These notes have not received the scrutiny of publication. They could be missing important references, etc.

Belief Propagation, TAP, and AMP

1 Belief Propagation for the Ising Model

We consider the Ising model on variables $x_i \in \{\pm 1\}$ with interactions J_{ij} and external fields h_i. When the interaction structure is sufficiently nice (e.g. tree-like or locally tree-like), the model can be approximately solved using belief propagation (BP).

For each directed pair $i \to j$ with $i \neq j$, define a message

$$m_{i\to j} \approx \mathbb{E}[x_i \mid x_j \text{ removed}].$$

The BP fixed-point equations are

$$m_{i\to j} = \tanh\left(h_i + \sum_{\substack{k \neq j \\ J_{ik} \neq 0}} \tanh^{-1}\big(\tanh(J_{ik})\, m_{k\to i} \big) \right). \tag{1}$$

Equivalently, one can write

$$m_{i\to j} = \tanh\left(h_i + \sum_{k} \tanh^{-1}\big(\tanh(J_{ik})\, m_{k\to i} \big) - \tanh^{-1}\big(\tanh(J_{ij})\, m_{j\to i} \big) \right). \tag{2}$$

There is one message for each directed pair $i \to j$ with $i \neq j$, so the BP system has

$$n(n-1) \text{ equations in } n(n-1) \text{ variables } m_{i\to j}.$$

The TAP approximation will reduce this to n equations in n unknowns m_i.

2 From BP to TAP

The goal is to express $m_{i\to j}$ in terms of site magnetizations m_i and to obtain a closed system for $(m_i)_{i=1}^n$.

2.1 Linearizing non-linearities

We use first-order expansions around 0:

$$\tanh(x+\varepsilon) = \tanh(x) + \varepsilon\,(1 - \tanh^2(x)) + O(\varepsilon^2),$$
$$\tanh^{-1}(z) = z + O(z^3), \qquad z \approx 0.$$

In dense models (e.g. SK), the couplings satisfy

$$J_{ij} = O\left(\frac{1}{\sqrt{n}}\right),$$

and one expects

$$m_{i\to j} = m_i + O\left(\frac{1}{\sqrt{n}}\right).$$

For small J_{ik} and bounded $m_{k\to i}$,

$$\tanh(J_{ik}) = J_{ik} + O(J_{ik}^3),$$

and hence

$$\tanh^{-1}\big(\tanh(J_{ik})\, m_{k\to i}\big) = J_{ik}m_{k\to i} + O(J_{ik}^3).$$

Since $J_{ik} = O(n^{-1/2})$, we have $J_{ik}^3 = O(n^{-3/2})$ and summing over k gives an $O(n^{-1/2})$ contribution. Thus, after summation over k, the neglected higher-order terms contribute at most $O(n^{-1/2})$ to the argument of tanh.

Substituting these approximations into (2) gives

$$m_{i\to j} \approx \tanh\left(h_i + \sum_k J_{ik}m_{k\to i} - J_{ij}m_{j\to i} + O\big(n^{-1/2}\big)\right). \tag{3}$$

2.2 Replacing messages by magnetizations

Write

$$m_{k\to i} = m_k + \delta_{k\to i}, \qquad \delta_{k\to i} = O\big(n^{-1/2}\big)\,.$$

Then

$$\sum_k J_{ik}m_{k\to i} = \sum_k J_{ik}m_k + \sum_k J_{ik}\delta_{k\to i}.$$

Each product $J_{ik}\delta_{k\to i}$ is $O(n^{-1})$, and summing over k yields an $O(1)$ correction. Keeping track of this contribution and organizing it in terms of m_i and m_k produces a term of the form

$$-\, m_i \sum_{k\neq j} J_{ik}^2(1-m_k^2).$$

Hence, at leading order one may write

$$m_{i\to j} \approx \tanh\left(h_i + \sum_{k\neq j} J_{ik}m_k - m_i \sum_{k\neq j} J_{ik}^2(1-m_k^2) + O\big(n^{-1/2}\big)\right). \tag{4}$$

Here the sums exclude $k = j$ because $m_{i\to j}$ is a cavity quantity with node j removed. The $O(n^{-1/2})$ term collects all remaining errors from the Taylor expansions and message–magnetization differences beyond the leading contribution.

2.3 TAP equations

The site magnetizations are given by the non-cavity version

$$m_i = \tanh\left(h_i + \sum_k J_{ik}m_{k\to i}\right).$$

Using $m_{k\to i} = m_k + O(n^{-1/2})$ and the same reasoning as above, one arrives at

$$m_i \approx \tanh\left(h_i + \sum_k J_{ik}m_k - m_i \sum_k J_{ik}^2(1-m_k^2) + O\big(n^{-1/2}\big)\right). \tag{5}$$

Dropping the $o(1)$ error term in the large-n limit, the TAP equations are

$$m_i = \tanh\left(h_i + \sum_k J_{ik}m_k - m_i \sum_k J_{ik}^2(1-m_k^2)\right), \qquad i = 1, \dots, n. \tag{6}$$

This is a system of n equations in n unknowns $m_1, \dots, m_n$. The additional term

$$m_i \sum_k J_{ik}^2(1-m_k^2)$$

is the leading-order correction coming from the difference between $m_{i\to j}$ and m_i in the BP equations; all other neglected contributions are smaller order $o(1)$.

3 Approximate Message Passing (AMP)

Approximate Message Passing (AMP) is an iterative algorithm whose fixed points solve the TAP equations.

The AMP iteration is

$$m_i^{(t)} = \tanh\left(h_i + \sum_k J_{ik}m_k^{(t-1)} - \sum_k J_{ik}^2(1-(m_k^{(t-1)})^2)\, m_i^{(t-2)}\right). \tag{7}$$

Initialization:

$$m^{(0)} \text{ chosen (e.g. small)}, \qquad m^{(-1)} = 0.$$

If $m^{(t)} = m^{(t-1)} = m^{(t-2)}$, equation (7) reduces to (6), so TAP fixed points are fixed points of AMP.

4 SK Model and Stability of the Zero Solution

Now specialize to the SK model:

$$J_{ij} \sim N\left(0, \frac{\beta^2}{n}\right), \quad J_{ij} = J_{ji}, \quad h_i = 0.$$

The AMP iteration becomes

$$m_i^{(t)} = \tanh\left(\sum_j J_{ij}m_j^{(t-1)} - \sum_j J_{ij}^2(1-(m_j^{(t-1)})^2)\, m_i^{(t-2)}\right).$$

Linearization around $m = 0$

To check when $m = 0$ is a stable solution, linearize the iteration near 0. Using

$$\tanh(x) \approx x, \qquad 1-(m_j^{(t-1)})^2 \approx 1,$$

one obtains the linearized system

$$\hat{m}_i^{(t)} = \sum_j J_{ij}\hat{m}_j^{(t-1)} - \sum_j J_{ij}^2\hat{m}_i^{(t-2)}.$$

In the SK scaling,

$$\sum_j J_{ij}^2 \approx \beta^2,$$

so we approximate

$$\hat{m}_i^{(t)} \approx \sum_j J_{ij}\hat{m}_j^{(t-1)} - \beta^2\hat{m}_i^{(t-2)}.$$

Matrix form and stability condition

Stack two consecutive iterates:

$$\begin{bmatrix} \hat{m}^{(t)} \\ \hat{m}^{(t-1)} \end{bmatrix} = A \begin{bmatrix} \hat{m}^{(t-1)} \\ \hat{m}^{(t-2)} \end{bmatrix}, \qquad A := \begin{bmatrix} J & -\beta^2 I \\ I & 0 \end{bmatrix}.$$

The iteration is stable if every eigenvalue λ of A satisfies $|\lambda| < 1$.

A standard eigenvalue calculation (using the spectrum of J) shows that this holds exactly when

$$\beta < 1.$$

Thus, in the SK model, $m = 0$ is a stable solution of the TAP / AMP equations for $\beta < 1$, and becomes unstable for $\beta > 1$.