CSE 107 *Midterm 1*

Scoring: no answer for question: 0 ; correct answer: $+2$; **each** wrong answer: -1 .

Feb. 05, 2021 **50 Points Total** **Closed book** **Answer on this Exam** **Use private chat if you need clarification.**

Possibly helpful information appears on last two pages,

1 Multiple-Choice

1. **(2 pts.)** $\binom{6}{3} =$:

- [ ] (a) 12.
- [ ] (b) 15.
- [x] (c) 20.
- [ ] (d) 24.

2. **(2 pts.)** $\binom{6}{4} =$:

- [ ] (a) 10.
- [x] (b) 15.
- [ ] (c) 20.
- [ ] (d) 24.

3. **(2 pts.)** $\binom{6}{5} =$:

- [x] (a) 6.
- [ ] (b) 12.
- [ ] (c) 18.
- [ ] (d) 24.

4. **(2 pts.)** $\binom{7}{3} =:$

- [ ] (a) 14.
- [ ] (b) 21.
- [ ] (c) 28.
- [x] (d) 35.

5. **(2 pts.)** $\binom{7}{4} =:$

- [ ] (a) 14.
- [ ] (b) 21.
- [ ] (c) 28.
- [x] (d) 35.

6. **(2 pts.)** $\binom{7}{5} =:$

- [ ] (a) 14.
- [x] (b) 21.
- [ ] (c) 28.
- [ ] (d) 35.

7. **(2 pts.)** $b(6,\,1/2,\,3) =$:

- [ ] (a) $12/64$.
- [ ] (b) $15/64$.
- [x] (c) $20/64$.
- [ ] (d) $24/64$.

8. **(2 pts.)** $b(6,\,1/2,\,4) =$:

- [ ] (a) $10/64$.
- [x] (b) $15/64$.
- [ ] (c) $20/64$.
- [ ] (d) $24/64$.

9. **(2 pts.)** $b(5,\,1/2,\,3) =$:

- [x] (a) $10/32$.
- [ ] (b) $12/32$.
- [ ] (c) $15/32$.
- [ ] (d) $20/32$.

10. **(2 pts.)** $b(5,\ 1/2,\ 4) =$:

- [x] (a) $5/32$.
- [ ] (b) $10/32$.
- [ ] (c) $15/32$.
- [ ] (d) $20/32$.

11. **(2 pts.)** For what j is $b(8,\ 1/4,\ j)$ maximized?

- [x] (a) 2.
- [ ] (b) 4.
- [ ] (c) 6.
- [ ] (d) 8.

12. **(2 pts.)** For what j is $b(8,\ 3/4,\ j)$ maximized?

- [ ] (a) 2.
- [ ] (b) 4.
- [x] (c) 6.
- [ ] (d) 8.

13. **(2 pts.)** A “smart” coin has been invented, hoping to reduce unlucky streaks on flips. In a sequence of flips, the first flip is fair (H and T equally likely), but each succeeding flip is biased against being the same as the previous flip, with 0.4 probability to be the same outcome and 0.6 probability to be the opposite outcome. Which is the most accurate statement about its outcomes, compared to a fair coin, for a sequence of **three** flips?

- [ ] (a) The sequence of three is a fair game and the extremes (streaks of 3) are more likely than with a fair coin.
- [ ] (b) The sequence of three is a biased game and the extremes (streaks of 3) are more likely than with a fair coin.
- [ ] (c) The sequence of three is a fair game and the extremes (streaks of 3) are less likely than with a fair coin.
- [x] (d) The sequence of three is a biased game and the extremes (streaks of 3) are less likely than with a fair coin.

14. **(2 pts.)** Two fair dice, one red and one green, are rolled and you have a bet of 100 dollars in which you win 500 dollars (and keep your 100) if the total is 7. But the dice are rolled under a shelf so no one can see them immediately. The croupier makes this offer: If you pay one dollar as a fee, the croupier will peek and tell you the number on the die closest them. Now you are allowed to double your bet or cut it in half, or stay the same.

Which is the most accurate statement about this gamble?

- [ ] (a) The game is in the Casino’s favor without the peek option, but the extra information you pay the dollar for allows you to win more or lose less in the long run.
- [ ] (b) The game is in the Casino’s favor without the peek option, and the extra information you pay the dollar for does not help
- [ ] (c) The game is fair without the peek option, and the extra information you pay the dollar for allows you to win more or lose less in the long run. in the long run.
- [x] (d) The game is fair without the peek option, and the extra information you pay the dollar for does not help in the long run.

For the three problems on this page two real numbers B and C are chosen independently from the interval $[0, 1]$ with uniform density. Note that the point (B, C) is then chosen AT RANDOM in the unit square.

15. **(2 pts.)** The probability that $B + C < 1/2$ is:

- [x] (a) $1/8$.
- [ ] (b) $1/4$.
- [ ] (c) $1/2$.
- [ ] (d) $3/4$.

16. **(2 pts.)** The probability that $\min(B, C) < 1/2$ is:

- [ ] (a) $1/8$.
- [x] (b) $1/4$.
- [ ] (c) $1/2$.
- [ ] (d) $3/4$.

17. **(2 pts.)** The probability that $\max(B, C) < 1/2$ is:

- [ ] (a) $1/8$.
- [x] (b) $1/4$.
- [ ] (c) $1/2$.
- [ ] (d) $3/4$.

For the three problems on this page a real number X is chosen from the interval $[4, 10]$.

18. **(2 pts.)** Suppose the density function is of the form $f(x) = Cx$. Then the value of C is about:

- [x] (a) 0.083.
- [ ] (b) 0.143.
- [ ] (c) 4.500.
- [ ] (d) 6.667.

19. **(2 pts.)** Suppose the density function is of the form $f(x) = C/x^2$. Then the value of C is about:

- [ ] (a) 0.083.
- [x] (b) 0.143.
- [ ] (c) 4.500.
- [ ] (d) 6.667.

20. **(2 pts.)** Suppose the density function is of the form $f(x) = C/(x-1)^2$. Then the value of C is about:

- [ ] (a) 0.083.
- [x] (b) 0.143.
- [ ] (c) 4.500.
- [ ] (d) 6.667.

21. **(2 pts.)** A “smart” coin has been invented, hoping to reduce unlucky streaks on flips. In a sequence of flips, the first flip is fair (H and T equally likely), but each succeeding flip is biased against being the same as the previous flip, with 0.4 probability to be the same outcome and 0.6 probability to be the opposite outcome. Which is the most accurate statement about its outcomes, compared to a fair coin, for a sequence of **three** flips?

- [ ] (a) The sequence of three is a fair game and the extremes (streaks of 3) are more likely than with a fair coin.
- [ ] (b) The sequence of three is a biased game and the extremes (streaks of 3) are more likely than with a fair coin.
- [ ] (c) The sequence of three is a fair game and the extremes (streaks of 3) are less likely than with a fair coin.
- [d] (d) The sequence of three is a biased game and the extremes (streaks of 3) are less likely than with a fair coin.

22. **(2 pts.)** Two fair dice, one red and one green, are rolled and you have a bet of 100 dollars in which you win 500 dollars (and keep your 100) if the total is 7. But the dice are rolled under a shelf so no one can see them immediately. The croupier makes this offer: if you pay one dollar as a fee, the croupier will peek and tell you the number on the red die. Now you are allowed to double your bet or cut it in half, or stay the same.

    Which is the most accurate statement about this gamble?

- [ ] (a) The game is in the Casino’s favor without the peek option, but the extra information you pay the dollar for allows you to win more or lose less in the long run.
- [ ] (b) The game is fair without the peek option, and the extra information you pay the dollar for allows you to win more or lose less in the long run.
- [ ] (c) The game is in the Casino’s favor without the peek option, and the extra information you pay the dollar for does not help in the long run.
- [x] (d) The game is fair without the peek option, and the extra information you pay the dollar for does not help in the long run.

For the problems on this page, three fair dice are rolled and the three face-up numbers are added to get the random variable called **total**. We know the probability that total = 3 is the same as total = 18. We know the probability that total = 4 is the same as total = 17. Etc.

(Note that $2^3 = 8$, $3^3 = 27$, $4^3 = 64$, $5^3 = 125$, $6^3 = 216$.)

23. **(2 pts.)** What are the probabilities of the two lowest totals?

- [ ] (a) $1/216$ and $8/216$.
- [x] (b) $1/216$ and $3/216$.
- [ ] (c) $1/216$ and $6/216$.
- [ ] (d) $1/216$ and $1/125$.

24. **(2 pts.)** What is the average total?

- [ ] (a) 9.5.
- [ ] (b) 10.
- [x] (c) 10.5.
- [ ] (d) 11.

25. **(2 pts.)** What is the conditional probability that total < 5, **given that** total < 11?

- [ ] (a) $1/6$.
- [ ] (b) $1/9$
- [x] (c) $1/12$.
- [ ] (d) $1/18$.

26. **(2 pts.)** You are traveling to an appointment on a street with five traffic lights remainiing in the path. Experience tells you that each traffic light that you hit on red reduces your probability to be on time by 0.20. Assume it is fifty-fifty whether you hit a light red or green, and it does not depend on the earlier lights. You get green on the first two lights (Yay). What is the **conditional probability** you will be on time, given that that knowledge? If no choice is exact, which one is closest?

- [x] (a) 0.60.
- [ ] (b) 0.67.
- [ ] (c) 0.70.
- [ ] (d) 0.75.

MISC. THEOREMS, EQUATIONS AND NUMBERS — OK TO TEAR THIS PAGE OFF.

Theorem 1.4 If A and B are subsets of Ω, then

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).(1.1)$$

Definition 2.1 Let X be a continuous real-valued random variable. A density function for X is a real-valued function f which satisfies

$$P(a \leq X \leq b) = \int_a^b f(x)\,dx$$

for all a, $b \in R$. In other words for very small dx, $P(x \leq X \leq x + dx) = f(x)\,dx$.

Definition 2.2 Let X be a continuous real-valued random variable. Then the cumulative distribution function of X is defined by the equation $F_X(x) = P(X \leq x)$.

Theorem 2.1 Let X be a continuous real-valued random variable with density function $f(x)$. Then the function defined by

$$F(x) = \int_{-\infty}^{\infty} f(t)\,dt$$

is the cumulative distribution function of X. Furthermore, $\dfrac{d}{dx}\,F(x) = f(x)$.

$$(n)_r = (n)(n-1)\cdots(n-r+1) = \frac{n!}{r!}$$

is read as "n down r", or "n lower r", or "n to the r falling." Note that $(n)_n = n)_{n-1} = n!$.

Binomial coefficients, read "n choose r, represent the number of distinct sets of r elements that can be chosen from a set of n, and are given by:

$$\binom{n}{r} = \frac{n!}{r!\,(n-r)!} = \frac{(n)_r}{r!},$$

Bernoulli trials are repeated experiments where the outcome is either success or failure, and each trial is indepndent of the others. failure with probability $1 - p$.

Theorem 3.6 Given n Bernoulli trials with probability p of success on each experiment, the probability of exactly j successes is:

$$b(n, p, j) = \binom{n}{j} p^j\ (1-p)^{n-j},$$

where $(1 - p)$ is often abbreviated to q.

Stirling's formula approximates $n!$ and is useful for computing combinatorial probabilities such as $b(n, p, j)$ for large n and j:

$$n! \approx \sqrt{2\pi n}\ \left(\frac{n}{e}\right)^n.$$

Define $m(\omega_k \mid E) = \dfrac{m(\omega_k)}{P(E)}$ for $\omega_k \in E$. We call this new distribution the **conditional distribution** given E. For a general event F, this gives

$$P(F \mid E) = \sum_{F \cap E} m(\omega_k \mid E) = \sum_{F \cap E} \frac{m(\omega_k)}{P(E)} = \frac{P(F \cap E)}{P(E)}.$$

We call $P(F \mid E)$ the **conditional probability** of F occurring **given that** E occurs.

Definition 4.1 Let E and F be two events with positive probability. We say that they are **independent** if

$$P(E \mid F) = P(E) \text{ and } P(F \mid E) = P(F).$$

Theorem 4.1 (Bayes) Two events E and F are **independent** if and only if $P(E \cap F) = P(E)P(F)$.

Definition 4.2 A set of events $\{A_1, A_2, \dots, A_n\}$ is said to be **mutually independent** if for **any** subset $\{A_i, A_j, \dots, A_m\}$ of these events we have

$$P(A_i \cap A_j \cap \cdots \cap A_m) = P(A_i)P(A_j) \cdots P(A_m).$$

Some integrals

$$\int x\,dx = (1/2)\,x^2 + c \qquad \int x^{-1}\,dx = \ln(x) + c \qquad \int x^{-2}\,dx = -x^{-1} + c$$