在 $\mathbb{R}^2$ 连续, 上式中的两个积分可以交换次续, 得

$$|I(y_1) - I(y_2)| < 4\varepsilon + \left| \int_{y_1}^{y_2} \left[\int_{-G'}^{G'} \frac{f(x)}{x} \sin(xy)\mathrm{d}x \right] \mathrm{d}y \right|.$$

令

$$J(y) = \int_{-G'}^{G'} \frac{f(x)}{x} \sin(xy)\mathrm{d}x,$$

则 $J(y)$ 一致收敛, 因而有界, 设 $|J(y)| \leqslant M$. 所以,

$$|I(y_1) - I(y_2)| < 4\varepsilon + \left| \int_{y_1}^{y_2} J(y)\mathrm{d}y \right| \leqslant 4\varepsilon + M|y_2 - y_1| = (4 + M)\varepsilon,$$

即得 $I(y)$ 在 $(-\infty, +\infty)$ 一致连续. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

    10. 利用积分 $\int_0^{+\infty} e^{-x^2}\mathrm{d}x = \frac{\sqrt{\pi}}{2}$.

方法一. 由分部积分法,

$$\int_0^{+\infty} \frac{e^{-ax^2} - e^{-bx^2}}{x^2}\mathrm{d}x = -\int_0^{+\infty} (e^{-ax^2} - e^{-bx^2})\mathrm{d}\left(\frac{1}{x}\right)$$

$$= -\left.\frac{e^{-ax^2} - e^{-bx^2}}{x}\right|_0^{+\infty} - 2\int_0^{+\infty} (ae^{-ax^2} - be^{-bx^2})\mathrm{d}x$$

$$= -2\int_0^{+\infty} \sqrt{a}e^{-(\sqrt{a}x)^2}\mathrm{d}(\sqrt{a}x) + 2\int_0^{+\infty} \sqrt{b}e^{-(\sqrt{b}x)^2}\mathrm{d}(\sqrt{b}x)$$

$$= -2\sqrt{a}\frac{\sqrt{\pi}}{2} + 2\sqrt{b}\frac{\sqrt{\pi}}{2} = \sqrt{\pi}(\sqrt{b} - \sqrt{a}).$$

方法二. 注意到 $\frac{e^{-ax^2} - e^{-bx^2}}{x^2} = \int_a^b e^{-x^2 y}\mathrm{d}y$, 并且 $|e^{-x^2 y}| \leqslant e^{-ax^2}$, 而 $\int_0^{+\infty} e^{-ax^2}\mathrm{d}x$ 收敛, 所以 $\int_0^{+\infty} e^{-x^2 y}\mathrm{d}x$ 在 $[a, b]$ 一致收敛, 因而,

$$\int_0^{+\infty} \frac{e^{-ax^2} - e^{-bx^2}}{x^2}\mathrm{d}x = \int_0^{+\infty}\mathrm{d}x\int_a^b e^{-x^2 y}\mathrm{d}y = \int_a^b\mathrm{d}y\int_0^{+\infty} e^{-x^2 y}\mathrm{d}x$$

$$= \int_a^b \left(\frac{1}{\sqrt{y}}\int_0^{+\infty} e^{-x^2 y}\mathrm{d}(\sqrt{y}x)\right)\mathrm{d}y = \int_a^b \frac{\sqrt{\pi}}{2\sqrt{y}}\mathrm{d}y$$

$$= \sqrt{\pi}(\sqrt{b} - \sqrt{a}).$$

方法三. 记

$$I(t) = \int_0^{+\infty} \frac{e^{-tx^2} - e^{-bx^2}}{x^2}\mathrm{d}x,$$

则 $I(b) = 0$, $I(a)$ 为所求. 由于 $\left|\frac{e^{-tx^2}}{x^2}\right| \leqslant \left|\frac{1}{x^2}\right|, \left|\frac{e^{-bx^2}}{x^2}\right| \leqslant \left|\frac{1}{x^2}\right|$, 而 $\int_0^{+\infty} \frac{1}{x^2}\mathrm{d}x$ 收敛, 因此 $\int_0^{+\infty} \frac{e^{-tx^2}}{x^2}\mathrm{d}x$ 和 $\int_0^{+\infty} \frac{e^{-bx^2}}{x^2}\mathrm{d}x$ 均收敛, 从而 $I(t)$ 收敛. 又因为 $|e^{-x^2 y}| \leqslant e^{-ax^2}$, 而 $\int_0^{+\infty} e^{-ax^2}\mathrm{d}x$ 收敛, 所以 $\int_0^{+\infty} e^{-x^2 y}\mathrm{d}x$ 在 $[a, b]$ 一致收敛. 因此,

$$I'(t) = \int_0^{+\infty} \frac{\partial}{\partial t}\frac{e^{-tx^2} - e^{-bx^2}}{x^2}\mathrm{d}x = -\int_0^{+\infty} e^{-tx^2}\mathrm{d}x = -\frac{\sqrt{\pi}}{2\sqrt{t}},$$

上式两端从 a 到 b 积分并注意到 $I(b) = 0$, 得
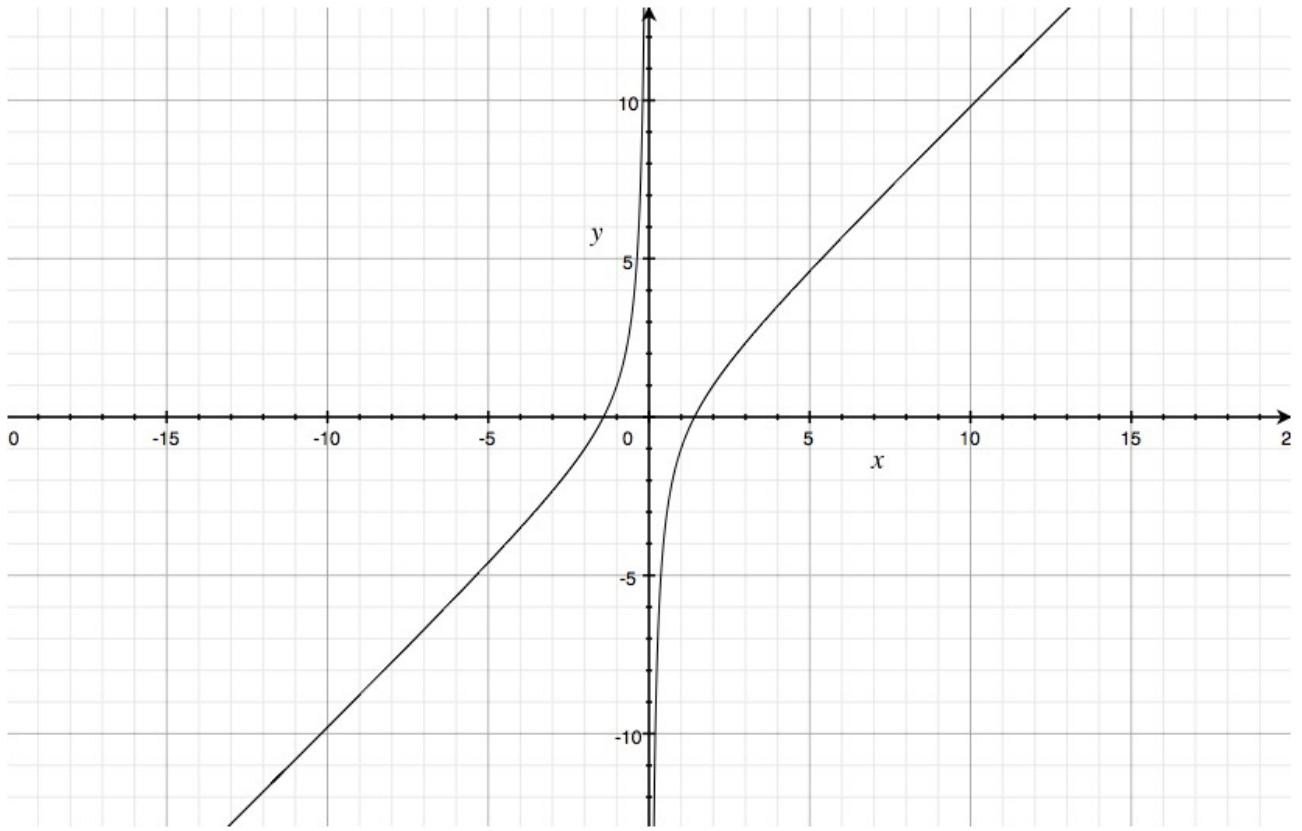
$$I(a) = \int_a^b \frac{\sqrt{\pi}}{2\sqrt{t}}\mathrm{d}t = \sqrt{\pi}(\sqrt{b} - \sqrt{a}).$$

$\square$

11. 易知

$$I = \int_0^{+\infty} e^{-x^2 - a^2 x^{-2}}\mathrm{d}x = e^{-2a}\int_0^{+\infty} e^{-(x - a/x)^2}\mathrm{d}x,$$

令 $t = x - a/x$, 则 $x = \frac{1}{2}(t + \sqrt{t^2 + 4a})$, $x > 0$, 于是 (见图 4),

图 4: $t = x - 2/x$

$$I = \frac{1}{2}e^{-2a}\int_{-\infty}^{+\infty} e^{-t^2}\left(1 + \frac{t}{\sqrt{t^2 + 4a}}\right)\mathrm{d}t$$
$$= \frac{1}{2}e^{-2a}\int_{-\infty}^{0} e^{-t^2}\left(1 + \frac{t}{\sqrt{t^2 + 4a}}\right)\mathrm{d}t + \frac{1}{2}e^{-2a}\int_{0}^{+\infty} e^{-t^2}\left(1 + \frac{t}{\sqrt{t^2 + 4a}}\right)\mathrm{d}t$$
$$\triangleq I_1 + I_2.$$

令 $u = -t$, 得

$$I_1 = -\frac{1}{2}e^{-2a}\int_{+\infty}^{0} e^{-u^2}\left(1 - \frac{u}{\sqrt{u^2 + 4a}}\right)\mathrm{d}u = \frac{1}{2}e^{-2a}\int_{0}^{+\infty} e^{-t^2}\left(1 - \frac{t}{\sqrt{t^2 + 4a}}\right)\mathrm{d}t,$$

所以,

$$I = e^{-2a}\int_{0}^{+\infty} e^{-t^2}\mathrm{d}t = e^{-2a}\frac{\sqrt{\pi}}{2}.$$

$\square$

第 16 章 重积分

习题 16.1 二重积分的概念

1. 仿定理 8.1.2 的证明. □

2. 由定理 16.1.4, 有界函数 $f(x,y)$ 在 D 可积的充要条件是 $\lim_{\|T\|\to 0} S(T) = \lim_{\|T\|\to 0} s(T)$, 即 $\lim_{\|T\|\to 0}(S(T)-s(T)) = 0$, 亦即, 对 $\forall \varepsilon > 0, \exists \delta > 0$, 对任意分割 $T : \|T\| < \delta, S(T)-s(T) < \varepsilon$. 即得定理 16.1.5. □

3. 用反证法. 假设 $f(x,y) \not\equiv 0, (x,y) \in D$, 则 $\exists P_0(x_0, y_0) \in D^\circ$, 使得 $f(x_0, y_0) \neq 0$, 不妨设 $f(x_0, y_0) > 0$. 由连续函数的保号性, $\exists \delta_0 > 0$, 使得 $\bar{B}_{\delta_0}(P_0) \subset D^\circ$ 且 $f(x,y) \geqslant \frac{f(x_0,y_0)}{2} > 0, (x,y) \in \bar{B}_{\delta_0}(P_0)$, 即 $f(x,y)$ 在 $\bar{B}_{\delta_0}(P_0)$ 非负, 故

$$\iint_{\bar{B}_{\delta_0}(P_0)} f(x,y)\mathrm{d}x\mathrm{d}y \geqslant \frac{f(x_0,y_0)}{2} \iint_{\bar{B}_{\delta_0}(P_0)} \mathrm{d}x\mathrm{d}y = \frac{f(x_0,y_0)}{2} \cdot \pi \delta_0^2 > 0,$$

这与已知矛盾, 从而 $f(x,y) \equiv 0, (x,y) \in D$. □

4. (1) 函数 $f(x,y)$ 在 D 有界, 在 $\{(x,y)|y=1, x \in [0,2]\}$ 处不连续, 由定理 16.1.7, $f(x,y)$ 在 D 可积.

(2) 一个有界函数在 D 上不可积意味着: $\exists \varepsilon_0 > 0$, 使得对 D 的任意分割 T 均有 $S(T)-s(T) \geqslant \varepsilon_0$. 对 $f(x,y)$, 取 $\varepsilon_0 = 1$ 和 D 的任意分割 $T = \{\sigma_i\}_{i=1}^n$.

1° 先假设 $\{y=1\} \subset T$, 则 $y=1$ 将分割 T 分为两部分: 位于 $D_1 = [0,2] \times [0,1]$ 内的部分记为 $T_1 = \{\sigma_i\}_{i=1}^m$, 位于 $D_2 = [0,2] \times [1,2]$ 内的部分记为 $T_2 = \{\sigma_i\}_{i=m+1}^n$. 由有理数的稠密性知, 在 $[0,2] \times [0,1]$ 上, $M_i = \sup_{(x,y)\in\sigma_i} f(x,y) = 1, m_i = \inf_{(x,y)\in\sigma_i} f(x,y) = 0, i = 1,2,...,m$, 所以 $S(T) - s(T) = [S(T_1)-s(T_1)] + [S(T_2)-s(T_2)] \geqslant S(T_1)-s(T_1) = \sum_{i=1}^m (1-0)\Delta\sigma_i = \sum_{i=1}^m \Delta\sigma_i = S_{D_2} = 2 > \varepsilon_0$, 即 $f(x,y)$ 在 D 上不可积.

2° 如果 $\{y=1\} \not\subset T$, 则添加 $\{y=1\}$ 后构成新的分割 $T^* = T \cup \{y=1\}$. 显然, $S(T) \geqslant S(T^*), s(T) \leqslant s(T^*)$. 于是, $S(T) - s(T) \geqslant S(T^*) - s(T^*)$, 再对分割 T^* 重复 1° 的过程即得证. □

5. 由积分中值定理, 存在 $(\xi, \eta) \in D = \{(x,y)|x^2 + y^2 \leqslant \rho^2\}$, 使得

$$\iint_{x^2+y^2\leqslant\rho^2} f(x,y)\mathrm{d}x\mathrm{d}y = f(\xi,\eta)S_D = \pi\rho^2 f(\xi,\eta),$$

于是, 由 $f(x,y)$ 在原点的邻域连续, 有

$$\lim_{\rho\to 0} \frac{1}{\pi\rho^2} \iint_{x^2+y^2\leqslant\rho^2} f(x,y)\mathrm{d}x\mathrm{d}y = \lim_{\rho\to 0} f(\xi,\eta) = f(0,0).$$

□

6. 因为 $f(x)$ 在 $[a,b]$ 可积, $g(y)$ 在 $[c,d]$ 可积, 由定理 8.4.3, 对 $\forall \varepsilon > 0$, 分别存在 $[a,b]$ 的分割 $T_1 = \{a = x_0 < x_1 < ... < x_s = b\}$ 和 $[c,d]$ 的分割 $T_2 = \{c = y_0 < y_1 < ... < y_r = d\}$, 使得

$$S(T_1) - s(T_1) = \sum_{i=1}^{s} \omega_i^{(f)} \Delta x_i < \varepsilon, \ \ S(T_2) - s(T_2) = \sum_{j=1}^{r} \omega_j^{(g)} \Delta y_j < \varepsilon,$$

其中

$$\omega_i^{(f)} = M_i^{(f)} - m_i^{(f)}, \ \ S(T_1) = \sum_{i=1}^{s} M_i^{(f)} \Delta x_i, \ \ s(T_1) = \sum_{i=1}^{s} m_i^{(f)} \Delta x_i,$$

$$M_i^{(f)} = \sup_{x \in [x_{i-1}, x_i]} f(x), m_i^{(f)} = \inf_{x \in [x_{i-1}, x_i]} f(x), \Delta x_i = x_i - x_{i-1},$$

$$\omega_j^{(g)} = M_j^{(g)} - m_j^{(g)}, \ \ S(T_2) = \sum_{j=1}^{r} M_j^{(g)} \Delta y_j, \ \ s(T_2) = \sum_{j=1}^{r} m_j^{(g)} \Delta y_j,$$

$$M_j^{(g)} = \sup_{y \in [y_{j-1}, y_j]} g(y), m_j^{(g)} = \inf_{y \in [y_{j-1}, y_j]} g(y), \Delta y_j = y_j - y_{j-1}.$$

直线网 $T = T_1 \cup T_2$ 恰好构成 $D = [a,b] \times [c,d]$ 的一个分割, 对此分割,

$$\begin{aligned}
\omega_{ij}^{(fg)} &= \sup_{\substack{x \in [x_{i-1}, x_i] \\ y \in [y_{j-1}, y_j]}} f(x)g(y) - \inf_{\substack{x \in [x_{i-1}, x_i] \\ y \in [y_{j-1}, y_j]}} f(x)g(y) \\
&= \sup_{\substack{x_1, x_2 \in [x_{i-1}, x_i] \\ y_1, y_2 \in [y_{j-1}, y_j]}} |f(x_1)g(y_1) - f(x_2)g(y_2)| \\
&\leqslant \sup_{\substack{x_1, x_2 \in [x_{i-1}, x_i] \\ y_1, y_2 \in [y_{j-1}, y_j]}} \{|f(x_1)| \cdot |g(y_1) - g(y_2)| + |g(y_2)| \cdot |f(x_1) - f(x_2)|\},
\end{aligned}$$

因为 $f(x)$ 在 $[a,b]$ 可积, $g(y)$ 在 $[c,d]$ 可积, 所以 $\exists A_1, A_2 > 0$, 使得 $|f(x)| \leqslant A_1, |g(y)| \leqslant A_2$, 因此,

$$\omega_{ij}^{(fg)} \leqslant A_1 \omega_j^{(g)} + A_2 \omega_i^{(f)}.$$

于是,

$$S(T) - s(T) = \sum_{i=1}^{s} \sum_{j=1}^{r} \left[\sup_{\substack{x \in [x_{i-1}, x_i] \\ y \in [y_{j-1}, y_j]}} f(x)g(y) - \inf_{\substack{x \in [x_{i-1}, x_i] \\ y \in [y_{j-1}, y_j]}} f(x)g(y) \right] \Delta x_i \Delta y_j$$

$$= \sum_{i=1}^{s} \sum_{j=1}^{r} \omega_{ij}^{(fg)} \Delta x_i \Delta y_j$$

$$\leqslant \sum_{i=1}^{s} \sum_{j=1}^{r} [A_1 \omega_j^{(g)} + A_2 \omega_i^{(f)}] \Delta x_i \Delta y_j$$

$$= A_1 \sum_{i=1}^{s} \Delta x_i \sum_{j=1}^{r} \omega_j^{(g)} \Delta y_j + A_2 \sum_{i=1}^{s} \omega_i^{(f)} \Delta x_i \sum_{j=1}^{r} \Delta y_j$$

$$= A_1(b-a) \sum_{j=1}^{r} \omega_j^{(g)} \Delta y_j + A_2(d-c) \sum_{i=1}^{s} \omega_i^{(f)} \Delta x_i$$

$$< [A_1(b-a) + A_2(d-c)]\varepsilon,$$

由定理 16.1.5, $f(x)g(y)$ 在 $D = [a,b] \times [c,d]$ 可积. 对 $\forall \xi_i \in [x_{i-1}, x_i], \eta_j \in [y_{j-1}, y_j]$, 有

$$\iint_D f(x)g(y)\mathrm{d}x\mathrm{d}y = \lim_{\|T\| \to 0} \sum_{i=1}^{s} \sum_{j=1}^{r} f(\xi_i)g(\eta_j) \Delta x_i \Delta y_j$$

$$= \lim_{\|T\| \to 0} \sum_{i=1}^{s} f(\xi_i) \Delta x_i \cdot \lim_{\|T\| \to 0} \sum_{j=1}^{r} g(\eta_j) \Delta y_j$$

$$= \lim_{\|T_1\| \to 0} \sum_{i=1}^{s} f(\xi_i) \Delta x_i \cdot \lim_{\|T_2\| \to 0} \sum_{j=1}^{r} g(\eta_j) \Delta y_j$$

$$= \int_a^b f(x)\mathrm{d}x \int_c^d g(y)\mathrm{d}y.$$

得证. □

7. 考察积分 $I = \iint_{[a,b]^2} [f(x)g(y) - f(y)g(x)]^2 \mathrm{d}x\mathrm{d}y,$

$$0 \leqslant I = \iint_{[a,b]^2} [f(x)g(y) - f(y)g(x)]^2 \mathrm{d}x\mathrm{d}y$$

$$= \iint_{[a,b]^2} f^2(x)g^2(y)\mathrm{d}x\mathrm{d}y - 2\iint_{[a,b]^2} f(x)g(x)f(y)g(y)\mathrm{d}x\mathrm{d}y + \iint_{[a,b]^2} f^2(y)g^2(x)\mathrm{d}x\mathrm{d}y$$

$$= \int_a^b f^2(x)\mathrm{d}x \int_a^b g^2(y)\mathrm{d}y - 2\int_a^b f(x)g(x)\mathrm{d}x \int_a^b f(y)g(y)\mathrm{d}y + \int_a^b f^2(y)\mathrm{d}y \int_a^b g^2(x)\mathrm{d}x$$

$$= 2\int_a^b f^2(x)\mathrm{d}x \int_a^b g^2(x)\mathrm{d}x - 2\left[\int_a^b f(x)g(x)\mathrm{d}x\right]^2,$$

移项即得证. □

习题 16.2 直角坐标系下二重积分的计算

1. (1)

$$
\begin{aligned}
\iint_D (x^3 + xy + y^2)\mathrm{d}x\mathrm{d}y &= \int_0^1 \mathrm{d}x \int_0^1 (x^3 + xy + y^2)\mathrm{d}y \\
&= \int_0^1 \left(x^3 y + \frac{xy^2}{2} + \frac{y^3}{3} \right)\Big|_{y=0}^{y=1} \mathrm{d}x \\
&= \int_0^1 \left(x^3 + \frac{x}{2} + \frac{1}{3} \right)\mathrm{d}x \\
&= \frac{x^4}{4} + \frac{x^2}{4} + \frac{x}{3}\Big|_0^1 \\
&= \frac{5}{6}.
\end{aligned}
$$

(2) 设 $D_1 = [0,\pi]^2 \cap \{(x,y)|x+y \leqslant \pi\}$, $D_2 = [0,\pi]^2 \backslash D_1$, 则

$$
\begin{aligned}
\iint_D |\sin(x+y)|\mathrm{d}x\mathrm{d}y &= \iint_{D_1} |\sin(x+y)|\mathrm{d}x\mathrm{d}y + \iint_{D_2} |\sin(x+y)|\mathrm{d}x\mathrm{d}y \\
&= \int_0^\pi \mathrm{d}x \int_0^{\pi-x} \sin(x+y)\mathrm{d}y - \int_0^\pi \mathrm{d}x \int_{\pi-x}^\pi \sin(x+y)\mathrm{d}y \\
&= -\int_0^\pi \cos(x+y)|_{y=0}^{y=\pi-x}\mathrm{d}x + \int_0^\pi \cos(x+y)|_{y=\pi-x}^{y=\pi}\mathrm{d}x \\
&= \int_0^\pi 1 + \cos x\mathrm{d}x + \int_0^\pi 1 + \cos(x+\pi)\mathrm{d}x \\
&= 2\pi.
\end{aligned}
$$

(3)

$$
\begin{aligned}
\iint_D (x^2 + y^2)\mathrm{d}x\mathrm{d}y &= \int_0^1 \mathrm{d}x \int_{\sqrt{x}}^{2\sqrt{x}} (x^2 + y^2)\mathrm{d}y = \int_0^1 \left(x^2 y + \frac{y^3}{3} \right)\Big|_{y=\sqrt{x}}^{y=2\sqrt{x}} \mathrm{d}x \\
&= \int_0^1 \left(x^{\frac{5}{2}} + \frac{7}{3}x^{\frac{3}{2}} \right)\mathrm{d}x = \frac{2}{7}x^{\frac{7}{2}} + \frac{14}{15}x^{\frac{5}{2}}\Big|_0^1 = \frac{128}{105}.
\end{aligned}
$$

(4)

$$
\begin{aligned}
\iint_D xy^2\mathrm{d}x\mathrm{d}y &= \int_0^{\frac{p}{2}} \mathrm{d}x \int_{-\sqrt{2px}}^{\sqrt{2px}} xy^2\mathrm{d}y = \int_0^{\frac{p}{2}} \frac{xy^3}{3}\Big|_{y=-\sqrt{2px}}^{y=\sqrt{2px}} \mathrm{d}x \\
&= \int_0^{\frac{p}{2}} \frac{4p}{3}\sqrt{2p}x^{\frac{5}{2}}\mathrm{d}x = \frac{8p}{21}\sqrt{2p}x^{\frac{7}{2}}\Big|_0^{\frac{p}{2}} = \frac{p^5}{21}.
\end{aligned}
$$

(5)

$$\iint_D \sqrt{x}\mathrm{d}x\mathrm{d}y = \int_0^2 \mathrm{d}x \int_{-\sqrt{2x-x^2}}^{\sqrt{2x-x^2}} \sqrt{x}\mathrm{d}y = \int_0^2 \sqrt{x}y|_{y=-\sqrt{2x-x^2}}^{y=\sqrt{2x-x^2}}\mathrm{d}x$$
$$=2\int_0^2 x\sqrt{2-x}\mathrm{d}x \stackrel{t=\sqrt{2-x}}{=\!=\!=} 2\int_{\sqrt{2}}^0 (2-t^2)t(-2t)\mathrm{d}t = \frac{32}{15}\sqrt{2}.$$

□

4. $D = \{(x,y)|x \in [0,2], y \in [0,4], x+y \leqslant 5\}$, 因此

$$V = \iint_D z\mathrm{d}x\mathrm{d}y = \int_0^1 \mathrm{d}x \int_0^4 (5-x-y)\mathrm{d}y + \int_1^2 \mathrm{d}x \int_0^{5-x} (5-x-y)\mathrm{d}y$$
$$= \int_0^1 (12-4x)\mathrm{d}x + \int_1^2 \left(\frac{x^2}{2} - 5x + \frac{25}{2}\right)\mathrm{d}x$$
$$=10 + \frac{37}{6} = \frac{97}{6}.$$

□

5. 设 $D = \{(x,y)|1-4x^2-y^2 \geqslant 0, x \geqslant 0, y \geqslant 0\} = \{(x,y)|4x^2+y^2 \geqslant 1, x \geqslant 0, y \geqslant 0\}$, 由对称性 (见图 5),

$$V = 4\iint_D z\mathrm{d}x\mathrm{d}y = 4\int_0^{1/2} \mathrm{d}x \int_0^{\sqrt{1-4x^2}} (1-4x^2-y^2)\mathrm{d}y$$
$$= \frac{8}{3}\int_0^{1/2} (1-4x^2)\sqrt{1-4x^2}\mathrm{d}x$$
$$\stackrel{x=\frac{1}{2}\sin t}{=\!=\!=} \frac{4}{3}\int_0^{\pi/2} \cos^4 t\,\mathrm{d}t$$
$$= \frac{1}{6}\int_0^{\pi/2} (3+4\cos 2t + \cos 4t)\mathrm{d}t$$
$$= \frac{\pi}{4}.$$

□

6. 考察积分 $I = \iint_{[a,b]^2}[f(x)-f(y)]^2\mathrm{d}x\mathrm{d}y,$

$$0 \leqslant I = \iint_{[a,b]^2} [f(x)-f(y)]^2\mathrm{d}x\mathrm{d}y$$
$$= \iint_{[a,b]^2} f^2(x)\mathrm{d}x\mathrm{d}y - 2\iint_{[a,b]^2} f(x)f(y)\mathrm{d}x\mathrm{d}y + \iint_{[a,b]^2} f^2(y)\mathrm{d}x\mathrm{d}y$$
$$= \int_a^b f^2(x)\mathrm{d}x \int_a^b \mathrm{d}y - 2\int_a^b f(x)\mathrm{d}x \int_a^b f(y)\mathrm{d}y + \int_a^b f^2(y)\mathrm{d}y \int_a^b \mathrm{d}x$$
$$=2(b-a)\int_a^b f^2(x)\mathrm{d}x - 2\left[\int_a^b f(x)\mathrm{d}x\right]^2,$$
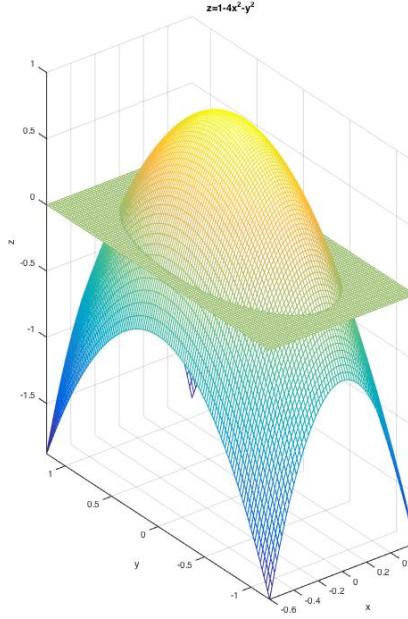
图 5: 第 5 题图

移项即得

$$\left[\int_a^b f(x)\mathrm{d}x\right]^2 \leqslant (b-a)\int_a^b f^2(x)\mathrm{d}x,$$

等号成立当且仅当在 $[a,b]$ 上 $f(x) = f(y)$, 即 $f(x)$ 是常值函数. □

7. 由 $t \in [0, \pi/2], a > 0$ 知 D 位于第一象限 (见图 6), 于是

$$I = \iint_D xy\mathrm{d}x\mathrm{d}y = \int_0^a \mathrm{d}x \int_0^{a(1-(x/a)^{2/3})^{3/2}} xy\mathrm{d}y$$
$$= \frac{a^2}{2} \int_0^a x(1 - a^{-2/3}x^{2/3})^3 \mathrm{d}x = \frac{a^4}{80}.$$

□

习题 16.3 二重积分的变量变换

1. (1) 由 $\begin{cases} u = \frac{3}{5}x + \frac{4}{5}y, \\ v = -\frac{4}{5}x + \frac{3}{5}y \end{cases}$ 得到 $\begin{cases} x = \frac{3}{5}u - \frac{4}{5}v, \\ y = \frac{4}{5}u + \frac{3}{5}v, \end{cases}$ 同时 D 变换为 $\Delta = \{(u,v)|u^2 + v^2 \leqslant 1\}$.

Jacobi 行列式

$$J(u,v) = \begin{vmatrix} \frac{3}{5} & -\frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{vmatrix} = 1 > 0.$$

$$y=\left(1-x^{\frac{2}{3}}\right)^{\frac{3}{2}}$$

图 6: 第 7 题图

因此,
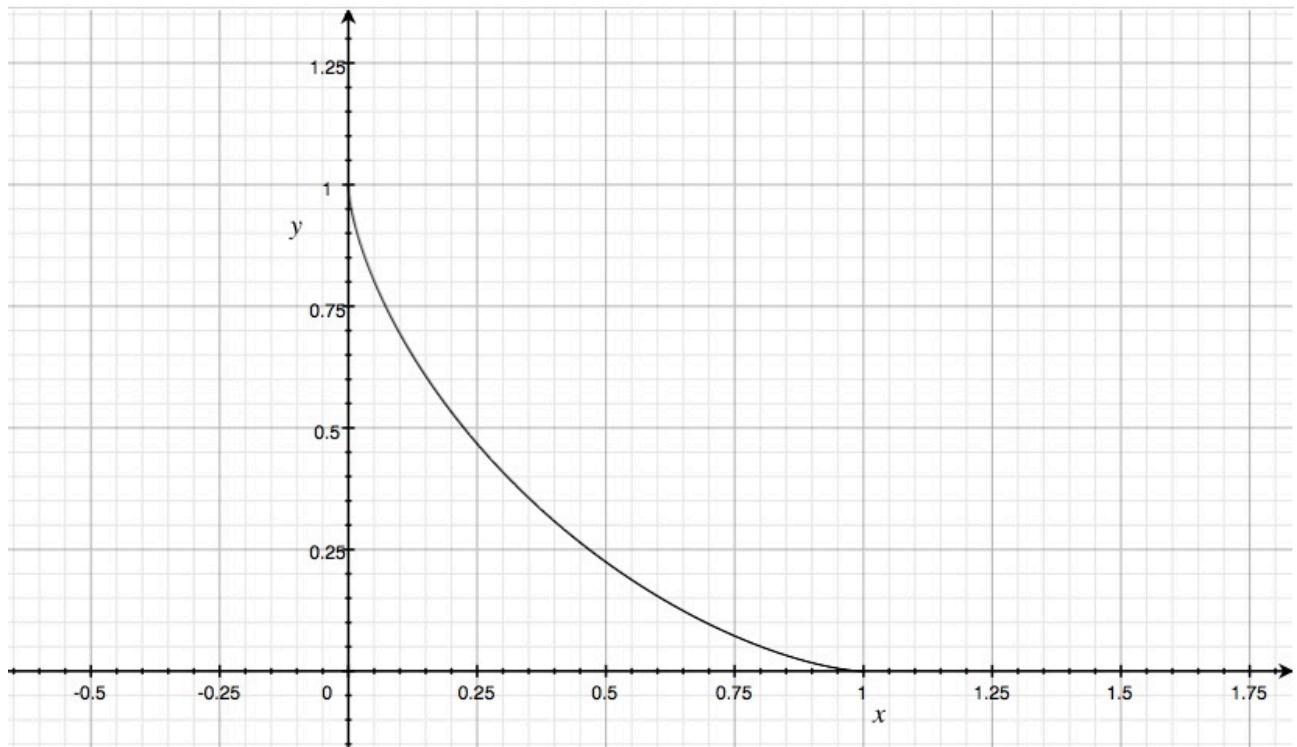
$$\iint_D f(3x + 4y)\mathrm{d}x\mathrm{d}y = \iint_\Delta f(5u)\mathrm{d}u\mathrm{d}v$$

$$= \int_{-1}^1 \mathrm{d}u \int_{-\sqrt{1-u^2}}^{\sqrt{1-u^2}} f(5u)\mathrm{d}v$$

$$= \int_{-1}^1 \mathrm{d}v \int_{-\sqrt{1-v^2}}^{\sqrt{1-v^2}} f(5u)\mathrm{d}u.$$

(2) 区域 $D = \{(x,y)|0 \leqslant x \leqslant 2, 1-2x \leqslant y \leqslant 4-2x\}$. 由 $\begin{cases} u = 2x + y, \\ v = x - y \end{cases}$ 得到 $\begin{cases} x = \frac{1}{3}(u+v), \\ y = \frac{1}{3}(u-2v), \end{cases}$
同时 D 变换为 $\Delta = \{(u,v)|1 \leqslant u \leqslant 4, -u \leqslant v \leqslant 6 - u\}$. Jacobi 行列式

$$|J(u,v)| = \pm \left| \begin{array}{cc} \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{2}{3} \end{array} \right| = \frac{1}{3}.$$

因此,

$$\int_0^2 \mathrm{d}x \int_{1-2x}^{4-2x} f(x,y)\mathrm{d}y = \frac{1}{3}\int_1^4 \mathrm{d}u \int_{-u}^{6-u} f\left(\frac{1}{3}(u+v), \frac{1}{3}(u-2v)\right)\mathrm{d}v$$

$$= \frac{1}{3}\int_{-4}^{-1} \mathrm{d}v \int_{-v}^4 f\left(\frac{1}{3}(u+v), \frac{1}{3}(u-2v)\right)\mathrm{d}u$$

$$+ \frac{1}{3}\int_{-1}^2 \mathrm{d}v \int_1^4 f\left(\frac{1}{3}(u+v), \frac{1}{3}(u-2v)\right)\mathrm{d}u$$

$$+ \frac{1}{3}\int_2^5 \mathrm{d}v \int_1^{6-v} f\left(\frac{1}{3}(u+v), \frac{1}{3}(u-2v)\right)\mathrm{d}u.$$

$\square$

2. (1) 做极坐标变换 $x = r\cos\theta, y = r\sin\theta$, 区域 D 转化为 $\Delta = \{(r,\theta)|0 \leqslant r \leqslant 2\cos\theta, -\pi/2 \leqslant \theta \leqslant \pi/2\}$. 因此,

$$\iint_D (2x - 3y)\mathrm{d}x\mathrm{d}y = \iint_\Delta r^2(2\cos\theta - 3\sin\theta)\mathrm{d}r\mathrm{d}\theta$$

$$= \int_{-\pi/2}^{\pi/2} \mathrm{d}\theta \int_0^{2\cos\theta} r^2(2\cos\theta - 3\sin\theta)\mathrm{d}r$$

$$= \frac{8}{3}\int_{-\pi/2}^{\pi/2} \cos^3\theta(2\cos\theta - 3\sin\theta)\mathrm{d}\theta$$

$$= \frac{8}{3}\int_{-\pi/2}^{\pi/2} \frac{1}{2}\left(1 + 2\cos 2\theta + \frac{1 + \cos 4\theta}{2}\right)\mathrm{d}\theta + \frac{8}{3}\int_{-\pi/2}^{\pi/2} 3\cos^3\theta \mathrm{d}\cos\theta$$

$$= 2\pi.$$

(2) 做极坐标变换 $x = r\cos\theta, y = r\sin\theta$, 区域 D 转化为 $\Delta = \{(r,\theta)|1 \leqslant r \leqslant \pi, 0 \leqslant \theta \leqslant 2\pi\}$.

因此,

$$\iint_D \cos(x^2+y^2)\mathrm{d}x\mathrm{d}y = \iint_\Delta r\cos r^2 \mathrm{d}r\mathrm{d}\theta = \int_0^{2\pi} \mathrm{d}\theta \int_1^\pi r\cos r^2 \mathrm{d}r$$
$$= \frac{1}{2}\int_0^{2\pi} \sin r^2 |_1^\pi \mathrm{d}\theta = \pi(\sin \pi^2 - \sin 1).$$

(3) 区域 $D = \{(r,\theta)|0 \leqslant r \leqslant \theta, 0 \leqslant \theta \leqslant \pi\}$. 因此,

$$\iint_D y\mathrm{d}x\mathrm{d}y = \iint_D r^2 \sin\theta \mathrm{d}r\mathrm{d}\theta = \int_0^\pi \mathrm{d}\theta \int_0^\theta r^2 \sin\theta \mathrm{d}r = \frac{1}{3}\int_0^\pi \theta^3 \sin\theta \mathrm{d}\theta,$$

连续使用分部积分可得 $\iint_D y\mathrm{d}x\mathrm{d}y = \frac{1}{3}\pi^3 - 2\pi$. $\qquad\qquad\square$

    3. (1) 做变换 $\begin{cases} u = x+y, \\ v = y-x, \end{cases}$ 从而 $\begin{cases} x = \frac{1}{2}(u-v), \\ y = \frac{1}{2}(u+v), \end{cases}$ 同时 D 变换为 $\Delta = \{(u,v)| -2\pi \leqslant u \leqslant 2\pi, -2\pi \leqslant v \leqslant 2\pi\}$. Jacobi 行列式

$$J(u,v) = \begin{vmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{vmatrix} = \frac{1}{2} > 0.$$

因此,

$$\iint_D |\cos(x+y)|\mathrm{d}x\mathrm{d}y = \frac{1}{2}\iint_\Delta |\cos u|\mathrm{d}u\mathrm{d}v = \frac{1}{2}\int_{-2\pi}^{2\pi} \mathrm{d}u \int_{-2\pi}^{2\pi} |\cos u|\mathrm{d}v$$
$$= 2\pi \int_{-2\pi}^{2\pi} |\cos u|\mathrm{d}u = 16\pi \int_0^{\pi/2} \cos u\mathrm{d}u = 16\pi.$$

(2) 做变换 $\begin{cases} u = x+y, \\ v = x-y, \end{cases}$ 从而 $\begin{cases} x = \frac{1}{2}(u+v), \\ y = \frac{1}{2}(u-v), \end{cases}$ 同时 D 变换为 $\Delta = \{(u,v)|0 \leqslant u \leqslant \pi, 0 \leqslant v \leqslant \pi\}$. Jacobi 行列式

$$|J(u,v)| = \pm \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = \frac{1}{2}.$$

因此,

$$\iint_D (x-y)\cos(x+y)\mathrm{d}x\mathrm{d}y = \frac{1}{2}\iint_\Delta v\cos u\mathrm{d}u\mathrm{d}v = \frac{1}{2}\int_0^\pi \mathrm{d}u \int_0^\pi v\cos u\mathrm{d}v$$
$$= \frac{\pi^2}{4}\int_0^\pi \cos u\mathrm{d}u = 0.$$

(3) 做变换 $\begin{cases} u = x+y, \\ v = x-y, \end{cases}$ 从而 $\begin{cases} x = \frac{1}{2}(u+v), \\ y = \frac{1}{2}(u-v), \end{cases}$ 同时 $D = \{(x,y)|x \geqslant 0, y \geqslant 0, x+y \leqslant 1\}$ 变
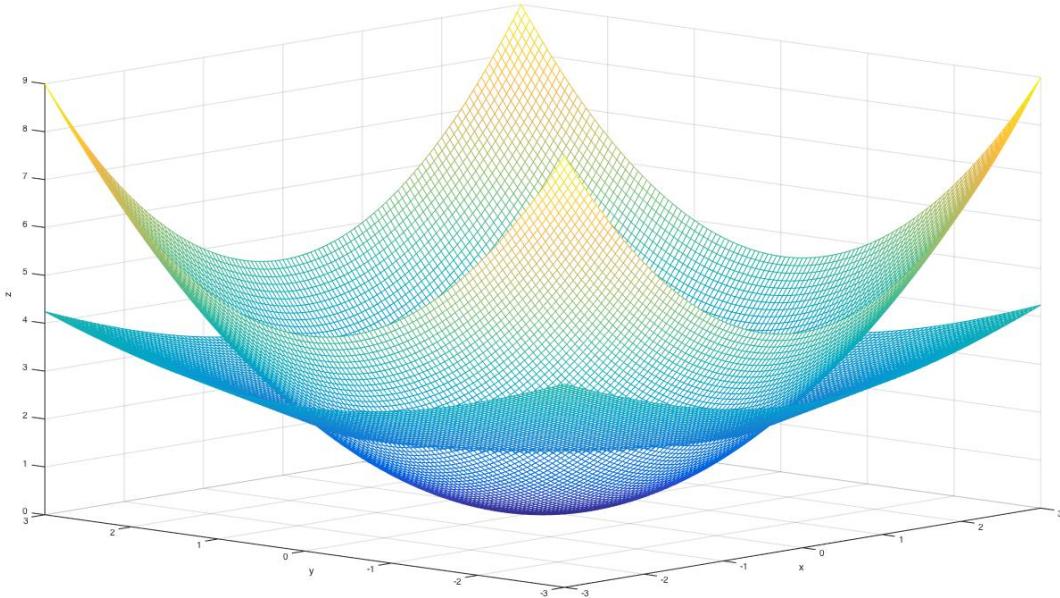
图 7: 第 4 题 (1)

换为 $\Delta = \{(u,v)|0 \leqslant u \leqslant 1, -u \leqslant v \leqslant u\}$. Jacobi 行列式

$$|J(u,v)| = \pm \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = \frac{1}{2}.$$

因此,

$$\iint_D e^{\frac{x-y}{x+y}} \mathrm{d}x\mathrm{d}y = \frac{1}{2} \iint_\Delta e^{\frac{v}{u}} \mathrm{d}u\mathrm{d}v = \frac{1}{2} \int_0^1 \mathrm{d}u \int_{-u}^u e^{\frac{v}{u}} \mathrm{d}v$$
$$= \frac{1}{2} \int_0^1 u(e - e^{-1})\mathrm{d}u = \frac{e - e^{-1}}{4}.$$

$\square$

4. (1) 两曲面的交线是 $\begin{cases} z = 2, \\ x^2 + y^2 = 4, \end{cases}$ (见图 7). 因此 V 在 xy 平面上投影为 $D = \{(x,y)|x^2 + y^2 \leqslant 4\}$. 做极坐标变换 $x = r\cos\theta, y = r\sin\theta$, 区域 D 转化为 $\Delta = \{(r,\theta)|0 \leqslant r \leqslant 2, 0 \leqslant \theta \leqslant 2\pi\}$, Jacobi 行列式 $J(r,\theta) = r$. 因此,

$$V = \iint_D \sqrt{x^2 + y^2}\mathrm{d}x\mathrm{d}y - \iint_D \frac{x^2 + y^2}{2}\mathrm{d}x\mathrm{d}y$$
$$= \iint_\Delta \sqrt{x^2 + y^2} - \frac{x^2 + y^2}{2}\mathrm{d}x\mathrm{d}y$$
$$= \int_0^{2\pi} \mathrm{d}\theta \int_0^2 (r - r^2/2)r\mathrm{d}r = \frac{4}{3}\pi.$$
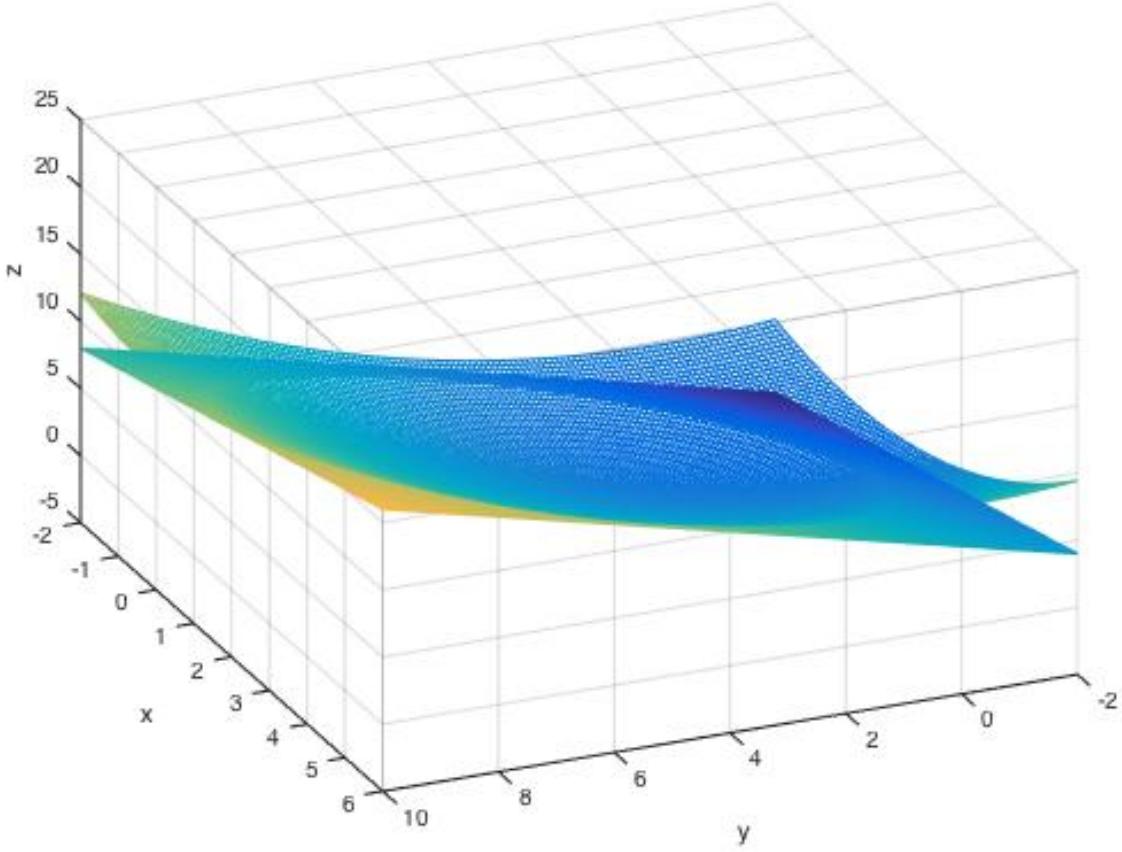
图 8: 第 4 题 (2)

(2) 两曲面的交线是 $\begin{cases} z = x + y, \\ x^2/4 + y^2/9 = x + y, \end{cases}$ (见图 8). 因此 V 在 xy 平面上投影为 $D = \{(x,y)|\frac{(x-2)^2}{13} + \frac{(y-9/2)^2}{117/4} \leqslant 1\}$. 做变换 $x = 2 + \sqrt{13}r\cos\theta, y = \frac{9}{2} + \frac{3}{2}\sqrt{13}r\sin\theta$, 区域 D 转化为 $\Delta = \{(r,\theta)|0 \leqslant r \leqslant 1, 0 \leqslant \theta \leqslant 2\pi\}$, Jacobi 行列式 $J(r,\theta) = \frac{39}{2}r$. 因此,

$$\begin{aligned}
V &= \iint_D (x+y)\mathrm{d}x\mathrm{d}y - \iint_D (x^2/4 + y^2/9)\mathrm{d}x\mathrm{d}y \\
&= \iint_D \left[\frac{13}{4} - \frac{1}{4}(x-2)^2 - \frac{1}{9}(y-9/2)^2 \right] \mathrm{d}x\mathrm{d}y \\
&= \iint_\Delta \left(\frac{13}{4} - \frac{13r^2\cos^2\theta}{4} - \frac{13r^2\sin^2\theta}{4} \right) \frac{39}{2}r\mathrm{d}r\mathrm{d}\theta \\
&= \frac{507}{8} \int_0^{2\pi} \mathrm{d}\theta \int_0^1 (r - r^3)\mathrm{d}r = \frac{507}{16}\pi.
\end{aligned}$$

$\square$

5. (1) 做极坐标变换 $x = r\cos\theta, y = r\sin\theta$, 区域 D 转化为 $\Delta = \{(r,\theta)|1 \leqslant r \leqslant 2, 0 \leqslant \theta \leqslant$

$2\pi\}$, Jacobi 行列式 $J(r,\theta)=r$. 因此,

$$\iint_D f(x^2+y^2)\mathrm{d}x\mathrm{d}y = \iint_\Delta rf(r^2)\mathrm{d}r\mathrm{d}\theta = \int_0^{2\pi}\mathrm{d}\theta\int_1^2 rf(r^2)\mathrm{d}r = 2\pi\int_1^2 rf(r^2)\mathrm{d}r.$$

(2) 做极坐标变换 $x=r\cos\theta, y=r\sin\theta$, 区域 D 转化为 $\Delta=\{(r,\theta)|0\leqslant r\leqslant\cos\theta, -\frac{\pi}{2}\leqslant\theta\leqslant\frac{\pi}{2}\}$, Jacobi 行列式 $J(r,\theta)=r$. 因此,

$$\iint_D f(y/x)\mathrm{d}x\mathrm{d}y = \iint_\Delta rf(\tan\theta)\mathrm{d}r\mathrm{d}\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(\tan\theta)\mathrm{d}\theta\int_0^{\cos\theta} r\mathrm{d}r = \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^2\theta f(\tan\theta)\mathrm{d}\theta.$$

(3) 作变换 $u=ax+by, v=-bx+ay$, 则 $x=\frac{au-bv}{a^2+b^2}, y=\frac{bu+av}{a^2+b^2}$, 区域 D 转化为 $\Delta=\{(u,v)|u^2+v^2\leqslant a^2+b^2\}$, Jacobi 行列式 $J(u,v)=1$. 因此,

$$\iint_D f(ax+by)\mathrm{d}x\mathrm{d}y = \iint_\Delta f(u)\mathrm{d}u\mathrm{d}v = \int_{-\sqrt{a^2+b^2}}^{\sqrt{a^2+b^2}}\mathrm{d}u\int_{-\sqrt{a^2+b^2-u^2}}^{\sqrt{a^2+b^2-u^2}} f(u)\mathrm{d}u$$
$$=2\int_{-\sqrt{a^2+b^2}}^{\sqrt{a^2+b^2}}\sqrt{a^2+b^2-u^2}f(u)\mathrm{d}u.$$

(4) 作变换 $u=xy, v=y/x$, 则 $x=\sqrt{u/v}, y=\sqrt{uv}$, 区域 D 转化为 $\Delta=\{(u,v)|1\leqslant u\leqslant 3, 1/e\leqslant v\leqslant e\}$, Jacobi 行列式 $J(u,v)=1/(2v)$. 因此,

$$\iint_D f(xy)\mathrm{d}x\mathrm{d}y = \iint_\Delta f(u)/(2v)\mathrm{d}u\mathrm{d}v = \int_1^3 f(u)\mathrm{d}u\int_{1/e}^e 1/(2v)\mathrm{d}v = \int_1^3 f(u)\mathrm{d}u.$$

$\square$

6. 作变换 $u=a_1x+b_1y+c_1, v=a_2x+b_2y+c_2$, 则

$$x=\frac{b_2(u-c_1)-b_1(v-c_2)}{a_1b_2-a_2b_1},\ y=\frac{-a_2(u-c_1)+a_1(v-c_2)}{a_1b_2-a_2b_1},$$

区域 $D=\{(x,y)|(a_1x+b_1y+c_1)^2+(a_2x+b_2y+c_2)^2\leqslant 1\}$ 转化为 $\Delta=\{(u,v)|u^2+v^2\leqslant 1\}$, Jacobi 行列式 $J(u,v)=1/(a_1b_2-a_2b_1)$. 因此,

$$S_D=\iint_D\mathrm{d}x\mathrm{d}y = \iint_\Delta\frac{1}{|a_1b_2-a_2b_1|}\mathrm{d}u\mathrm{d}v$$
$$=\frac{1}{|a_1b_2-a_2b_1|}\iint_\Delta\mathrm{d}u\mathrm{d}v = \frac{1}{|a_1b_2-a_2b_1|}S_\Delta$$
$$=\frac{\pi}{|a_1b_2-a_2b_1|}.$$

$\square$

习题 16.4 三重积分

1. (1)

$$\iiint_V \sin x \cos^2 y \tan z \mathrm{d}x\mathrm{d}y\mathrm{d}z = \int_0^3 \sin x \mathrm{d}x \int_0^{\frac{\pi}{2}} \cos^2 y \mathrm{d}y \int_0^{\frac{\pi}{4}} \tan z \mathrm{d}z$$

$$= -\cos x|_0^3 + \left(\frac{y}{2} + \frac{\sin 2y}{4}\right)\bigg|_0^{\frac{\pi}{2}} - \ln|\cos z||_0^{\frac{\pi}{4}}$$

$$= \frac{\pi}{8} \ln 2(1 - \cos 3).$$

(2)

$$\iiint_V (x^2 - y + 2z)\mathrm{d}x\mathrm{d}y\mathrm{d}z = \int_{-3}^3 \mathrm{d}x \int_0^2 \mathrm{d}y \int_{-4}^1 (x^2 - y + 2z)\mathrm{d}z$$

$$= \int_{-3}^3 \mathrm{d}x \int_0^2 (x^2 z - yz + z^2)|_{z=-4}^{z=1}\mathrm{d}y$$

$$= 5\int_{-3}^3 \mathrm{d}x \int_0^2 (x^2 - y - 3)\mathrm{d}y$$

$$= 5\int_{-3}^3 (x^2 y - y^2/2 - 3y)|_{y=0}^{y=2}\mathrm{d}x$$

$$= 10\int_{-3}^3 (x^2 - 4)\mathrm{d}x = -60.$$

$\square$

复习题

1. (1) 记 $D_1 = \{(x,y)||x| + |y| \leqslant 1, x + y \geqslant 0\}, D_2 = \{(x,y)||x| + |y| \leqslant 1, x + y \leqslant 0\}$, 则 $\iint_D |x + y|\mathrm{d}x\mathrm{d}y = \iint_{D_1}(x + y)\mathrm{d}x\mathrm{d}y - \iint_{D_2}(x + y)\mathrm{d}x\mathrm{d}y$, 其中

$$\iint_{D_1}(x + y)\mathrm{d}x\mathrm{d}y = \int_0^1 \mathrm{d}x \int_0^{1-x}(x + y)\mathrm{d}y + \int_{-\frac{1}{2}}^0 \mathrm{d}x \int_{-x}^{x+1}(x + y)\mathrm{d}y + \int_{-\frac{1}{2}}^0 \mathrm{d}y \int_{-y}^{y+1}(x + y)\mathrm{d}x$$

$$= \frac{1}{3} + \frac{1}{12} + \frac{1}{12} = \frac{1}{2},$$

$$\iint_{D_2}(x + y)\mathrm{d}x\mathrm{d}y = \int_{-1}^0 \mathrm{d}x \int_0^{-1-x}(x + y)\mathrm{d}y + \int_0^{\frac{1}{2}} \mathrm{d}x \int_{x-1}^{-x}(x + y)\mathrm{d}y + \int_0^{\frac{1}{2}} \mathrm{d}y \int_{y-1}^{-y}(x + y)\mathrm{d}x$$

$$= -\frac{1}{3} - \frac{1}{12} - \frac{1}{12} = -\frac{1}{2},$$

所以 $\iint_D |x + y|\mathrm{d}x\mathrm{d}y = 1$.

(2)

$$\iint_D \mathrm{sgn}(3 - x^2)\mathrm{d}x\mathrm{d}y = -\int_{-2}^{-\sqrt{3}} \mathrm{d}y \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} \mathrm{d}x + \int_{-\sqrt{3}}^{\sqrt{3}} \mathrm{d}y \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} \mathrm{d}x - \int_{\sqrt{3}}^2 \mathrm{d}y \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} \mathrm{d}x$$

$$= (\sqrt{3} - 2\pi/3) + (4\sqrt{3} + 4\pi/3) + (-\sqrt{3} + 2\pi/3) = 4\sqrt{3} + 4\pi/3.$$

(3) 作变换 $\begin{cases} u = \frac{y}{x^3}, & a \leqslant u \leqslant b, \\ v = \frac{y^2}{x}, & p \leqslant v \leqslant q, \end{cases}$ 即 $\begin{cases} x = \sqrt[5]{\frac{v}{u^2}}, \\ y = \sqrt[5]{\frac{v^3}{u}}, \end{cases}$ 则 xy 平面上区域 D 与 uv 平面上矩形区域 $\Delta = [a,b] \times [p,q]$ ——对应, 并且 $J(u,v) = -\frac{1}{5u\sqrt[5]{u^3 v}}$, 所以

$$\iint_D xy \mathrm{d}x\mathrm{d}y = \frac{1}{5} \iint_\Delta u^{-\frac{11}{5}} v^{\frac{3}{5}} \mathrm{d}u\mathrm{d}v = \frac{1}{5} \int_a^b u^{-\frac{11}{5}} \mathrm{d}u \int_p^q v^{\frac{3}{5}} \mathrm{d}v = \frac{5}{48}(a^{-\frac{6}{5}} - b^{-\frac{6}{5}})(q^{\frac{8}{5}} - p^{\frac{8}{5}}).$$

$\square$

2. 作变换 $\begin{cases} u = a_1 x + b_1 y + c_1 z, \\ v = a_2 x + b_2 y + c_2 z, \\ w = a_3 x + b_3 y + c_3 z, \end{cases}$ 即 $(u,v,w)^\top = A(x,y,z)^\top$, 其中 $A = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$,

而 $\Delta = \det A \neq 0$. 则 $(x,y,z)^\top = A^{-1}(u,v,w)^\top$, 因此 Jacobi 行列式 $J(u,v,w) = \frac{\partial(x,y,z)}{\partial(u,v,w)} = \det A^{-1}$. 区域 $V = \{(x,y,z)|(a_1 x + b_1 y + c_1 z)^2 + (a_2 x + b_2 y + c_2 z)^2 + (a_3 x + b_3 y + c_3 z)^2 \leqslant h^2\}$ 转化为 $V' = \{(u,v,w)|u^2 + v^2 + w^2 \leqslant h^2\}$, 因此,

$$V_{椭球} = \iiint_V \mathrm{d}V = \iiint_{V'} |\det A^{-1}|\mathrm{d}V' = \frac{1}{|\Delta|} \iiint_{V'} \mathrm{d}V' = \frac{4\pi h^3}{3|\Delta|}.$$

$\square$

3. 用反证法. 假设 $f(x,y) \not\equiv 0$, 则 $\exists P_0(x_0, y_0) \in D^\circ$, 使得 $f(x_0, y_0) \neq 0$. 取 $g(x,y) = f(x,y)$, 则 $f(x_0, y_0)g(x_0, y_0) = f^2(x_0, y_0) > 0$. 由连续函数的保号性, $\exists \delta_0 > 0$, 使得 $B_{\delta_0}(P_0) \subset D^\circ$ 且

$$f(x,y)g(x,y) = f^2(x,y) \geqslant \frac{f^2(x_0, y_0)}{2}, \ (x,y) \in B_{\delta_0}(P_0).$$

再由 $f(x,y)g(x,y) = f^2(x,y)$ 非负有

$$\iint_D f(x,y)g(x,y)\mathrm{d}x\mathrm{d}y \geqslant \iint_{B_{\delta_0}(P_0)} f(x,y)g(x,y)\mathrm{d}x\mathrm{d}y \geqslant \frac{f^2(x_0, y_0)}{2} \cdot \pi \delta_0^2 > 0,$$

与 $\iint_D f(x,y)g(x,y)\mathrm{d}x\mathrm{d}y = 0$ 矛盾. $\square$

4. 令 $\begin{cases} u = 1 - x, \\ v = 1 - y, \end{cases}$ 则 $\begin{cases} x = 1 - u, \\ y = 1 - v, \end{cases}$ Jacobi 行列式 $J(u,v) = 1 > 0$, 区域 $D = \{(x,y)|0 \leqslant x \leqslant 1, 0 \leqslant y \leqslant x\}$ 变换为 $\Delta = \{(u,v)|0 \leqslant v \leqslant 1, 0 \leqslant u \leqslant v\}$, 再由 $f(x,y) = f(y,x)$ 可得,

$$\int_0^1 \mathrm{d}x \int_0^x f(x,y)\mathrm{d}y = \int_0^1 \mathrm{d}v \int_0^v f(1-u, 1-v)\mathrm{d}u = \int_0^1 \mathrm{d}v \int_0^v f(1-v, 1-u)\mathrm{d}u$$
$$= \int_0^1 \mathrm{d}x \int_0^x f(1-x, 1-y)\mathrm{d}y.$$

$\square$

5. (1) 令 $x = ut, y = vt$, 则 Jacobi 行列式 $J(u,v) = t^2 > 0$, 区域 $D = [0,t]^2$ 变换为 $\Delta = [0,1]^2$, 于是

$$F(t) = t^2 \iint_{[0,1]^2} e^{-\frac{u}{v^2}} \mathrm{d}u\mathrm{d}v,$$

因此,

$$F'(t) = 2t \iint_{[0,1]^2} e^{-\frac{u}{v^2}} \mathrm{d}u\mathrm{d}v = \frac{2}{t} F(t).$$

(2) 令 $x = ut, y = vt, z = wt$, 则 Jacobi 行列式 $J(u,v,w) = t^3$, 区域 $D = [0,t]^3$ 变换为 $\Delta = [0,1]^3$, 于是

$$F(t) = t^3 \iiint_{[0,1]^3} f(uvwt^3)\mathrm{d}u\mathrm{d}v\mathrm{d}w,$$

因此,

$$\begin{aligned}
F'(t) &= 3t^2 \iiint_{[0,1]^3} f(uvwt^3)\mathrm{d}u\mathrm{d}v\mathrm{d}w + 3t^5 \iiint_{[0,1]^3} f'(uvwt^3)\mathrm{d}u\mathrm{d}v\mathrm{d}w \\
&= \frac{3}{t} F(t) + \frac{3}{t} \iiint_{[0,t]^3} xyzf(xyz)\mathrm{d}x\mathrm{d}y\mathrm{d}z.
\end{aligned}$$

$\square$

6. 仿定理 16.4.1 的证明. $\square$

7. $f(x,y)$ 在 $[0,\pi]^2$ 上连续且恒为正, 由连续函数的最值定理, 存在 m, M, 使得 $0 \leqslant m \leqslant f(x,y) \leqslant M, \forall (x,y) \in [0,\pi]^2$, 因此

$$\iint_{[0,\pi]^2} m^{\frac{1}{n}} \sin x\mathrm{d}x\mathrm{d}y \leqslant \iint_{[0,\pi]^2} (f(x,y))^{\frac{1}{n}} \sin x\mathrm{d}x\mathrm{d}y \leqslant \iint_{[0,\pi]^2} M^{\frac{1}{n}} \sin x\mathrm{d}x\mathrm{d}y,$$

而

$$\lim_{n\to\infty} \iint_{[0,\pi]^2} m^{\frac{1}{n}} \sin x\mathrm{d}x\mathrm{d}y = \lim_{n\to\infty} m^{\frac{1}{n}} \int_0^\pi \mathrm{d}y \int_0^\pi \sin x\mathrm{d}x = \lim_{n\to\infty} 2\pi m^{\frac{1}{n}} = 2\pi,$$

$$\lim_{n\to\infty} \iint_{[0,\pi]^2} M^{\frac{1}{n}} \sin x\mathrm{d}x\mathrm{d}y = \lim_{n\to\infty} M^{\frac{1}{n}} \int_0^\pi \mathrm{d}y \int_0^\pi \sin x\mathrm{d}x = \lim_{n\to\infty} 2\pi M^{\frac{1}{n}} = 2\pi,$$

由迫敛性定理, $\lim_{n\to\infty} \iint_{[0,\pi]^2} (f(x,y))^{\frac{1}{n}} \sin x\mathrm{d}x\mathrm{d}y = 2\pi$. $\square$

第 17 章 曲线积分和曲面积分

习题 17.1 第一型曲线积分

1. (1)

$$
\begin{aligned}
\int_{\Gamma}(x+2y+3z)\mathrm{d}s &= \int_0^{2\pi}(a\cos t + 2a\sin t + 3bt)\sqrt{a^2\sin^2 t + a^2\cos^2 t + b^2}\mathrm{d}t \\
&= \sqrt{a^2+b^2}\left.\left(a\sin t - 2a\cos t + \frac{3}{2}bt^2\right)\right|_0^{2\pi} \\
&= 6\pi^2 b\sqrt{a^2+b^2}.
\end{aligned}
$$

(2)

$$
\begin{aligned}
\int_{\Gamma}xy\mathrm{d}s &= \frac{2}{3}\int_0^1 t^{\frac{5}{2}}\sqrt{1+t+t^2/4}\,\mathrm{d}t \\
&= \frac{1}{3}\int_0^1 (t+2)t^{\frac{5}{2}}\mathrm{d}t \\
&= \frac{50}{189}.
\end{aligned}
$$

(3) Γ 各段的参数方程: $OA:\begin{cases} x=x, \\ y=0, \end{cases}$ $AB:\begin{cases} x=3-3y, \\ y=y, \end{cases}$ $BO:\begin{cases} x=0, \\ y=y. \end{cases}$ 因此,

$$
\begin{aligned}
\int_{\Gamma}(x+3y)\mathrm{d}s &= \int_{OA}(x+3y)\mathrm{d}s + \int_{AB}(x+3y)\mathrm{d}s + \int_{BO}(x+3y)\mathrm{d}s \\
&= \int_0^3 x\mathrm{d}x + \int_0^1 3\sqrt{1+9}\,\mathrm{d}y + \int_0^1 3y\mathrm{d}y \\
&= 6 + 3\sqrt{10}.
\end{aligned}
$$

(4) Γ 的参数方程: $\begin{cases} x=R\cos t, \\ y=R\sin t, \end{cases}$ $t \in [-\pi/2, \pi/2]$, 因此,

$$
\begin{aligned}
\int_{\Gamma}(x^2+y^2)^{\frac{1}{2}}\mathrm{d}s &= \int_{-\pi/2}^{\pi/2} R^2\mathrm{d}t \\
&= \pi R^2.
\end{aligned}
$$

(5) Γ 的参数方程: $\begin{cases} x=a\cos t, \\ y=b\sin t, \end{cases}$ $t \in [0, \pi/2]$, 因此,

$$\int_\Gamma xy\mathrm{d}s = \int_0^{\pi/2} ab\sin t\cos t\sqrt{a^2\sin^2 t + b^2\cos^2 t}\,\mathrm{d}t$$

$$=\frac{ab}{2}\int_0^{\pi/2}\sin 2t\sqrt{\frac{a^2+b^2}{2}-\frac{a^2-b^2}{2}\cos 2t}\,\mathrm{d}t$$

$$=-\frac{ab}{4}\int_0^{\pi/2}\sqrt{\frac{a^2+b^2}{2}-\frac{a^2-b^2}{2}\cos 2t}\,\mathrm{d}\cos 2t$$

$$=\frac{ab}{3(a^2-b^2)}\left(\frac{a^2+b^2}{2}-\frac{a^2-b^2}{2}\cos 2t\right)^{3/2}\Bigg|_0^{\pi/2}$$

$$=\frac{ab(a^3-b^3)}{3(a^2-b^2)}.$$

(6) 由 $\begin{cases} x^2+y^2+z^2=a^2, \\ x=y, \end{cases}$ 得 $2x^2+z^2=a^2$. 令 $\begin{cases} x=\frac{a}{\sqrt{2}}\cos t, \\ z=a\sin t, \end{cases}$ $t\in[0,2\pi]$, 则 $y=x=$

$\frac{a}{\sqrt{2}}\cos t$, 从而得到 Γ 的参数方程: $\begin{cases} x=\frac{a}{\sqrt{2}}\cos t, \\ y=\frac{a}{\sqrt{2}}\cos t, \quad t\in[0,2\pi], \text{ 因此,} \\ z=a\sin t, \end{cases}$

$$\int_\Gamma \sqrt{x^2+2y^2}\,\mathrm{d}s = \int_0^{2\pi}\sqrt{\frac{3}{2}}a|\cos t|\sqrt{\frac{a^2}{2}\sin^2 t+\frac{a^2}{2}\sin^2 t+a^2\cos^2 t}\,\mathrm{d}t$$

$$=\sqrt{\frac{3}{2}}a^2\int_0^{2\pi}|\cos t|\,\mathrm{d}t$$

$$=2\sqrt{6}a^2.$$

$\square$

2. 该金属线的线密度 $\rho(x,y,z)=\frac{k}{x^2+y^2+z^2}$, 其中 k 是常系数. 由其参数方程可得 $\rho(x,y,z)=\frac{k}{2z^2}$, 又因为在 $(1,0,1)$ 处线密度为 1, 可得 $k=2$, 因此, $\rho(x,y,z)=\frac{1}{z^2}$, 所以

$$M=\int_\Gamma \rho(x,y,z)\mathrm{d}s = \int_0^1 e^{-2t}\sqrt{2e^{2t}}\,\mathrm{d}t = \sqrt{2}(1-e^{-1}).$$

$\square$

3. 曲线 Γ 由三部分组成: $\Gamma=\Gamma_{xy}\cup\Gamma_{yz}\cup\Gamma_{xz}$, 其中

$$\Gamma_{xy}:\begin{cases} x=a\cos t, \\ y=a\sin t, \\ z=0, \end{cases} \quad \Gamma_{yz}:\begin{cases} x=0, \\ y=a\cos t, \\ z=a\sin t, \end{cases} \quad \Gamma_{xz}:\begin{cases} x=a\cos t, \\ y=0, \\ z=a\sin t, \end{cases} \quad t\in[0,\pi/2].$$

设曲线 Γ 的线密度为 ρ, 则其质量

$$m = \int_\Gamma \rho \mathrm{d}s = \rho \int_\Gamma \mathrm{d}s = \rho \cdot \Gamma\text{的周长} = \frac{3}{2}\pi a \rho.$$

Γ 的重心坐标

$$\begin{aligned}
x_0 =& \frac{1}{m}\int_\Gamma x\rho \mathrm{d}s = \frac{2}{3\pi a}\int_\Gamma x\mathrm{d}s = \frac{2}{3\pi a}\left(\int_{\Gamma_{xy}} x\mathrm{d}s + \int_{\Gamma_{yz}} x\mathrm{d}s + \int_{\Gamma_{xz}} x\mathrm{d}s\right)\\
=& \frac{2}{3\pi a}\left(\int_0^{\pi/2} a^2\cos t\mathrm{d}t + 0 + \int_0^{\pi/2} a^2\cos t\mathrm{d}t\right)\\
=& \frac{4a}{3\pi},\\
y_0 =& \frac{1}{m}\int_\Gamma y\rho \mathrm{d}s = \frac{2}{3\pi a}\int_\Gamma y\mathrm{d}s = \frac{2}{3\pi a}\left(\int_{\Gamma_{xy}} y\mathrm{d}s + \int_{\Gamma_{yz}} y\mathrm{d}s + \int_{\Gamma_{xz}} y\mathrm{d}s\right)\\
=& \frac{2}{3\pi a}\left(\int_0^{\pi/2} a^2\sin t\mathrm{d}t + \int_0^{\pi/2} a^2\cos t\mathrm{d}t + 0\right)\\
=& \frac{4a}{3\pi},\\
z_0 =& \frac{1}{m}\int_\Gamma z\rho \mathrm{d}s = \frac{2}{3\pi a}\int_\Gamma z\mathrm{d}s = \frac{2}{3\pi a}\left(\int_{\Gamma_{xy}} z\mathrm{d}s + \int_{\Gamma_{yz}} z\mathrm{d}s + \int_{\Gamma_{xz}} z\mathrm{d}s\right)\\
=& \frac{2}{3\pi a}\left(0 + \int_0^{\pi/2} a^2\sin t\mathrm{d}t + \int_0^{\pi/2} a^2\sin t\mathrm{d}t\right)\\
=& \frac{4a}{3\pi}.
\end{aligned}$$

$\square$

4. 曲线 Γ 化为 $\begin{cases} \frac{(x-y)^2}{x+y} = 3, \\ (x-y)(x+y) = \frac{9}{8}z^2, \end{cases}$ 因而 $\begin{cases} x - y = \frac{3}{2}z^{\frac{2}{3}}, \\ x + y = \frac{3}{4}z^{\frac{4}{3}}, \end{cases}$ 从而得 Γ 的参数方程

$\begin{cases} x = \frac{3}{8}z^{\frac{4}{3}} + \frac{3}{4}z^{\frac{2}{3}}, \\ y = \frac{3}{8}z^{\frac{4}{3}} - \frac{3}{4}z^{\frac{2}{3}}, \\ z = z, \end{cases}$ 所求弧长

$$\begin{aligned}
\int_{\widehat{OA}} \mathrm{d}s =& \int_0^{z_0} \sqrt{\left(\frac{1}{2}z^{-\frac{1}{3}} + \frac{1}{2}z^{\frac{1}{3}}\right)^2 + \left(\frac{1}{2}z^{\frac{1}{3}} - \frac{1}{2}z^{-\frac{1}{3}}\right)^2 + 1}\mathrm{d}z\\
=& \frac{\sqrt{2}}{2}\int_0^{z_0}(z^{\frac{1}{3}} + z^{-\frac{1}{3}})\mathrm{d}z\\
=& \sqrt{2}\left(\frac{3}{8}z_0^{\frac{4}{3}} + \frac{3}{4}z_0^{\frac{2}{3}}\right)\\
=& \sqrt{2}x_0.
\end{aligned}$$

$\square$

5. 设摆线 Γ 的线密度为 ρ, 则其质量

$$m = \int_\Gamma \rho \mathrm{d}s = \rho \int_\Gamma \mathrm{d}s = \rho \int_0^\pi \sqrt{a^2(1-\cos t)^2 + a^2\sin^2 t}\,\mathrm{d}t = 2\rho a\int_0^\pi \sin\frac{t}{2}\mathrm{d}t = 4\rho a.$$

摆线的重心坐标

$$
\begin{aligned}
x_0 =& \frac{1}{m} \int_\Gamma x\rho \mathrm{d}s = \frac{1}{4a} \int_\Gamma x \mathrm{d}s = \frac{a}{2} \int_0^\pi (t - \sin t) \sin \frac{t}{2} \mathrm{d}t \\
=& \frac{a}{2} \left(\int_0^\pi t \sin \frac{t}{2} \mathrm{d}t - \int_0^\pi \sin t \sin \frac{t}{2} \mathrm{d}t \right) \\
=& \frac{a}{2} \left[-2 \left(t \cos \frac{t}{2} \Big|_0^\pi - 2 \sin \frac{t}{2} \Big|_0^\pi \right) - 2 \int_0^\pi \cos \frac{t}{2} \sin^2 \frac{t}{2} \mathrm{d}t \right] \\
=& \frac{4a}{3}, \\
y_0 =& \frac{1}{m} \int_\Gamma y\rho \mathrm{d}s = \frac{1}{4a} \int_\Gamma y \mathrm{d}s = \frac{a}{2} \int_0^\pi (1 - \cos t) \sin \frac{t}{2} \mathrm{d}t \\
=& \frac{a}{2} \left(\int_0^\pi \sin \frac{t}{2} \mathrm{d}t - \int_0^\pi \cos t \sin \frac{t}{2} \mathrm{d}t \right) \\
=& \frac{4a}{3}.
\end{aligned}
$$

$\Box$

习题 17.2 第一型曲面积分

1. (1) 单位球面 $S = S_1 \cup S_2$, 其中 S_1 是上半球面 $z = \sqrt{1-x^2-y^2}$, S_2 是下半球面 $z = -\sqrt{1-x^2-y^2}$, S_1, S_2 在 xy 平面上的投影 $D = \{(x,y)|x^2+y^2 \leqslant 1\}$, $\mathrm{d}S_1 = \mathrm{d}S_2 = \sqrt{1+z_x^2+z_y^2}\mathrm{d}x\mathrm{d}y = \frac{1}{\sqrt{1-x^2-y^2}}\mathrm{d}x\mathrm{d}y$. 由 $(x+y+z)^2 = 1 + 2xy + 2(x+y)z$, 于是

$$
\begin{aligned}
\iint_S (x+y+z)^2 \mathrm{d}S =& \iint_{S_1} (x+y+z)^2 \mathrm{d}S_1 + \iint_{S_2} (x+y+z)^2 \mathrm{d}S_2 \\
=& \iint_D \left[\frac{1+2xy}{\sqrt{1-x^2-y^2}} + 2(x+y) \right] \mathrm{d}x\mathrm{d}y \\
&+ \iint_D \left[\frac{1+2xy}{\sqrt{1-x^2-y^2}} - 2(x+y) \right] \mathrm{d}x\mathrm{d}y \\
=& 2 \iint_D \frac{1+2xy}{\sqrt{1-x^2-y^2}} \mathrm{d}x\mathrm{d}y.
\end{aligned}
$$

令 $\begin{cases} x = r\cos t, \\ y = r\sin t, \end{cases}$ $r \in [0,1], t \in [0,2\pi]$, 则

$$
\begin{aligned}
\iint_S (x+y+z)^2 \mathrm{d}S =& 2 \int_0^1 \mathrm{d}r \int_0^{2\pi} \frac{1 + 2r^2 \sin t \cos t}{\sqrt{1-r^2}} r \mathrm{d}t \\
=& 2 \int_0^1 \left[\frac{r}{\sqrt{1-r^2}} t \Big|_0^{2\pi} - \frac{r^3}{2\sqrt{1-r^2}} \cos 2t \Big|_0^{2\pi} \right] \mathrm{d}r \\
=& 4\pi \int_0^1 \frac{r}{\sqrt{1-r^2}} \mathrm{d}r = -4\pi \sqrt{1-r^2} \Big|_0^1 \\
=& 4\pi.
\end{aligned}
$$

(2) $S = S_1 \cup S_2$, 其中 S_1 是 $z = \sqrt{x^2 + y^2}$, S_2 是 $z = -\sqrt{x^2 + y^2}$, S_1, S_2 在 xy 平面上的投影 $D = \{(x,y)|x^2 + y^2 \leqslant 2x\}$, $\mathrm{d}S_1 = \mathrm{d}S_2 = \sqrt{1 + z_x^2 + z_y^2}\mathrm{d}x\mathrm{d}y = \sqrt{2}\mathrm{d}x\mathrm{d}y$. 于是,

$$\iint_S (x^4 - y^4 + y^2 z^2 - z^2 x^2 + 1)\mathrm{d}S = 2\sqrt{2}\iint_D [x^4 - y^4 + (y^2 - x^2)(x^2 + y^2) + 1]\mathrm{d}x\mathrm{d}y$$
$$= 2\sqrt{2}\iint_D \mathrm{d}x\mathrm{d}y = \sqrt{2}S_D = 2\sqrt{2}\pi.$$

(3) 由对称性, 只需计算第一卦限的积分. 在第一卦限, S_0 为 $z = 1 - x - y$, $\mathrm{d}S = \sqrt{1 + z_x^2 + z_y^2} = \sqrt{3}$, S_0 在 xy 平面上的投影 $D = \{(x,y)|0 \leqslant x \leqslant 1, 0 \leqslant y \leqslant 1 - x\}$, 于是,

$$\iint_S |xyz|\mathrm{d}S = 8\iint_{S_0} |xyz|\mathrm{d}S = 8\sqrt{3}\iint_{S_0} xy(1 - x - y)\mathrm{d}x\mathrm{d}y$$
$$= 8\sqrt{3}\int_0^1 \mathrm{d}x \int_0^{1-x} xy(1 - x - y)\mathrm{d}y$$
$$= \frac{\sqrt{3}}{15}.$$

(4) 由 $\begin{cases} z = \sqrt{x^2 + y^2}, \\ x^2 + y^2 + z^2 = R^2, \end{cases}$ 得 S 在 xy 平面上的投影 $D = \{(x,y)|x^2 + y^2 \leqslant \frac{R^2}{2}\}$, $\mathrm{d}S = \sqrt{1 + z_x^2 + z_y^2}\mathrm{d}x\mathrm{d}y = \sqrt{2}\mathrm{d}x\mathrm{d}y$. 于是,

$$\iint_S z^2\mathrm{d}S = \sqrt{2}\iint_D (x^2 + y^2)\mathrm{d}x\mathrm{d}y.$$

令 $\begin{cases} x = r\cos t, \\ y = r\sin t, \end{cases}$ $0 \leqslant r \leqslant \frac{R}{\sqrt{2}}, 0 \leqslant t \leqslant 2\pi$, 所以,

$$\iint_S z^2\mathrm{d}S = \sqrt{2}\int_0^{2\pi} \mathrm{d}t \int_0^{\frac{R}{\sqrt{2}}} r^3\mathrm{d}r = \frac{\sqrt{2}}{8}\pi R^4.$$

$\square$

2. 球面 $x^2 + y^2 + z^2 = t^2$ 被 $z = \sqrt{x^2 + y^2}$ 截成一大一小两份, 分别记为 S_0 和 S. 在 S_0 上 $f(x,y,z) = 0$, 在 S 上 $f(x,y,z) = x^2 + y^2$, 因此只需考虑在 S 上的积分. 由 $\begin{cases} z = \sqrt{x^2 + y^2}, \\ z = \sqrt{t^2 - x^2 - y^2}, \end{cases}$ 得 S 在 xy 平面上的投影 $D = \{(x,y)|x^2 + y^2 \leqslant \frac{t^2}{2}\}$, $\mathrm{d}S = \sqrt{1 + z_x^2 + z_y^2}\mathrm{d}x\mathrm{d}y = \frac{t}{\sqrt{t^2 - x^2 - y^2}}\mathrm{d}x\mathrm{d}y$. 于是,

$$F(t) = \iint_S (x^2 + y^2)\mathrm{d}S = \iint_D \frac{t(x^2 + y^2)}{\sqrt{t^2 - x^2 - y^2}}\mathrm{d}x\mathrm{d}y.$$

令 $\begin{cases} x = r\cos u, \\ y = r\sin u, \end{cases} 0 \leqslant r \leqslant \frac{t}{\sqrt{2}}, 0 \leqslant u \leqslant 2\pi$, 所以,

$$F(t) = \int_0^{2\pi} \mathrm{d}u \int_0^{\frac{t}{\sqrt{2}}} \frac{tr^2}{\sqrt{t^2 - r^2}} \mathrm{d}r = (4/3 - 5\sqrt{2}/6)\pi t^4.$$

$\square$

3. 记 S 为上半球面 $z = \sqrt{a^2 - x^2 - y^2}$ 被柱面 $x^2 + y^2 = ax$ 所截部分, 则 S 在 xy 平面上的投影为 $D = \{(x, y)|x^2 + y^2 \leqslant ax\}$, $\mathrm{d}S = \sqrt{1 + z_x^2 + z_y^2}\mathrm{d}x\mathrm{d}y = \frac{a}{\sqrt{a^2 - x^2 - y^2}}\mathrm{d}x\mathrm{d}y$. 于是, S 的面积

$$\Delta S = \iint_S \mathrm{d}S = \iint_D \frac{a}{\sqrt{a^2 - x^2 - y^2}}\mathrm{d}x\mathrm{d}y.$$

令 $\begin{cases} x = r\cos t, \\ y = r\sin t, \end{cases} 0 \leqslant r \leqslant a\cos t, -\frac{\pi}{2} \leqslant t \leqslant \frac{\pi}{2}$, 则

$$\Delta S = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \mathrm{d}t \int_0^{a\cos t} \frac{ar}{\sqrt{a^2 - r^2}}\mathrm{d}r = (\pi - 2)a^2.$$

球面的面密度 $\rho = 1$, 所以 S 的质量 $m = \iint_S \rho\mathrm{d}S = \rho \iint_S \mathrm{d}S = \rho\Delta S = (\pi - 2)\rho a^2$, 重心坐标

$$
\begin{aligned}
x_0 =& \frac{1}{m}\iint_S \rho x\mathrm{d}S = \frac{1}{(\pi - 2)a^2}\iint_S x\mathrm{d}S = \frac{1}{(\pi - 2)a}\iint_D \frac{x}{\sqrt{a^2 - x^2 - y^2}}\mathrm{d}x\mathrm{d}y \\
=& \frac{1}{(\pi - 2)a}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \mathrm{d}t \int_0^{a\cos t} \frac{r^2\cos t}{\sqrt{a^2 - r^2}}\mathrm{d}r \\
=& \frac{1}{(\pi - 2)a}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos t\left[\frac{a^2}{2}\arcsin\frac{r}{a} - \frac{r}{2}\sqrt{a^2 - r^2}\Big|_0^{a\cos t}\right]\mathrm{d}t \\
=& \frac{a}{\pi - 2}\int_0^{\frac{\pi}{2}} (\cos t\arcsin\cos t - \cos^2 t\sin t)\mathrm{d}t \\
=& \frac{a}{\pi - 2}\int_0^{\frac{\pi}{2}} [\cos t(\pi/2 - t) - \cos^2 t\sin t]\mathrm{d}t \\
=& \frac{a}{\pi - 2}\left[\frac{\pi}{2}\sin t\Big|_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} t\cos t\mathrm{d}t + \frac{\cos^3 t}{3}\Big|_0^{\frac{\pi}{2}}\right] = \frac{2a}{3(\pi - 2)},
\end{aligned}
$$

$$y_0 = \frac{1}{m} \iint_S \rho y \mathrm{d}S = \frac{1}{(\pi-2)a^2} \iint_S y \mathrm{d}S = \frac{1}{(\pi-2)a} \iint_D \frac{y}{\sqrt{a^2-x^2-y^2}} \mathrm{d}x\mathrm{d}y$$

$$= \frac{1}{(\pi-2)a} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \mathrm{d}t \int_0^{a\cos t} \frac{r^2\sin t}{\sqrt{a^2-r^2}} \mathrm{d}r$$

$$= \frac{1}{(\pi-2)a} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin t \left[\frac{a^2}{2}\arcsin\frac{r}{a} - \frac{r}{2}\sqrt{a^2-r^2} \Big|_0^{a\cos t} \right] \mathrm{d}t = 0,$$

$$z_0 = \frac{1}{m} \iint_S \rho z \mathrm{d}S = \frac{1}{(\pi-2)a^2} \iint_S z \mathrm{d}S = \frac{1}{(\pi-2)a} \iint_D \frac{z}{\sqrt{a^2-x^2-y^2}} \mathrm{d}x\mathrm{d}y$$

$$= \frac{1}{(\pi-2)a} \iint_D \mathrm{d}x\mathrm{d}y$$

$$= \frac{1}{(\pi-2)a} S_D = \frac{\pi a}{\pi-2}.$$

$\square$

习题 17.3 第二型曲线积分

1. (1) 记 $O(0,0), A(2,0), B(2,1), C(0,1)$, 则 $\Gamma = \overrightarrow{OA} \cup \overrightarrow{AB} \cup \overrightarrow{BC} \cup \overrightarrow{CO}$, 各段的参数方程分别表示如下:

$$\overrightarrow{OA}: \begin{cases} x = x, \\ y = 0, \end{cases} x \in [0,2], \quad \overrightarrow{AB}: \begin{cases} x = 2, \\ y = y, \end{cases} y \in [0,1],$$

$$\overrightarrow{CB}: \begin{cases} x = x, \\ y = 1, \end{cases} x \in [0,2], \quad \overrightarrow{OC}: \begin{cases} x = 0, \\ y = y, \end{cases} y \in [0,1].$$

因此,

$$\int_\Gamma xy\mathrm{d}x + ye^x\mathrm{d}y$$

$$= \int_{\overrightarrow{OA}} xy\mathrm{d}x + ye^x\mathrm{d}y + \int_{\overrightarrow{AB}} xy\mathrm{d}x + ye^x\mathrm{d}y + \int_{\overrightarrow{BC}} xy\mathrm{d}x + ye^x\mathrm{d}y + \int_{\overrightarrow{CO}} xy\mathrm{d}x + ye^x\mathrm{d}y$$

$$= 0 + \int_0^1 ye^2\mathrm{d}y + \int_2^0 x\mathrm{d}x + \int_1^0 y\mathrm{d}y$$

$$= \frac{1}{2}(e^2 - 5).$$

(2) $\Gamma: \begin{cases} x = x, \\ y = x^2, \end{cases} x \in [0,1]$, 所以,

$$\int_\Gamma y\mathrm{d}x + x\mathrm{d}y = \int_0^1 (x^2 + 2x^2)\mathrm{d}x = 1.$$

(3)

$$\int_\Gamma \frac{x}{y}\mathrm{d}x + \frac{1}{y-a}\mathrm{d}y = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \left[\frac{t-\sin t}{1-\cos t}a(1-\cos t) + \frac{1}{a(1-\cos t)-a}a\sin t\right]\mathrm{d}t$$

$$=\frac{1}{2}at^2 + a\cos t + \ln\cos t\Big|_{\frac{\pi}{6}}^{\frac{\pi}{3}}$$

$$=\frac{\pi^2}{24}a + \frac{a}{2}(1-\sqrt{3}) - \frac{1}{2}\ln 3.$$

(4)

$$\int_\Gamma x\mathrm{d}x + y\mathrm{d}y + z\mathrm{d}z = \int_0^1 (t + 2t^3 + 3t^5)\mathrm{d}t$$

$$= \frac{t^2}{2} + \frac{t^4}{2} + \frac{t^6}{2}\Big|_0^1$$

$$=\frac{3}{2}.$$

(5) θ 为参数, 因此,

$$\int_\Gamma y\mathrm{d}x + z\mathrm{d}y + x\mathrm{d}z$$

$$= \int_0^{2\pi}[R\sin\phi\sin\theta \cdot (-R\sin\phi\sin\theta) + R\cos\phi \cdot R\sin\phi\cos\theta + 0]\mathrm{d}\theta$$

$$= R^2\sin\phi\left[\cos\phi\sin\theta - \frac{1}{2}\sin\phi\left(\theta - \frac{1}{2}\sin 2\theta\right)\right]\Big|_0^{2\pi}$$

$$= -\pi R^2\sin^2\phi.$$

$\square$

2. 设 Γ 的参数方程

$$x = \varphi(t), y = \psi(t), z = f(\varphi(t), \psi(t)), t \in [0, 2\pi],$$

则 γ 的参数方程为

$$x = \varphi(t), y = \psi(t), t \in [0, 2\pi],$$

因此,

$$\oint_\Gamma p(x, y, z)\mathrm{d}x = \int_0^{2\pi} p(\varphi(t), \psi(t), f(\varphi(t), \psi(t)))\varphi'(t)\mathrm{d}t,$$

$$\oint_\gamma p(x, y, f(x, y))\mathrm{d}x = \int_0^{2\pi} p(\varphi(t), \psi(t), f(\varphi(t), \psi(t)))\varphi'(t)\mathrm{d}t,$$

显然, $\oint_\Gamma p(x, y, z)\mathrm{d}x = \oint_\gamma p(x, y, f(x, y))\mathrm{d}x.$

$\square$

3. (1) 由曲线 $\Gamma : \begin{cases} x^2 + y^2 + z^2 = 1, \\ y = z, \end{cases}$ 得 $\begin{cases} x^2 + 2y^2 = 1, \\ y = z, \end{cases}$ 从而得 Γ 得参数方程

$$x = \cos t, y = \frac{1}{\sqrt{2}} \sin t, z = \frac{1}{\sqrt{2}} \sin t, t \in [0, 2\pi],$$

因此,

$$\int_{\Gamma} xyz \mathrm{d}z = \frac{1}{2\sqrt{2}} \int_0^{2\pi} \cos^2 t \sin^2 t \mathrm{d}t = \frac{\pi}{8\sqrt{2}}.$$

(2) 设 $\Gamma = \Gamma_{xy} \cup \Gamma_{yz} \cup \Gamma_{zx}$, 其中

$$\Gamma_{xy} : \begin{cases} x = \cos t, \\ y = \sin t, \quad t \in [0, \pi/2], \\ z = 0, \end{cases} \Gamma_{yz} : \begin{cases} x = 0, \\ y = \cos t, \quad t \in [0, \pi/2], \\ z = \sin t, \end{cases} \Gamma_{zx} : \begin{cases} z = \cos t, \\ x = \sin t, \quad t \in [0, \pi/2], \\ y = 0, \end{cases}$$

因此,

$$\int_{\Gamma} (y^2 - z^2)\mathrm{d}x + (z^2 - x^2)\mathrm{d}y + (x^2 - y^2)\mathrm{d}z$$
$$= \int_{\Gamma_{xy}} (y^2 - z^2)\mathrm{d}x + (z^2 - x^2)\mathrm{d}y + (x^2 - y^2)\mathrm{d}z$$
$$+ \int_{\Gamma_{yz}} (y^2 - z^2)\mathrm{d}x + (z^2 - x^2)\mathrm{d}y + (x^2 - y^2)\mathrm{d}z$$
$$+ \int_{\Gamma_{zx}} (y^2 - z^2)\mathrm{d}x + (z^2 - x^2)\mathrm{d}y + (x^2 - y^2)\mathrm{d}z$$
$$= -3 \int_0^{\pi/2} (\sin^3 t + \cos^3 t)\mathrm{d}t$$
$$= -4.$$

$\square$

复习题

2. (1) 设 $\Gamma = \Gamma_1 \cup \Gamma_2$, 其中

$$\Gamma_1 : \begin{cases} x = y^2, \\ y = y, \end{cases} y \in [-2, 1], \Gamma_2 : \begin{cases} x = x, \\ y = 2 - x, \end{cases} x \in [1, 4],$$

因此,

$$\int_{\Gamma} y\mathrm{d}s = \int_{\Gamma_1} y\mathrm{d}s + \int_{\Gamma_2} y\mathrm{d}s = \int_{-2}^1 y\sqrt{1 + 4y^2}\mathrm{d}y + \int_1^4 (2 - x)\sqrt{1 + 1}\mathrm{d}x = \frac{1}{12}(5^{3/2} - 17^{3/2}) - \frac{3}{\sqrt{2}}.$$

(2) 令 $x = a\cos^3 t, y = a\sin^3 t, t \in [0, 2\pi]$, 则

$$\int_{\Gamma}\left(x^{\frac{4}{3}} + y^{\frac{4}{3}}\right)\mathrm{d}s = a^{\frac{4}{3}}\int_0^{2\pi}(\cos^4 t + \sin^4 t)\sqrt{9a^2(\sin^2 t\cos^4 t + \sin^4 t\cos^2 t)}\mathrm{d}t$$

$$= 3a^{\frac{7}{3}}\int_0^{2\pi}(\cos^4 t + \sin^4 t)|\sin t\cos t|\mathrm{d}t$$

$$= 3a^{\frac{7}{3}}\int_0^{\pi/2}(\sin t\cos^5 t + \sin^5 t\cos t)\mathrm{d}t$$

$$- 3a^{\frac{7}{3}}\int_{\pi/2}^{\pi}(\sin t\cos^5 t + \sin^5 t\cos t)\mathrm{d}t$$

$$+ 3a^{\frac{7}{3}}\int_{\pi}^{3\pi/2}(\sin t\cos^5 t + \sin^5 t\cos t)\mathrm{d}t$$

$$- 3a^{\frac{7}{3}}\int_{3\pi/2}^{2\pi}(\sin t\cos^5 t + \sin^5 t\cos t)\mathrm{d}t$$

$$= 4a^{\frac{7}{3}}.$$

(3)

$$\int_{\Gamma} z\mathrm{d}s = \int_0^{t_0} t\sqrt{(\cos t - t\sin t)^2 + (\sin t + t\cos t)^2 + 1}\,\mathrm{d}t$$

$$= \int_0^{t_0} t\sqrt{2 + t^2}\,\mathrm{d}t$$

$$= \frac{1}{3}[(2 + t_0^2)^{3/2} - 2^{3/2}].$$

$\square$

3. (1) 易知 S 为曲面 $z = \sqrt{1 - x^2 - y^2}, (x, y) \in D$, 其中 $D = \{(x, y)|x^2 + y^2 \leqslant 1, x \geqslant 1, y \geqslant 1\}$. 因此,

$$I = \iint_S z^{99}(x^2 + y^2)\mathrm{d}S = \iint_D (1 - x^2 - y^2)^{49}\sqrt{1 - x^2 - y^2}(x^2 + y^2)\sqrt{1 + z_x^2 + z_y^2}\mathrm{d}x\mathrm{d}y.$$

令 $x = r\cos t, y = r\sin t, 0 \leqslant r \leqslant 1, 0 \leqslant t \leqslant \pi/2$, 则

$$I = \iint_D (1 - x^2 - y^2)^{49}(x^2 + y^2)\mathrm{d}x\mathrm{d}y$$

$$= \int_0^{\pi/2}\mathrm{d}t\int_0^1 r^3(1 - r^2)^{49}\mathrm{d}r$$

$$= \frac{\pi}{4}\int_0^1 u(1 - u)^{49}\mathrm{d}u$$

$$= \frac{\pi}{200}\int_0^1 u\mathrm{d}(1 - u)^{50}$$

$$= \frac{\pi}{200 \times 51}.$$

(2) 设 $S = S_1 \cup S_2$, 其中 $S_1 : z = \sqrt{R^2 - x^2 - y^2}, (x, y) \in D, S_2 : z = -\sqrt{R^2 - x^2 - y^2}, (x, y) \in$

D, 而 $D = \{(x,y)|x^2 + y^2 \leqslant R^2\}$. 因此,

$$\iint_S \sqrt{x^2+y^2}\mathrm{d}S = \iint_{S_1} \sqrt{x^2+y^2}\mathrm{d}S + \iint_{S_2} \sqrt{x^2+y^2}\mathrm{d}S$$
$$= 2R \iint_D \sqrt{x^2+y^2}\frac{1}{\sqrt{R^2-x^2-y^2}}\mathrm{d}x\mathrm{d}y.$$

令 $x = r\cos t, y = r\sin t, 0 \leqslant r \leqslant R, 0 \leqslant t \leqslant \pi/2$, 则

$$\iint_S \sqrt{x^2+y^2}\mathrm{d}S = 2R\int_0^{2\pi}\mathrm{d}t\int_0^R \frac{r^2}{\sqrt{R^2-r^2}}\mathrm{d}r = -4\pi R\int_0^R r\mathrm{d}\sqrt{R^2-r^2}$$
$$= -4\pi R\left(r\sqrt{R^2-r^2}\Big|_0^R - \int_0^R \sqrt{R^2-r^2}\mathrm{d}r\right)$$
$$= 4\pi R^3\int_0^{\pi/2}\cos^2\theta\mathrm{d}\theta$$
$$= \pi^2 R^3.$$

(3) 设 $S = S_1 \cup S_2$, 其中 $S_1 : x = \sqrt{R^2-y^2}, (y,z) \in D_{yz}$, $S_2 : x = -\sqrt{R^2-y^2}, (y,z) \in D_{yz}$, 而 $D_{yz} = \{(y,z)| - R \leqslant y \leqslant R, 0 \leqslant z \leqslant h\} = [-R,R]\times[0,h]$. 因此,

$$\iint_S \frac{1}{\sqrt{x^2+y^2+z^2}}\mathrm{d}S = \iint_{S_1} \frac{1}{\sqrt{x^2+y^2+z^2}}\mathrm{d}S + \iint_{S_2} \frac{1}{\sqrt{x^2+y^2+z^2}}\mathrm{d}S$$
$$= 2\iint_{D_{yz}} \frac{1}{\sqrt{R^2+z^2}}\sqrt{1+x_y^2+x_z^2}\mathrm{d}y\mathrm{d}z$$
$$= 2R\iint_{D_{yz}} \frac{1}{\sqrt{R^2+z^2}}\frac{1}{\sqrt{R^2-y^2}}\mathrm{d}y\mathrm{d}z$$
$$= 4R\int_0^R \frac{1}{\sqrt{R^2-y^2}}\mathrm{d}y\int_0^h \frac{1}{\sqrt{R^2+z^2}}\mathrm{d}z$$
$$= 2\pi R[\ln(h+\sqrt{h^2+R^2}) - \ln R].$$

$\square$

4. (1) $\Gamma : \begin{cases} x = x, \\ y = x^2 - 4, \end{cases} x \in [0,2]$, 所以,

$$\int_\Gamma \frac{\mathrm{d}y - \mathrm{d}x}{x-y} = \int_0^2 \frac{2x-1}{x-x^2+4}\mathrm{d}x = -\int_0^2 \frac{1}{x-x^2+4}\mathrm{d}(x-x^2+4) = \ln 2.$$

(2) (见图 9) 由 $x^2 + y^2 = ax$, 可设 $x = \frac{a}{2} + \frac{a}{2}\cos t, y = \frac{a}{2}\sin t, t \in [0,2\pi]$, 则

$$z = \sqrt{a^2-x^2-y^2} = a\sin\frac{t}{2}.$$

从而, Viviani 曲线的参数方程为
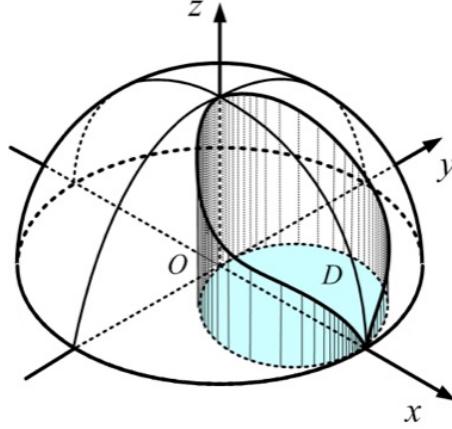
图 9: Viviani 曲线

$$\begin{cases} x = a\cos^2\frac{t}{2}, \\ y = \frac{a}{2}\sin t, \quad t \in [0, 2\pi]. \\ z = a\sin\frac{t}{2}, \end{cases}$$

于是,

$$\int_\Gamma y^2\mathrm{d}x + z^2\mathrm{d}y + x^2\mathrm{d}z$$
$$= \int_0^{2\pi} \left(\frac{a^3\sin^3 t}{8} + \frac{a^3\sin^2\frac{t}{2}\cos t}{8} + \frac{a^3\cos^4\frac{t}{2}}{2} \right)\mathrm{d}t$$
$$= -\frac{\pi a^3}{4}.$$

$\square$

6. 椭圆的参数方程: $\Gamma : \begin{cases} x = 2\cos t, \\ y = \sqrt{3}\sin t, \end{cases} t \in [0, 2\pi]$. 做功

$$W = \int_\Gamma \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r} = \int_\Gamma (3x - 4y)\mathrm{d}x + (4x + 2y)\mathrm{d}y$$
$$= \int_0^{2\pi} [(6\cos t - 4\sqrt{3}\sin t)(-2\sin t) + (8\cos t + 2\sqrt{3}\sin t)(\sqrt{3}\cos t)]\mathrm{d}t$$
$$= \int_0^{2\pi} (8\sqrt{3} - 3\sin 2t)\mathrm{d}t$$
$$= 16\sqrt{3}\pi.$$

$\square$

8. 设螺旋线 Γ 的线密度为 ρ, 则其质量

$$m = \int_\Gamma \rho \mathrm{d}s = \rho \int_\Gamma \mathrm{d}s = \rho \int_0^T \sqrt{a^2 \sin^2 t + a^2 \cos^2 t + h^2} \mathrm{d}t = T\rho\sqrt{a^2 + h^2}.$$

Γ 的重心坐标

$$x_0 = \frac{1}{m} \int_\Gamma x\rho \mathrm{d}s = \frac{1}{T\sqrt{a^2 + h^2}} \int_\Gamma x \mathrm{d}s = \frac{1}{T\sqrt{a^2 + h^2}} \int_0^T a\cos t\sqrt{a^2 + h^2}\mathrm{d}t = \frac{a\sin T}{T},$$

$$y_0 = \frac{1}{m} \int_\Gamma y\rho \mathrm{d}s = \frac{1}{T\sqrt{a^2 + h^2}} \int_\Gamma y \mathrm{d}s = \frac{1}{T\sqrt{a^2 + h^2}} \int_0^T a\sin t\sqrt{a^2 + h^2}\mathrm{d}t = \frac{a(1 - \cos T)}{T},$$

$$z_0 = \frac{1}{m} \int_\Gamma z\rho \mathrm{d}s = \frac{1}{T\sqrt{a^2 + h^2}} \int_\Gamma z \mathrm{d}s = \frac{1}{T\sqrt{a^2 + h^2}} \int_0^T ht\sqrt{a^2 + h^2}\mathrm{d}t = \frac{hT}{2}.$$

$\square$

9. 因为 $x = \rho(\theta)\cos\theta, y = \rho(\theta)\sin\theta, \theta \in [\theta_1, \theta_2]$, 所以

$$\mathrm{d}s = \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}\theta}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}\theta}\right)^2}\mathrm{d}\theta = \sqrt{(\rho(\theta))^2 + (\rho'(\theta))^2}\mathrm{d}\theta,$$

所以,

$$\int_\Gamma f(x, y)\mathrm{d}s = \int_{\theta_1}^{\theta_2} f(\rho(\theta)\cos\theta, \rho(\theta)\sin\theta)\sqrt{(\rho(\theta))^2 + (\rho'(\theta))^2}\mathrm{d}\theta.$$

(1) $\rho = a, \theta \in [0, \pi/4]$, 所以,

$$\int_\Gamma e^{\sqrt{x^2 + y^2}}\mathrm{d}s = \int_0^{\frac{\pi}{4}} ae^a\mathrm{d}\theta = \frac{\pi}{4}ae^a.$$

(2) 对数螺线 $\rho = ae^{k\theta}$ 在圆 $\rho = a$ 内的部分是 $\rho = ae^{k\theta}, \theta \in (-\infty, 0]$, 所以,

$$\int_\Gamma x\mathrm{d}s = \int_{-\infty}^0 ae^{k\theta}\cos\theta\sqrt{a^2e^{2k\theta} + a^2k^2e^{2k\theta}}\mathrm{d}\theta = a^2\sqrt{1 + k^2} \int_{-\infty}^0 e^{2k\theta}\cos\theta\mathrm{d}\theta$$

$$= a^2\sqrt{1 + k^2}\frac{2k}{1 + 4k^2} = \frac{2ka^2\sqrt{1 + k^2}}{1 + 4k^2}.$$

$\square$

10. $f(x(t), y(t))$ 在 $t \in [\alpha, \beta]$ 连续, 因而存在最值, 可设 $m \leqslant f(x(t), y(t)) \leqslant M$, 由第一型曲线积分的单调性, 有

$$m\Delta L = \int_\Gamma m\mathrm{d}s \leqslant \int_\Gamma f(x, y)\mathrm{d}s \leqslant \int_\Gamma M\mathrm{d}s = M\Delta L,$$

即

$$m \leqslant \frac{1}{\Delta L} \int_\Gamma f(x, y)\mathrm{d}s \leqslant M,$$

由连续函数介值性质, $\exists t_0 \in (\alpha, \beta)$, 从而 $\exists (x_0, y_0) \in L$, 使得

$$f(x_0, y_0) = \frac{1}{\Delta L} \int_\Gamma f(x, y) \mathrm{d}s,$$

即有

$$\int_\Gamma f(x, y) \mathrm{d}s = f(x_0, y_0) \Delta L.$$

$\square$

12. 第一型曲线积分和第二型曲线积分有如下关系:

$$\int_\Gamma P \mathrm{d}x + Q \mathrm{d}y = \int_\Gamma (P \cos \alpha + Q \cos \beta) \mathrm{d}s,$$

其中 $(\cos \alpha, \cos \beta)$ 是曲线的切方向的方向余弦, 它的方向与曲线的正向一致. 因此,

$$\left| \int_\Gamma P \mathrm{d}x + Q \mathrm{d}y \right| = \left| \int_\Gamma (P \cos \alpha + Q \cos \beta) \mathrm{d}s \right| \leqslant \int_\Gamma |P \cos \alpha + Q \cos \beta| \mathrm{d}s.$$

又由 Cauchy-Schwartz 不等式, 注意到 $\cos^2 \alpha + \cos^2 \beta = 1$, 有

$$|P \cos \alpha + Q \cos \beta| = |(P, Q) \cdot (\cos \alpha, \cos \beta)| \leqslant \sqrt{P^2 + Q^2} \sqrt{\cos^2 \alpha + \cos^2 \beta} = \sqrt{P^2 + Q^2},$$

因此,

$$\left| \int_\Gamma P \mathrm{d}x + Q \mathrm{d}y \right| \leqslant \int_\Gamma \sqrt{P^2 + Q^2} \mathrm{d}s \leqslant M \int_\Gamma \mathrm{d}s = LM.$$

取 $\Gamma : x^2 + y^2 = R^2, P = \dfrac{y}{(x^2 + xy + y^2)^2}, Q = -\dfrac{x}{(x^2 + xy + y^2)^2}$, 有

$$P^2 + Q^2 = \frac{x^2 + y^2}{(x^2 + xy + y^2)^4} = \frac{R^2}{(R^2 + xy)^4} \leqslant \frac{R^2}{(R^2 - |xy|)^4} \leqslant \frac{R^2}{\left(R^2 - \frac{x^2 + y^2}{2}\right)^4} = \frac{16}{R^6},$$

于是 $M \leqslant \frac{4}{R^3}$, 得

$$|I_R| \leqslant \frac{4}{R^3} \cdot 2\pi R = \frac{8\pi}{R^2},$$

进一步有

$$\lim_{R \to +\infty} I_R = 0.$$

$\square$

(END)