作业 10 月 25 日

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练习 1. 证明 n 维向量 $Y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$ 和 $n \times n$ 矩阵 $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$ 的范数有如下性质:

(1) $\left\|AY\right\| \leq \left\|A\right\| \cdot \left\|Y\right\|$;

(2) $\left\|AB\right\| \leq \left\|A\right\| \cdot \left\|B\right\|$.

其中,

$$\|Y\| = \sum_{i=1}^{n} |y_i|, \quad \|A\| = \sum_{i,j=1}^{n} |a_{ij}|.$$

解.

$$\begin{aligned}
\left\|AY\right\| &= \left\| \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} \right\| \\
&= \left\| \begin{pmatrix} \sum_{k=1}^{n} a_{1k} y_k \\ \sum_{k=1}^{n} a_{2k} y_k \\ \vdots \\ \sum_{k=1}^{n} a_{nk} y_k \end{pmatrix} \right\| \\
&= \sum_{i=1}^{n} \left| \sum_{k=1}^{n} a_{ik} y_k \right| \\
&\leq \sum_{i=1}^{n} \sum_{k=1}^{n} |a_{ik}| \, |y_k| \\
&= \sum_{k=1}^{n} \left(\sum_{i=1}^{n} |a_{ik}| \right) |y_k| \\
&\leq \left\|A\right\| \cdot \left\|Y\right\|;
\end{aligned}$$

$$
\begin{aligned}
\left\|AB\right\| &= \left\| \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{pmatrix} \right\| \\
&\le \sum_{i=1}^{n} \sum_{j=1}^{n} \left| \sum_{k=1}^{n} a_{ik} b_{kj} \right| \\
&\le \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} |a_{ik}| \, |b_{kj}| \\
&\le \sum_{j=1}^{n} \sum_{i=1}^{n} \sum_{k=1}^{n} \left(\sum_{l=1}^{n} |a_{il}| \right) |b_{kj}| \\
&= \sum_{i=1}^{n} \sum_{l=1}^{n} |a_{il}| \sum_{k=1}^{n} \sum_{j=1}^{n} |b_{kj}| \\
&\le \left\|A\right\| \cdot \left\|B\right\|
\end{aligned}
$$

练习 2. 设 $A(x) = (a_{ij}(x))_{n\times n}$, 其中 $a_{ij}(x), 1 \le i, j \le n$ 关于 x 可导. 设 $D(x) = \det(A(x))$, 则有

$$
\frac{dD(x)}{dx} = \sum_{k=1}^{n} \begin{vmatrix} a_{11}(x) & a_{12}(x) & \cdots & a_{1n}(x) \\ \vdots & \vdots & & \vdots \\ a'_{k1}(x) & a'_{k2}(x) & \cdots & a'_{kn}(x) \\ \vdots & \vdots & & \vdots \\ a_{n1}(x) & a_{n2}(x) & \cdots & a_{nn}(x) \end{vmatrix}
$$

解. 根据行列式的定义, 得

$$
\begin{aligned}
D(x) &= \begin{vmatrix} a_{11}(x) & a_{12}(x) & \cdots & a_{1n}(x) \\ a_{21}(x) & a_{22}(x) & \cdots & a_{2n}(x) \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}(x) & a_{n2}(x) & \cdots & a_{nn}(x) \end{vmatrix} \\
&= \sum_{m_1 m_2 \cdots m_n} (-1)^{N(m_1 m_2 \cdots m_n)} a_{1m_1}(x) a_{2m_2}(x) \cdots a_{nm_n}(x);
\end{aligned}
$$

于是,

$$
\begin{aligned}
\frac{dD(x)}{dx} &= \frac{d}{dx}\left(\sum_{m_1m_2\cdots m_n}(-1)^{N(m_1m_2\cdots m_n)}a_{1m_1}(x)a_{2m_2}(x)\cdots a_{nm_n}(x)\right)\\
&= \sum_{m_1m_2\cdots m_n}(-1)^{N(m_1m_2\cdots m_n)}\frac{d}{dx}\left(a_{1m_1}(x)a_{2m_2}(x)\cdots a_{nm_n}(x)\right)\\
&= \sum_{m_1m_2\cdots m_n}(-1)^{N(m_1m_2\cdots m_n)}\sum_{k=1}^{n}\left(a_{1m_1}(x)\cdots a'_{km_k}(x)\cdots a_{nm_n}(x)\right)\\
&= \sum_{k=1}^{n}\sum_{m_1m_2\cdots m_n}(-1)^{N(m_1m_2\cdots m_n)}\left(a_{1m_1}(x)\cdots a'_{km_k}(x)\cdots a_{nm_n}(x)\right),\\
&= \sum_{k=1}^{n}\begin{vmatrix} a_{11}(x) & a_{12}(x) & \cdots & a_{1n}(x)\\ \vdots & \vdots & & \vdots\\ a'_{k1}(x) & a'_{k2}(x) & \cdots & a'_{kn}(x)\\ \vdots & \vdots & & \vdots\\ a_{n1}(x) & a_{n2}(x) & \cdots & a_{nn}(x)\end{vmatrix}.
\end{aligned}
$$

练习 3. 如果方程组 $\dfrac{dY}{dx}=A(x)Y$ 与 $\dfrac{dY}{dx}=B(x)Y$ 有一个相同的基本解矩阵, 则 $A(x)\equiv B(x)$

证明. 设相同的基本解矩阵为 $\Phi(x)$, 则

$$A(x)\Phi(x)=B(x)\Phi(x).$$

因 $\Phi(x)$ 可逆, 两边同乘以其逆, 即得

$$A(x)\equiv B(x).$$

练习 4. 求解常系数齐次线性微分方程组

(1) $\frac{dy}{dx}=\begin{pmatrix}-2 & 1\\ -1 & 2\end{pmatrix}y,\quad y=\begin{pmatrix}y_1\\ y_2\end{pmatrix}$;

(2) $\frac{dy}{dx}=\begin{pmatrix}2 & -1\\ 3 & 4\end{pmatrix}y,\quad y=\begin{pmatrix}y_1\\ y_2\end{pmatrix}$;

(3) $\frac{dy}{dx}=\begin{pmatrix}2 & -3 & 3\\ 4 & -5 & 3\\ 4 & -4 & 2\end{pmatrix}y,\quad y=\begin{pmatrix}y_1\\ y_2\\ y_3\end{pmatrix}$;

(4) $\frac{dy}{dx}=\begin{pmatrix}2 & -1 & 2\\ 1 & 0 & 2\\ -2 & 1 & -1\end{pmatrix}y,\quad y=\begin{pmatrix}y_1\\ y_2\\ y_3\end{pmatrix}$;

解. (1) 特征方程为

$$\det(\lambda I-A)=\begin{vmatrix}\lambda+2 & -1\\ 1 & \lambda-2\end{vmatrix}=\lambda^2-3=0.$$

特征根 $\lambda_1=-\sqrt{3},\lambda_2=\sqrt{3}$ 为两个相异实根. 下面计算特征向量
(i) $\lambda_1=-\sqrt{3}$ 时, 考虑 $(\lambda_1 I-A)r=0$, 即由

$$\begin{cases}(2-\sqrt{3})u_1-u_2=0,\\ u_1-(2+\sqrt{3})u_2=0,\end{cases}$$

求得特征向量

$$r_2 = \begin{pmatrix} 2+\sqrt{3} \\ 1 \end{pmatrix}.$$

(ii) $\lambda_2 = \sqrt{3}$ 时, 考虑 $(\lambda_2 I - A)r = 0$, 即由

$$\begin{cases} (2+\sqrt{3})u_1 - u_2 = 0, \\ u_1 - (2-\sqrt{3})u_2 = 0, \end{cases}$$

求得特征向量

$$r_2 = \begin{pmatrix} 2-\sqrt{3} \\ 1 \end{pmatrix}$$

所以原方程的通解为:

$$y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = C_1 e^{-\sqrt{3}x} \begin{pmatrix} 2+\sqrt{3} \\ 1 \end{pmatrix} + C_2 e^{\sqrt{3}x} \begin{pmatrix} 2-\sqrt{3} \\ 1 \end{pmatrix},$$

其中 C_1, C_2 是任意常数.

(2) 设系数矩阵为 A. 特征方程为

$$\det(A - \lambda I) = \det \begin{vmatrix} 2-\lambda & -1 \\ 3 & 4-\lambda \end{vmatrix} = \lambda^2 - 6\lambda + 11 = 0.$$

得特征根 $\lambda_1 = 3 + \sqrt{2}i, \lambda_2 = 3 - \sqrt{2}i$. 对应于 $\lambda_1 = 3 + \sqrt{2}i,$, 由特征方程组

$$(A - \lambda_1 I)r = \begin{pmatrix} -1-\sqrt{2}i & -1 \\ 3 & 1-\sqrt{2}i \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = 0,$$

求得特征向量

$$r_1 = \begin{pmatrix} -1 \\ -1-\sqrt{2}i \end{pmatrix},$$

所以原方程有解

$$e^{(3+\sqrt{2}\,i)x} \begin{pmatrix} -1 \\ -1-\sqrt{2}i \end{pmatrix}.$$

它的实部和虚部

$$e^{3x} \begin{pmatrix} \cos\sqrt{2}x \\ -\cos\sqrt{2}x + \sqrt{2}\sin\sqrt{2}x \end{pmatrix}, \quad e^{3x} \begin{pmatrix} \sin\sqrt{2}x \\ -\sin\sqrt{2}x - \sqrt{2}\cos\sqrt{2}x \end{pmatrix}$$

是方程的两个实的线性无关解, 所以该方程组的通解为

$$y = C_1 e^{3x} \begin{pmatrix} \cos\sqrt{2}x \\ -\cos\sqrt{2}x + \sqrt{2}\sin\sqrt{2}x \end{pmatrix} + C_2 e^{3x} \begin{pmatrix} \sin\sqrt{2}x \\ -\sin\sqrt{2}x - \sqrt{2}\cos\sqrt{2}x \end{pmatrix},$$

其中 C_1, C_2 是任意的常数.

(3) 特征值为 $\lambda_1 = 2, \lambda_2 = -2, \lambda_3 = -1$. 对应的特征向量为

$$r_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, r_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, r_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

通解为

$$y = C_1 e^{2x} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + C_2 e^{-2x} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + C_3 e^{-x} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix},$$

(4) 特征方程是

$$\begin{vmatrix} 2-\lambda & -1 & 2 \\ 1 & -\lambda & 2 \\ -2 & 1 & -1-\lambda \end{vmatrix} = -(\lambda-1)(\lambda^2+1) = 0.$$

得特征根 $\lambda_1 = 1, \lambda_2 = i, \lambda = -i$. 下求特征向量.
(i) 对 $\lambda_1 = 1$, 考虑特征方程组 $(A - \lambda_1 I)r = 0$, 即

$$\begin{cases} u_1 - u_2 + 2u_3 = 0, \\ u_1 - u_2 + 2u_3 = 0, \\ -2u_1 + u_2 - 2u_3 = 0. \end{cases}$$

求得 $u_1 = 0, u_2 = 2u_3$. 于是求出特征向量 $\begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$. 所以, 对应于一重特征根 $\lambda_1 = 1$, 求出了一个解 $e^x \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$.
(ii) 对于 $\lambda_2 = i$, 考虑特征方程组 $(A - \lambda_2 I)r = 0$, 即

$$\begin{cases} (2-i)u_1 - u_2 + 2u_3 = 0, \\ u_1 - iu_2 + 2u_3 = 0, \\ -2u_1 + u_2 - (1+i)u_3 = 0. \end{cases}$$

求得 $u_1 = u_2 = -(1+i)u_3$. 于是求出特征向量 $\begin{pmatrix} 1+i \\ 1+i \\ -1 \end{pmatrix}$. 所以, 对应于一重特征根 $\lambda_2 = i$, 求出了一个复值解 $e^{ix} \begin{pmatrix} 1+i \\ 1+i \\ -1 \end{pmatrix}$. 由欧拉公式,

$$e^{ix} \begin{pmatrix} 1+i \\ 1+i \\ -1 \end{pmatrix} = \begin{pmatrix} \cos x - \sin x \\ \cos x - \sin x \\ -\cos x \end{pmatrix} + i \begin{pmatrix} \cos x + \sin x \\ \cos x + \sin x \\ -\sin x \end{pmatrix},$$

它的实部和虚部 $\begin{pmatrix} \cos x - \sin x \\ \cos x - \sin x \\ -\cos x \end{pmatrix}$ $\begin{pmatrix} \cos x + \sin x \\ \cos x + \sin x \\ -\sin x \end{pmatrix}$ 是方程组的两个线性无关的实数解. 所以, 原方程的通解为

$$y = C_1 e^x \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} + C_2 \begin{pmatrix} \cos x - \sin x \\ \cos x - \sin x \\ -\cos x \end{pmatrix} + C_3 \begin{pmatrix} \cos x + \sin x \\ \cos x + \sin x \\ -\sin x \end{pmatrix},$$

其中 C_1, C_2, C_3 是任意常数.