Homework 1

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I. Baker-Campbell-Hausdorff formula

Let A and B be two operators that commute with $C = [A, B] = AB - BA$, i.e. they commute with their commutator,

$$[A, [A, B]] = [A, C] = [B, [A, B]] = [B, C] = 0. \tag{1}$$

Prove that

$$\exp(A + B) = \exp(A)\exp(B)\exp\left(-\frac{1}{2}[A, B]\right). \tag{2}$$

Let us define the following operators parametrized by a real number t:

$$U_1(t) = \exp(t\{A + B\}), \tag{3}$$

$$U_2(t) = \exp(tA)\exp(tB)\exp\left(-\frac{t^2}{2}[A, B]\right). \tag{4}$$

It is straightforward to see that:

$$\frac{d}{dt}U_1(t) = (A + B)e^{t(A+B)} = (A + B)U_1(t). \tag{5}$$

For $U_2(t)$, the calculation is more subtle because A and B don't commute:

$$\frac{d}{dt}U_2(t) = Ae^{tA}e^{tB}e^{-\frac{t^2}{2}[A,B]} + e^{tA}Be^{tB}e^{-\frac{t^2}{2}[A,B]} - t[A, B]e^{tA}e^{tB}e^{-\frac{t^2}{2}[A,B]} \tag{6}$$

$$= Ae^{tA}e^{tB}e^{-\frac{t^2}{2}[A,B]} + e^{tA}B\underbrace{e^{-tA}e^{tA}}_{=\mathbb{I}}e^{tB}e^{-\frac{t^2}{2}[A,B]} - t[A, B]e^{tA}e^{tB}e^{-\frac{t^2}{2}[A,B]} \tag{7}$$

$$= \left(A + e^{tA}Be^{-tA} - [A, B]t\right)e^{tA}e^{tB}e^{-\frac{t^2}{2}[A,B]} \tag{8}$$

$$= \left(A + e^{tA}Be^{-tA} - [A, B]t\right)U_2(t). \tag{9}$$

To obtain (7), we introduced the identity $e^{-tA}e^{tA} = \mathbb{I}$ and used the fact that $[A, B]$ commutes with both A and B to permute it with $e^{tA}e^{tB}$. We will now show that $e^{tA}Be^{-tA} = B + [A, B]t$.

$$e^{tA}Be^{-tA} = \left(1 + tA + \frac{t^2A^2}{2} + \frac{t^3A^3}{6} + ...\right)B\left(1 - tA + \frac{t^2A^2}{2} - \frac{t^3A^3}{6} + ...\right) \tag{10}$$

$$= B + (AB - BA)t + (A^2B - 2ABA + BA^2)\frac{t^2}{2} + (A^3B - 3A^2BA + 3ABA^2 - BA^3)\frac{t^3}{6} + ... \tag{11}$$

$$\begin{aligned} &= B + [A, B]t + (A\underbrace{(AB - BA)}_{=[A,B]} - (AB - BA)A)\frac{t^2}{2} \\ &\quad + (A\underbrace{(A^2B - 2ABA + BA^2)}_{=[A,[A,B]]} - (A^2B - 2ABA + BA^2)A)\frac{t^3}{6} + ... \end{aligned} \tag{12}$$

$$= B + [A, B]t + [A, [A, B]]\frac{t^2}{2} + [A, [A, [A, B]]]\frac{t^3}{6} + ... \tag{13}$$

To obtain (10), we used the Taylor series of the exponentials. To obtain (11), we multiplied the series and grouped the powers of t. Since $[A,[A,B]]=0$, all the terms of the summation will be null except the first two. Thus, we are left with:

$$e^{tA}Be^{-tA}=B+[A,B]t. \tag{14}$$

Inserting (14) in (9), we obtain:

$$\frac{d}{dt}U_2(t)=(A+B+[A,B]t-[A,B]t)\,U_2(t)=(A+B)\,U_2(t). \tag{15}$$

From results (5) and (15), we have shown that the two operators satisfy the same differential equation

$$\frac{d}{dt}U_i(t)=(A+B)U_i(t) \tag{16}$$

for $i\in\{1,2\}$. For $t=0$, it is straightforward to see that $U_1(0)=\exp(0)=U_2(0)=\exp(0)\exp(0)\exp(0)=\mathbb{I}$, where $\mathbb{I}$ is the identity operator.

Since the operators satisfy the same differential equation and have the same initial value at $t=0$ then it holds that $U_1(t)=U_2(t)$ for $t>0$. Thus, we substitute $t=1$ in (3) and (4) to recover the result:

$$e^{(A+B)}=e^Ae^Be^{-\frac{1}{2}[A,B]}. \tag{17}$$

The proof is concluded.

II. Another useful operator identity

Prove that for two operators the following is satisfied

$$e^ABe^{-A}=B+[A,B]+\frac{1}{2}[A,[A,B]]+...+\frac{1}{n!}\underbrace{[A,[A,...[A}_{n-\text{fold commutator with }A},B]]]. \tag{18}$$

In the previous question, we have derived

$$e^{tA}Be^{-tA}=\left(1+tA+\frac{t^2A^2}{2}+\frac{t^3A^3}{6}+...\right)B\left(1-tA+\frac{t^2A^2}{2}-\frac{t^3A^3}{6}+...\right) \tag{19}$$

$$=B+(AB-BA)t+(A^2B-2ABA+BA^2)\frac{t^2}{2}+(A^3B-3A^2BA+3ABA^2-BA^3)\frac{t^3}{6}+... \tag{20}$$

$$\begin{aligned}&=B+[A,B]t+(A\underbrace{(AB-BA)}_{=[A,B]}-(AB-BA)A)\frac{t^2}{2}\\&\quad+(A\underbrace{(A^2B-2ABA+BA^2)}_{=[A,[A,B]]}-(A^2B-2ABA+BA^2)A)\frac{t^3}{6}+...\end{aligned} \tag{21}$$

$$=B+t[A,B]+\frac{t^2}{2}[A,[A,B]]+\frac{t^3}{3!}[A,[A,[A,B]]]+...+\frac{t^n}{n!}\underbrace{[A,[A,...[A}_{n-\text{fold commutator with }A},B]]]. \tag{22}$$

By substituting $t=1$ in (22), we find the following

$$e^ABe^{-A}=B+[A,B]+\frac{1}{2}[A,[A,B]]+...+\frac{1}{n!}\underbrace{[A,[A,...[A}_{n-\text{fold commutator with }A},B]]]. \tag{23}$$

The proof is concluded.

Define the unitary rotation operator to be $R(\theta)=e^{i\theta a^\dagger a}$. Verify that

$$R(\theta)^\dagger aR(\theta)=ae^{i\theta}. \tag{24}$$

Let us define $A=-i\theta a^\dagger a$ and $B=a$. By substituting these expressions in (23), we find:

$$R(\theta)^\dagger a R(\theta) = e^{-i\theta a^\dagger a} a e^{i\theta a^\dagger a} \tag{25}$$

$$= a + [-i\theta a^\dagger a, a] + \frac{1}{2}[-i\theta a^\dagger a, [-i\theta a^\dagger a, a]] + \frac{1}{3!}[-i\theta a^\dagger a, [-i\theta a^\dagger a, [-i\theta a^\dagger a, a]]] + ... \tag{26}$$

$$= a + i\theta(-a^\dagger a a + \underbrace{a a^\dagger}_{=a^\dagger a + 1} a) + \frac{1}{2}[-i\theta a^\dagger a, [-i\theta a^\dagger a, a]] + \frac{1}{3!}[-i\theta a^\dagger a, [-i\theta a^\dagger a, [-i\theta a^\dagger a, a]]] + ... \tag{27}$$

$$= a + i\theta a + \frac{i\theta}{2}(-a^\dagger a \underbrace{[-i\theta a^\dagger a, a]}_{=i\theta a} + [-i\theta a^\dagger a, a] a^\dagger a) + \frac{1}{3!}[-i\theta a^\dagger a, [-i\theta a^\dagger a, [-i\theta a^\dagger a, a]]] + ... \tag{28}$$

$$= a + i\theta a + \frac{(i\theta)^2}{2}(-a^\dagger a a + \underbrace{a a^\dagger}_{=a^\dagger a + 1} a) + \frac{1}{3!}[-i\theta a^\dagger a, [-i\theta a^\dagger a, [-i\theta a^\dagger a, a]]] + ... \tag{29}$$

$$= a + i\theta a + \frac{(i\theta)^2}{2} a + \frac{i\theta}{3!}(-a^\dagger a \underbrace{[-i\theta a^\dagger a, [-i\theta a^\dagger a, a]]}_{=(i\theta)^2 a} + [-i\theta a^\dagger a, [-i\theta a^\dagger a, a]] a^\dagger a) + ... \tag{30}$$

$$= a + i\theta a + \frac{(i\theta)^2}{2} a + \frac{(i\theta)^3}{3!}(-a^\dagger a a + \underbrace{a a^\dagger}_{=a^\dagger a + 1} a) + ... \tag{31}$$

$$= a + i\theta a + \frac{(i\theta)^2}{2} a + \frac{(i\theta)^3}{3!} a + ... \tag{32}$$

$$= a(1 + i\theta + \frac{(i\theta)^2}{2} + \frac{(i\theta)^3}{3!} + ...) \tag{33}$$

$$= a e^{i\theta}. \tag{34}$$

To obtain (26), we used the formula (23). In many parts of the derivation, we used the commutator $[a, a^\dagger] = 1$. To obtain (34), we used the Taylor series of the exponential. Therefore, we have verified that

$$R(\theta)^\dagger a R(\theta) = a e^{i\theta}. \tag{35}$$

III. Overlap of two coherent states

We can express a coherent state in the Fock basis as

$$|\alpha\rangle = D(\alpha)|0\rangle = e^{-\frac{1}{2}|\alpha|^2} \sum_n \frac{\alpha^n}{\sqrt{n!}} |n\rangle. \tag{36}$$

A. Show that the overlap between two coherent states $|\alpha\rangle = D(\alpha)|0\rangle$ and $|\beta\rangle = D(\beta)|0$ is given by

$$\langle\beta|\alpha\rangle = \exp\left(-\frac{|\alpha|^2}{2} - \frac{|\beta|^2}{2} + \beta^*\alpha\right). \tag{37}$$

We have

$$|\alpha\rangle = D(\alpha)|0\rangle = e^{-\frac{1}{2}|\alpha|^2} \sum_{n_a} \frac{\alpha^{n_a}}{\sqrt{n_a!}} |n_a\rangle \tag{38}$$

$$\langle\beta| = \langle 0|D(\beta)^\dagger = e^{-\frac{1}{2}|\beta|^2} \sum_{n_b} \frac{(\beta^*)^{n_b}}{\sqrt{n_b!}} \langle n_b| \tag{39}$$

Thus, the overlap between the two coherent states is

$$\langle\beta|\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2}e^{-\frac{1}{2}|\beta|^2}\sum_{n_b}\sum_{n_a}\frac{(\beta^*)^{n_b}}{\sqrt{n_b!}}\frac{\alpha^{n_a}}{\sqrt{n_a!}}\langle n_b|n_a\rangle \tag{40}$$

$$= e^{-\frac{1}{2}|\alpha|^2}e^{-\frac{1}{2}|\beta|^2}\sum_{n_b}\sum_{n_a}\frac{(\beta^*)^{n_b}}{\sqrt{n_b!}}\frac{\alpha^{n_a}}{\sqrt{n_a!}}\delta_{n_b n_a} \tag{41}$$

$$= e^{-\frac{1}{2}|\alpha|^2}e^{-\frac{1}{2}|\beta|^2}\sum_{n}\frac{(\beta^*\alpha)^n}{n!} \tag{42}$$

$$= e^{-\frac{1}{2}|\alpha|^2}e^{-\frac{1}{2}|\beta|^2}e^{\beta^*\alpha} \tag{43}$$

$$= \exp\left(-\frac{|\alpha|^2}{2}-\frac{|\beta|^2}{2}+\beta^*\alpha\right). \tag{44}$$

To obtain (41), we used the orthonormality of the Fock basis, i.e. $\langle n_b|n_a\rangle = \delta_{n_b n_a}$. To obtain (42), we used the Kronecker delta to merge the two summations as all the cross terms $n_b \neq n_a$ vanish. To obtain (43), we used the Taylor series of the exponential. Therefore, we have shown the expected result.

B. Write $|\langle\beta|\alpha\rangle|^2$ as a function of $|\alpha-\beta|$ only. We have

$$|\langle\beta|\alpha\rangle|^2 = \langle\alpha|\beta\rangle\langle\beta|\alpha\rangle = \exp\left(-|\alpha|^2-|\beta|^2+\alpha^*\beta+\beta^*\alpha\right). \tag{45}$$

We also have

$$|\alpha-\beta|^2 = (\alpha^*-\beta^*)(\alpha-\beta) = |\alpha|^2+|\beta|^2-\alpha^*\beta-\beta^*\alpha. \tag{46}$$

Substituting (46) in (45), we easily find:

$$|\langle\beta|\alpha\rangle|^2 = \exp\left(-|\alpha-\beta|^2\right). \tag{47}$$

Therefore, we have expressed $|\langle\beta|\alpha\rangle|^2$ as a function of $|\alpha-\beta|$ only.

IV. Beamsplitters and coherent states

A classical interferometer transforms its two input amplitudes ($\alpha_{1,\text{in}}$, $\alpha_{2,\text{in}}$) as follows

$$\alpha_{1,\text{out}} = T_{11}\alpha_{1,\text{in}} + T_{12}\alpha_{2,\text{in}}, \tag{48}$$

$$\alpha_{2,\text{out}} = T_{21}\alpha_{1,\text{in}} + T_{22}\alpha_{2,\text{in}}. \tag{49}$$

We demand that energy is conserved, which implies that the matrix

$$T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} \tag{50}$$

must be unitary. Now consider the quantum case: we now demand that the annihilation operators transform precisely as the classical amplitudes in the Heisenberg picture

$$a_{1,\text{out}} = \mathcal{U}^\dagger a_{1,\text{in}}\mathcal{U} = T_{11}a_{1,\text{in}} + T_{12}a_{2,\text{in}}, \tag{51}$$

$$a_{2,\text{out}} = \mathcal{U}^\dagger a_{2,\text{in}}\mathcal{U} = T_{21}a_{1,\text{in}} + T_{22}a_{2,\text{in}}. \tag{52}$$

In the last equation $\mathcal{U}$ is the unitary evolution operator. We would now like to investigate what is the evolution of two coherent states sent into an interferometer. The input ket describing the system is thus

$$|\psi_{\text{in}}\rangle = D_1(\alpha_{1,\text{in}})D_2(\alpha_{2,\text{in}})|00\rangle \tag{53}$$

where $D_i(\alpha) = \exp\left(\alpha a_{i,\text{in}}^\dagger - \alpha^* a_{i,\text{in}}\right)$ is a single mode displacement operator in mode i.

Show that

$$|\psi_{\text{out}}\rangle = \mathcal{U}|\psi_{\text{in}}\rangle = D_1(\alpha_{1,\text{out}})D_2(\alpha_{2,\text{out}})|00\rangle \tag{54}$$

where the output amplitudes $\alpha_{i,\text{out}}$ are linked to the input amplitudes $\alpha_{i,\text{in}}$ precisely as in (48) and (49). From the annihilation transforms (51) and (52), we derive the creation operators transform

$$\mathcal{U}a_{1,\text{in}}^\dagger\mathcal{U}^\dagger = T_{11}a_{1,\text{in}}^\dagger + T_{21}a_{2,\text{in}}^\dagger, \tag{55}$$

$$\mathcal{U}a_{2,\text{in}}^\dagger\mathcal{U}^\dagger = T_{12}a_{1,\text{in}}^\dagger + T_{22}a_{2,\text{in}}^\dagger. \tag{56}$$

The output ket describing the system is

$$|\psi_\text{out}\rangle = \mathcal{U}|\psi_\text{in}\rangle = \mathcal{U}D_1(\alpha_{1,\text{in}})D_2(\alpha_{2,\text{in}})|00\rangle \tag{57}$$

$$= \mathcal{U}\exp\left(-\frac{1}{2}|\alpha_{1,\text{in}}|^2 - \frac{1}{2}|\alpha_{2,\text{in}}|^2\right)\exp\left(\alpha_{1,\text{in}}a_{1,\text{in}}^\dagger + \alpha_{2,\text{in}}a_{2,\text{in}}^\dagger\right)\underbrace{\exp\left(-\alpha_{1,\text{in}}^*a_{1,\text{in}} - \alpha_{2,\text{in}}^*a_{2,\text{in}}\right)|00\rangle}_{=|00\rangle} \tag{58}$$

$$= \exp\left(-\frac{1}{2}|\alpha_{1,\text{in}}|^2 - \frac{1}{2}|\alpha_{2,\text{in}}|^2\right)\mathcal{U}\exp\left(\alpha_{1,\text{in}}a_{1,\text{in}}^\dagger + \alpha_{2,\text{in}}a_{2,\text{in}}^\dagger\right)\underbrace{\mathcal{U}^\dagger\mathcal{U}}_{=\mathbb{I}}|00\rangle \tag{59}$$

$$= \exp\left(-\frac{1}{2}|\alpha_{1,\text{in}}|^2 - \frac{1}{2}|\alpha_{2,\text{in}}|^2\right)\exp\left(\alpha_{1,\text{in}}\mathcal{U}a_{1,\text{in}}^\dagger\mathcal{U}^\dagger + \alpha_{2,\text{in}}\mathcal{U}a_{2,\text{in}}^\dagger\mathcal{U}^\dagger\right)\underbrace{\mathcal{U}|00\rangle}_{=|00\rangle} \tag{60}$$

$$= \exp\left(-\frac{1}{2}|\alpha_{1,\text{in}}|^2 - \frac{1}{2}|\alpha_{2,\text{in}}|^2\right)\exp\left(\alpha_{1,\text{in}}\left(T_{11}a_{1,\text{in}}^\dagger + T_{21}a_{2,\text{in}}^\dagger\right) + \alpha_{2,\text{in}}\left(T_{12}a_{1,\text{in}}^\dagger + T_{22}a_{2,\text{in}}^\dagger\right)\right)|00\rangle \tag{61}$$

$$= \exp\underbrace{\left(-\frac{1}{2}|\alpha_{1,\text{in}}|^2 - \frac{1}{2}|\alpha_{2,\text{in}}|^2\right)}_{=-\frac{1}{2}|\alpha_{1,\text{out}}|^2-\frac{1}{2}|\alpha_{2,\text{out}}|^2}\exp\left(\underbrace{\left(T_{11}\alpha_{1,\text{in}} + T_{12}\alpha_{2,\text{in}}\right)}_{=\alpha_{1,\text{out}}}a_{1,\text{in}}^\dagger + \underbrace{\left(T_{21}\alpha_{1,\text{in}} + T_{22}\alpha_{2,\text{in}}\right)}_{=\alpha_{2,\text{out}}}a_{2,\text{in}}^\dagger\right)|00\rangle \tag{62}$$

$$= \exp\left(-\frac{1}{2}|\alpha_{1,\text{out}}|^2 - \frac{1}{2}|\alpha_{2,\text{out}}|^2\right)\exp\left(\alpha_{1,\text{out}}a_{1,\text{in}}^\dagger + \alpha_{2,\text{out}}a_{2,\text{in}}^\dagger\right)\underbrace{\exp\left(-\alpha_{1,\text{out}}^*a_{1,\text{in}} - \alpha_{2,\text{out}}^*a_{2,\text{in}}\right)|00\rangle}_{=|00\rangle} \tag{63}$$

$$= D_1(\alpha_{1,\text{out}})D_2(\alpha_{2,\text{out}})|00\rangle. \tag{64}$$

In many parts of the derivation, we used $[a_i, a_j^\dagger] = \delta_{ij}$ to permute the exponentials and regroup them. To obtain (59), we used $a|0\rangle = 0$. To obtain (60), we used the fact that $\mathcal{U}e^A\mathcal{U}^\dagger = e^{\mathcal{U}A\mathcal{U}^\dagger}$ holds for any unitary operator $\mathcal{U}$. To obtain (61), we used the creation transforms (55) and (56). To obtain (63), we used the energy conservation $|\alpha_{1,\text{in}}|^2 + |\alpha_{2,\text{in}}|^2 = |\alpha_{1,\text{out}}|^2 + |\alpha_{2,\text{out}}|^2$ from the fact that T is unitary; we used (48) and (49) to link the input amplitudes to the output amplitudes; and we resolved the vanishing annihilation operators. Therefore, we have shown the expected result.

V. Hong-Ou-Mandel with a different beamsplitter

In class we saw that if we send a single photon in each of the input of an interferometer with transfer matrix

$$T = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & i\\ i & 1\end{pmatrix} \tag{65}$$

then the output state is

$$|\psi_\text{out}\rangle = \mathcal{U}|\psi_\text{in} = \frac{i}{\sqrt{2}}(|20\rangle + |02\rangle). \tag{66}$$

A. Your first task in this problem is to calculate the output state if the same input state with one photon in each input is sent into an interferometer with transfer matrix

$$T' = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix} \tag{67}$$

$$|\psi_{\text{out}}\rangle = \mathcal{U}|\psi_{\text{in}}\rangle = \mathcal{U}a^\dagger_{1,\text{in}}a^\dagger_{2,\text{in}}|00\rangle \tag{68}$$

$$= \mathcal{U}a^\dagger_{1,\text{in}}\mathcal{U}^\dagger\mathcal{U}a^\dagger_{2,\text{in}}\mathcal{U}^\dagger\underbrace{\mathcal{U}|00\rangle}_{=|00\rangle} \tag{69}$$

$$= \left(T_{11}a^\dagger_{1,\text{in}} + T_{21}a^\dagger_{2,\text{in}}\right)\left(T_{12}a^\dagger_{1,\text{in}} + T_{22}a^\dagger_{2,\text{in}}\right)|00\rangle \tag{70}$$

$$= \left(T_{11}T_{12}a^{\dagger 2}_{1,\text{in}} + \left(T_{11}T_{22} + T_{21}T_{12}\right)a^\dagger_{1,\text{in}}a^\dagger_{2,\text{in}} + T_{21}T_{22}a^{\dagger 2}_{2,\text{in}}\right)|00\rangle \tag{71}$$

$$= \frac{1}{\sqrt{2}}\left(-|20\rangle + |02\rangle\right). \tag{72}$$

B. Repeat the calculation you performed in part A of this question but with this transmission matrix:

$$T'' = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \tag{73}$$

That is, find the output state $|\psi_{\text{out}}\rangle$ if you used the matrix T'' instead of T. Substituting the coefficients of T'' in (71), we obtain:

$$|\psi_{\text{out}} = \frac{1}{\sqrt{2}}|20\rangle + |11\rangle + \frac{1}{\sqrt{2}}|02\rangle. \tag{74}$$

What is wrong with the matrix T'' and with the state $|\psi_{\text{out}}\rangle$ obtained by transforming the input state using T''? The matrix T'' is not unitary, and thus the energy is not conserved. The state $|\psi_{\text{out}}\rangle$ is not normalized.

VI. Interferometers

In this problem, we will generalize beamsplitters to consider more general interferometers. Classically, an interferometer takes l input amplitudes and produces l output amplitudes in the following way

$$\alpha_{i,\text{out}} = \sum_{j=1}^{l} T_{ij}\alpha_{j,\text{in}}. \tag{75}$$

In quantum mechanics the matrix T transforms the operators in the Heisenberg picture

$$\mathcal{U}^\dagger a_{i,\text{in}}\mathcal{U} = \sum_{j=1}^{l} T_{ij}a_{j,\text{in}} \tag{76}$$

where $\mathcal{U}$ is the time evolution operator.

A. What condition does the matrix T need to satisfy so that the input energy $I_{in} = \sum_{i=1}^{l} |\alpha_{i,\text{in}}|^2$ equals the output energy $I_{out} = \sum_{i=1}^{l} |\alpha_{i,\text{out}}|^2$.

$$\sum_{i=1}^{l} |\alpha_{i,\text{in}}|^2 = \sum_{i=1}^{l} |\alpha_{i,\text{out}}|^2 \tag{77}$$

$$= \sum_{i=1}^{l} \Big| \sum_{j=1}^{l} T_{ij}\alpha_{j,\text{in}} \Big|^2 \tag{78}$$

$$= \sum_{i=1}^{l} \sum_{k=1}^{l} T_{ki}^{\dagger} \alpha_{k,\text{in}}^{*} \sum_{j=1} T_{ij}\alpha_{j,\text{in}} \tag{79}$$

$$= \sum_{k=1}^{l} \sum_{j=1}^{l} \underbrace{\sum_{i=1}^{l} T_{ki}^{\dagger} T_{ij}}_{=\delta_{kj}} |\alpha_{j,\text{in}}|^2 \tag{80}$$

$$= \sum_{k=1}^{l} |\alpha_{k,\text{in}}|^2. \tag{81}$$

For the equality to hold, we had to impose $\sum_{i=1}^{l} T_{ki}^{\dagger} T_{ij} = \delta_{kj}$. This is equivalent to $T^{\dagger}T = I$, and therefore the matrix T needs to satisfy the condition of unitarity.

B. What condition need to satisfy the matrix T so that the output operator satisfy the same canonical commutation relations as the input operators, i.e. so that:

$$[a_{i,\text{in}}, a_{j,\text{in}}] = [a_{i,\text{out}}, a_{j,\text{out}}] = 0 \tag{82}$$

$$[a_{i,\text{in}}^{\dagger}, a_{j,\text{in}}^{\dagger}] = [a_{i,\text{out}}^{\dagger}, a_{j,\text{out}}^{\dagger}] = 0 \tag{83}$$

$$[a_{i,\text{in}}, a_{j,\text{in}}^{\dagger}] = [a_{i,\text{out}}, a_{j,\text{out}}^{\dagger}] = \delta_{ij} \tag{84}$$

Let us observe that the two first relations are already satisfied, no matter the matrix T. To satisfy (84), we derive:

$$[a_{i,\text{out}}, a_{j,\text{out}}^{\dagger}] = \left[\sum_{k} T_{ik} a_{k,\text{in}}, \sum_{l} T_{lj}^{\dagger} a_{l,\text{in}}^{\dagger}\right] \tag{85}$$

$$= \sum_{k}\sum_{l} T_{ik} T_{lj}^{\dagger} [a_{k,\text{in}}, a_{l,\text{in}}^{\dagger}] \tag{86}$$

$$= \sum_{k}\sum_{l} T_{ik} T_{lj}^{\dagger} \delta_{kl} \tag{87}$$

$$= \sum_{k} T_{ik} T_{kj}^{\dagger} \tag{88}$$

$$= \delta_{ij}. \tag{89}$$

For the last equality to hold, we have $\sum_{k} T_{ik} T_{kj}^{\dagger} = \delta_{ij}$. This is equivalent to $TT^{\dagger} = I$, and therefore the matrix T needs to satisfy the condition of unitarity.

C. Consider the total number of photons observable

$$N_{\text{in}} = \sum_{i} a_{i,\text{in}}^{\dagger} a_{i,\text{in}} \tag{90}$$

Verify that if the condition you found in part A and B for the matrix T is satisfied, then

$$N_{\text{out}} = \mathcal{U}^\dagger N_{\text{in}} \mathcal{U} = N_{\text{in}}, \tag{91}$$

thus showing that the total number of photons is conserved in an interferometer.

$$N_{\text{out}} = \mathcal{U}^\dagger N_{\text{in}} \mathcal{U} \tag{92}$$

$$= \sum_i \mathcal{U}^\dagger a^\dagger_{i,\text{in}} \underbrace{\mathcal{U}\mathcal{U}^\dagger}_{=I} a_{i,\text{in}} \mathcal{U} \tag{93}$$

$$= \sum_i \sum_j T^\dagger_{ji} a^\dagger_{j,\text{in}} \sum_k T_{ik} a_{k,\text{in}} \tag{94}$$

$$= \sum_j \sum_k \underbrace{\sum_i T^\dagger_{ji} T_{ik}}_{\delta_{jk}} \, a^\dagger_{j,\text{in}} a_{k,\text{in}} \tag{95}$$

$$= \sum_j \sum_k \delta_{jk} a^\dagger_{j,\text{in}} a_{k,\text{in}} \tag{96}$$

$$= \sum_j a^\dagger_{j,\text{in}} a_{j,\text{in}} \tag{97}$$

$$= N_{\text{in}}. \tag{98}$$

In (95), we introduced the condition of unitarity of matrix T. Therefore, we have shown the conservation of the total number of photons.

D. Consider now a three mode interferometer where a single photon is injected in each of the three ports

$$|\psi_{\text{in}}\rangle = |111\rangle = a^\dagger_{1,\text{in}} a^\dagger_{2,\text{in}} a^\dagger_{3,\text{in}} |000\rangle. \tag{99}$$

The output state after the interferometer is

$$|\psi_{\text{out}}\rangle = \mathcal{U}|\psi_{\text{in}}\rangle \sum_{i,j,k} c_{ijk} |ijk\rangle. \tag{100}$$

In the last equation we left the limit of the sum unspecified and $|ijk\rangle$ is a Fock state with i photons in mode 1, j photons in mode 2 and k photons in mode k. List all the possible non-zero coefficients present in the sum above just by considering that the number of particles is conserved.

$$\begin{gathered} c_{111}, \\ c_{012}, \\ c_{021}, \\ c_{102}, \\ c_{120}, \\ c_{201}, \\ c_{210}, \\ c_{003}, \\ c_{030}, \\ c_{300} \end{gathered}$$

E. Calculate $\langle 210|\mathcal{U}|111\rangle$ for an arbitrary unitary transmission matrix T of size 3×3. We start by computing the output state:

$$\mathcal{U}|111\rangle = \mathcal{U}a^\dagger_{1,\text{in}}a^\dagger_{2,\text{in}}a^\dagger_{3,\text{in}}|000\rangle \tag{101}$$

$$= \sum_i T_{i1}a^\dagger_{i,\text{in}} \sum_j T_{j2}a^\dagger_{j,\text{in}} \sum_k T_{k3}a^\dagger_{k,\text{in}}|000\rangle \tag{102}$$

$$= \left(T_{11}T_{12}T_{23}a_1^{\dagger 2}a_2^\dagger + T_{13}\left(T_{11}T_{22} + T_{21}T_{12}\right)a_1^{\dagger 2}a_2^\dagger + ...\right)|000\rangle \tag{103}$$

$$= \sqrt{2}\left(T_{11}T_{12}T_{23} + T_{13}\left(T_{11}T_{22} + T_{21}T_{12}\right)\right)|210\rangle + ... \tag{104}$$

The output state has more terms than computed above, but since we're interested in the overlap $\langle 210|\mathcal{U}|111\rangle$, these terms are sufficient. Therefore, we have:

$$\langle 210|\mathcal{U}|111\rangle = \sqrt{2}\left(T_{11}T_{12}T_{23} + T_{13}\left(T_{11}T_{22} + T_{21}T_{12}\right)\right). \tag{105}$$

F. Verify that the following matrix

$$T = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix} \tag{106}$$

satisfies the conditions you found in parts A and B if $\omega = e^{\frac{2\pi}{3}i}$. We found that T needs to be a unitary matrix. Therefore, we need to verify that $T^\dagger T = I$.

$$T^\dagger T = \frac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega^* & \omega^{2*} \\ 1 & \omega^{2*} & \omega^* \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix} \tag{107}$$

$$= \frac{1}{3}\begin{pmatrix} 3 & 1+\omega+\omega^2 & 1+\omega^2+\omega \\ 1+\omega^*+\omega^{2*} & 3 & 1+\omega+\omega^* \\ 1+\omega^{2*}+\omega^* & 1+\omega^*+\omega & 3 \end{pmatrix} \tag{108}$$

$$= \frac{1}{3}\begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix} \tag{109}$$

$$= I. \tag{110}$$

By observing that $\omega = \omega^{2*}$ and $\omega^2 = \omega^*$, and that $1 + \omega + \omega^2 = 0$, it is easy to see that all the off-diagonal terms are zero. We have verified that T is a unitary matrix.

G. Specialize the results from part E of this section to the matrix written down in part F.

$$\langle 210|\mathcal{U}|111\rangle = \sqrt{2}\left(T_{11}T_{12}T_{23} + T_{13}\left(T_{11}T_{22} + T_{21}T_{12}\right)\right) \tag{111}$$

$$= \frac{\sqrt{2}}{3\sqrt{3}}\left(\omega^2 + \omega + 1\right) \tag{112}$$

$$= 0. \tag{113}$$

VII. WHISKERS OF THE CAT

Consider the cat state

$$|\text{cat}_\alpha\rangle = \mathcal{N}\left(|\alpha\rangle + e^{i\phi}|-\alpha\rangle\right) \tag{114}$$

where $|\pm\alpha\rangle$ are coherent states with amplitude $\pm\alpha$.

A. Find the constant $\mathcal{N}$ so that the cat state $|\text{cat}_\alpha\rangle$ is normalized.

$$\langle\text{cat}_\alpha|\text{cat}_\alpha\rangle = |\mathcal{N}|^2\left(\langle\alpha| + e^{-i\phi\langle-\alpha|}\right)\left(|\alpha\rangle + e^{i\phi}|-\alpha\rangle\right) \tag{115}$$

$$= |\mathcal{N}|^2\left(\langle\alpha|\alpha\rangle + e^{i\phi}\langle\alpha|-\alpha\rangle + e^{-i\phi}\langle-\alpha|\alpha\rangle + \langle-\alpha|-\alpha\rangle\right) \tag{116}$$

$$= |\mathcal{N}|^2\left(1 + e^{i\phi}e^{-2|\alpha|^2} + e^{-i\phi}e^{-2|\alpha|^2} + 1\right) \tag{117}$$

$$= |\mathcal{N}|^2\left(2 + 2e^{-2|\alpha|^2}\cos\phi\right) \tag{118}$$

$$= 1. \tag{119}$$

To compute $\langle\alpha|-\alpha\rangle$ and $\langle-\alpha|\alpha\rangle$, we used the result (44). For the last equality to hold, i.e. such that the cat state is normalized, we have:

$$\mathcal{N} = \frac{1}{\sqrt{2 + 2e^{-2|\alpha|^2}\cos\phi}}. \tag{120}$$

B. Write down the Fock space coefficients of the cat state, i.e. write down the c_n that give $|\text{cat}_\alpha\rangle = \sum_n c_n|n\rangle$ where $|n\rangle$ is a Fock state.

$$|\text{cat}_\alpha\rangle = \mathcal{N}\left(|\alpha\rangle + e^{i\phi}|-\alpha\rangle\right) \tag{121}$$

$$= \mathcal{N}\left(e^{-\frac{1}{2}|\alpha|^2}\sum_n \frac{\alpha^n}{\sqrt{n!}}|n\rangle + e^{i\phi}e^{-\frac{1}{2}|\alpha|^2}\sum_n \frac{(-\alpha)^n}{\sqrt{n!}}|n\rangle\right) \tag{122}$$

$$= \sum_n \mathcal{N}e^{-\frac{1}{2}|\alpha|^2}\frac{\alpha^n}{\sqrt{n!}}\left(1 + (-1)^n e^{i\phi}\right)|n\rangle. \tag{123}$$

Therefore, the Fock space coefficients of the cat state are

$$c_n = \mathcal{N}e^{-\frac{1}{2}|\alpha|^2}\frac{\alpha^n}{\sqrt{n!}}\left(1 + (-1)^n e^{i\phi}\right), \tag{124}$$

with the normalization constant $\mathcal{N}$ given by (120).

C. Imagine that a cat state with amplitude α is sent into a beamsplitter with transmission matrix

$$T = \frac{1}{\sqrt{2}}\begin{pmatrix} \sqrt{\eta} & \sqrt{1-\eta} \\ -\sqrt{1-\eta} & \sqrt{\eta} \end{pmatrix} \tag{125}$$

with a second mode prepared in a vacuum state. Find the output state:

$$|\psi_{\text{out}}\rangle = \mathcal{U}|\text{cat}_\alpha\rangle \otimes |0\rangle. \tag{126}$$

$$= \mathcal{U}\mathcal{N}\left(D_1(\alpha) + e^{i\phi}D_1(-\alpha)\right)\mathcal{U}^\dagger\mathcal{U}|00\rangle \tag{127}$$

$$= \mathcal{N}e^{-\frac{1}{2}|\alpha|^2}\left(\exp\left(\mathcal{U}\alpha a_1^\dagger\mathcal{U}^\dagger\right) + e^{i\phi}\exp\left(-\mathcal{U}\alpha a_1^\dagger\mathcal{U}^\dagger\right)\right)|00\rangle \tag{128}$$

$$= \mathcal{N}e^{-\frac{1}{2}|\alpha|^2}\left(\exp\left(\alpha T_{11}a_1^\dagger + \alpha T_{21}a_2^\dagger\right) + e^{i\phi}\exp\left(-\alpha T_{11}a_1^\dagger - \alpha T_{21}a_2^\dagger\right)\right)|00\rangle \tag{129}$$

$$= \mathcal{N}e^{-\frac{1}{2}|\alpha|^2(1-|T_{11}|^2-|T_{21}|^2)}\left(D_1(\alpha T_{11})D_2(\alpha T_{21}) + e^{i\phi}D_1(-\alpha T_{11})D_2(-\alpha T_{21})\right)|00\rangle \tag{130}$$

$$= \mathcal{N}e^{-\frac{1}{2}|\alpha|^2(1-|T_{11}|^2-|T_{21}|^2)}\left(|\alpha T_{11}, \alpha T_{21}\rangle + e^{i\phi}|-\alpha T_{11}, -\alpha T_{21}\rangle\right) \tag{131}$$

$$= \mathcal{N}e^{-\frac{1}{4}|\alpha|^2}\left(|\alpha T_{11}, \alpha T_{21}\rangle + e^{i\phi}|-\alpha T_{11}, -\alpha T_{21}\rangle\right) \tag{132}$$

with $T_{11} = \sqrt{\frac{\eta}{2}}$ and $T_{21} = -\sqrt{\frac{1-\eta}{2}}$.

D. Construct the reduced density matrix of mode 1 after passing through the beamsplitter.

$$\rho_1 = \text{Tr}_2\left(|\psi_{\text{out}}\rangle\langle\psi_{\text{out}}|\right) \tag{133}$$

$$\begin{aligned} &= |\mathcal{N}|^2 e^{-\frac{1}{2}|\alpha|^2}\,\text{Tr}_2\Big(|\alpha T_{11}, \alpha T_{21}\rangle\langle\alpha T_{11}, \alpha T_{21}| + e^{-i\phi}|\alpha T_{11}, \alpha T_{21}\rangle\langle-\alpha T_{11}, -\alpha T_{21}| \\ &\quad + e^{i\phi}|-\alpha T_{11}, -\alpha T_{21}\rangle\langle\alpha T_{11}, \alpha T_{21}| + |-\alpha T_{11}, -\alpha T_{21}\rangle\langle-\alpha T_{11}, -\alpha T_{21}|\Big) \end{aligned} \tag{134}$$

$$= |\mathcal{N}|^2 e^{-\frac{1}{2}|\alpha|^2}\Big(|\alpha T_{11}\rangle\langle\alpha T_{11}| + e^{-i\phi}e^{-2|\alpha T_{21}|^2}|\alpha T_{11}\rangle\langle-\alpha T_{11}| \tag{135}$$

$$+ e^{i\phi}e^{-2|\alpha T_{21}|^2}|-\alpha T_{11}\rangle\langle\alpha T_{11}| + |-\alpha T_{11}\rangle\langle-\alpha T_{11}|\Big). \tag{136}$$

If we compare it to the form:

$$\rho = c_+|\alpha'\rangle\langle\alpha'| + c_-|-\alpha'\rangle\langle-\alpha'| + c_0|\alpha'\rangle\langle-\alpha'| + c_0^*|-\alpha'\rangle\langle\alpha'|, \tag{137}$$

the coefficients c_+, c_-, c_0 and the amplitude α' can be expressed as:

$$c_+ = |\mathcal{N}|^2 e^{-\frac{1}{2}|\alpha|^2} \tag{138}$$

$$c_- = |\mathcal{N}|^2 e^{-\frac{1}{2}|\alpha|^2} \tag{139}$$

$$c_0 = |\mathcal{N}|^2 e^{-\frac{1}{2}|\alpha|^2}e^{-i\phi}e^{-2|\alpha T_{21}|^2} = |\mathcal{N}|^2 e^{-\frac{1}{2}|\alpha|^2}e^{-i\phi}e^{-|\alpha|^2(1-\eta)} \tag{140}$$

$$\alpha' = \alpha T_{11} = \alpha\sqrt{\frac{\eta}{2}} \tag{141}$$

E. Consider the case where $\eta = 1/2$ and an input cat state with mean photon number $|\alpha|^2 = 10$. Evaluate the coefficients from the previous question. What is more dramatic, the decay in the amplitude parameter α or in the coefficient c_0. The coefficient c_0 has an exponential decay:

$$c_0 \propto e^{-|\alpha|^2(1-\eta)} = e^{-10(1-1/2)} = e^{-5}, \tag{142}$$

whereas the amplitude α' has a decay that goes like:

$$\alpha' = \alpha\sqrt{\frac{\eta}{2}} = \frac{\alpha}{2}. \tag{143}$$

Since $e^{-5} \ll 1/2$, the decay in the coefficient c_0 is more dramatic than the amplitude α.

VIII. Difference between mixture and coherent superposition

The output of sending a single photon $|\psi_{\text{in}}\rangle = |1\rangle$ through a loss channel with energy transmission η is the mixed state

$$\rho_{\text{out}} = (1-\eta)|0\rangle\langle 0| + \eta|1\rangle\langle 1|. \tag{144}$$

What observable would you measure to distinguish the mixed state above from the pure state

$$|\psi\rangle = \sqrt{1-\eta}|0\rangle + e^{i\phi}\sqrt{\eta}|1\rangle. \tag{145}$$

If we look at the density matrix of both states, the only difference lies in the off-diagonal terms. By performing state tomography, one can measure its Wigner function. The Wigner function is the 2D Fourier transform of the characteristic function. Therefore, we will show that the characteristic functions of both states are different. Let us first compute a result that will be useful:

$$\langle 1|D(\lambda)|1\rangle = \langle 1|e^{-\frac{1}{2}|\lambda|^2} e^{\lambda a^\dagger} \underbrace{e^{-\lambda^* a}|1\rangle}_{|1\rangle - \lambda^*|0\rangle} \tag{146}$$

$$= \langle 1|e^{-\frac{1}{2}|\lambda|^2} \underbrace{e^{\lambda a^\dagger}\left(|1\rangle - \lambda^*|0\rangle\right)}_{=|1\rangle - |\lambda|^2|1\rangle + \ldots} \tag{147}$$

$$= e^{-\frac{1}{2}|\lambda|^2}(1-|\lambda|^2). \tag{148}$$

Then, the characteristic function (with $\lambda = \frac{1}{\sqrt{2}}(x+iy)$) of the mixed state is:

$$C_{\text{mixed}}(\lambda) = \text{Tr}(D(\lambda)\rho_{\text{out}}) \tag{149}$$

$$= (1-\eta)\langle 0|D(\lambda)|0\rangle + \eta\langle 1|D(\lambda)|1\rangle \tag{150}$$

$$= (1-\eta)e^{-\frac{1}{2}|\lambda|^2} + \eta e^{-\frac{1}{2}|\lambda|^2}(1-|\lambda|^2) \tag{151}$$

$$= e^{-\frac{1}{2}|\lambda|^2}(1-\eta|\lambda|^2). \tag{152}$$

Similarly, the characteristic function of the pure state is:

$$C_{\text{pure}}(\lambda) = \text{Tr}(D(\lambda)|\psi\rangle\langle\psi|) = \langle\psi|D(\lambda)|\psi\rangle \tag{153}$$

$$\begin{aligned} &= (1-\eta)\langle 0|D(\lambda)|0\rangle + e^{i\phi}\sqrt{\eta(1-\eta)}\langle 0|D(\lambda)|1\rangle \\ &\quad + e^{-i\phi}\sqrt{\eta(1-\eta)}\langle 1|D(\lambda)|0\rangle + \eta\langle 1|D(\lambda)|1\rangle \end{aligned} \tag{154}$$

$$= (1-\eta)e^{-\frac{1}{2}|\lambda|^2} - e^{i\phi}\lambda^*\sqrt{\eta(1-\eta)}e^{-\frac{1}{2}|\lambda|^2} + e^{-i\phi}\lambda\sqrt{\eta(1-\eta)}e^{-\frac{1}{2}|\lambda|^2} + \eta e^{-\frac{1}{2}|\lambda|^2}(1-|\lambda|^2) \tag{155}$$

$$= e^{-\frac{1}{2}|\lambda|^2}\left(1-\eta|\lambda|^2 - e^{i\phi}\lambda^*\sqrt{\eta(1-\eta)} + e^{-i\phi}\lambda\sqrt{\eta(1-\eta)}\right) \tag{156}$$

Therefore, the mixed state and the pure state do not have the same characteristic functions, and observing the Wigner function for both states would yield different results. For then we would be able to distinguish the mixed state from the pure state.

IX. Thermal states

In statistical mechanics you are taught that the density matrix of a system with Hamiltonian H weakly coupled to an environment in thermal equilibrium with temperature T is given by

$$\rho = \frac{\exp(-\beta H)}{Z} \tag{157}$$

where $\beta = \frac{1}{k_B T}$ is the coolness (inverse temperature) of the system, k_B is the Boltzmann constant and $Z = \text{Tr}(\exp(-\beta H))$ is a normalization constant known as the partition function.

A. Verify that the trace of the density matrix in the equation above is 1.

$$\mathrm{Tr}(\rho) = \mathrm{Tr}\left(\frac{\exp(-\beta H)}{Z}\right) = \frac{\mathrm{Tr}(\exp(-\beta H))}{Z} = \frac{\mathrm{Tr}(\exp(-\beta H))}{\mathrm{Tr}(\exp(-\beta H))} = 1. \tag{158}$$

B. Consider a mode of the electromagnetic field with frequency ω and Hamiltonian $H = \hbar\omega\left(\hat{n} + \frac{1}{2}\right)$. Write its density matrix in the Fock basis, i.e. find c_{nm} as function of $x = \frac{\hbar\omega}{k_B T}$ such that $\rho = \sum_{n,m} c_{nm}|n\rangle\langle m|$. Let us first derive an expression for Z:

$$Z = \mathrm{Tr}(\exp(-\beta H)) \tag{159}$$

$$= \sum_{n=0}^{\infty}\langle n|\exp(-\beta H)|n\rangle \tag{160}$$

$$= \sum_{n=0}^{\infty}\exp(-x(n+1/2)) \tag{161}$$

$$= \exp(-x/2)\sum_{n=0}^{\infty}\exp(-xn) \tag{162}$$

$$= \exp(-x/2)\frac{1}{1-\exp(-x)}, \tag{163}$$

where we used the fact that $\exp(-x) < 1$ to obtain the last equality (by considering that the sum is a geometric series).

$$\rho = \sum_{n,m=0}^{\infty}|n\rangle\langle n|\rho|m\rangle\langle m| \tag{164}$$

$$= \sum_{n,m=0}^{\infty}\frac{1}{Z}\exp(-x(m+1/2))|n\rangle\langle n|m\rangle\langle m| \tag{165}$$

$$= \sum_{n,m=0}^{\infty}(1-\exp(-x))\exp(-xm)\delta_{nm}|n\rangle\langle m|. \tag{166}$$

Hence, the c_{nm} are:

$$c_{nm} = (1-\exp(-x))\exp(-xm)\delta_{nm}. \tag{167}$$

C. Calculate the mean number of photons in the thermal state from the last question as a function of x:

$$\langle n\rangle = \mathrm{Tr}(\hat{n}\rho) \tag{168}$$

$$= \sum_{n=0}^{\infty}\langle n|\hat{n}\rho|n\rangle \tag{169}$$

$$= \sum_{n=0}^{\infty} n\langle n|\rho|n\rangle \tag{170}$$

$$= (1-\exp(-x))\sum_{n=0}^{\infty} n\exp(-xn) \tag{171}$$

$$= (1-\exp(-x))\,\frac{\exp(-x)}{(1-\exp(-x))^2} \tag{172}$$

$$= \frac{\exp(-x)}{(1-\exp(-x))} \tag{173}$$

$$= \frac{1}{\exp(x)-1}. \tag{174}$$

D. Evaluate the variance of the number of photons $\Delta^2 n = \langle n^2\rangle - \langle n\rangle^2$. Let us compute $\langle n^2\rangle$:

$$\langle n^2\rangle = \mathrm{Tr}(\hat{n}^2\rho) \tag{175}$$

$$= \sum_{n=0}^{\infty}\langle n|\hat{n}^2\rho|n\rangle \tag{176}$$

$$= \sum_{n=0}^{\infty} n^2\langle n|\rho|n\rangle \tag{177}$$

$$= (1-\exp(-x))\sum_{n=0}^{\infty} n^2\exp(-xn) \tag{178}$$

$$= (1-\exp(-x))\,\frac{\exp(-x)(\exp(-x)+1)}{(1-\exp(-x))^3} \tag{179}$$

$$= \frac{\exp(-x)(\exp(-x)+1)}{(1-\exp(-x))^2} \tag{180}$$

$$= \frac{\exp(x)+1}{(\exp(x)-1)^2} \tag{181}$$

The variance is then:

$$\Delta^2 n = \langle n^2\rangle - \langle n\rangle^2 \tag{182}$$

$$= \frac{\exp(x)+1}{(\exp(x)-1)^2} - \frac{1}{(\exp(x)-1)^2} \tag{183}$$

$$= \frac{\exp(x)}{(\exp(x)-1)^2} \tag{184}$$

$$= \langle n\rangle + \langle n\rangle^2 \tag{185}$$

For large, but equal, mean number of photons $\langle n\rangle \gg 1$ which photon number distribution is more spread, the one of a coherent state or the one of a thermal state? For a coherent state, we have the expected values:

$$\langle n\rangle = \langle\alpha|\hat{n}|\alpha\rangle \tag{186}$$

$$= \langle\alpha|a^\dagger a|\alpha\rangle \tag{187}$$

$$= \langle\alpha|\alpha^*\alpha|\alpha\rangle \tag{188}$$

$$= |\alpha|^2. \tag{189}$$

and

$$\langle n^2\rangle = \langle\alpha|a^\dagger a a^\dagger a|\alpha\rangle \tag{190}$$

$$= |\alpha|^2\langle\alpha|aa^\dagger|\alpha\rangle \tag{191}$$

$$= |\alpha|^2\langle\alpha|\left(a^\dagger a+1\right)|\alpha\rangle \tag{192}$$

$$= |\alpha|^2\left(|\alpha|^2+1\right) \tag{193}$$

$$= |\alpha|^4 + |\alpha|^2. \tag{194}$$

Therefore, the variance is:

$$\Delta^2 n = \langle n^2\rangle - \langle n\rangle^2 \tag{195}$$

$$= |\alpha|^4 + |\alpha|^2 - |\alpha|^4 \tag{196}$$

$$= |\alpha|^2 \tag{197}$$

$$= \langle n\rangle. \tag{198}$$

For the thermal state the variance is $\Delta^2 n = \langle n\rangle + \langle n\rangle^2$, whereas for the coherent the variance is $\Delta^2 n = \langle n\rangle$. Therefore, the photon number distribution is more spread for a thermal state.

E. Evaluate the mean number of thermal photons at room temperature $T = 300$ K for

(a) An optical cavity with a frequency corresponding to a wavelength $\lambda = 1550$ nm in vacuum.

$$\langle n\rangle = \frac{1}{\exp(x)-1} = \frac{1}{\exp\left(\frac{(6.626\times10^{-34})(3\times10^{8})}{(1.38\times10^{-23})(300)(1550\times10^{-9})}\right)-1} = 3.52\times10^{-14} \tag{199}$$

(b) A microwave cavity with $\omega = 2\pi$ GHz.

$$\langle n\rangle = \frac{1}{\exp(x)-1} = \frac{1}{\exp\left(\frac{(6.626\times10^{-34})(1\times10^{9})}{(1.38\times10^{-23})(300)}\right)-1} = 6248 \tag{200}$$

F. To what temperature do you need to cool down the microwave cavity from point (b) above so that it has the same mean number of thermal photons as the optical cavity in (a). It requires:

$$T_b = \frac{\omega_b}{\omega_a}T_a = \frac{(1\times10^{9})(1550\times10^{-9})}{(3\times10^{8})}(300) = 1.55\text{ mK}. \tag{201}$$

G. Two thermal states with the same temperature and frequency are sent into a 50:50 beamsplitter. We label the destruction operators of the two input modes as a_1 and a_2.

(a) Write down the input state to the interferometer.

$$\rho_{\text{in}} = \rho_1\otimes\rho_2 \tag{202}$$

$$= (1-\exp(-x))^2\sum_{n,m=0}^{\infty}\exp(-x(n+m))|nm\rangle\langle nm| \tag{203}$$

(b) Write down the output state of the interferometer. Let us first compute $\mathcal{U}|nm\rangle\langle nm|\mathcal{U}^\dagger$.

$$\mathcal{U}|nm\rangle\langle nm|\mathcal{U}^\dagger = \frac{1}{n!m!}\mathcal{U}a_1^{\dagger n}a_2^{\dagger m}|00\rangle\langle 00|a_1^n a_2^m\mathcal{U}^\dagger \tag{204}$$

$$= \frac{1}{n!m!}\left(T_{11}a_1^\dagger + T_{21}a_2^\dagger\right)^n\left(T_{12}a_1^\dagger + T_{22}a_2^\dagger\right)^m|00\rangle\langle 00|\left(T_{11}^*a_1 + T_{21}^*a_2\right)^n\left(T_{12}^*a_1 + T_{22}^*a_2\right)^m \tag{205}$$

$$= \frac{1}{n!m!}\sum_{k=0}^{n}\sum_{l=0}^{m}\binom{n}{k}\binom{m}{l} \tag{206}$$

$$(T_{11}a_1^\dagger)^k(T_{21}a_2^\dagger)^{n-k}(T_{12}a_1^\dagger)^l(T_{22}a_2^\dagger)^{m-l}|00\rangle\langle 00|(T_{11}^*a_1)^k(T_{21}^*a_2)^{n-k}(T_{12}^*a_1)^l(T_{22}^*a_2)^{m-l} \tag{207}$$

$$= \frac{1}{n!m!}\sum_{k=0}^{n}\sum_{l=0}^{m}(k+l)!(n+m-k-l)!\binom{n}{k}\binom{m}{l} \tag{208}$$

$$|T_{11}|^{2k}|T_{21}|^{2(n-k)}|T_{12}|^{2l}|T_{22}|^{2(m-l)}|k+l,n+m-k-l\rangle\langle k+l,n+m-k-l| \tag{209}$$

$$= \frac{(n+m)!}{n!m!}\sum_{k=0}^{n}\sum_{l=0}^{m}\frac{\binom{n}{k}\binom{m}{l}}{\binom{n+m}{k+l}}\left(\frac{1}{2}\right)^{n+m}|k+l,n+m-k-l\rangle\langle k+l,n+m-k-l|, \tag{210}$$

where we used the fact that it is a 50:50 beamsplitter to substitute the matrix T coefficients in the last equality. Hence, the output state of the interferometer is

$$\rho_{\text{out}} = \mathcal{U}\rho_{\text{in}}\mathcal{U}^\dagger \tag{211}$$

$$= (1-\exp(-x))^2\sum_{n,m=0}^{\infty}\exp(-x(n+m))\mathcal{U}|nm\rangle\langle nm|\mathcal{U}^\dagger \tag{212}$$

$$= (1-\exp(-x))^2\sum_{n,m=0}^{\infty}\exp(-x(n+m))\frac{(n+m)!}{n!m!}\sum_{k=0}^{n}\sum_{l=0}^{m}\frac{\binom{n}{k}\binom{m}{l}}{\binom{n+m}{k+l}}\left(\frac{1}{2}\right)^{n+m}|k+l,n+m-k-l\rangle\langle k+l,n+m-k-l|. \tag{213}$$

X. ORTHOGONALITY OF D AND B IN FREE SPACE

Consider one particular plane-wave mode of the form

$$\mathbf{D}_{I,\mathbf{k}}(\mathbf{r},t) = i\sqrt{\frac{\hbar cn|\mathbf{k}|\epsilon_0}{2V}}a_{I\mathbf{k}}(t)\hat{\mathbf{e}}_{I\mathbf{k}}e^{i\mathbf{k}\cdot\mathbf{r}} + h.c., \tag{214}$$

$$\mathbf{B}_{I,\mathbf{k}}(\mathbf{r},t) = \sqrt{\frac{\hbar c|\mathbf{k}|\mu_0}{2nV}}a_{I\mathbf{k}}(t)s_I\hat{\mathbf{e}}_{I\mathbf{k}}e^{i\mathbf{k}\cdot\mathbf{r}} + h.c. \tag{215}$$

A. Pick the wavevector to be along the z direction. Use the definitions of the circular polarization vectors to obtain the cartesian components of B and D. We have:

$$\mathbf{k}\cdot\mathbf{r} = \frac{2\pi}{L}m_z z. \tag{216}$$

We define the vectors perpendicular to $\mathbf{k}$:

$$\hat{\mathbf{e}}_{1\mathbf{k}} = \hat{\mathbf{x}} \tag{217}$$

$$\hat{\mathbf{e}}_{2\mathbf{k}} = \hat{\mathbf{y}}. \tag{218}$$

Therefore, the circular polarization vectors are:

$$\hat{\mathbf{e}}_{I\mathbf{k}} = -\frac{1}{\sqrt{2}}\left(s_I\hat{\mathbf{e}}_{1\mathbf{k}} + i\hat{\mathbf{e}}_{2\mathbf{k}}\right) = -\frac{1}{\sqrt{2}}\left(s_I\hat{\mathbf{x}} + i\hat{\mathbf{y}}\right), \tag{219}$$

with I to denote either L or R, $s_L = 1$ and $s_R = -1$. Therefore, we have:

$$\mathbf{D}_{I,\mathbf{k}}(\mathbf{r},t) = -\frac{i}{\sqrt{2}}\sqrt{\frac{\hbar cn|\mathbf{k}|\epsilon_0}{2V}}a_{I\mathbf{k}}(t)\left(s_I\hat{\mathbf{x}} + i\hat{\mathbf{y}}\right)e^{i\frac{2\pi}{L}m_z z} + h.c., \tag{220}$$

$$\mathbf{B}_{I,\mathbf{k}}(\mathbf{r},t) = -\frac{1}{\sqrt{2}}\sqrt{\frac{\hbar c|\mathbf{k}|\mu_0}{2nV}}a_{I\mathbf{k}}(t)s_I\left(s_I\hat{\mathbf{x}} + i\hat{\mathbf{y}}\right)e^{i\frac{2\pi}{L}m_z z} + h.c. \tag{221}$$

To obtain the cartesian components of B and D, we take the dot product with $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$, respectively:

$$\mathbf{D}_{I,\mathbf{k}}(x,t) = \hat{\mathbf{x}}\cdot\mathbf{D}_{I,\mathbf{k}}(\mathbf{r},t) = -\frac{i}{\sqrt{2}}\sqrt{\frac{\hbar cn|\mathbf{k}|\epsilon_0}{2V}}a_{I\mathbf{k}}(t)s_I e^{i\frac{2\pi}{L}m_z z} + h.c. \tag{222}$$

$$\mathbf{D}_{I,\mathbf{k}}(y,t) = \hat{\mathbf{y}}\cdot\mathbf{D}_{I,\mathbf{k}}(\mathbf{r},t) = \frac{1}{\sqrt{2}}\sqrt{\frac{\hbar cn|\mathbf{k}|\epsilon_0}{2V}}a_{I\mathbf{k}}(t)e^{i\frac{2\pi}{L}m_z z} + h.c. \tag{223}$$

$$\mathbf{D}_{I,\mathbf{k}}(z,t) = \hat{\mathbf{z}}\cdot\mathbf{D}_{I,\mathbf{k}}(\mathbf{r},t) = 0 \tag{224}$$

$$\mathbf{B}_{I,\mathbf{k}}(x,t) = \hat{\mathbf{x}}\cdot\mathbf{B}_{I,\mathbf{k}}(\mathbf{r},t) = -\frac{1}{\sqrt{2}}\sqrt{\frac{\hbar c|\mathbf{k}|\mu_0}{2nV}}a_{I\mathbf{k}}(t)s_I^2 e^{i\frac{2\pi}{L}m_z z} + h.c. \tag{225}$$

$$\mathbf{B}_{I,\mathbf{k}}(y,t) = \hat{\mathbf{y}}\cdot\mathbf{B}_{I,\mathbf{k}}(\mathbf{r},t) = -\frac{i}{\sqrt{2}}\sqrt{\frac{\hbar c|\mathbf{k}|\mu_0}{2nV}}a_{I\mathbf{k}}(t)s_I e^{i\frac{2\pi}{L}m_z z} + h.c. \tag{226}$$

$$\mathbf{B}_{I,\mathbf{k}}(z,t) = \hat{\mathbf{z}}\cdot\mathbf{B}_{I,\mathbf{k}}(\mathbf{r},t) = 0 \tag{227}$$

B. Verify that B and D are orthogonal.

$$\mathbf{D}_{I,\mathbf{k}}(\mathbf{r},t)\cdot\mathbf{B}_{I,\mathbf{k}}(\mathbf{r},t) = \frac{hc|\mathbf{k}|}{4V}\sqrt{\epsilon_0\mu_0}a_{I\mathbf{k}}^2(t)(is_I^3 - is_I)e^{2i\frac{2\pi}{L}m_z z} + h.c. \tag{228}$$

Since

$$(is_I^3 - is_I) = is_I(s_I^2 - 1) = is_I((\pm1)^2 - 1) = 0, \tag{229}$$

it is verified that B and D are orthogonal.