Stat 31512 hw 1

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For all problems, you are allowed to use any reference/tool for assistance, but you should aim to do as much as possible by yourself (so that it is more interesting and educational). Please write up the solution in your own words even if you use a reference (you should cite the reference you used).

1 Sampling spanning trees

1. Suppose that $G = (V, E)$ is a graph with $V = [n]$ and that vertices 1 and 2 are neighbors in G. If we want to compute the number of spanning trees which do not include the edge $(1, 2)$, how can we write a formula for this in terms of a single determinant?

2. If we want to compute the number of spanning trees which *do* include the edge $(1, 2)$, how can we write a formula for this in terms of a single determinant?

3. Explain how 1 and 2 give formulas related to the generating polynomial

$$g(x_1, \dots, x_{|E|}) = \sum_T x_T$$

where T ranges over the set of spanning trees (viewed as subsets of E). (What is the connection to e.g. differentiation)

4. Using 1 and 2, give an algorithm to sample a uniformly random spanning tree.

2 Problem 2

Prove that for any distribution and any function f, $\mathrm{Var}(f(X)) = \frac{1}{2}\mathbb{E}(f(X) - f(Y))^2$ where Y is an iid copy of X.

3 Problem 3

Prove that the spectral gap $1 - \lambda_2$ of the simple random walk on a path of length N is size $\Theta(1/N^2)$ (prove both upper and lower bounds).

4 Problem 4

Prove that the spectral gap $1-\lambda_2$ of the simple random walk on the hypercube $\{\pm 1\}^n$ (where two vertices are neighbors iff they differ in exactly one coordinate) is size $\Theta(1/N)$.

5 Problem 5

Prove that for $Z \sim N(0, I)$,

$$\mathrm{Var}(f(Z)) \leq \mathbb{E}\|\nabla f\|^2$$

6 Problem 6

Using the result of problem 5, prove that if J is a symmetric matrix with zero diagonal and upper triangular entries given by $J_{ij} \sim N(0, \beta^2/n)$ for $\beta > 0$ fixed (independent of n), then

$$\mathrm{Var}(\log \sum_{x \in \{\pm 1\}^n} \exp(\langle x, Jx \rangle)) = O_\beta(n).$$

(The notation O_β means that the constant factor can depend in an arbitrary way on β.)

7 Problem 7

Using the result of problem 5, prove that for any probability measure with density $q(x) = \exp(H(x) - \|x\|^2/2)$ where $|H(x)| \leq 1$ everywhere, that

$$\mathrm{Var}_q(f) = O(\mathbb{E}_q\|\nabla f\|_2^2).$$

8 Problem 8

Read through Chapter 4.4 of `https://link.springer.com/book/10.1007/978-3-319-00227-9` (Analysis and Geometry of Markov Diffusion Operators, Bakry-Gentil-Ledoux, you can access pdf copy with uchicago id). Reproduce in your own words self-contained proofs of (a) the Poincaré inequality for exponential measure, and (b) the exponential concentration inequality (4.4.6) at the start of chapter 4.4.3.

(Sidenote: the argument for (b) is fairly general.)

9 Problem 9

Read through any proof of the trickle-down theorem (see, e.g., `https://frkoehle.github.io/stat31512-s2024/lecture15.pdf`, `https://simons.berkeley.edu/sites/default/files/docs/14119/ecctalkii.pdf`, `https://arxiv.org/pdf/2307.13826`) and write down a self-contained proof in your own words. Try to make the proof as clear/intuitive as possible and remember to cite your reference(s).