习题 9.2

1.(1) 交点:(-2,2),(2,2).(X型), $S_上=2\int_0^2(\sqrt{8-x^2}-\frac{1}{2}x^2)dx=2\pi+\frac{4}{3}$,

$$S_下=8\pi-S_上=6\pi-\frac{4}{3}$$

(2) 交点:(1,1),$(2,\frac{1}{2})$,(2,2) (X型) $S=\int_1^2[x-\frac{1}{x}]dx=\frac{3}{2}-\ln 2$.

(3) 交点(0,0),(1,e),$(1,e^{-1})$ (X型), $S=\int_0^1(e^x-e^{-x})dx=e+e^{-1}-2$

(4) Y型: $\begin{cases}\ln a\le y\le \ln b\\ 0\le x\le e^y\end{cases}$ $\Rightarrow S=\int_{\ln a}^{\ln b}(e^y-0)dy=b-a$

2. 交点(0,2),(12,-4), Y型 $S=\int_{-4}^{2}[(4-2y)-(y^2-4)]dy=36$

3. $S=4\int_0^{\frac{\pi}{2}}|a\sin^3t\cdot(-3a\cos^2t\sin t)|dt=12a^2\int_0^{\frac{\pi}{2}}\sin^4t\cos^2t\,dt$

$=3a^2\int_0^{\frac{\pi}{2}}\frac{1-\cos 2t}{2}\sin^2 2t\,dt=3a^2\left(\int_0^{\frac{\pi}{2}}\frac{1-\cos 4t}{4}dt-\int_0^{\frac{\pi}{2}}\sin^2 2t\cdot\cos 2t\,dt\right)$

$=\cdots=\frac{3}{8}\pi a^2$

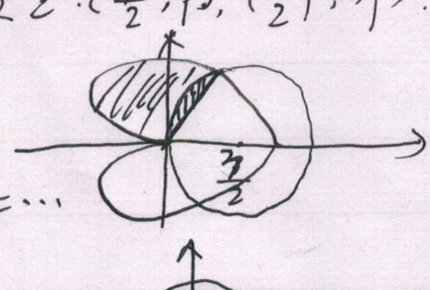
4. $S=4\int_0^{\frac{\pi}{4}}\frac{1}{2}r^2(\theta)d\theta=2a^2\int_0^{\frac{\pi}{4}}\cos 2\theta\, d\theta=a^2$

5. $S=2\int_0^{\pi}\frac{1}{2}[a(1+\cos\theta)]^2d\theta=\frac{3}{2}\pi a^2$

6. 略. 7. 注意到点$(\frac{p}{2},p)$处法线斜率为 $-\frac{dx}{dy}\Big|_{y=p}=-1$,

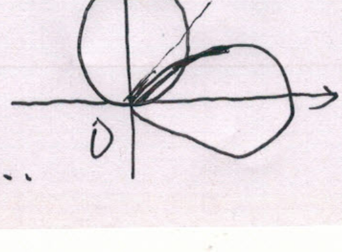
法线方程为 $y-p=-(x-\frac{p}{2})$, 交点:$(\frac{p}{2},p)$, $(\frac{9}{2}p,3p)$. Y型.

8.(1) $\begin{cases}r=3\cos\theta\\ r=1+\cos\theta\end{cases}\Rightarrow(\frac{3}{2},\frac{\pi}{3}),(\frac{3}{2},\frac{4}{3}\pi)$



$S=2\left[\int_{\frac{\pi}{3}}^{\pi}\frac{1}{2}(1+\cos\theta)^2d\theta-\int_{\frac{\pi}{3}}^{\frac{\pi}{2}}\frac{1}{2}(3\cos\theta)^2d\theta\right]=\cdots$

(2) $\begin{cases}r=\sqrt{2}\sin\theta\\ r^2=\cos 2\theta=1-2\sin^2\theta\end{cases}\Rightarrow\theta=\frac{\pi}{6}$ 或 $\frac{5}{6}\pi$

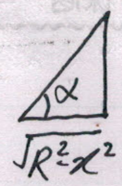


$S=2\left[\int_{\frac{\pi}{6}}^{\frac{\pi}{4}}\frac{1}{2}\cos 2\theta\, d\theta+\int_0^{\frac{\pi}{6}}\frac{1}{2}(\sqrt{2}\sin\theta)^2d\theta\right]=\cdots$

习题9.3

1. 依题设 $A(x)=\frac{1}{2}\sqrt{R^2-x^2}\cdot\sqrt{R^2-x^2}\,\frac{h}{R}=\frac{1}{2}\frac{h}{R}(R^2-x^2)$

$V=4\int_0^R A(x)\,dx=2\frac{h}{R}\left(R^2-\frac{1}{3}x^3\Big|_0^R\right)=\frac{4}{3}R^2h$ (?)



2. 截面为正方形, $A(x)=(\sqrt{R^2-x^2})^2=R^2-x^2$, $0\le x\le a$.

$V=8\int_0^a(R^2-x^2)\,dx=\frac{16}{3}a^3$.

3. $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1-\frac{z^2}{c^2}\iff\frac{x^2}{[a\,d(z)]^2}+\frac{y^2}{[b\,d(z)]^2}=1$, 即以 $z=z$ 截取椭球

所得椭圆的面积为 $\pi ab\,d^2(z)$ (P67, 例4), 从而

$$V=2\int_0^c\pi ab\,d^2(z)\,dz=2\int_0^c\pi ab\left[\sqrt{1-\frac{z^2}{c^2}}\right]^2dz=\frac{4\pi}{3}abc$$

4. $V_x=2\int_0^a\pi y^2\,dx=2\int_0^a b^2\left(1-\frac{x^2}{a^2}\right)dx=\frac{4}{3}\pi ab^2$, 同理 $V_y=\frac{4}{3}\pi a^2b$

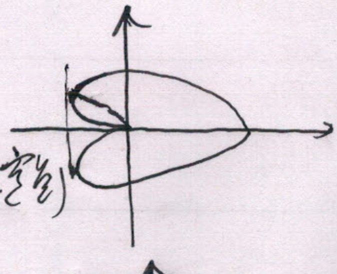
5. 取截面为平行 xy 平面的圆, 则 $A(z)=\pi(\sqrt{R^2-z^2})^2$, $R-H\le z\le R$.

$$V=\int_{R-H}^R\pi(R^2-z^2)\,dz=\pi H^3\left(R-\frac{H}{3}\right)$$

6. $V=2\int_0^a\pi y^2\,dx=2\pi\int_0^{\frac{\pi}{2}}|(a\sin^3t)^2\cdot(-3a\cos^2t\sin t)|\,dt$

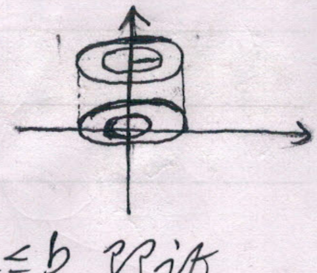
$=6\pi a^3\int_0^{\frac{\pi}{2}}\sin^7t\cos^2t\,dt=-6\pi a^3\int_0^{\frac{\pi}{2}}(1-\cos^2t)^3\cos^2t\,d\cos t=\cdots=\frac{32\pi}{105}a^3$

7. $V=2\left[\int_0^{\frac{2\pi}{3}}\pi r^2(\theta)\,d\theta-\int_{\frac{2\pi}{3}}^{\pi}\pi r^2(\theta)\,d\theta\right]=\frac{8}{3}\pi a^3$



(这里 $\theta=\frac{2}{3}\pi$ 由 $r(\theta)=0$ 求得, 类似 $f'(x)=0$, 即驻点)

8. 该结论的使用容易作用. 注意到



以 dx 为底, $f(x)$ 为高的矩形绕 y 轴旋转

所得小圆环体积 $dV=2\pi x\cdot f(x)\,dx$, $a\le x\le b$, 即证.

习题9.4

1. $L=\int_0^a\sqrt{1+y'^2}\,dx=\int_0^a\sqrt{1+\left(\frac{e^x-e^{-x}}{2}\right)^2}\,dx=\int_0^a\frac{e^x+e^{-x}}{2}\,dx=\frac{e^a-e^{-a}}{2}$

2. (1) $L=2\int_0^1\sqrt{1+y'^2}\,dx=\frac{2}{27}(13\sqrt{13}-8)$

(2) 考虑 $x=e^y$, $0\le y\le\ln\sqrt{3}$, $L=\int_0^{\ln\sqrt{3}}\sqrt{1+e^{2y}}\,dy=?$

$L=\int_1^{\sqrt{3}}\sqrt{1+y'^2}\,dx=\int_1^{\sqrt{3}}\sqrt{1+\left(\frac{1}{x}\right)^2}\,dx=\int_1^{\sqrt{3}}\frac{\sqrt{1+x^2}}{x}\,dx$

$\overset{x=\tan t}{=\!=}\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}\frac{\sec t}{\tan t}\cdot\sec^2 t\,dt=\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}\frac{1}{\sin t\cos^2 t}\,dt$

$=\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}\frac{-d\cos t}{(1-\cos^2 t)\cos^2 t}=\cdots$

(3) $L=4\int_0^{\frac{\pi}{2}}\sqrt{x'^2(t)+y'^2(t)}\,dt=\cdots=12\int_0^{\frac{\pi}{2}}\sin t\cos t\,dt=6a.$

(4) $L=\int_0^{2\pi}\sqrt{x'^2(t)+y'^2(t)}\,dt=a\int_0^{2\pi}t\,dt=2a\pi^2$

(5) $L=\int_0^{3\pi}\sqrt{r^2(\theta)+r'^2(\theta)}\,d\theta=\cdots=a\int_0^{3\pi}\sin^2\frac{\theta}{3}\,d\theta=\frac{1}{2}a\pi$

(6) $L=2\int_0^{\pi}\sqrt{r^2(\theta)+r'^2(\theta)}\,d\theta=4a\int_0^{\pi}\sin\frac{\theta}{2}\,d\theta=8a.$

习题9.5

1. (1) $V=\int_0^{\pi} 2\pi y\sqrt{1+y'^2}\,dx = 2\pi\int_0^{\pi}\cos x\sqrt{1+\sin^2 x}\,dx = 2\pi\int_0^{\pi}\sqrt{1+\sin^2 x}\,d\sin x$

利用公式

$=2\pi[\sqrt{2}+\ln(\sqrt{2}+1)]$.

(2) $V=2\int_0^{\frac{\pi}{2}}2\pi a\sin^3 t\sqrt{[(a\cos^3 t)']^2+[(a\sin^3 t)']^2}\,dt$

$=12\pi a^2\int_0^{\frac{\pi}{2}}\sin^3 t\sqrt{\cos^4 t\sin^2 t+\sin^4 t\cos^2 t}\,dt$

$=12\pi a^2\int_0^{\frac{\pi}{2}}\sin^4 t\,\underline{\cos t\,dt} = \frac{12}{5}\pi a^2$

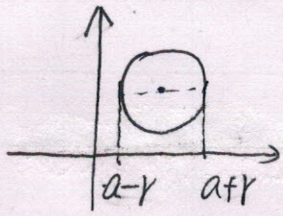
(3) $S_x=2\int_0^a 2\pi y\sqrt{1+y'^2}\,dx = 4\pi\int_0^a b\sqrt{1-\frac{x^2}{a^2}}\sqrt{1+\frac{b^2}{a^2}\left(\frac{-x}{\sqrt{a^2-x^2}}\right)^2}\,dx=\cdots$

利用 $\begin{cases} y=b\sin t \\ x=a\cos t\end{cases}$, $0\le t\le 2\pi$, $S_x=2\int_0^{\frac{\pi}{2}}2\pi y(t)\sqrt{x'^2(t)+y'^2(t)}\,dt$

$=\cdots$

需分 $a>b$, $a<b$, $a=b$ 讨论.

(4) $S=S_上+S_下$ (直角坐标下计算复杂)



$\begin{cases} x=r\cos t+a \\ y=r\sin t+b\end{cases}$ $0\le t\le \pi$.

$S_上=2\pi\int_0^{\pi}(r\sin t+b)\sqrt{x'^2(t)+y'^2(t)}\,dt=\cdots=$

$S_下=\cdots$

2. (1) $S=\int_0^{\pi}2\pi\,\underbrace{r(\theta)\sin\theta}\sqrt{r^2(\theta)+r'^2(\theta)}\,d\theta=2\pi a^2\int(1+\cos\theta)\sin\theta\cdot 2\cos\frac{\theta}{2}\,d\theta$

$=\cdots=\frac{32}{5}\pi a^2$

(2) $S=2\cdot\int_0^{\frac{\pi}{4}}2\pi r(\theta)\sin\theta\sqrt{r^2(\theta)+r'^2(\theta)}\,d\theta$

$=8\pi a^2\int_0^{\frac{\pi}{4}}\sin\theta\,d\theta=8\left(1-\frac{\sqrt{2}}{2}\right)\pi a^2$.

习题 9.7

1. (1) $\int_1^{+\infty}\frac{1}{x^3}dx=-\frac{1}{2x^2}\Big|_1^{+\infty}=\frac{1}{2}$；(2) $\int_0^{+\infty}e^{-ax}dx=-\frac{1}{a}e^{-ax}\Big|_0^{+\infty}=\frac{1}{a}$

(3) $\int_0^{+\infty}e^{-ax}\sin bx\,dx=\frac{-a\sin bx-b\cos bx}{a^2+b^2}e^{-ax}\Big|_0^{+\infty}=\frac{b}{a^2+b^2}$

(4) $\int_{-\infty}^{+\infty}\frac{1}{x^2+2x+2}dx=\int_{-\infty}^{0}\frac{d(x+1)}{(x+1)^2+1}+\int_0^{+\infty}\frac{d(x+1)}{(x+1)^2+1}=\pi/2+\frac{\pi}{2}=\pi$

(5) $I=\int_0^1\frac{-d(1-x^2)}{2\sqrt{1-x^2}}=-\sqrt{1-x^2}\Big|_0^1=1$

(6) $\because \lim\limits_{x\to0^+}x\cdot\frac{1}{x\sqrt{(1-x)^2}}=1$, $\therefore \int_0^{\frac{1}{2}}\frac{dx}{x\sqrt{(1-x)^2}}$ 发散, 故原积分发散.

(7) 由于 $\int_1^{+\infty}\frac{1}{\sqrt{x}}dx$ 发散, 故原积分发散.

(8) $I=\int_1^e\frac{d\ln x}{\sqrt{1-(\ln x)^2}}=\arcsin(\ln x)\Big|_1^e=\frac{\pi}{2}$

(9) $I=\left(\int_0^{\frac{1}{2}}+\int_{\frac{1}{2}}^1\right)\frac{d(x-\frac{1}{2})}{\sqrt{(\frac{1}{2})^2-(x-\frac{1}{2})^2}}=\arcsin(2x-1)\Big|_0^{\frac{1}{2}}+\arcsin(2x-1)\Big|_{\frac{1}{2}}^1=\pi$

(10) $I\overset{t=-\ln x}{=\!=\!=}\int_0^{+\infty}\frac{1}{t^p}dt=\left(\int_0^1+\int_1^{+\infty}\right)\frac{1}{t^p}dx$ 发散

2. $\int_0^1\frac{1}{\sqrt{x}}dx$ 收敛但 $\int_0^1\left(\frac{1}{\sqrt{x}}\right)^2dx$ 发散.

3. 反证法. 假设 $A\neq0$, 不妨假设 $A>0$, 则 $\lim\limits_{x\to+\infty}f(x)=A\Rightarrow\exists X>0$, 当 $x>X$ 时, $f(x)>\frac{A}{2}$, 则 $\int_a^{+\infty}f(x)dx=+\infty$, 与已知矛盾! 即证.

4.
$$f(x)=\begin{cases}2^{n+1}(x-n), & n\le x\le n+\frac{1}{2^{n+1}}\\ 2+2^{n+1}(n-x), & n+\frac{1}{2^{n+1}}\le x\le n+\frac{1}{2^n},\quad n=0,1,2,3,\cdots\\ 0, & n+\frac{1}{2^n}\le x\le n+1,\end{cases}$$

习题9.8

1.(1)(该积分为无穷积分) $\because \lim\limits_{x\to+\infty} x^{\frac{3}{5}}\dfrac{1}{\sqrt[5]{x^3+1}}=1$，$\therefore$ 原积分发散.

(2) $\because \lim\limits_{x\to+\infty} x^3\left|\dfrac{x\arctan x}{1+x^4}\right|=\dfrac{\pi}{2}$，$\therefore$ 原积分收敛.

(3) $\because \lim\limits_{x\to1^-}(x-1)^2\dfrac{1}{(x-1)^2}=1$，$\therefore \int_0^1\dfrac{1}{(1-x)^2}dx$ 发散，原积分发散.

(4) 仅有一个瑕点 $x=0$. $\because \sqrt{x}\dfrac{\ln x}{1-x}\to0$，$x\to0$，$\therefore$ 原积分收敛.

(5) $I=\left(\int_0^{\frac{1}{2}}+\int_{\frac{1}{2}}^1+\int_1^2+\int_2^{+\infty}\right)\dfrac{1}{\sqrt[3]{x^2(x-1)^2}}dx$

注意到 $\lim\limits_{x\to0^+}x^{\frac{2}{3}}\dfrac{1}{\sqrt{x^2(x-1)^2}}=1$，$\lim\limits_{x\to1^-}(1-x)^{\frac{2}{3}}\dfrac{1}{\sqrt[3]{x^2(x-1)^2}}=1$

$\lim\limits_{x\to1^+}(x-1)^{\frac{2}{3}}\dfrac{1}{\sqrt[3]{x^2(x-1)^2}}=1$，$\lim\limits_{x\to+\infty}x^{\frac{4}{3}}\dfrac{1}{\sqrt{x^2(x-1)^2}}=1$

即可知原积分收敛.

(6) $I=\underbrace{\int_0^1\dfrac{dx}{x^p(1+x^2)}}_{p<1\text{ 收敛}}+\underbrace{\int_1^{+\infty}\dfrac{1}{x^p(1+x^2)}dx}_{p+2>1\text{ 时收敛}}$ （即 $-1<p<1$ 时收敛；$|p|\geq1$ 时发散）

(7) 注意到 $1-\cos x\sim\dfrac{x^2}{2}$，$x\to0$. 易知 $p-2<1$ 即 $p<3$ 时收敛，$p\geq3$ 时发散（事实上：当 $p<3$ 时，$\lim\limits_{x\to0^+}x^{p-2}\left|\dfrac{1-\cos x}{x^p}\right|=\dfrac{1}{2}$；当 $p\geq3$ 时，$\lim\limits_{x\to0}x^{p-2}\left|\dfrac{1-\cos x}{x^p}\right|=\dfrac{1}{2}$）

(8) $I=\underbrace{\int_0^1\dfrac{x^q}{1+x^p}dx}_{q<1\text{ 收敛?}}+\underbrace{\int_1^{+\infty}\dfrac{x^q}{1+x^p}dx}_{p-q>1\text{ 收敛}✓}$ 即当 $p-q>1$ 时收敛，当 $p-q\leq1$ 时发散.

× 非瑕积分！

习题9.8

2.(1) $\because$ (i) $\left|\int_0^u \sin x\,dx\right| \le 0$, (ii) $\lim\limits_{x\to+\infty}\frac{\sqrt{x}}{100+x}=0$ 且 $\frac{\sqrt{x}}{100+x}$ 在 $[0,+\infty)$ 上单调.

$\therefore \int_0^{+\infty}\frac{\sqrt{x}\sin x}{100+x}dx$ 收敛.

又 $\because \left|\frac{\sqrt{x}\sin x}{100+x}\right| \ge \frac{\sqrt{x}\sin^2 x}{100+x} = \frac{\sqrt{x}}{2(100+x)} - \frac{\sqrt{x}\cos 2x}{2(100+x)}$.

且易知 $\int_0^{+\infty}\frac{\sqrt{x}\,dx}{2(100+x)}$ 发散, $\int_0^{+\infty}\frac{\sqrt{x}\cos 2x}{2(100+x)}dx$ 收敛

$\therefore \int_0^{+\infty}\left|\frac{\sqrt{x}\sin x}{100+x}\right|dx$ 发散, 从而原级数条件收敛.

(2) $I \xlongequal{t=\frac{1}{x}} \int_1^{+\infty}\frac{\sin t}{t^{2-\alpha}}dt$ 按 $\begin{cases}2-\alpha>1\\0<2-\alpha\le 1\\2-\alpha\le 0\end{cases}$ 讨论即可.

3、利用 $|fg| \le \frac{1}{2}(f^2+g^2)$, $|f+g|^2 \le 2(f^2+g^2)$

4、利用 $\int_a^u h(x)dx \le \int_a^u f(x)dx \le \int_a^u g(x)dx$.

5.(1) $I=(\int_0^1+\int_1^{+\infty})f(x)dx$. 由于 $p>1$, 故 $\int_1^{+\infty}f(x)dx$ 绝对收敛.

注意到 $x\to0$, $\sin x\sim x$, 故 $\lim\limits_{x\to0^+}\frac{\sin x}{x^p+\sin x}=1$ $(p>1)$, 即 0 非瑕点.

(2) D判别法可证其自身收敛, 再由 $\left|\frac{\ln(\ln x)}{\ln x}\sin x\right| > \frac{1}{2x}(1-\cos 2x)$

可证非绝对收敛.

(3) $\because \lim\limits_{x\to0}\sqrt{x}|(\ln x)^n|=0$, $\therefore$ 原级数绝对收敛.

(4) $I=\underbrace{\int_0^{\frac{1}{2}}\frac{x^\alpha}{\sqrt{1-x}}dx}_{\begin{cases}\alpha>-1, \text{绝对收敛}\\ \alpha\le-1, \text{发散}\end{cases}}+\underbrace{\int_{\frac{1}{2}}^1 x^\alpha\frac{1}{\sqrt{1-x}}dx}_{\text{绝对收敛}}$