Coherence and entanglement in a two-qubit system

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Entanglement is a fundamental concept in quantum mechanics. In this review, we study various aspects of coherence and entanglement, illustrated by several examples. We relate the concepts of loss of coherence and disentanglement, via a model of two two-level atoms in different types of reservoir, including cases of both independent and common baths. Finally, we relate decoherence and disentanglement, by focusing on the sudden death of the entanglement and the dependence of the death time with the distance of our initial condition from the decoherence-free subspace. In particular, we study the sudden death of the entanglement in a two-atom system with a common reservoir. © 2010 Optical Society of America

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1. Introduction

Quantum theory, formalized in the 1920's, contains elements which are completely different from the classical description of Nature. In particular, quantum correlations play a fundamental role. This was recognized early on by Einstein, Podolsky, Rosen, and Schrödinger.

During the past decades, these nonclassical states, also referred to as entangled states, have provided us with more than the old paradoxes. As a matter of fact, they may be used to perform tasks that cannot possibly be achieved with their classical counterparts.

In this work, we review, in Section 2, the concepts of entanglement, for both pure and mixed states, entanglement measures, such as the Schmidt decomposition, the Schmidt number, and the entropy of entanglement, for the pure case, and the Peres–Horodecki criteron of positive partial transpose (PPT), the entanglement of formation, and the concurrence for two qubits in a mixed state. We also discuss, at the end of the section, the methods on how to increase the entanglement in a bipartite system, using, for example, the Bennett protocol.

Section 3 is devoted to the important subject of decoherence, starting with a historical introduction; then we refer to the dynamics of decoherence, on how long it takes for an electromagnetic cat to decohere. In the second part of the section, we refer to corners in Hilbert space that are free from decoherence, usually referred to as decoherence-free subspaces (DFSs). We give simple examples of such subspaces, such as collective dephasing, and, towards the end of the section, we give more formal definitions of the DFSs, based on Hamiltonian and Lindblad semigroup approaches.

In Section 4 we study the entanglement evolution. We first discuss a mathematical model, based on a general Markovian master equation, that models a pair of qubits in various types of reservoirs, such as the vacuum, thermal, or squeezed. This general master equation includes both cases of independent and common reservoirs for both atoms. We treat these two cases separately, focusing on the recently observed phenomena of *entanglement sudden death* (ESD).

However, in the case of two qubits having a common reservoir, one encounters some additional features. Besides finding the sudden death effect, we also find a *sudden birth*; that is, some time after the entanglement vanishes in a finite time, it revives again.

A curious and related effect is the delayed sudden birth of entanglement; that is,

with initially unentangled qubits, creation of entanglement is found, where the only coupling of the two qubits is via the common reservoir.

In the common reservoir subsection we discuss an important feature. Coherence and entanglement are closely related phenomena. Decoherence is related to the entanglement of each qubit with the reservoir, and, as a consequence, coherence is diminished or lost from the system. On the other hand, entanglement, that is, the quantum correlations between the two qubits, is also fragile under the effects of the environment, and it is eventually destroyed.

In the common reservoir case, we take an example where one has two qubits in a common squeezed bath, a system for which a two-dimensional DFS exists. Therefore, here we have an interesting system where the connection between decoherence and disentanglement can be made. In fact, we study, for various initial situations, the effect of the distance of our initial condition to the DFS on the sudden death time. In Section 7 we discuss a different model that uses the Lehmberg–Agarwal master equation, which describes the two qubits in a reservoir, with a dipole–dipole interaction. In the particular case of a small separation between the atoms, one obtains the sudden birth of entanglement, qualitatively similar to the common reservoir case in the previous section.

The last section is devoted to discussion and conclusions.

2. Entanglement

Quantum correlations are among the most surprising effects in Quantum Mechanics. It was Schrödinger [1], who coined the term "entanglement" to describe

> *When two systems, of which we know the states by their respective representatives, enter into temporary physical interaction due to known forces between them, and when after a time of mutual influence the systems separate again, then they can no longer be described in the same way as before, viz. by endowing each of them with a representative of its own. I would not call that one but rather the characteristic trait of quantum mechanics, the one that enforces its entire departure from classical lines of thought. By the interaction the two representatives [the quantum states] have become entangled.*

this peculiar connection between quantum systems: This property refers to a quantum system composed of two or more parts or objects, in which the quantum states of these objects are such that each one of them cannot be described without referring to the other one, even in the case when both objects are spatially far apart. This interaction leads to nonclassical correlations for the physical properties of the systems.

Previous to Schrödinger, the concept of quantum entanglement was considered a mere curiosity until it was pointed out by Einstein, Podolsky, and Rosen, in their

famous EPR paper [2], that quantum mechanics may not be a complete theory. We cite Einstein [3]:

> *In the quantum theory, the state of the system is characterized by the ψ function which, in its turn, represents the solution of the Schrödinger equation. Each of these solutions (ψ-functions) has to be regarded within the sense of the theory, as a description of a physically possible state of the system. The question is in what sense does the ψ function describe the state of the system? My assertion is this: the ψ-function cannot be regarded as a complete description of the system, only as an incomplete one. In other words: there are attributes of the individual system whose reality no-one doubts, but which the description by means of the ψ function does not include.*

Almost three decades later, when in 1964 John Bell [4] published his well known theorem, it was hardly noticed by the international community. Bell showed that, for a classical theory based on local hidden variables, certain correlations were upper bounded, known today as the Bell inequalities. This theory is based on the following assumptions:

- Measurement results are determined by the properties the particles carry prior to and independent of the measurements (realism).
- The results obtained at a particular location are independent of any actions performed in locations spatially separated from the first one (locality).

Bell proved that the above assumptions impose constraints on the statistical correlations in experiments involving bipartite systems, in the form of the well-known Bell inequalities. Quantum mechanics predicts that these inequalities can be violated, thus giving rise to possible experimental tests that could decide either in favor of the EPR argument or of quantum mechanics.

Experiments agree with quantum mechanics [5], although some researchers still find loopholes in the experimental assumptions [6,7]. More recently, entanglement has become an essential resource for various applications of quantum mechanics, such as quantum teleportation [8], quantum cryptography [9,10], and superdense coding [11]. Thus, at present, quantum entanglement is not only a subject of philosophical debate but a key resource for important applications, such as the ones mentioned above.

2.1. Entanglement of Pure States

The simplest quantum mechanical system is the *qubit*. A qubit is the fundamental concept of quantum computation and information, and it is the quantum analog of the *bit* of classical information. The qubits have a two-dimensional state space. Suppose that $|0\rangle$ and $|1\rangle$ form an orthonormal basis for that state space;

then an arbitrary state vector in the state space can be written as $a|0\rangle + b|1\rangle$, where a and b are complex numbers.

For quantum computation purposes, it is necessary give some qubits a robust physical representation. An attractive physical representation for qubits are two-level atoms, or optical photons, among others.

Consider a system A in a state $|\psi\rangle_A$, and a second system B in state $|\psi\rangle_B$, with respective Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$. The Hilbert space of the composite system is given by $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$. Traditionally these systems are associated with two distant observers called Alice and Bob.

If a pure state $|\psi\rangle \in H_A \otimes H_B$ can be written as a product state

$$|\psi\rangle = |\psi\rangle_A \otimes |\phi\rangle_B, \tag{1}$$

it is said to be separable. Otherwise we say that $|\psi\rangle$ is an *entangled state*.

Example 1.

- **Separable State.** Let us consider the state $|1,0\rangle \in \mathcal{H}$. We can easily see that system A is in the state $|1\rangle \in \mathcal{H}_A$, and system B is in the state $|0\rangle \in \mathcal{H}_B$. Thus, the state $|1,0\rangle = |1\rangle_A \otimes |0\rangle_B$ is separable, and therefore nonentangled.
- **Nonseparable State.** A composite quantum state with two parts is called a bipartite state, and a multipartite state consists of more than two parts. For example, given two basis vectors $\{|0\rangle_A, |1\rangle_A\} \in \mathcal{H}_A$ and two basis vectors $\{|0\rangle_B, |1\rangle_B\} \in \mathcal{H}_B$, the state

$$\frac{1}{\sqrt{2}}(|0\rangle_A \otimes |1\rangle_B - |1\rangle_A \otimes |0\rangle_B) \tag{2}$$

is entangled, since it is impossible to attribute to either system A or system B a definite pure state. The most important bipartite entangled states are

$$|\Phi^{\pm}\rangle \equiv \frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle), \tag{3}$$

$$|\Psi^{\pm}\rangle \equiv \frac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle). \tag{4}$$

These four states are called *Bell states* and form an orthonormal basis for the sate space of the two qubits, called the *Bell basis*. The Bell states are maximally entangled, and one can be converted into another by applying a unitary transformation on any of the subsystems. In the above example we have used $|ij\rangle$ as a shorthand notation for $|i\rangle \otimes |j\rangle$. This notation will be used throughout this review.

Example 2. In this second example, suppose that we know that the two particles A and B are photons coming from a two-photon source, with opposite polarization, horizontal H and vertical V, without us knowing which is which. This situation can be described in a quasi-classical and in a quantum mechanical way:

$$\rho_C = \frac{1}{2}(|H_A V_B\rangle\langle H_A V_B|) + \frac{1}{2}(|V_A H_B\rangle\langle V_A H_B|),$$

$$\rho_{QM} = |\psi^+\rangle\langle\psi^+|, \tag{5}$$

with

$$|\psi^+\rangle = \frac{1}{\sqrt{2}}(|H_A V_B\rangle + |V_A H_B\rangle).$$

If we observe, for example, an H polarization for photon A, expressed as

$$\frac{Tr_A\{\rho|H_A\rangle\langle H_A|\}}{Tr\{\rho|H_A\rangle\langle H_A|\}} = |V_B\rangle\langle V_B|, \tag{6}$$

this gives certain V polarization of photon B in both cases. As we shall see very soon, ρ_C is not entangled, while ρ_{QM} is.

To appreciate the difference between the two cases, we write these two density matrices as rows and columns ordered as HH, HV, VH, VV. The result is

$$\rho_C = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \rho_{QM} = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{7}$$

We notice that the nonzero off-diagonal elements, or coherences, are present in ρ_{QM} only in the mixed positions $HV-VH$ and $VH-HV$. This fact, however, has no consequence for the particles individually; thus, for both cases, the reduced density matrix, say ρ_A, yields

$$(\rho_C)_A = (\rho_{QM})_A = \frac{1}{2}I, \tag{8}$$

ρ being the identity matrix [12].

As we can see, the off-diagonal elements of ρ_{QM} do not have an effect in individual particles, and therefore we get the same reduced density matrix. The difference between the two cases shows up when doing violations of Bells inequalities, which are purely quantum effects [13,14].

2.1a. Schmidt Decomposition

An interesting tool when we are working with bipartite pure state entanglement, i.e., system A and system B, is the Schmidt decomposition, which is a diagonal decomposition in a biorthogonal basis [14,15].

If we have a pure state $|\psi_{AB}\rangle$, we can write it in terms of some product basis, orthonormal in both systems,

$$|\psi_{AB}\rangle = \sum_{i=1}^{d_A} \sum_{\mu=1}^{d_B} M(i,\mu)|i\rangle \otimes |\mu\rangle, \tag{9}$$

where $M(i,\mu)$ are the elements in a $d_A \times d_B$ matrix M:

$$M = \sum_{i=1}^{d_A} \sum_{\mu=1}^{d_B} M(i,\mu)|i\rangle\langle\mu|. \tag{10}$$

One can always diagonalize M with a couple of unitary matrices $U(d_A \times d_A)$ and $V(d_B \times d_B)$ such that M can now be written as $M = UDV$, with D being a $d_A \times d_B$ matrix whose elements are zero except for the diagonal elements, which are real and positive. Thus, M can be written as

$$M(i,\mu) = \sum_{k=1}^{\min(d_A,d_B)} u_{ik}d_{kk}v_{k\mu}, \tag{11}$$

thus yielding for the state $|\psi_{AB}\rangle$

$$|\psi_{AB}\rangle = \sum_{i=1}^{d_A} \sum_{\mu=1}^{d_B} \sum_{k=1}^{\min(d_A,d_B)} u_{ik}d_{kk}v_{k\mu}|i\rangle \otimes |\mu\rangle$$

$$= \sum_{k=1}^{\min(d_A,d_B)} d_{kk}\left(\sum_{i=1}^{d_A} u_{ik}|i\rangle\right) \otimes \left(\sum_{\mu=1}^{d_B} v_{k\mu}|\mu\rangle\right). \tag{12}$$

We now define a new couple of basis vectors

$$|a_k\rangle = \left(\sum_{i=1}^{d_A} u_{ik}|i\rangle\right),$$

$$|b_k\rangle = \left(\sum_{\mu=1}^{d_B} v_{k\mu}|\mu\rangle\right), \tag{13}$$

and we have the Schmidt decomposition

$$|\psi_{AB}\rangle = \sum_{k=1}^{\min(d_A,d_B)} d_{kk}|a_k\rangle \otimes |b_k\rangle. \tag{14}$$

What we have achieved is to write any state as a linear combination of $\min(d_A, d_B)$ product of biorthogonal basis vectors.

We note that the original $|\psi_{AB}\rangle$ was written in terms of a double sum, while the present diagonal form has been reduced to a single sum.

In terms of the density operator, one can write

$$\rho = \sum_{i,k} \sqrt{\lambda_i\lambda_k}|a_i\rangle\langle a_k| \otimes |b_i\rangle\langle b_k|, \tag{15}$$

where $d_{kk} \equiv \sqrt{\lambda_k}$.

It is simple to see that the reduced density matrices are now diagonal,

$$\rho_A = \sum_k \lambda_k|a_k\rangle\langle a_k|,$$

$$\rho_B = \sum_k \lambda_k |b_k\rangle\langle b_k|, \tag{16}$$

because the λ_k are eigenvalues of the reduced density matrices and they obey the following conditions:

$$0 \le \lambda_k \le 1 \text{ and } \sum_k \lambda_k = 1. \tag{17}$$

In the case of a product state, there is only one Schmidt term different from 0, and the corresponding eigenvalue is 1. Conversely, if we have a state with only one Schmidt coefficient, it must be a product state.

Thus, $|\psi_{AB}\rangle$ is a product state if and only if the corresponding reduced density matrices correspond to pure states. This implies that if we have an entangled state, the corresponding reduced density operators must correspond to a mixed state, with more than one Schmidt eigenvalue different from zero. We see that entanglement of a state is directly related to the mixedness of the reduced density operator.

A simple way of measuring the degree of entanglement in a two-partite pure state is via the Schmidt number, defined as the reciprocal of the purity of the reduced density matrix [16]:

$$K = \frac{1}{Tr_A \rho_A^2} = \frac{1}{Tr_B \rho_B^2} = \frac{1}{\sum_n \lambda_n^2}. \tag{18}$$

If only one eigenvalue is 1, and the rest are 0, then $K=1$, and we have a product state. On the other hand, if all the λ_s are equal, implying that all the N terms are equally important in the Schmidt decomposition, then $\lambda_s = 1/N$ and $K=N$.

So, if D is the dimension of the space, then

$$1 \le K \le D. \tag{19}$$

2.1b. Measure of Entanglement, Entropy of Entanglement

One of the most popular measure of mixedness of the density operator is the von Neumann entropy $S(\rho) = -tr(\rho \log_2 \rho)$. For a pure state, this entropy vanishes, and for a maximally mixed state, gives $\log_2 d$, d being the dimension of the Hilbert space.

The entropy is a convex function; that is, for $p \in [0,1]$,

$$S[p\rho_1 + (1-p)\rho_2] \ge pS(\rho_1) + (1-p)S(\rho_2),$$

which implies that it always increases by further mixing. This motivates the next definition.

Given a state $|\psi\rangle$, we define the *entropy of entanglement* $E(\psi)$ as the von Neumann entropy of the reduced density operator. So, using the above discussion,

$$E(\psi) = S(\rho_A) = S(\rho_B) = -\sum_{k=1}^{d} \lambda_k \log_2(\lambda_k). \qquad (20)$$

Thus, once more we see that the more mixed the reduced density operator is, the more entangled the original state is.

As we will see in the next section, this definition is valid only for pure states. For mixed states, the quantification of entanglement becomes more complex.

The entropy of entanglement depends on the Schmidt coefficients, and it is independent of the basis. This means that it is invariant under local unitary operations; in other words, if $|\psi'\rangle = U_A \otimes U_B |\psi\rangle$, then $E(\psi') = E(\psi)$.

Similarly, it is possible to show that the entropy of entanglement cannot increase, on the average, by local operations [17]. That is, if we perform independent measurements in A and B and obtain $|\psi_k\rangle$ with probability p_k, then we have that

$$E(\psi) \geq \sum_k p_k E(\psi_k).$$

We notice, however, that the above inequality does not imply that none of the $E(\psi_k)$ can be larger than $E(\psi)$, which is an interesting conclusion that will be used in the process of distillation.

2.2. Entanglement of Mixed States

The states considered in the previous subsection are idealized ones, since real physical systems always interact with some environment. Thus, a mixed state may arise when an initially pure state interacts, intentionally or unexpectedly, with this environment (other quantum degrees of freedom), resulting in a non-unitary evolution of the pure state into a mixed state. For a detailed discussion of this effect, we refer the reader to Section 3, on decoherence.

Though there are several criteria for the separability of mixed states [18], the theory of mixed state entanglement is more complicated and less well understood than that of pure state entanglement.

A mixed state is called *separable* if it can be prepared by the two parties in a "classical way," which means agreeing by direct communication on the local preparation of states. The corresponding density matrix of a separable state should have only classical correlations, or mathematically should be of the form [19]

$$\rho = \sum_i p_i |a_i\rangle\langle a_i| \otimes |b_i\rangle\langle b_i|; \qquad (21)$$

otherwise it is *entangled*.

Here, the coefficients p_i are probabilities with $0 \leq p_i \leq 1$, and $\Sigma_i p_i = 1$. This decomposition is not unique.

Example 3. An example of a mixed state that has classical but not quantum correlations is $\rho = \frac{1}{2}(|00\rangle\langle00| + |11\rangle\langle11|)$. Another example is the state $\rho = \frac{1}{2}(|\phi^+\rangle\langle\phi^+| + |\phi^-\rangle\langle\phi^-|)$, which is separable because it can be written as the previous example, as can be seen from the definitions of the Bell states. However, we should warn the reader that a criterion for separability like Eq. (21) is not easy to use. Or, in other words, finding for ρ a form like this or proving that it does not exist is not a simple

task. Therefore, we must find a simpler criterion to detect entanglement.

2.2a. Peres–Horodecki Criterion. Positive Partial Transpose

The Peres–Horodecki criterion is a necessary condition for a joint density matrix ρ, of two systems A and B, to be separable [20,21]. It is also called the PPT criterion (for positive partial transpose). In the 2×2 and 2×3 dimensional cases, it is also sufficient.

The partial transpose of a composite density matrix is given by transposing only one of the two subsystems. For example, the partial transposition with respect to Alice is

$$\rho^{T_A}_{m\mu,n\nu} = \rho_{n\mu,m\nu}, \tag{22}$$

where we are using Latin subindices for the Alice subsystem and Greek for Bob.

Thus, for any separable state one can write the partial transpose as

$$\rho^{T_A}_{\text{sep}} = \sum_i p_i (|a_i\rangle\langle a_i|)^T \otimes |b_i\rangle\langle b_i|. \tag{23}$$

Since the $(|a_i\rangle\langle a_i|)^T$ are again valid density matrices for Alice, one immediately finds that $\rho^{T_A}_{sep} \geq 0$ implying nonnegative eigenvalues.

The same holds for the partial transposition with respect to Bob. In summary, the partial transpose of a separable density matrix is positive. This means that it has only positive nonvanishing eigenvalues (or equivalently, a positive operator has a positive or zero expectation value with *any* state).

For the converse, that is, if $\rho^{T_A} \geq 0$, then ρ is separable only for low-dimensional systems, namely, for composite 2×2 and 2×3 systems. In these cases, the positivity of the partial transpose (PPT) is a necessary and sufficient condition for separability. For higher dimensions, the PPT condition is only necessary.

The *partial transposition criterion* for detecting entanglement is simple: given a bipartite state ρ_{AB}, find the eigenvalues of any of its partial transpositions. A negative eigenvalue immediately implies that the state is entangled. Examples of such states include the singlet state.

The partial transposition criterion allows us to detect in a simple way all entangled states that are negative partial transpose (NPT) density matrices with at least a negative eigenvalue, which is a large class of states. However, in higher dimensions, there are PPT states that are not separable, called "bound entangled states" [22,23].

2.2b. Entanglement of Formation, Concurrence

The *entanglement of formation* was originally proposed by Bennett *et al.* in 1996 [24], and it is a direct generalization of entropy of entanglement applied to mixed states.

For a given ensemble of pure states $\{p_i, |\psi_i\rangle\}$, the entanglement of formation is the average of the entropy of entanglement for the states of the ensemble

$$E(\rho) = \sum_i p_i E(|\psi_i\rangle).$$

A mixed state can be realized by a large number of pure state ensembles, with different entanglement of formation.

Thus, the natural definition of the entanglement of formation for a mixed state ρ is the average entropy of entanglement over a set of states that minimizes this average over all possible decompositions of ρ:

$$E(\rho) = \min \sum_i p_i E(\psi_i),$$

where the entanglement $E(\psi)$ is defined as the entropy of either of two subsystems A and B, i.e.,

$$E(\psi) = -Tr(\rho_A \log \rho_A) = -Tr(\rho_B \log \rho_B). \tag{24}$$

Here ρ_A is the reduced density matrix obtained by partial trace of $|\psi\rangle\langle\psi|$ over subsystem B, and similarly for ρ_B.

A concept closely related to the entanglement of formation is the concurrence [25,26].

For a general mixed state ρ_{AB} of two qubits, we define $\tilde\rho$ to be the spin-flipped state

$$\tilde\rho_{AB} = (\sigma_y \otimes \sigma_y)\rho_{AB}^*(\sigma_y \otimes \sigma_y), \tag{25}$$

where ρ^* is the complex conjugate of ρ and σ_y is the Pauli matrix.

The concurrence is defined as

$$C(\rho) = \max\{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\}, \tag{26}$$

where $\{\lambda_i\}$ are the square roots, in decreasing order of the eigenvalues of the non-Hermitian matrix $\rho\tilde\rho$.

For separable qubits $C=0$, and for maximally entangled ones $C=1$.

In the particular case of a pure state, $C(\Psi) = |\langle\Psi|\tilde\Psi\rangle|$, where we define the spin-flipped state $|\tilde\Psi\rangle_{AB} = (\sigma_y^A \otimes \sigma_y^B)|\Psi^*\rangle_{AB}$, $|\Psi^*\rangle_{AB}$ is the complex conjugate of $|\Psi\rangle_{AB}$, σ_y^i, and $i=(A,B)$ is the Pauli operator, both taken in the standard basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$.

The concurrence has the great advantage that is easily computable and is directly related to the entanglement of formation, providing an explicit formula for the entanglement of formation in the case of two qubits.

The entanglement is given by

$$E(\rho_{AB}) = \mathcal{E}(C(\rho_{AB})), \tag{27}$$

with

$$\mathcal{E}(C) = H\left[\frac{1}{2} + \frac{1}{2}\sqrt{1 - C^2}\right],$$

$$H(x) = -x\log_2 x - (1-x)\log_2(1-x).$$

A common initial condition is represented by X matrices. In the standard two-qubit product basis $\{|11\rangle_{AB}, |10\rangle_{AB}, |01\rangle_{AB}, |00\rangle_{AB}\}$ an X matrix is

$$\rho(0) = \begin{pmatrix} \rho_{11} & 0 & 0 & \rho_{14} \\ 0 & \rho_{22} & \rho_{23} & 0 \\ 0 & \rho_{32} & \rho_{33} & 0 \\ \rho_{41} & 0 & 0 & \rho_{44} \end{pmatrix}, \tag{28}$$

where $\rho_{11} + \rho_{22} + \rho_{33} + \rho_{44} = 1$.

The concurrence for a state of the form given by Eq. (28) can be easily computed as

$$C = 2 \max\{0, \sqrt{\rho_{23}\rho_{32}} - \sqrt{\rho_{11}\rho_{44}}, \sqrt{\rho_{14}\rho_{41}} - \sqrt{\rho_{22}\rho_{33}}\}. \tag{29}$$

2.3. Entanglement Distillation

As mentioned above, in practice, one has at one's disposal not pure, but mixed states. For example, if we want to perform an experiment on teleportation of the polarization state of a photon, over a rather long distance, even if we start from a pure state, at the end of the process, there will be a considerable entanglement with the environment, and therefore the state will invariably be mixed. However, there is a way one can make the state more pure and even more entangled.

The central idea is to use several copies of the state. Using local operations and classical communication, it is possible to sacrifice some of the copies and get fewer states that may be nearly maximally entangled. This process we call entanglement distillation.

2.3a. One-Particle Purification

We consider a collection of qubits, in a mixture of the form

$$\rho = f|0\rangle\langle 0| + (1 - f)|1\rangle\langle 1|,$$

where f denotes the entanglement fidelity of the ensemble. We want to purify this state, that is, to increase f. We could, for example, directly measure the spin, say, by having a Stern Gerlach apparatus with an inhomogeneous magnetic field and selecting one particular path, say, the one with spin down. In this way, we could think of the particles as having been purified to the desired pure state. If we measure, we do get $|0\rangle\langle 0|$, but we have destroyed the qubit in the process.

A better idea is to have two copies of the state. The first copy we call the source, and the second, the target:

$$\rho = (f|0\rangle\langle 0| + (1 - f)|1\rangle\langle 1|)_S \otimes (f|0\rangle\langle 0| + (1 - f)|1\rangle\langle 1|)_T,$$

and now we apply a C-NOT (controlled NOT) operation,

$$\text{C-NOT}(|0\rangle\langle 0|)_C(|0\rangle\langle 0|)_T \Rightarrow (|0\rangle\langle 0|)_C(|0\rangle\langle 0|)_T,$$

$$\text{C-NOT}(|1\rangle\langle 1|)_C(|0\rangle\langle 0|)_T \Rightarrow (|1\rangle\langle 1|)_C(|1\rangle\langle 1|)_T.$$

Notice that in the C-NOT operation the target changes only when the control is in the $|1\rangle\langle 1|$ state, and nothing happens otherwise.

As a result of the above operation on the control–target system, we get

$$\rho = (f^2|0\rangle\langle 0| + (1-f)^2|1\rangle\langle 1|)_S \otimes |0\rangle\langle 0|_T + f(1-f)(|0\rangle\langle 0| + |1\rangle\langle 1|)_S \otimes |1\rangle\langle 1|_T.$$

If we now measure on the target qubit and find, for example, the $|0\rangle$ state, then we obtain the ensemble of entangled states, with a renormalized density matrix

$$\rho' = \frac{f^2}{f^2+(1-f)^2}|0\rangle\langle 0| + \frac{(1-f)^2}{f^2+(1-f)^2}|1\rangle\langle 1| = f'|0\rangle\langle 0| + (1-f')|1\rangle\langle 1|,$$

where $f' = f^2/f^2 + (1-f)^2$.

It is evident, from Fig. 1, that for $f > \frac{1}{2}$, the remaining control qubits (those for which the target qubit was measured in the state $|0\rangle$) form a purified ensemble, since $f' > f$, and therefore there is an increased fraction of particles in the state $|0\rangle$.

If we iterate this process, the ensemble can be distilled arbitrarily close to a pure state $\|0\rangle$; however, we would require an infinite number of copies.

After the nth iteration, the fidelity will be
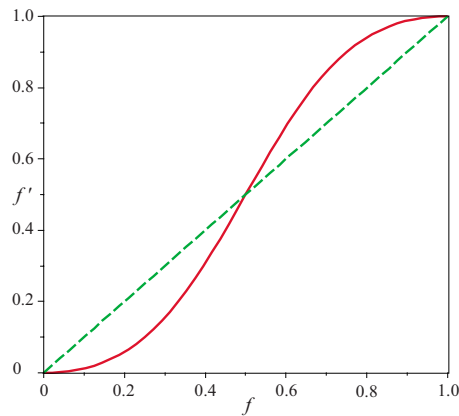
$$f_{n+1} = \frac{f_n^2}{f_n^2+(1-f_n)^2}.$$

2.3b. Two-Particle Purification

Let us consider the two-qubit Werner states defined as

$$\rho_G = G|\psi^-\rangle\langle\psi^-| + \frac{(1-G)}{3}[|\psi^+\rangle\langle\psi^+| + |\phi^-\rangle\langle\phi^-| + |\phi^+\rangle\langle\phi^+|], \tag{30}$$

and we assume that both Bob and Alice share two states like those in Eq. (30), so that the complete density matrix is $\rho_G^S \otimes \rho_G^T$. We let A_1, A_2 and B_1, B_2 denote the

Figure 1

Fidelity after an iteration of the purification algorithm. Note that $f' > f$ for $f > \frac{1}{2}$.

two particles (source and target) of Alice and Bob, respectively.

In the distillation procedure, we proceed as follows: first, Alice applies a unitary transformation σ_y to her qubits. The net effect is to transform $\psi^\pm \leftrightarrow \phi^\mp$; so, after the rotation, we get

$$\rho'_G = G|\phi^+\rangle\langle\phi^+| + \frac{(1-G)}{3}[|\phi^-\rangle\langle\phi^-| + |\psi^+\rangle\langle\psi^+| + |\psi^-\rangle\langle\psi^-|].$$

Next, we apply the C-NOT operation to their two particles, where A_1 and B_1 act as sources and A_2 and B_2 as targets.

The C-NOT operation, in this case, acts as follows:

$$|0\rangle_{A_1}|0\rangle_{A_2} \rightarrow |0\rangle_{A_1}|0\rangle_{A_2},$$

$$|0\rangle_{A_1}|1\rangle_{A_2} \rightarrow |0\rangle_{A_1}|1\rangle_{A_2},$$

$$|1\rangle_{A_1}|0\rangle_{A_2} \rightarrow |1\rangle_{A_1}|1\rangle_{A_2},$$

$$|1\rangle_{A_1}|1\rangle_{A_2} \rightarrow |1\rangle_{A_1}|0\rangle_{A_2},$$

and the same for Bob. After the bilateral C-NOT operation, the states of the particles are transformed as shown in Table 1.

The density matrix (source and target), after the bilateral C-NOT gate, reads as

$$\rho''_T = F_1|\phi^+\rangle_T\langle\phi^+| + F_2|\phi^-\rangle_T\langle\phi^-| + F_3|\psi^+\rangle_T\langle\psi^+| + F_4|\psi^-\rangle_T\langle\psi^-|,$$
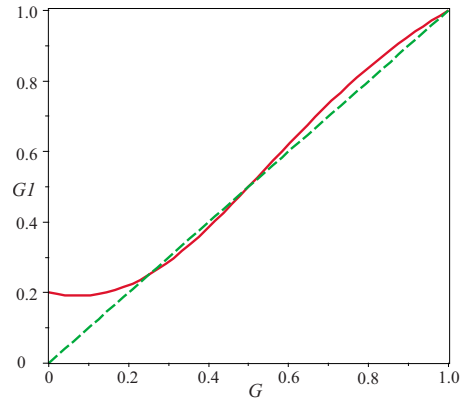
where

$$F_1 = G^2|\phi^+\rangle_C\langle\phi^+| + \frac{G(1-G)}{3}|\phi^-\rangle_C\langle\phi^-| + \frac{(1-G)^2}{9}|\psi^+\rangle_C\langle\psi+|$$

$$+ \frac{(1-G)^2}{9}|\psi^-\rangle_C\langle\psi^-|,$$

Table 1. Effect of Bilateral C-NOT Gate on Input Source and Target, Yielding Final States

Initial Source	Initial Target	Final Source	Final Target
$\phi^\pm$	ϕ^+	$\phi^\pm$	ϕ^+
$\phi^\pm$	ϕ^-	$\phi^\mp$	ϕ^-
$\psi^\pm$	ψ^+	$\psi^\pm$	ϕ^+
$\psi^\pm$	ψ^-	$\psi^\mp$	ϕ^-
$\phi^\pm$	ψ^+	$\phi^\pm$	ψ^+
$\phi^\pm$	ψ^-	$\phi^\mp$	ψ^-
$\psi^\pm$	ϕ^+	$\psi^\pm$	ψ^+
$\psi^\pm$	ϕ^-	$\psi^\mp$	ψ^-

Figure 2

Fidelity after one iteration of Bennett's algorithm. Note that $G_1 > G$ for $G > \frac{1}{2}$.

$$F_2 = \frac{(1-G)^2}{9}|\phi^+\rangle_C\langle\phi^+| + \frac{G(1-G)}{3}|\phi^-\rangle_C\langle\phi^-| + \frac{(1-G)^2}{9}|\psi^+\rangle_C\langle\psi+|$$

$$+ \frac{(1-G)^2}{9}|\psi^-\rangle_C\langle\psi^-|,$$

$$F_3 = \frac{(1-G)^2}{9}|\phi^+\rangle_C\langle\phi^+| + \frac{G(1-G)}{3}|\phi^-\rangle_C\langle\phi^-| + \frac{(1-G)^2}{9}|\psi^+\rangle_C\langle\psi^+|$$

$$+ \frac{(1-G)^2}{9}|\psi^-\rangle_C\langle\psi^-|,$$

$$F_4 = \frac{(1-G)^2}{9}|\phi^+\rangle_C\langle\phi^+| + \frac{G(1-G)}{3}|\phi^-\rangle_C\langle\phi^-| + \frac{(1-G)^2}{9}|\psi^+\rangle_C\langle\psi+|$$

$$+ \frac{(1-G)^2}{9}|\psi^-\rangle_C\langle\psi^-|.$$

Finally, the target pairs are measured in the $|0\rangle, |1\rangle$ basis. If they measure the same result, we keep the source particles; otherwise, the source particles are discarded. So, after a successful measurement, we get $F_1 + F_2$, which after renormalization becomes

$$\rho_C = G_1|\phi^+\rangle_C\langle\phi^+| + G_2|\phi^-\rangle_C\langle\phi^-| + G_3|\psi^+\rangle_C\langle\psi^+| + G_4|\psi^-\rangle_C\langle\psi^-|,$$

with

$$G_1 = \frac{G^2 + \dfrac{(1-G)^2}{9}}{G^2 + \dfrac{2G(1-G)}{3} + \dfrac{5(1-G)^2}{9}}.$$

To finish the process, we again perform a unilateral σ_y rotation that finally leaves the state as

$$\rho_C^{\text{final}} = G_1|\psi^-\rangle_C\langle\psi^-| + G_2|\psi^+\rangle_C\langle\psi^+| + G_3|\phi^-\rangle_C\langle\phi^-| + G_4|\phi^+\rangle_C\langle\phi^+|.$$

The new fidelity is plotted against the old one in Fig. 2. We can see that if we have an initial fidelity $G > \frac{1}{2}$, then $G_1 > G$; then repeated iteration can lead to a high fidelity, sacrificing many qubits along the process [10,27].

2.4. Continuous Entanglement Distillation

In the previous subsection, we dealt with distillation in the static cases. It is possible to push this idea further and apply, for example, in a lossy system governed by a Markovian master equation, the Bennett or other protocols at the beginning of each small time interval during the dynamical evolution of the system. This is a method of protecting the quantum state of a system against the effects of the environment [28,29].

3. Decoherence

Decoherence is the irreversible formation of quantum correlations of a system with the environment. These correlations show up in completely new properties as compared with the isolated system. When we consider observations related to only one of the two subsystems, system–environment, we immediately notice several consequences of this entanglement.

The first and most important effect is that the system no longer obeys Schrödinger's equation. The dynamics of our system becomes complicated, and in some approximations, it leads to a master equation for the density matrix of our system.

If we start from a pure state, the system has become mixed by the interaction with the reservoir. To put it in a different way, in an open system, **the coherence leaks out of the system into the environment, and as a result, we have decoherence** [30–32].

If the coupling to the environment becomes very strong, the internal dynamics of the system slows down or may even be frozen, an effect known as the quantum Zeno effect.

On the other hand, the detailed dynamics depends on the type of interaction between the system and the environment. In many cases, this coupling is quite similar to a measurement process. Thus, it is appropriate to discuss here some elements of quantum theory of measurement.

Niels Bohr [33] proposed that, according to the Copenhagen interpretation of quantum mechanics, a classical apparatus was necessary to carry out the measurements, thus implying **a sharp borderline between the classical and the quantum world**. Traditionally, the classical systems are associated with the macroscopic and quantum with the microscopic [30], but this distinction is actually not very adequate considering recently studied effects of macroscopic systems that behave quantum mechanically. We also have the nonclassical squeezed states with a large number of photons, etc.

As opposed to Bohr, von Neumann [34] considered **quantum measurements**.

We take, now, a couple of states of our system, $|s_1\rangle$ and $|s_2\rangle$, and a meter that can be in states $|d_1\rangle$ and $|d_2\rangle$. If the detector is initially in the $|d_2\rangle$ state, we assume that it switches when the system is in the $|s_1\rangle$ state and does not change if the system is in the $|s_2\rangle$ state; that is,

$$|s_1\rangle|d_2\rangle \rightarrow |s_1\rangle|d_1\rangle,$$

$$|s_2\rangle|d_2\rangle \rightarrow |s_2\rangle|d_2\rangle. \tag{31}$$

If, on the other hand, we assume that the system is in a superposition state

$$|\psi_{\text{initial}}\rangle = \alpha|s_1\rangle + \beta|s_2\rangle, \tag{32}$$

with $|\alpha|^2 + |\beta|^2 = 1$, then

$$|\psi_{\text{initial}}\rangle = (\alpha|s_1\rangle + \beta|s_2\rangle)|d_2\rangle \overset{\text{after meas}}{\Rightarrow} \alpha|s_1\rangle|d_1\rangle + \beta|s_2\rangle|d_2\rangle \equiv |\Psi^c\rangle, \tag{33}$$

where the state $|\Psi^c\rangle$ is a correlated one, and this process can be achieved, as we will see soon, just with Schrödinger's equation, with an appropriate interaction.

Thus, if the detector is in the $|d_1\rangle$ state, one can be certain that the system is in the $|s_1\rangle$ state. However, we are ignorant about the quantum state of the system, and it is more realistic to approach the system in a statistical way, with the density matrix.

According to von Neumann, besides the unitary evolution that rules the dynamics of the quantum phenomena, there is also a **non-unitary reduction of the wavefunction** $|\Psi^c\rangle$ that takes the pure state density matrix $|\Psi^c\rangle\langle\Psi^c|$ and converts it into a mixed state by eliminating the off-diagonal elements,

$$\rho^c = |\Psi^c\rangle\langle\Psi^c| = |\alpha|^2|s_1\rangle\langle s_1||d_1\rangle\langle d_1| + |\beta|^2|s_2\rangle\langle s_2||d_2\rangle\langle d_2| + \alpha^*\beta|s_2\rangle\langle s_1||d_2\rangle\langle d_1|$$

$$+ \alpha\beta^*|s_1\rangle\langle s_2||d_1\rangle\langle d_2|$$

$$\overset{\text{Non}}{\text{unitary}} \Rightarrow \rho^r = |\alpha|^2|s_1\rangle\langle s_1||d_1\rangle\langle d_1| + |\beta|^2|s_2\rangle\langle s_2||d_2\rangle\langle d_2|. \tag{34}$$

The difference between the original ρ^c and the after-the-measurement reduced density matrix ρ^r is that, because in the latter case the off-diagonal elements are missing, one could safely describe the system with alternative states ruled by classical probabilities $|\alpha|^2$ and $|\beta|^2$.

On the other hand, in the quantum case (ρ^c), things are more complicated, because we may use a different basis, say

$$|s_3\rangle = \frac{1}{\sqrt{2}}(|s_1\rangle + |s_2\rangle),$$

$$|s_4\rangle = \frac{1}{\sqrt{2}}(|s_1\rangle - |s_2\rangle), \tag{35}$$

and, choosing $\alpha = -\beta = 1/\sqrt{2}$, we write

$$|\Psi^c\rangle = \frac{1}{\sqrt{2}}(|s_1\rangle|d_1\rangle - |s_2\rangle|d_2\rangle) = \frac{1}{\sqrt{2}}\left[\frac{1}{\sqrt{2}}(|s_3\rangle + |s_4\rangle)|d_1\rangle - \frac{1}{\sqrt{2}}(|s_3\rangle - |s_4\rangle)|d_2\rangle\right]$$

$$= \frac{1}{\sqrt{2}}[|s_3\rangle|d_3\rangle + |s_4\rangle|d_4\rangle], \tag{36}$$

where

$$|d_3\rangle = \frac{1}{\sqrt{2}}(|d_1\rangle - |d_2\rangle),$$

$$|d_4\rangle = \frac{1}{\sqrt{2}}(|d_1\rangle + |d_2\rangle). \tag{37}$$

We see that the diagonal elements of ρ^c give us different alternatives. In the first basis

$$(\rho^c)_{\text{diag}} = \frac{1}{2}|s_1\rangle\langle s_1||d_1\rangle\langle d_1| + \frac{1}{2}|s_2\rangle\langle s_2||d_2\rangle\langle d_2|, \tag{38}$$

while in the second basis

$$(\rho^c)_{\text{diag}} = \frac{1}{2}|s_3\rangle\langle s_3||d_3\rangle\langle d_3| + \frac{1}{2}|s_4\rangle\langle s_4||d_4\rangle\langle d_4|. \tag{39}$$

Here, we put in evidence, once more, that the problem is precisely our ignorance about the quantum state of our system.

Now, as we mentioned before, the first step of the measurement is to obtain the correlated wavefunction $|\Psi^c\rangle$, which can be achieved via a unitary operator. The second step, however, was the **von Neumann** nonunitary reduction. Can this step be achieved in a different way? Perhaps, by another unitary operator?

The answer to this question is, "yes" [30], and the way to do it is by coupling the system–detector pair to the environment, to dispose of the extra information.

We call the environment states $|\epsilon\rangle$. Then

$$|\Psi^c\rangle|\epsilon_0\rangle = [\alpha|s_1\rangle|d_1\rangle + \beta|s_2\rangle|d_2\rangle]|\epsilon_0\rangle \rightarrow (\alpha|s_1\rangle|d_1\rangle|\epsilon_1\rangle + \beta|s_2\rangle|d_2\rangle|\epsilon_2\rangle) = |\psi\rangle, \tag{40}$$

where the correlation has been extended from the system–detector to system–detector–environment, getting a chain of states.

If the environment states $|\epsilon_1\rangle$ and $|\epsilon_2\rangle$, corresponding to the detector states $|d_1\rangle$ and $|d_2\rangle$, respectively, are orthogonal, then we can trace (average) over the environment variables

$$\rho_{SD} = Tr_\epsilon|\psi\rangle\langle\psi| = \sum_i \langle\epsilon_i|\psi\rangle\langle\psi|\epsilon_i\rangle = \rho^r, \tag{41}$$

getting precisely the von Neumann reduced density matrix, but this time by only unitary transformations, without *ad hoc* assumptions.

3.1. Mechanism of Decoherence

In this subsection we will discuss the interaction between the "local" object, the system of interest, which we may call the "system," and a second system, which may be the environment. Since the interaction in many cases is strongly similar to a measurement, we will deal with the von Neumann measurement model [35–37].

To model the measurement process, an interaction that is diagonal in the eigenstates of the measured observable, and we must also be able to move the state of the measurement apparatus (pointer state) [38].

We assume that the system is coupled to the environment (apparatus) by a Hamiltonian of the following form:

$$H_{int} = \hbar \sum_n |m\rangle\langle m| \mathbf{D}_m, \tag{42}$$

where $\mathbf{D}_m$ are m-dependent operators acting on the Hilbert space of the environment (apparatus) and $|m\rangle$ is an eigenstate of a system observable to be measured.

The environment (apparatus) acquires the information about the state $|m\rangle$, in the sense that changes according to

$$|m\rangle|\phi_0\rangle \xrightarrow{t} \exp\left(-i\frac{H_{int}}{\hbar}t\right)|m\rangle|\phi_0\rangle = |m\rangle\exp(-iD_m t)|\phi_0\rangle = |m\rangle|\phi_m(t)\rangle. \tag{43}$$

We notice that here the measurement is made not in the sense of von Neumann but rather as a dynamical evolution of the joint system, according to Schrödinger's equation, with the appropriate coupling.

The resulting environment (apparatus) states $|\phi_m(t)\rangle$ are called **pointer states**. When the environment is interpreted as the measuring apparatus, they would correspond to particular apparatus states.

From the linearity of Schrödinger's equation, one can also write

$$\sum_m F_m|m\rangle|\phi_0\rangle \rightarrow \sum_m F_m|m\rangle|\phi_m(t)\rangle; \tag{44}$$

that is, we get a correlated state representing the superposition of all the possible measured results. From the above expression, we can calculate the local or subsystem reduced density matrix by tracing over the environment,

$$\rho_S = Tr_{envir}\sum_{m,p} F_m F_p^*|m\rangle\langle p||\phi_m(t)\rangle\langle\phi_p(t)| = \sum_{m,p} F_m F_p^*|m\rangle\langle p|\langle\phi_m(t)|\phi_p(t)\rangle, \tag{45}$$

and for orthogonal states

$$\langle\phi_m(t)|\phi_p(t)\rangle = \delta_{pm} \tag{46}$$

the reduced density matrix for the system becomes diagonal,

$$\rho_S \rightarrow \sum_m |F_m|^2|m\rangle\langle m|. \tag{47}$$

During this evolution, the interference was destroyed, and the system appears to be classical with respect to the quantum number m, which implies that the phase information has been destroyed. No interference between different $|m\rangle$ can be observed in the system.

If the above evolution is viewed as a model of system–apparatus coupling, unfortunately, the apparatus, being macroscopic, will invariably interact with the environment ϵ.

By the same mechanism, the information about the measurement is rapidly transferred to the environment, leading to the following state:

$$\sum_m F_m|m\rangle|\phi_m\rangle|\epsilon_0\rangle \rightarrow \sum_m F_m|m\rangle|\phi_m\rangle|\epsilon_m\rangle; \tag{48}$$

and if the environment states are orthogonal, then we obtain the diagonal density matrix for the system–apparatus,

$$\rho_{\text{system-apparatus}} = \sum_m |F_m|^2|m\rangle\langle m||\phi_m\rangle\langle\phi_m|. \tag{49}$$

Once more, we have defined the interaction of the apparatus with the environment by a Hamiltonian of the form given by Eq. (42), defining in this way the pointer states $|\phi_n\rangle$.

3.2. How Long Does It Take to Decohere?

As we saw in the previous examples, a clear manifestation of decoherence is that the nondiagonal element of the density matrix, in certain bases, vanish in time. Coherences associated with superpositions are strongly affected by damping. Early work on this subject was done by Joos and Zeh [39] and Caldeira and Leggett [40].

If we take a superposition of coherent states $\propto [|\alpha\rangle + |-\alpha\rangle]$, we will learn in this subsection that, while the coherent amplitudes decay as γ^{-1}, the off-diagonal elements of the density matrix decay in a time, which is the decoherence time, which under normal conditions is much smaller that the inverse damping,

$$t_c = \frac{\gamma^{-1}}{2|\alpha|^2}.$$

Let us assume an initial state given by [41]

$$|\psi(0)\rangle = N(|\alpha_1\rangle + |\alpha_2\rangle), \tag{50}$$

where N is just a normalization factor.

Now, we are interested to study the effects of damping. This will give us information about the characteristic decoherence time due to the interaction with the environment.

The master equation for the damped harmonic oscillator, at zero temperature, is given by

$$\frac{d\rho}{dt} = \frac{\gamma}{2}(2a\rho a^\dagger - a^\dagger a\rho - \rho a^\dagger a). \tag{51}$$

The normally ordered characteristic function is defined as

$$X_N(\eta,t) = Tr(\rho(t)\exp(\eta a^\dagger)\exp(-\eta^* a)). \tag{52}$$

One can write [42]

$$\begin{aligned}
\frac{\partial X_N(\eta,t)}{\partial t} &= Tr\left[\frac{d\rho}{dt}\exp(\eta a^\dagger)\exp(-\eta^* a)\right] \\
&= \frac{\gamma}{2}Tr[(2a\rho a^\dagger - a^\dagger a\rho - \rho a^\dagger a)\exp(\eta a^\dagger)\exp(-\eta^* a)] \\
&= \frac{\gamma}{2}Tr[2\rho a^\dagger \exp(\eta a^\dagger)\exp(-\eta^* a)a - \rho \exp(\eta a^\dagger)\exp(-\eta^* a)a^\dagger a \\
&\quad - \rho a^\dagger a \exp(\eta a^\dagger)\exp(-\eta^* a)] \\
&= -\frac{\gamma}{2}\{\eta Tr[\rho a^\dagger \exp(\eta a^\dagger)\exp(-\eta^* a)] - \eta^* Tr[\rho \exp(\eta a^\dagger) \\
&\quad \times \exp(-\eta^* a)a]\}\frac{\partial X_N(\eta,t)}{\partial t} \\
&= -\frac{\gamma}{2}\left[\eta\frac{\partial X_N(\eta,t)}{\partial \eta} + \eta^*\frac{\partial X_N(\eta,t)}{\partial \eta^*}\right]. \tag{53}
\end{aligned}$$

In the last steps, we used the following properties:

$$[a,f(a,a^\dagger)] = \frac{\partial f(a,a^\dagger)}{\partial a^\dagger}, \tag{54}$$

$$[a^\dagger,f(a,a^\dagger)] = -\frac{\partial f(a,a^\dagger)}{\partial a}, \tag{55}$$

$$\begin{aligned}
[a^\dagger a,\exp(\eta a^\dagger)\exp(-\eta^* a)] &= \eta a^\dagger \exp(\eta a^\dagger)\exp(-\eta^* a) \\
&\quad + \eta^* \exp(\eta a^\dagger)\exp(-\eta^* a)a. \tag{56}
\end{aligned}$$

One can show that the solution to Eq. (53) is

$$X_N(\eta,t) = X_N\left(\eta\exp\left(-\frac{\gamma t}{2}\right),0\right) = X_N(\eta(t),0). \tag{57}$$

We can confirm the above result as follows (considering η as a complex number):

$$\begin{aligned}
\frac{\partial X_N(\eta,t)}{\partial t} &= \frac{\partial X_N(\eta,t)}{\partial \eta(t)}\frac{\partial \eta(t)}{\partial t} + \frac{\partial X_N(\eta,t)}{\partial \eta^*(t)}\frac{\partial \eta^*(t)}{\partial t} \\
&= -\frac{\gamma}{2}\left[\eta\frac{\partial X_N(\eta,t)}{\partial \eta} + \eta^*\frac{\partial X_N(\eta,t)}{\partial \eta^*}\right].
\end{aligned}$$

Now, the initial condition is

$$X_N(\eta,0) = Tr(\rho(0) \ \exp(\eta a^\dagger)\exp(-\eta^* a)$$

$$= N^2 Tr \sum_{i,j} [|\alpha_i\rangle\langle\alpha_j|\exp(\eta a^\dagger)\exp(-\eta^* a)]$$

$$= N^2 \sum_{i,j} [\langle\alpha_j|\exp(\eta a^\dagger)\exp(-\eta^* a)|\alpha_i\rangle]$$

$$= N^2 \sum_{i,j} [\langle\alpha_j|\alpha_i\rangle\exp(\eta\alpha_j^* - \eta^*\alpha_i)]. \tag{58}$$

So

$$X_N(\eta,t) = N^2 \sum_{i,j} \left[\langle\alpha_j|\alpha_i\rangle\exp(\eta\alpha_j^* - \eta^*\alpha_i)\exp\left(-\frac{\gamma t}{2}\right) \right]. \tag{59}$$

It is not difficult to show that the corresponding density matrix ρ is given by

$$\rho = N^2 \sum_{i,j=1}^{2} \langle\alpha_i|\alpha_j\rangle^{(1-\exp(-\gamma t))} \left| \alpha_j \exp\left(-\frac{\gamma t}{2}\right)\right\rangle\left\langle \alpha_i \exp\left(-\frac{\gamma t}{2}\right)\right|. \tag{60}$$

One can confirm the above result as follows: if we replace the density matrix in the definition of the generating function (52), one can retrieve the result of Eq. (59).

In the particular case $\alpha_1 = \alpha$, $\alpha_2 = -\alpha$, we have

$$\langle\alpha|-\alpha\rangle = \exp(-2|\alpha|^2), \tag{61}$$

$$\rho = N^2 \left\{ \left| \alpha\exp\left(-\frac{\gamma t}{2}\right)\right\rangle\left\langle \alpha\exp\left(-\frac{\gamma t}{2}\right)\right| + \left| -\alpha\exp\left(-\frac{\gamma t}{2}\right)\right\rangle \right.$$

$$\times \left\langle -\alpha\exp\left(-\frac{\gamma t}{2}\right)\right| \right\} + N^2 \exp[-2|\alpha|^2(1-\exp(-\gamma t)] \left\{ \left| \alpha\exp\left(-\frac{\gamma t}{2}\right)\right\rangle \right.$$

$$\times \left\langle -\alpha\exp\left(-\frac{\gamma t}{2}\right)\right| + \left| -\alpha\exp\left(-\frac{\gamma t}{2}\right)\right\rangle\left\langle \alpha\exp\left(-\frac{\gamma t}{2}\right)\right| \right\}. \tag{62}$$

If $\gamma t \ll 1$, then the relevant exponential factor multiplying the crossed terms becomes

$$\exp(-2|\alpha|^2\gamma t) \equiv \exp\left(-\frac{t}{t_c}\right), \tag{63}$$

where

$$t_c = \frac{\gamma^{-1}}{2|\alpha|^2}. \tag{64}$$

As we can see, while the coherent amplitudes decay on a time scale of γ^{-1}, the coherences or off-diagnonal elements of the density matrix decay in a time, that

for macroscopic photon numbers $|\alpha|^2$, is much smaller than γ^{-1}. This may explain the difficulties in observing the quantum coherence in a macroscopic situation.

It has been shown [43] that the decay time of the quantum coherence, in a phase sensitive reservoir, for an initial superposition of $|\alpha\rangle$ and $|-\alpha\rangle$, is given by

$$t_c^{(sq)} = \frac{\gamma^{-1}}{2\left[N + 2\alpha^2\left(N - M + \frac{1}{2}\right)\right]}, \tag{65}$$

where M and N are squeezing parameters defined in Section 4.

In the vacuum reservoir, $M=N=0$, the result coincides with Eq. (64).

On the other hand, it is interesting to note that for an ideally squeezed reservoir $|M|^2=N(N+1)$, with $M>0$, the decay rate of the quantum coherence is significantly suppressed, and for large N is independent of α, namely,

$$t_c^{(sq)} = \frac{\gamma^{-1}}{2[N]}, \tag{66}$$

which means that the decay rate of the quantum coherence (off-diagonal terms in the density matrix) is of the same order of magnitude as the decay rate of the energy (diagonal part of the density matrix). Also, if $M<0$, the decay rate of the coherence increases.

As we can see, one could in principle control the decay rate of the quantum coherence by monitoring the phase of the squeezing parameter of the reservoir [43–46], which may have interesting applications in quantum computing. Finally, there are some additional publications on decoherence in the nonclassical motion of trapped ions [47–49].

3.3. Decoherence-Free Subspaces

As we have seen in the previous sections, decoherence is a consequence of the inevitable coupling of any quantum system to its environment, causing information loss from the system to the environment. In other words, we consider the decoherence a nonunitary dynamics that is a consequence of the system–environment coupling. This includes both dissipative and dephasing contributions.

Dissipation is a process by which the populations of the quantum states are modified by the interactions with the environment, while *dephasing* is a process of randomization of the relative phases of the quantum states. Both effects are caused by the entanglement of the system with the environment degrees of freedom, leading to the nonunitary dynamics of our system.

Lidar *et al.* [50,51] introduced the term "decoherence-free subspaces," referring to robust states against perturbations, in the context of Markovian master equations.

One uses the symmetry of the system–environment coupling to find a quiet corner in the Hilbert Space not experiencing this interaction. A more formal definition of the DFS is as follows:

A system with a Hilbert space H is said to have a decoherence-free subspace $\tilde{H} \subset H$, if the evolution inside $\tilde{H}$ is purely unitary.

3.3a. Simple Example: Collective Dephasing

We assume a collection of two-level atoms coupled to a common reservoir. Defining a qubit as a two-level system that in the basis $|0\rangle$ and $|1\rangle$ can be written as

$$|\psi\rangle = a|0\rangle + b|1\rangle, \tag{67}$$

we assume that the effect of the dephasing bath over these basis states is the following one:

$$|0\rangle_j \rightarrow |0\rangle_j,$$

$$|1\rangle_j \rightarrow \exp[i\phi]|1\rangle_j, \tag{68}$$

where ϕ is a random phase.

This transformation can be written as a matrix

$$R_z(\phi) = \begin{bmatrix} 1 & 0 \\ 0 & \exp(i\phi) \end{bmatrix} \tag{69}$$

acting on the $\{|0\rangle, |1\rangle\}$ basis.

Here, we assume in this particular example that this transformation is collective, implying that the phase ϕ is the same for all two-level systems.

Now, we study the effect of the bath on an initial qubit $|\psi\rangle_j = a|0\rangle_j + b|1\rangle_j$.

We can now write the average density matrix over all possible phases, distributed with a probability density $p(\phi)$, as

$$\rho_j = \int_{-\infty}^{\infty} R_z(\phi)|\psi\rangle_j \langle\psi|R_z(\phi)^{\mathsf{T}} p(\phi) d\phi, \tag{70}$$

and we assume that all the qubits are in the same state.

To be more specific, if we take a Gaussian distribution for the phase

$$p(\phi) = \frac{1}{\sqrt{4\pi\gamma}} \exp\left(-\frac{\phi^2}{4\gamma}\right), \tag{71}$$

then it is simple to show that

$$\rho_j = \begin{bmatrix} |a|^2 & ab^* \exp(-\gamma) \\ a^*b \exp(-\gamma) & |b|^2 \end{bmatrix}, \tag{72}$$

basically showing the decoherence as the exponential decay of the nondiagonal elements of the density matrix.

Now, let us go and hunt for DFSs, considering, for example, two and three particles.

Two particles. For the two-particle case, we have four basis states, and the effect of the bath is the following one:

$$|0\rangle_1 \otimes |0\rangle_2 = |00\rangle \rightarrow |00\rangle,$$

$$|0\rangle_1 \otimes |1\rangle_2 = |01\rangle \rightarrow \exp(i\phi)|0\rangle_1 \otimes |1\rangle_2 = \exp(i\phi)|01\rangle,$$

$$|1\rangle_1 \otimes |0\rangle_2 = |10\rangle \rightarrow \exp(i\phi)|1\rangle_1 \otimes |0\rangle_2 = \exp(i\phi)|10\rangle,$$

$$|1\rangle_1 \otimes |1\rangle_2 = |11\rangle \rightarrow \exp(2i\phi)|1\rangle_1 \otimes |1\rangle_2 = \exp(2i\phi)|11\rangle. \tag{73}$$

We note that the states $|10\rangle$ and $|01\rangle$ transform with the same phase factor $\exp(i\phi)$; so any combination of these states will have only a global phase

$$|\chi\rangle = \alpha|10\rangle + \beta|01\rangle \rightarrow \exp(i\phi)(\alpha|10\rangle + \beta|01\rangle) = \exp(i\phi)|\chi\rangle. \tag{74}$$

Thus, we have in this example a DFS of dimension 2,

$$\mathrm{DFS}_2 = \{|10\rangle, |01\rangle\}. \tag{75}$$

We also have a couple of trivial DFSs of dimension 1, such as $\{|00\rangle\}$ and $\{|11\rangle\}$. However, as the global phases of the various DFSs are different, there is decoherence between them.

Three particles. We have two three-dimensional DFSs, namely,

$$\{|001\rangle, |010\rangle, |100\rangle\} = \mathrm{DFS}_3^{(1)}, \quad \{|110\rangle, |101\rangle, |011\rangle\} = \mathrm{DFS}_3^{(2)}, \tag{76}$$

plus the trivial one-dimensional subspaces $\{|000\rangle\}$, $\{|111\rangle\}$. There has been experimental verification of DFS_2 by using trapped ions [52].

3.4. Formal Decoherence

We now present a more formal treatment of decoherence. Consider a closed quantum system composed of a system S interacting with bath B, and the full Hamiltonian (defined on a Hilbert space H) is given by

$$H = H_S \otimes I_B + I_S \otimes H_B + H_I, \tag{77}$$

where H_S, H_B, and H_I are, respectively, the system, bath, and system–bath interaction Hamiltonians, and I is the identity operator.

The time evolution of the whole system is given by

$$\rho_{SB}(t) = U(t)\rho_{SB}(0)U^\dagger(t), \tag{78}$$

where ρ_{SB} is the combined system–bath density operator and U is the usual time evolution operator $U = \exp[-(i/\hbar)Ht]$.

At this point, we assume that initially the system and the bath are decoupled, so

$$\rho_{SB}(t) = U(t)\rho_S(0) \otimes \rho_B(0)U^\dagger(t).$$

The interaction term can be written, quite generally as

$$H_I = \sum_\alpha S_\alpha \otimes B, \tag{79}$$

where S_α and B are system and bath operators, respectively. It is precisely this interaction of the system and the bath that causes the decoherence of the system, through the entanglement with the bath.

Assuming initially that the system is decoupled from the bath, and tracing over the bath variables, we get

$$\rho(t) = Tr_B[U(t)\rho(0) \otimes \rho_B(0)U^\dagger(t)]. \tag{80}$$

This reduced density matrix represents the system alone. If we diagonalize the density matrix of the bath, $\rho_B(0) = \sum_\nu \lambda_\nu |\nu\rangle\langle\nu|$, by performing the trace over the bath variables, we obtain

$$\rho = \sum \langle\mu|U(t)(\rho(0) \otimes \sum_\nu \lambda_\nu |\nu\rangle\langle\nu|)U^\dagger(t)|\mu\rangle = \sum_a A_a\rho(0)A_a^\dagger, \tag{81}$$

with the Kraus operators defined as

$$A_a = \sqrt{\lambda_\nu}\langle\mu|U(t)|\nu\rangle, \quad a = \mu, \nu. \tag{82}$$

As the density matrix is normalized, one can write

$$\sum_a A_a^\dagger A_a = I_S. \tag{83}$$

If only one term is present in the above summation, then the evolution is unitary.

A possible criterion for decoherence is the presence of several terms in Eq. (83).

3.5. Condition for Decoherence-Free Subspaces

3.5a. Hamiltonian Approach

As we remarked above, the decoherence is a direct result of the entanglement between the system and the bath, as a direct consequence of the system–bath interaction, caused in the present model, by the Hamiltonian H_I. Of course, if $H_I=0$, then there is no decoherence and the dynamics follows a unitary evolution. Unfortunately, in practice, one cannot switch off the interaction with the reservoir. We have to look for alternatives, such as a particular subspace that is free of decoherence.

It was first shown by Zanardi *et al.* [53,54] that such a subspace is found by assuming a set of states that are eigenvectors $\{|\tilde{k}\rangle\}$ of S_α such that

$$S_\alpha|\tilde{k}\rangle = c_\alpha|\tilde{k}\rangle, \qquad \forall\, \alpha, |\tilde{k}\rangle. \tag{84}$$

These are degenerate eigenvectors of the system operators whose eigenvalues depend only on the index α, but not on the state index k.

If H_S leaves the Hilbert space $\tilde{H}=\mathrm{span}\{|\tilde{k}\rangle\}$ invariant, and if we start within $\tilde{H}$, then the evolution of the system will be decoherence free.

3.5b. Lindblad Semigroup Approach

We now formulate the existence of the DFS in terms of the Lindblad semigroup master equation. Lindblad has shown that the most general evolution of a system density matrix ρ_s is governed by the master equation

$$\frac{d\rho}{dt} = -\frac{i}{\hbar}[H_s,\rho] + L_D(\rho), \tag{85}$$

with

$$L_D(\rho(t)) = \frac{1}{2}\sum_{\alpha,\beta=1}^{M} d_{\alpha,\beta}([F_\alpha,\rho F_\beta^\dagger] + [F_\alpha\rho,F_\beta^\dagger]), \tag{86}$$

where H_s is the system Hamiltonian, the F_α is a family of the 'Lindblad' operators in an M-dimensional space and $d_{\alpha,\beta}$ are elements of a positive Hermitian matrix.

All the nonunitary, decohering dynamics is accounted for by L_D. That is, in the Lindblad form, there is a clear separation between the unitary and decohering dynamics.

Let $\{|\tilde{k}\rangle_{k=1}^N\}$ be a basis for an N-dimensional subspace

$$\tilde{H}(\text{DFS}) \subseteq H(\text{total system Hilbert space}).$$

In this basis, we may express the density matrix as

$$\rho = \sum_{k,j=1}^{N} \rho_{kj}|\tilde{k}\rangle\langle\tilde{j}|. \tag{87}$$

Now, we consider the action of the Lindblad operators F_α on $|\tilde{k}\rangle$,

$$F_\alpha|\tilde{k}\rangle = \sum_{j=1}^{N} C_{kj}^\alpha|\tilde{j}\rangle.$$

Substituting into Eq. (86), we find

$$L_D(\rho) = \frac{1}{2}\sum_{\alpha,\beta=1}^{M} d_{\alpha,\beta} \sum_{k,j,m,n=1}^{N} \rho_{kj}(2C_{jm}^{\beta*}C_{kn}^\alpha|\tilde{n}\rangle\langle\tilde{m}| - C_{mn}^{\beta*}C_{kn}^\alpha|\tilde{m}\rangle\langle\tilde{j}| - C_{jm}^{\beta*}C_{nm}^\alpha|\tilde{k}\rangle\langle\tilde{n}|)$$

$$= 0. \tag{88}$$

Notice that we have used the condition $L_D(\rho)=0$, which is precisely the definition of the DFS.

The coefficients $d_{\alpha,\beta}$ represent information about the bath, which we assume is uncontrollable. So we require that each term in the α,β sum vanish separately. Furthermore, we expect no dependence on the initial conditions, i.e., no dependence on ρ_{kj}, which implies that each term in the parenthesis vanishes separately.

This can be done if all the projectors are the same, a requirement that is satisfied if

$$C_{kn}^\alpha = C_n^\alpha \delta_{k,n},$$

so Eq. (88) becomes

$$\sum_{k,j=1}^{N} \rho_{kj} |\tilde{k}\rangle\langle\tilde{j}| (2C_j^{\beta*} C_k^{\alpha} - C_k^{\beta*} C_k^{\alpha} - C_j^{\beta*} C_j^{\alpha}) = 0. \qquad (89)$$

Assuming that $C_k^{\alpha} \neq 0$, this yields

$$(2C_j^{\beta*} C_k^{\alpha} - C_k^{\beta*} C_k^{\alpha} - C_j^{\beta*} C_j^{\alpha}) = 0, \qquad (90)$$

or

$$2 = Z^* + Z^{-1}, \qquad (91)$$

with $Z \equiv C_j^{\alpha} / C_k^{\alpha}$.

Equation (91) has the unique solution $Z=1$. Thus, $C_j^{\alpha} = C_k^{\alpha} \equiv C^{\alpha}$

Thus, we have proved the following theorem.

Theorem 3.1 *The necessary and sufficient condition for a subspace $\tilde{H}$(DFS)* $= \{|\tilde{k}\rangle_{k=1}^N\}$ *to be decoherence free is that the basis states $|\tilde{k}\rangle$ are degenerate eigenstates of all Lindblad operators F_α:*

$$F_\alpha |\tilde{k}\rangle = C^{\alpha} |\tilde{k}\rangle \quad for \ \forall \ \alpha, k. \qquad (92)$$

The above condition can be also written as

$$[F_\alpha, F_\beta] |\tilde{k}\rangle = 0. \qquad (93)$$

If one can write

$$[F_\alpha, F_\beta] = \sum_{\gamma=1}^{M} f_{\alpha,\beta}^{\gamma} F_\gamma,$$

with $f_{\alpha,\beta}^{\gamma} \neq 0$ and linearly independent (the Fs forming a semi-simple Lie algebra). In this case, condition (93) can be written as

$$\sum_{\gamma=1}^{M} f_{\alpha,\beta}^{\gamma} C^{\gamma} = 0, \qquad (94)$$

which can be satisfied only for $C^{\gamma}=0$.

> *Thus, the condition for a DFS, for the semisimple case, is that the set of states $\{|\tilde{k}\rangle_{k=1}^N\}$ should be degenerate eigenstates of all Lindblad operators with zero eigenvalue:*
> $$F_\alpha |\tilde{k}\rangle = 0, \ for \forall \ \alpha, k. \qquad (95)$$

4. Entanglement Evolution

Decoherence and entanglement are closely related phenomena, mainly because decoherence is responsible for the fragility of the entanglement in systems interacting with reservoirs [55]. For this reason, in the current decade, many papers have extensively investigated the decoherence dynamics of entangled quantum systems under the influence of environmental noise by focusing mainly on the dynamical system of two parties.

Recently Yu and Eberly [56] investigated the dynamics of disentanglement of a bipartite qubit system due to spontaneous emission, where The two two-level atoms (qubits) were coupled individually to two cavities (environments). They found that the quantum entanglement may vanish in a finite time, while local decoherence takes a infinite time. They called this phenomena "entanglement sudden death" (ESD). In previous work, Diósi [57] demonstrated, using Werner's criteron for separability, that ESD can also occur in two-state quantum systems.

Since then, ESD has been examined in several model situations involving pairs of atomic, photonic, and spin qubits [58–60], continuous Gaussian states [61], and spin chains [62]. Also, ESD has been examined for different environments including random matrix environments [63,64], thermal reservoirs [65–67] and squeezed reservoirs [68]. ESD is not unique to systems of independent atoms. It can also occur for atoms coupled to a common reservoir, in which case we also observe the effect of the revival of an entanglement that has already been destroyed [69]. The effect of global noise on entanglement decay may depend on whether the initial two-party state belongs to a DFS or not.

In a recent experiment Almeida *et al.* [70,71] used correlated horizontally and vertically polarized photons to show evidence of the sudden death of entanglement under the influence of independent reservoirs. They created the initial state by using a downconversion process.

As opposed to the ESD and against our intuition, it has been shown that under certain conditions the process of spontaneous emission can entangle qubits that were initially unentangled [72–74], and in some cases the creation of entanglement can occur some time after the system–reservoir interaction has been turned on. The authors in [75] call this phenomenon "delayed sudden birth of entanglement."

4.1. Time Evolution of a Two-Atom System

In this section we study two-atom systems. However, in real systems, these are not isolated and do experience interactions with the outside world. To build useful quantum information systems, it is necessary to understand and control such noise processes.

A useful mathematical description for the qubits (system of interest) is given by the density matrix. When the system is exposed to environmental noise, the density matrix will change in time. Such a time evolution is traditionally studied via a master equation. In this approach, the dynamics is studied in terms of the reduced density operator $\hat{\rho}_s$ of the atomic system interacting with the quantized electromagnetic field regarded as a reservoir. The reservoirs have many possible realizations. A reservoir can be modeled as a many-mode vacuum, thermal, or

squeezed vacuum field. The major advantage of the master equation is that it allows us to consider the evolution of the atoms-plus-field system entirely in terms of average values of atomic operators.

We consider two situations. The first one consists of two two-level atoms initially entangled and coupled to uncorrelated reservoirs, and the second one of two two-level atoms initially entangled and interacting with a common reservoir. This coupling between the system and the reservoir originates the disentanglement.

We now write a general master equation for the reduced density matrix in the interaction picture, assuming that the correlation time between the atoms and the reservoirs is much shorter than the characteristic time of the dynamical evolution of the atoms, so that the Markov approximation is valid (for non-Markovian effects, see [76–80]):

$$
\frac{\partial \hat{\rho}}{\partial t} = \frac{\Gamma}{2} \sum_{i,j=1}^{2} [(N+1)(2\sigma_i \hat{\rho} \sigma_j^\dagger - \sigma_i^\dagger \sigma_j \hat{\rho} - \hat{\rho} \sigma_i^\dagger \sigma_j) + N(2\sigma_i^\dagger \hat{\rho} \sigma_j - \sigma_i \sigma_j^\dagger \hat{\rho} - \hat{\rho} \sigma_i \sigma_j^\dagger)
$$
$$
- M(2\sigma_i^\dagger \hat{\rho} \sigma_j^\dagger - \sigma_i^\dagger \sigma_j^\dagger \hat{\rho} - \hat{\rho} \sigma_i^\dagger \sigma_j^\dagger) - M^*(2\sigma_i \hat{\rho} \sigma_j - \sigma_i \sigma_j \hat{\rho} - \hat{\rho} \sigma_i \sigma_j)], \tag{96}
$$

where Γ is the decay constant of the qubits, and $\sigma_i^+ = |1\rangle_i \langle 0|$ and $\sigma_i^- = |0\rangle_i \langle 1|$ are the raising (+) and lowering (−) operators of the ith atom. It should be pointed out that in Eq. (96) the $i=j$ terms describe the atoms interacting with independent local reservoirs, while the $i \neq j$ terms denote the couplings between the modes induced by the common bath; see Fig. 3.

Equation (96) has at least three possible realizations. These are vacuum, thermal, and squeezed reservoirs.

For a **vacuum reservoir** we set $N \to 0$ and $M \to 0$ in Eq. (96).

For a **thermal reservoir** $N \to \bar{n}$, where $\bar{n}$ is the mean number of the thermal field (assumed to be the same for both qubits) and $M \to 0$. Additionally, this reservoir can cause excitation of the qubits. Thus the first term on the right-hand side of Eq. (96) corresponds to the depopulation of the atoms due to stimulated and spontaneous emission, and the second term describes the excitations caused by temperature.

For a **squeezed reservoir**, N represents the mean photon number of the reservoir, and it is defined as $N = \sinh^2 r$, and M is a parameter related to the phase correlations of the squeezed reservoir defined as $M = -e^{i\theta} \sinh r \cosh r$. r is the squeezing parameter of the reservoir, and θ is the squeezing angle. The Heisenberg

Figure 3
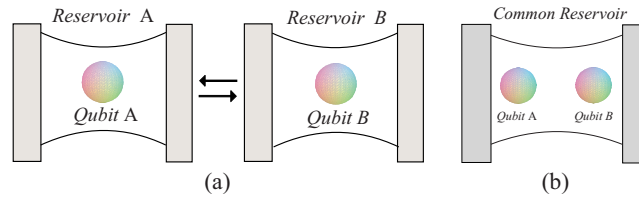
Schematic of an assembly of two qubits A and B, located in (a) two independent and spatially separated reservoirs and (b) a common reservoir.

uncertainty relation imposes the constraint $|M|^2 \leq N(N+1)$, where the equality holds for a minimum-uncertainty squeezed state.

5. Entanglement with Independent Reservoirs

In the next subsections we consider two two-level atoms A and B that represent a two-qubit system, each one interacting independently with their local environments, as is shown in Fig. 3(a). There is no direct interaction between the atoms. The correlation between the atoms results only from an initial quantum entanglement between them.

5.1. Vacuum Reservoir

The vacuum environment can still have a noisy degrading effect through its quantum fluctuations. As a consequence, the atoms lose their excitation at a rate Γ. So, their stationary state is their ground state $|--\rangle$. In other words, the atoms experience a disentanglement process. This process can be completed in a finite-time (ESD) or in an infinite time. These different behaviors exhibited in the disentanglement times depend on the initial state. Various publications have considered mixed and pure states [56,65,81,82], as well as quantum recoil effects [83].

Given the solution of master equation (96) for a vacuum reservoir, and considering as the initial condition the matrix shown in Eq. (28), in the standard basis the concurrence is given by [82] $C(t) = 2 \max\{0, C_1(t), C_2(t)\}$, where

$$C_1(t) = 2e^{-\Gamma t}[|\rho_{23}| - \sqrt{\rho_{11}(\rho_{44} + (\rho_{22} + \rho_{33})\omega^2 + \rho_{11}\omega^4)}],$$

$$C_2(t) = 2e^{-\Gamma t}[|\rho_{14}| - \sqrt{\rho_{22}\rho_{33} + \rho_{11}(\rho_{22} + \rho_{33})\omega^2 + \rho_{11}^2\omega^4}], \qquad (97)$$

with $\omega = \sqrt{1 - e^{-\Gamma t}}$.

When $C_1(t)$ is maximal, then if $\rho_{11} > |\rho_{23}|^2$, the concurrence decays to zero in a finite time

$$t_{d-1} = \frac{1}{\Gamma} \ln\left(\frac{\rho_{11}(2\rho_{11} + \rho_{22} + \rho_{33} + \sqrt{(\rho_{22} + \rho_{33})^2 - 4(\rho_{11}\rho_{44} - |\rho_{23}|^2)})}{2(\rho_{11} - |\rho_{23}|^2)}\right). \qquad (98)$$

However, if $\rho_{11} \leq |\rho_{23}|^2$, then the equation $C_1(t) = 0$ has no solution, and the concurrence decays asymptotically to zero.

Similarly, when $C_2(t)$ is maximal, then if $\rho_{22}\rho_{33} + \rho_{11}(1 - \rho_{44}) > |\rho_{14}|^2$, the concurrence vanishes in a finite time,

$$t_{d-2} = \frac{1}{\Gamma} \ln\left(\frac{\rho_{11}(2\rho_{11} + \rho_{22} + \rho_{33} + \sqrt{(\rho_{22} - \rho_{33})^2 + 4|\rho_{23}|^2})}{2(\rho_{22}\rho_{33} + \rho_{11}(\rho_{11} + \rho_{22} + \rho_{33}) - |\rho_{14}|^2)}\right), \qquad (99)$$

and if $\rho_{22}\rho_{33} + \rho_{11}(1 - \rho_{44}) \leq |\rho_{14}|^2$, then we again have the exponential behavior.

To summarize, the ranges of the respective initial conditions are

$$\rho_{11} \leq |\rho_{23}|^2, \quad \text{the entanglement decays asymptotically;}$$

$$\rho_{11} > |\rho_{23}|^2, \quad \text{the entanglement decays in a finite time,} \tag{100}$$

for the case of Eq. (98), and

$$\rho_{22}\rho_{33} + \rho_{11}(1 - \rho_{44}) \leq |\rho_{14}|^2, \quad \text{the entanglement decays asymptotically;}$$

$$\rho_{22}\rho_{33} + \rho_{11}(1 - \rho_{44}) > |\rho_{14}|^2, \quad \text{the entanglement decays in a finite time,} \tag{101}$$

for Eq. (99).

Let us show some examples.

(a) Consider the initial state of the form $\alpha|00\rangle + \beta|11\rangle$, with $|\alpha|^2 + |\beta|^2 = 1$. For this initial state, the concurrence is

$$C = \max\{0, 2|\beta|e^{-\Gamma t}(|\alpha| - |\beta|(e^{-\Gamma t} - 1))\}, \tag{102}$$

and the disentanglement time occurs for $t_d = -1/\Gamma(1 - \alpha/\beta)$. Only the states with $|\beta| > |\alpha|$, i.e., $|\beta| > 1/\sqrt{2}$, have a finite disentanglement time. Consider now the initial state $\gamma|01\rangle + \delta|10\rangle$, with $|\gamma|^2 + |\delta|^2 = 1$. Its concurrence is $C = \max\{0, 2e^{-\Gamma t}\delta\gamma\}$. In this case the concurrence goes asymptotically to zero, for all values of δ.

(b) The first and more traditional case of ESD mentioned in the literature contains the double excitation and the ground state components, and also includes the state in which one of the atoms is excited. This initial condition is

$$\rho(0) = \frac{1}{3} \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1-a \end{pmatrix}. \tag{103}$$

Yu and Eberly studied the ESD in the case $a=1$. They showed that for $\frac{1}{3} \leq a \leq 1$, there is ESD. The concurrence is given by

$$C = \max\left\{0, \frac{2}{3}e^{-\Gamma t}(1 - \sqrt{a(3 - 2(1+a)e^{-\Gamma t} + ae^{-2\Gamma t}))}\right\}, \tag{104}$$

and the time of disentanglement is

$$t_d = -\frac{1}{\Gamma} \ln\left(\frac{a + 1 - \sqrt{2 - a + a^2}}{a}\right). \tag{105}$$

For the case $a=1$ the results obtained in [56] are recovered. The time evolution of the concurrence for all values of a is shown in Fig. 4(a). Also, in Fig. 4(b) we show the typical concurrence versus time for initial conditions leading to sudden death and asymptotic behavior.

Clearly, to destroy the entanglement in a finite time, the spontaneous emission is not enough, and the sudden death of entanglement results from the decay of a mixed state with a doubly excited state component.

Figure 4
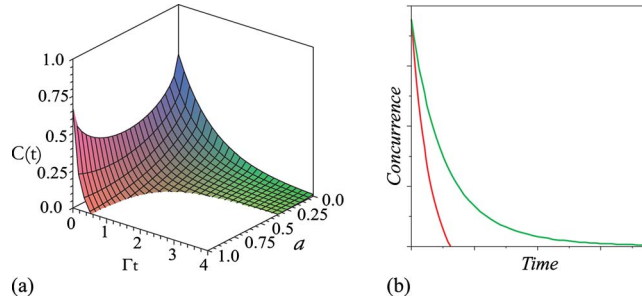
(a) Time evolution of the concurrence in a vacuum reservoir when the atoms are initially in the entangled mixed state [Eq. (103)]. (b) Two typical behaviors, sudden death and asymptotic decay in time, for qubits interacting with two independent vacuum reservoirs.

5.2. Thermal Reservoir

The interactions with a thermal reservoir lead typically to very rapid decoherence; thus one might expect the destruction of quantum entanglement. But, how quickly does it occur? In a vacuum reservoir, the disentanglement time depends explicitly on the initial state of the atoms; specifically, the ESD depends on the double excitation state $|++\rangle$. On the other hand, in the case of a thermal reservoir, the decay of the entanglement *always* occurs in a finite time. This conjecture has been recently proposed for all atoms initially entangled in the form of two-qubit X states [65,67].

Let us consider the initial density matrix (28) in the standard basis. The concurrence is given by $C(t) = 2 \max\{0, C_1(t), C_2(t)\}$, where C_1 and C_2 are given by

$$
\begin{aligned}
C_1(t) = 2|\rho_{23}|e^{-(2n+1)t} - \frac{2}{(2n+1)^2}\Bigg\{ \Bigg[&((\rho_{11} - \rho_{22} - \rho_{33} + \rho_{44})n^2 \\
&+ (2\rho_{11} - \rho_{22} - \rho_{33})n + \rho_{11})e^{-2(2n+1)t} + 2n\bigg((\rho_{11} - \rho_{44})n + \frac{1}{2}(2\rho_{11} + \rho_{22} \\
&+ \rho_{33})\bigg)e^{-(2n+1)t} + n^2 \Bigg]\Bigg[((\rho_{11} - \rho_{22} - \rho_{33} + \rho_{44})n^2 + (2\rho_{11} - \rho_{22} - \rho_{33})n \\
&+ \rho_{11})e^{-2(2n+1)t} - 2(n+1)\bigg((\rho_{11} - \rho_{44})n + \frac{1}{2}(2\rho_{11} + \rho_{22} + \rho_{33})\bigg)e^{-(2n+1)t} \\
&+ (n+1)^2 \Bigg]\Bigg\}^{1/2},
\end{aligned}
\tag{106}
$$

$$
\begin{aligned}
C_2(t) = 2|\rho_{14}|e^{-(2n+1)t} - \frac{2}{(2n+1)^2}\{[&-((\rho_{11} - \rho_{22} - \rho_{33} + \rho_{44})n^2 + (2\rho_{11} - \rho_{22} \\
&- \rho_{33})n + \rho_{11})e^{-2(2n+1)t} + (2(\rho_{22} - \rho_{33})n^2 + (\rho_{11} + 2\rho_{22} - 2\rho_{33} - \rho_{44})n \\
&+ \rho_{11} + \rho_{22})e^{-(2n+1)t} + n(n+1)][-((\rho_{11} - \rho_{22} - \rho_{33} + \rho_{44})n^2 + (2\rho_{11} - \rho_{22}
\end{aligned}
$$

$$- \rho_{33})n + a_0)e^{-2(2n+1)t} + (2(\rho_{33} - \rho_{22})n^2 + (\rho_{11} - 2\rho_{22} + 2\rho_{33} - \rho_{44})n$$

$$+ \rho_{11} + \rho_{33})e^{-(2n+1)t} + n(n+1)]\}^{1/2}, \tag{107}$$

n being the average number of thermal photons.

To get the disentanglement time, we solved analytically the equations $C_1(t)=0$ when $w(0)=0$ and $C_2(t)=0$ when $z(0)=0$. In Fig. 5, we plotted the time evolution of the concurrence for $n=0.01$ and $n=1$ when the atoms are initially in the state $\alpha|00\rangle \pm \beta|11\rangle$. The $\pm$ sign does not make any difference in the entanglement evolution.

Also, we considered the atoms initially entangled in a Werner state, Fig. 6. In this case, the initial condition is

$$\rho(0) = \frac{1}{4} \begin{pmatrix} 1-a & 0 & 0 & 0 \\ 0 & 1+a & -2a & 0 \\ 0 & -2a & 1+a & 0 \\ 0 & 0 & 0 & 1-a \end{pmatrix}. \tag{108}$$

In Fig. 7 we show the disentanglement time for different values of n. The main observed effect is that the death time of the entanglement decreases with the mean thermal photon number.
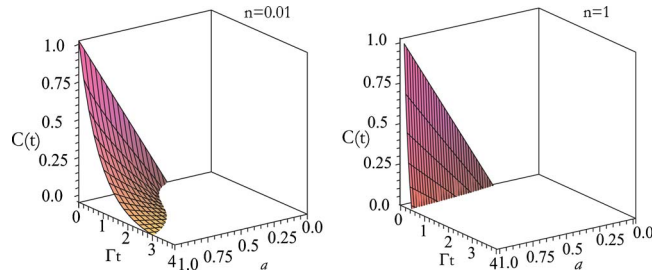
As we can see, at finite (nonzero) temperatures, for various initial mixed states, the phenomena of ESD is always present.
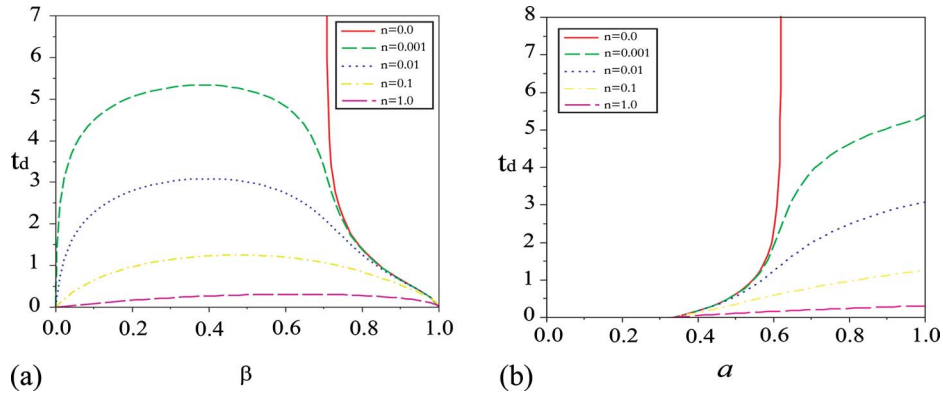
Figure 5
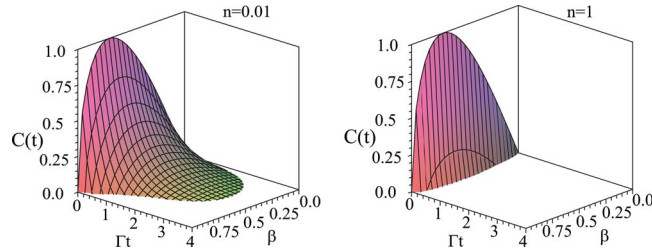
Entanglement evolution for the initial state $\alpha|00\rangle \pm \beta|11\rangle$ with (left) $n=0.01$ and (right) $n=1$. In both cases, ESD occurs for all ranges of $0 < \beta < 1$.

Figure 6

Concurrence for Werner states, Eq. (108), with (left) $n=0.01$ and (right) $n=1$.

Figure 7

Death time for (a) state $\alpha|00\rangle \pm \beta|11\rangle$ and (b) a Werner state. We plotted the disentanglement time for different values of the parameter n. In the case of $n=0$, there exists a range in the initial conditions for which the ESD is not permitted, decaying asymptotically. For larger values of n, the disentanglement time decreases; i.e., the sudden death occurs faster.

6. Entanglement in a Common Reservoir

In the next subsections, we will explore the relation between the sudden death (and revival) of the entanglement between the two two-level atoms in a squeezed bath and the normal decoherence via the DFS, which in this case is a two-dimensional plane [68]. We will also look at the special cases of the vacuum and thermal reservoirs. We should point out that, since we have a common bath, it would imply that the atoms are rather close to each other, that is, at a distance that is smaller that the correlation length of the reservoir. This would mean that we cannot neglect some direct interaction between them, like a dipole or Ising-type coupling. However, one can show that this type of coupling does not damage the DFS, in this case a plane [84].

The case we present next is a good example, where we are able to study in detail the effects of decoherence and disentanglement. Furthermore, we can relate the two phenomena by studying the effect of the distance from the DFS of the sudden death and revival times.

6.1. Model

We consider two two-level atoms that interact with a common squeezed reservoir, and we will focus on the evolution of the entanglement between them, using as a basis the DFS states, as defined in [84,85].

The master equation in the interaction picture for a two two-level atom system in a broadband squeezed vacuum bath is given in Eq. (96). It is simple to show that this master equation can also be written in the Lindblad form with a single Lindblad operator S,

$$\frac{\partial \rho}{\partial t} = \frac{1}{2}\Gamma(2S\rho S^\dagger - S^\dagger S\rho - \rho S^\dagger S), \tag{109}$$

with

$$S = \sqrt{N+1}(\sigma_1 + \sigma_2) - \sqrt{N}e^{i\Psi}(\sigma_1^\dagger + \sigma_2^\dagger)$$
$$= \cosh(r)(\sigma_1 + \sigma_2) - \sinh(r)e^{i\Psi}(\sigma_1^\dagger + \sigma_2^\dagger), \tag{110}$$

where $\sigma^\dagger, \sigma$ are the usual Pauli raising and lowering matrices and Γ is the spontaneous emission rate. The squeeze parameters are Ψ, and $N=\sinh^2 r$. Here we consider $M=\sqrt{N(N+1)}$.

The DFS consists of the eigenstates of S with zero eigenvalue. The states defined in this way form a two-dimensional plane in Hilbert space. Two orthogonal vectors in this plane are

$$|\phi_1\rangle = \frac{1}{\sqrt{N^2+M^2}}(N|++\rangle + Me^{-i\Psi}|--\rangle), \tag{111}$$

$$|\phi_2\rangle = \frac{1}{\sqrt{2}}(|-+\rangle - |+-\rangle). \tag{112}$$

We can also define the states $|\phi_3\rangle$ and $|\phi_4\rangle$ orthogonal to the $\{|\phi_1\rangle, |\phi_2\rangle\}$ plane:

$$|\phi_3\rangle = \frac{1}{\sqrt{2}}(|-+\rangle + |+-\rangle), \tag{113}$$

$$|\phi_4\rangle = \frac{1}{\sqrt{N^2+M^2}}(M|++\rangle - Ne^{-i\Psi}|--\rangle). \tag{114}$$

We solve analytically the master equation by using the $\{|\phi_1\rangle, |\phi_2\rangle, |\phi_3\rangle, |\phi_4\rangle\}$ basis; however, we use the standard basis to calculate the concurrence. For simplicity we will consider $\Gamma=1$ throughout this section.

6.2. Solution for Initial States in Decoherence-Free Subspaces

(a) Consider $|\Phi_1(0)\rangle = |\phi_1\rangle$ the initial state. The solution of the master equation is $\rho_1(t) = |\phi_1\rangle\langle\phi_1|$. This corresponds to an invariant state, and its concurrence is

$$C(\rho_1(t)) = \frac{2\sqrt{N(N+1)}}{2N+1},$$

which is a constant in time. The concurrence depends only on N. For $N=0$, $|\Phi_1(0)\rangle = |--\rangle$, we have a factorized state at all times, but as we increase N, we get a maximally entangled state in the large N limit.

(b) If we now consider $|\Phi_2(0)\rangle = |\phi_2\rangle$ the initial state, the solution of the master equation is $\rho_2(t) = |\phi_2\rangle\langle\phi_2|$. This initial state is also an invariant state, and its concurrence is independent of time,

$$C(\rho_2(t)) = 1.$$

6.3. General Solution for Vacuum Reservoir, $N\to 0$

(a) The third initial state considered is $|\Phi_3(0)\rangle=|\phi_3\rangle$. Initially its concurrence is $C(\rho_3(0))=1$. It corresponds to a maximally entangled state. The solution of the master equation for this initial condition and $N=0$ is given by

$$\rho_3(t) = \begin{pmatrix} (e^{2t}-1)e^{-2t} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & e^{-2t} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{115}$$

Since the matrix $\rho(t)\tilde{\rho}(t)$ has only one nonzero eigenvalue, in this case we use the separability criterion [20]. According to this criterion, the necessary condition for separability is that the matrix ρ^{PT}, obtained by partial transposition of ρ, should have only nonnegative eigenvalues. In this particular case, we observe a negative eigenvalue for all times, so the state stays entangled (Fig. 8).

(b) Consider the initial state $|\Phi_4(0)\rangle=|\phi_4\rangle$. When $N=0$, $|\Phi_4(0)\rangle=|++\rangle$ and $C(\rho_4(0))=0$, since this is a factorized state. The solution of the master equation for this initial condition is given by

$$\rho_4(t) = \begin{pmatrix} (-1-2t+e^{2t})e^{-2t} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2te^{-2t} & 0 \\ 0 & 0 & 0 & e^{-2t} \end{pmatrix}, \tag{116}$$

and its concurrence is $C(\rho_4(t))=0$. In the following, we will consider superpositions with one component in the DFS and one orthogonal (to the DFS), of the form

$$|\Psi_a\rangle = \epsilon|\phi_1\rangle + \sqrt{1-\epsilon^2}|\phi_4\rangle, \tag{117}$$

Figure 8
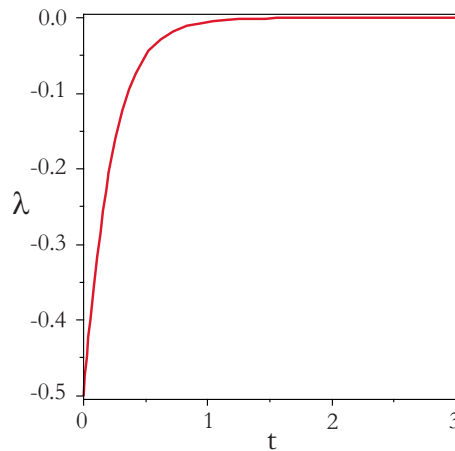
Negative eigenvalue (λ) of ρ^{PT}, for the separability criterion, for $|\phi_3\rangle$ as the initial state. This eigenvalue is always negative, indicating entanglement at all times.

$$|\Psi_b\rangle = \epsilon|\phi_2\rangle + \sqrt{1-\epsilon^2}|\phi_3\rangle, \tag{118}$$

where ϵ varies from zero to one. The idea is to increase ϵ and study the effect of having an increased component in the DFS on the death time of the entanglement.

(c) We consider an initial superposition of $|\phi_1\rangle$ and $|\phi_4\rangle$, i.e., $|\Psi_a\rangle$ given by Eq. (117). For $N=0$ we have

$$|\Psi_a(0)\rangle = \epsilon|--\rangle + \sqrt{1-\epsilon^2}|++\rangle, \tag{119}$$

and its initial concurrence is $C(\Psi_a(0))=2\epsilon\sqrt{1-\epsilon^2}$. The solution of the master equation for this initial condition is given by

$$\rho_a(t) = \begin{pmatrix} \dfrac{(-2t-1+2t\epsilon^2+\epsilon^2+e^{2t})}{e^{2t}} & 0 & 0 & \dfrac{\epsilon\sqrt{1-\epsilon^2}}{e^t} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \dfrac{2t(1-\epsilon^2)}{e^{2t}} & 0 \\ \dfrac{\epsilon\sqrt{1-\epsilon^2}}{e^t} & 0 & 0 & \dfrac{(1-\epsilon^2)}{e^{2t}} \end{pmatrix}, \tag{120}$$
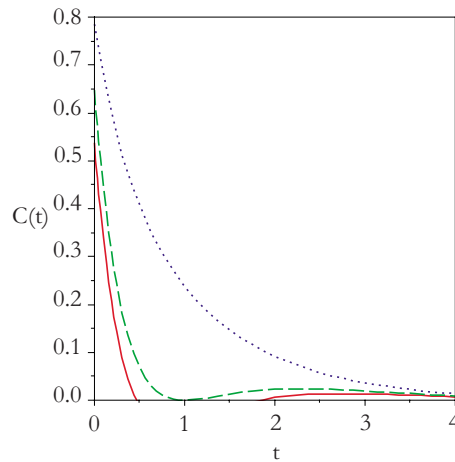
and the corresponding concurrence is given by

$$C(\rho_a) = \max\{0, 2((\epsilon\sqrt{1-\epsilon^2})e^{-t} - te^{-2t}(1-\epsilon^2))\}, \tag{121}$$

which is shown in Fig. 9 for various values of ϵ. For $\epsilon > 0$, the initial entanglement decreases in time, and the system becomes disentangled (sudden death) at a time satisfying the relation

$$te^{-t} = \frac{\epsilon}{\sqrt{1-\epsilon^2}}. \tag{122}$$

Figure 9

Time evolution of the concurrence for initial $|\Psi_1(0)\rangle$ with $\epsilon=0.28$ (solid curve), $\epsilon=0.345$ (dashed curve), $\epsilon=0.9$ (dotted curve),

- For $\epsilon=0$ and $\epsilon=1$, the concurrence is zero, therefore we have a non-entangled state.
- For $0<\epsilon<0.34525$ Eq. (122) has two solutions, namely, t_d when the system becomes separable, and $t_r \geq t_d$ when the entanglement revives. It should be noted that there is a critical ϵ for which $t_d=t_r$.
- For $0.34525<\epsilon<1$ the above equation has no solution, and the concurrence vanishes asymptotically in time.Thus, when we are not far from $|\phi_4\rangle$ we observe a sudden death and revival, but when we get near $|\phi_1\rangle$ this phenomenon disappears. Figure 10 shows the behavior of the death and revival time as function of ϵ.

(d) Finally, we consider an initial superposition of $|\phi_2\rangle$ and $|\phi_3\rangle$: $|\Psi_b(0)\rangle = \epsilon|\phi_2\rangle + \sqrt{1-\epsilon^2}|\phi_3\rangle$ [Eq. (118)], which is independent of N. Here, as in the previous cases, as we increase ϵ, starting from $\epsilon=0$, we increase the initial projection onto the DFS. For $\epsilon=1$ the initial state is in the DFS plane. For $N=0$ we have

$$|\Psi_b(0)\rangle = \frac{1}{\sqrt{2}}[(\epsilon + \sqrt{1-\epsilon^2})|-+\rangle - (\epsilon - \sqrt{1-\epsilon^2})|+-\rangle], \qquad (123)$$
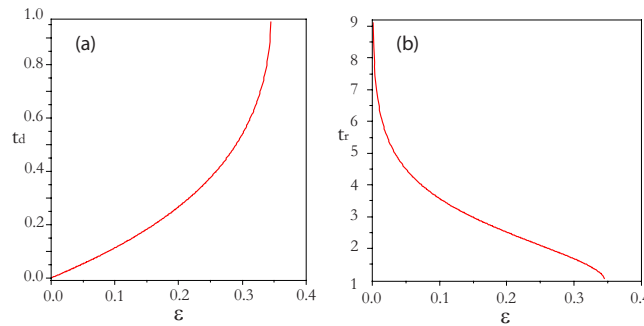
and its initial concurrence is $C(\Psi_b(0))=|2\epsilon^2-1|$.The solution of the master equation for this initial condition is given by

$$\rho_b(t) = \begin{pmatrix} \dfrac{(e^{2t} - \epsilon^2 e^{2t} - 1 + \epsilon^2)}{e^{2t}} & 0 & 0 & 0 \\ 0 & \epsilon^2 & \dfrac{\epsilon\sqrt{1-\epsilon^2}}{e^t} & 0 \\ 0 & \dfrac{\epsilon\sqrt{1-\epsilon^2}}{e^t} & \dfrac{(1-\epsilon^2)}{e^{2t}} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad (124)$$

and the corresponding concurrence is

$$C(\rho_b(t)) = \max\{0, e^{-2t}|\epsilon^2 e^{2t} - 1 + \epsilon^2|\}, \qquad (125)$$

Figure 10

(a) Death time. (b) Revival time of the entanglement as a function of ϵ, with initial $|\Psi_a\rangle$.

which is shown in Fig. 11(a).For $0 < \epsilon < 0.707$ the initial entanglement decreases in time, and the system becomes disentangled at a time given by (see Fig. 11(b))

$$t = \frac{1}{2} \ln\left(\frac{1 - \epsilon^2}{\epsilon^2}\right). \tag{126}$$

However, at the same time, the entanglement revives, asymptotically reaching its stationary value. This means that the sudden death and revival happen simultaneously. The phenomena of one or periodical revivals have been obtained before, but always in the context of one single reservoir connecting both atoms, as in the present case [69,86,87]. When we approach the DFS this phenomenon disappears.

Next, we treat the cases with $N > 0$.

6.4. General Solution for $N \neq 0$

In general, for both $|\Psi_a\rangle$ and $|\Psi_b\rangle$ as initial states, the evolution of the concurrence $C(\rho(t))$ is calculated in the standard basis but written in terms of density matrix $\rho'(t)$ in the $\{\phi_i\}$ basis as $C(\rho'(t)) = \max\{0, C_1(\rho'(t)), C_2(\rho'(t))\}$, where

$$C_1(\rho'(t)) = |\rho'_{33}(t) - \rho'_{22}(t)|$$
$$- 2\sqrt{\frac{(N(\rho'_{11}(t) + \rho'_{44}(t)) + \rho'_{44}(t) + 2\rho'_{14}(t)\sqrt{N(N+1)})}{2N+1}}$$
$$\times \sqrt{\frac{(N(\rho'_{11}(t) + \rho'_{44}(t)) + \rho'_{11}(t) - 2\rho'_{14}(t)\sqrt{N(N+1)})}{2N+1}},$$
$$\tag{127}$$

$$C_2(\rho'(t)) = \frac{2}{2N+1}|\sqrt{N(N+1)}(\rho'_{11}(t) - \rho'_{44}(t)) + \rho'_{14}(t)|$$
$$- \sqrt{(\rho'_{22}(t) - 2\rho'_{23}(t) + \rho'_{33}(t))(\rho'_{22}(t) + 2\rho^e_{23}(t) + \rho'_{33}(t))}, \tag{128}$$

where $\rho'_{ij}(t)$ are the density matrix elements in the $\{\phi_i\}$ basis.
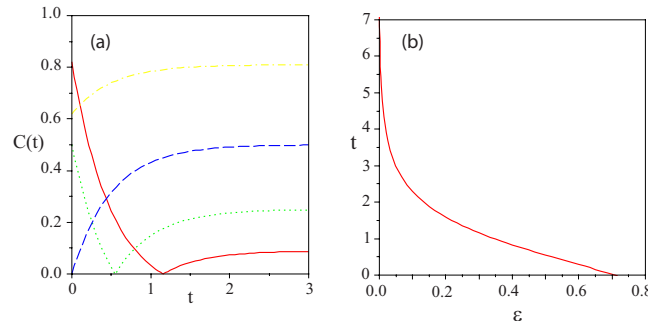
(a) Time evolution of the concurrence with initial $|\Psi_b\rangle$ for $\epsilon = 0.3$ (solid curve), $\epsilon = 0.5$ (dotted), $\epsilon = 0.707$ (dashed), $\epsilon = 0.9$ (dashed–dotted). (b) Death–revival time, as given by Eq. (130), versus ϵ.

(a) Next, we again consider the case for initial $|\phi_3\rangle$. In Fig. 12, we show the evolution of the concurrence for various values of N. We always observe sudden death in a finite time; then the concurrence remains zero for a period of time until the entanglement revives, and the concurrence asymptotically reaches its stationary value. Notice that this time period increases with N. In Fig. 13 we show the death and revival times versus N. They decrease and increase with N, respectively.

(b) We consider $|\phi_4\rangle$ as an initial state. The behavior of concurrence is similar to that in the $|\phi_3\rangle$ case. The initial entanglement quickly decays to zero, getting disentanglement for a finite time interval; then the entanglement revives, and it asymptotically reaches its stationary value. However, unlike the case with initial state $|\phi_3\rangle$, the death time first increases reaching a maximum for $N=0.421$, and subsequently it decreases, as shown in Fig 14. The revival time has the same behavior as in $|\phi_3\rangle$.

(c) In the following case, we consider the superposition

$$|\Psi_a(0)\rangle = \epsilon|\phi_1\rangle + \sqrt{1-\epsilon^2}|\phi_4\rangle \tag{129}$$

as the initial state. The solution of the master equation for this initial condition depends on ϵ and N and also its concurrence. In this case, $\rho'_{23}(t)=\rho'_{22}(t)=0$; thus, the concurrence is given by $C(\rho'_a(t))=\max\{0,C_2(\rho'_a(t))\}$, where $C_2(\rho'_a(t))$ given in Eq. (128).Initially, the concurrence is given by

$$C(\rho'_a(0)) = \frac{\left|2\epsilon\sqrt{1-\epsilon^2} + 4\sqrt{N(N+1)}\left(\epsilon^2 - \frac{1}{2}\right)\right|}{2N+1}. \tag{130}$$

We see that for certain pairs of N and ϵ, our initial state will be a nonseparable one. In Fig. 15 we show the time evolution of the concurrence for $N=0.1$ and several values of ϵ.For $\epsilon=0$ and $\epsilon=1$ we retrieve $|\phi_4\rangle$ and $|\phi_1\rangle$ respectively. For $0 < \epsilon < 0.5$ the concurrence dies in a finite time, stays zero for a time interval, and subsequently revives, going asymptotically to its stationary value. For values larger than

Figure 12
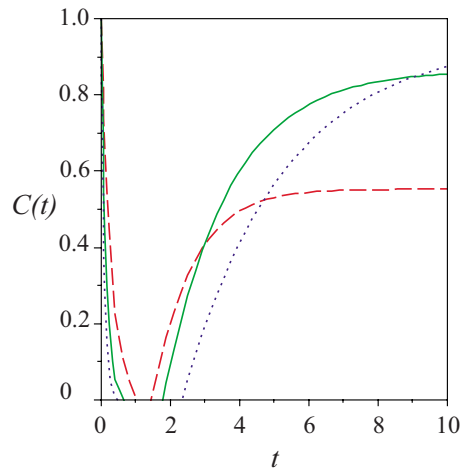
Time evolution of concurrence for initial $|\phi_3\rangle$, with $N=0.1$ (dashed curve), $N=0.5$ (solid), $N=1$ (dotted).

Figure 13
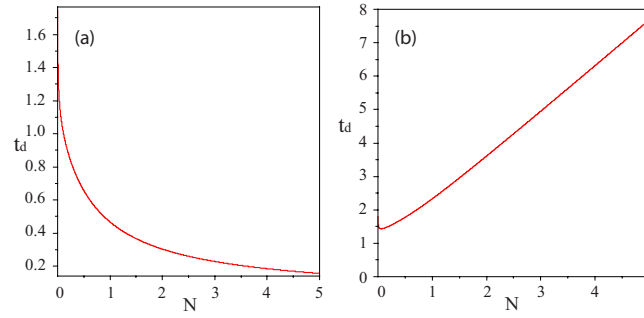
(a) Death time. (b) Revival time versus N for the initial state $|\phi_3\rangle$.

$\epsilon=0.5$, there is no more sudden death, since we are getting close to the DFS, and the concurrence goes asymptotically to its stationary value. Figure 16(a) shows the death times versus ϵ for $N=\{0,0.1,0.2\}$. In the case $N=0$, we notice a steady increase of the death time up to some critical value of ϵ, where the death time becomes infinite. There is a curious effect, that for $N\neq0$, as we increase ϵ: the death time first decreases up to the value $\epsilon=\sqrt{N/2N+1}$, and subsequently it behaves as usual, increasing with ϵ. In Fig. 16(b) we show the revival time as a function of ϵ for the same values of N. In all cases the revival time decreases with ϵ.

(d) Finally, we consider the case with initial

$$|\Psi_b(0)\rangle = \epsilon|\phi_2\rangle + \sqrt{1-\epsilon^2}|\phi_3\rangle. \tag{131}$$

Its concurrence is $C(\rho_b'(t))=\max\{0,C_1(\rho_b'(t)),C_2(\rho_b'(t))\}$, with C_1 and C_2 defined in Eqs. (127) and (128), and its initial value: $C(\rho_b'(0))=|2\epsilon^2-1|$. In Fig. 17, we show the time evolution of the concurrence with $N=0.1$ for several values of ϵ. As we can see from Fig. 17, this case is more complex, since there are more than one death and revival before reaching the critical value of ϵ. Such a situation has been described previously [69,86]. As in the previous cases, above a certain critical ϵ, when we get close to the DFS, these effects disappear and $C(t)$ goes asymptotically to its stationary value.
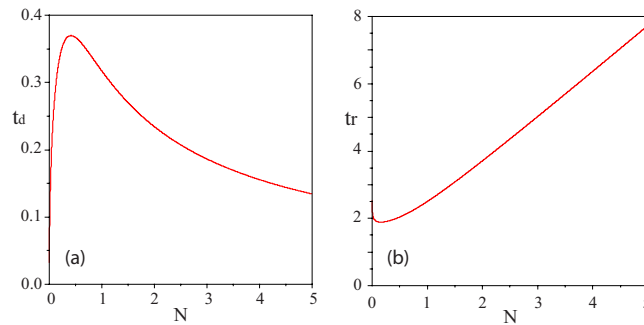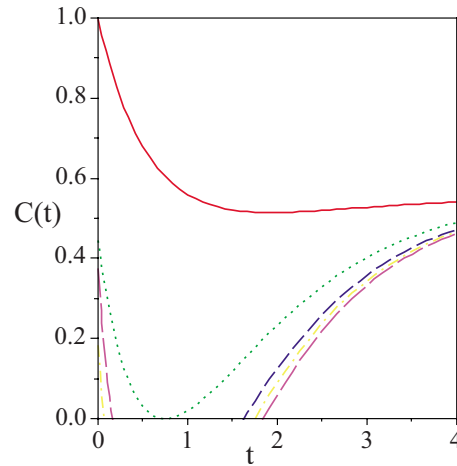
Figure 14

(a) Death time. (b) Revival time versus N for the initial state $|\phi_4\rangle$.

Time evolution of concurrence for $|\Psi_a(t)\rangle$ as initial state and $N=0.1$: $\epsilon=0.1$ (long dashed curve), $\epsilon=0.2$ (dashed–dotted), $\epsilon=0.29$ (dashed), $\epsilon=0.5$ (dotted), $\epsilon=0.9$ (solid).

6.5. Discussion

The first observation is that if we start from an initial state that is in the DFS plane, the local and nonlocal coherences are not affected by the environment; thus the system experiences no decoherence, and the concurrence stays constant in time.

In the case of initial $|\phi_1\rangle$, the concurrence does increase with the squeeze parameter N, getting maximum entanglement for $N\rightarrow\infty$. So this reservoir is not acting as a thermal one, in the sense that introduces randomness. On the contrary, a common squeezed bath tends to enhance the entanglement, as we increase the parameter N.
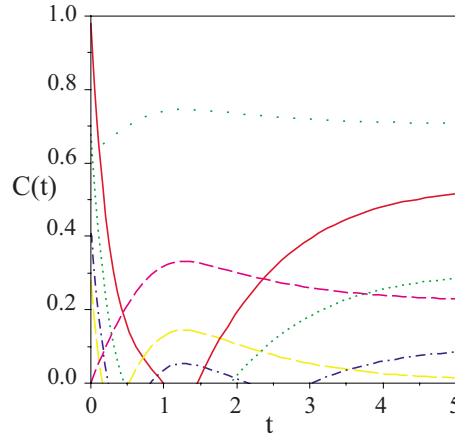
This is clear if we observe that for $N\rightarrow\infty$
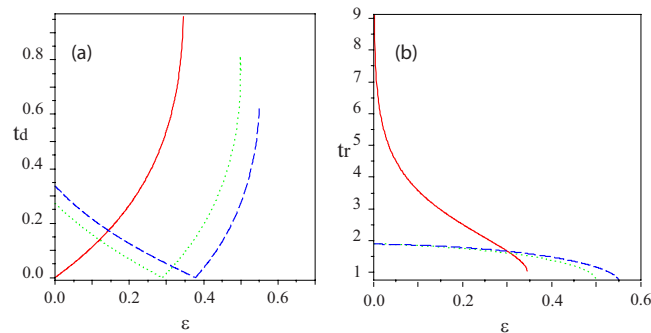
(a) Death time. (b) Revival time, with initial state $|\Psi_a\rangle$ and $N=0$ (solid curve), $N=0.1$ (dotted), $N=0.2$ (dashed).

Figure 17

Time evolution of the concurrence for $|\Psi_2(t)\rangle$ as initial state and $N=0.1$: $\epsilon=0$ (solid curve), $\epsilon=0.4$ (closely spaced dotted), $\epsilon=0.54$ (dashed-dotted line), $\epsilon=0.6$ (long dashed), $\epsilon_c=0.707$ (dashed), $\epsilon=0.9$ (widely spaced dotted).

$$|\phi_1\rangle \to \frac{1}{\sqrt{2}}(|++\rangle + |--\rangle), \tag{132}$$

which is a Bell state.

On the other hand, if we start with the initial state $|\phi_2\rangle$, this state is independent of N, and it is also maximally entangled, so $C=1$ for all times and all numbers N'.

Now, we consider other situations with initial states outside the DFS. We consider as initial states the superpositions given in (117) and (118), where we vary ϵ between 0 and 1 for a fixed value of the parameter N. We observe that

- When $\epsilon=0$, we recover $|\phi_4\rangle$ and $|\phi_3\rangle$. In the $|\phi_4\rangle$ case, the population of the $|++\rangle$ component goes down with N, meaning that the interaction with the reservoir goes also down with N, and therefore the death time will necessarily increase with N, which qualitatively describes the first part of the curve [Fig 14(a)]. Furthermore, as we increase the average photon number N, other processes such as the two-photon absorption will be favored; since there will be more photons and the $|--\rangle$ population tends to increase with N, this will enhance the system–bath interaction, and therefore the death of the entanglement will occur faster, or the death time will decrease. In the $|\phi_3\rangle$ case, initially there is no $|++\rangle$; thcomponentus we expect a higher initial death time. However, this case is different from the previous one in the sense that the state is independent of N, so there is no initial increase. However, as the state evolves in time, the $|++\rangle$ and $|--\rangle$ components will build up, and the argument for the decrease of the death time with N follows the same logic as in the previous case. See Fig. 13(a).
- For the interval $0 \le \epsilon < \epsilon_c$ the initial entanglement decays to zero in a finite time, t_d. After a finite period of time during which the concurrence stay null, the entanglement revives at time t_r, asymptotically

reaching its steady-state value. This death and revival cycle occurs once for the initial state $|\Psi_a\rangle$ and twice for $|\Psi_b\rangle$. For $|\Psi_a\rangle$ as initial state, when ϵ is equal to the critical value ϵ_c, the entanglement dies and revives simultaneously and eventually goes to its steady-state value. For the initial state $|\Psi_b\rangle$, the critical value of ϵ is $\epsilon_c=1/\sqrt{2}$, and, unlike the $|\Psi_a\rangle$ case, it is independent of N.

- When we get *near* the DFS ($\epsilon_c < \epsilon \leq 1$), the system shows no disentanglement, and this phenomenon of sudden death and revival disappears.

The squeezed vacuum has only nonzero components for even number of photons, so the interaction between our system and the reservoir goes by pairs of photons. Now, for very small N, the average photon number is also small, so the predominant interaction with the reservoir will be the doubly excited state that would tend to decay via two-photon spontaneous emission.

Let us consider again $|\Psi_a\rangle$, but now in terms of the standard basis

$$|\Psi_a\rangle = k_1|++\rangle + k_2|--\rangle,$$

with

$$k_1 = \frac{\epsilon N + M\sqrt{1-\epsilon^2}}{\sqrt{N^2+M^2}}, \quad k_2 = \frac{\epsilon M - N\sqrt{1-\epsilon^2}}{\sqrt{N^2+M^2}}. \tag{133}$$

We plotted k_1 versus ϵ for N between 0 to 2 in Fig. 18. Initially, k_1 increases with ϵ, thus favoring the coupling with the reservoir, or equivalently, producing a decrease in the death time. This is true up to $\epsilon=\sqrt{N/2N+1}$, where the curve shows a maxima. Beyond this point, k_1 starts to decrease, and therefore our system is slowly decoupling from the bath, and therefore the death time shows a steady increase.

6.6. A Curious Effect: Creation of Delayed Entanglement

In some cases, if two systems do not interact directly, but share a common heat bath in thermal equilibrium, entanglement can be created some time after the in-

Figure 18
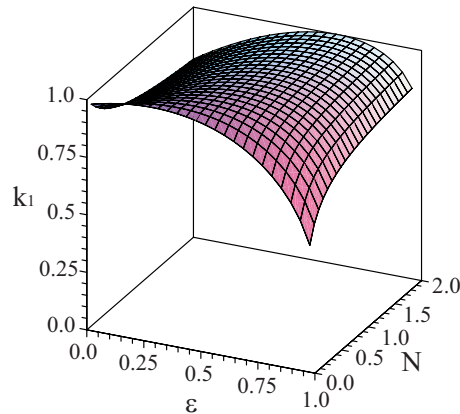
k_1 versus ϵ for N between 0 and 2.

teraction is turned on. Furthermore, this entanglement may persist in time. So, contrary to the intuition that spontaneous emission should have a destructive effect on the entanglement, it has been shown by several authors that under certain conditions this irreversible process can even entangle initially unentangled qubits [72–74,88–93].

Also, more recently, it was shown that a suddenness in the creation of entanglement exists, in the sense that it takes some finite time after the system is connected to the bath for the creation of entanglement to take place [75].

In the following we considered the simplest composite quantum system of two qubits. These qubits are coupled to a common thermal reservoir at zero temperature. The qubits do not interact directly with each other, but through the common environment; in this case there is a probability that a photon emitted by one atom will be absorbed by the other, and this photon exchange process can produce entanglement between atoms, which is larger than the decoherence caused by spontaneous emission. According to this assumption, it is crucial to have one atom excited and the other in the ground state to create entanglement. The dynamics of the system is given by Eq. (96) with N and M equal to zero.

For a given initial state $\rho(0) = \rho_{ij}(0)$ the solution of master equation (96) in the $\{e_1 = |11\rangle, e_2 = |10\rangle, e_3 = |01\rangle, e_4 = |00\rangle\}$ basis is given by the following matrix elements:

$$\rho_{11}(t) = \rho_{11}e^{-2\Gamma t},$$

$$\rho_{22}(t) = \frac{1}{4}(\rho_{22} + \rho_{33} - \rho_{23} - \rho_{32} + 2(\rho_{22} - \rho_{33})e^{-\Gamma t} + (4\rho_{11}t + \rho_{22} + \rho_{33} + \rho_{23}$$
$$+ \rho_{32})e^{-2\Gamma t}),$$

$$\rho_{33}(t) = \frac{1}{4}(\rho_{22} + \rho_{33} - \rho_{23} - \rho_{32} + 2(\rho_{33} - \rho_{22})e^{-\Gamma t} + (4\rho_{11}t + \rho_{22} + \rho_{33} + \rho_{23}$$
$$+ \rho_{32})e^{-2\Gamma t}),$$

$$\rho_{44}(t) = \frac{1}{2}(\rho_{23} + \rho_{32} + \rho_{11} + \rho_{44} + 1 - (4\rho_{11}t + 2\rho_{11} + \rho_{22} + \rho_{33} + \rho_{23} + \rho_{32})e^{-2\Gamma t}),$$

$$\rho_{12}(t) = \frac{1}{2}((\rho_{12} - \rho_{13})e^{-\Gamma t} + (\rho_{12} + \rho_{13})e^{-2\Gamma t}),$$

$$\rho_{13}(t) = \frac{1}{2}(-(\rho_{12} - \rho_{13})e^{-\Gamma t} + (\rho_{12} + \rho_{13})e^{-2\Gamma t}),$$

$$\rho_{14}(t) = \rho_{14}e^{-\Gamma t},$$

$$\rho_{22}(t) = \frac{1}{4}(\rho_{22} + \rho_{33} - \rho_{23} - \rho_{32} + 2(\rho_{22} - \rho_{33})e^{-\Gamma t} + (4\rho_{11}t + \rho_{22} + \rho_{33} + \rho_{23}$$
$$+ \rho_{32})e^{-2\Gamma t}),$$

$$\rho_{23}(t) = \frac{1}{4}(\rho_{23} + \rho_{32} - \rho_{22} - \rho_{33} + 2(\rho_{23} - \rho_{32})e^{-\Gamma t} + (4\rho_{11}t + \rho_{22} + \rho_{33} + \rho_{23}$$
$$+ \rho_{32})e^{-2\Gamma t}),$$

$$\rho_{24}(t) = \frac{1}{2}(\rho_{24} - \rho_{34} - 2(\rho_{12} + \rho_{13})e^{-2\Gamma t} + (2\rho_{12} + 2\rho_{13} + \rho_{34} + \rho_{24})e^{\Gamma t}),$$

$$\rho_{34}(t) = \frac{1}{2}(\rho_{34} - \rho_{24} - 2(\rho_{12} + \rho_{13})e^{-2\Gamma t} + (2\rho_{12}t + 2\rho_{13} + \rho_{34} + \rho_{24})e^{-\Gamma t}).$$

$$(134)$$

For simplicity, we write each matrix element $\rho_{ij}(0)$ as ρ_{ij} and take $\Gamma = 1$. The remaining matrix elements can be obtained from $\rho_{ij} = \rho_{ji}^*$.

To obtain a condition that determines when a state that was initially separable becomes nonseparable after a certain period of time, we will study the stationary state. In the limit $t \to \infty$, the density matrix is given by

$$\rho_{st} = \frac{1}{4}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \rho_{22} + \rho_{33} - (\rho_{23} + \rho_{32}) & \rho_{23} + \rho_{32} - (\rho_{22} + \rho_{33}) & 2(\rho_{24} - \rho_{34}) \\ 0 & \rho_{23} + \rho_{32} - (\rho_{22} + \rho_{33}) & \rho_{22} + \rho_{33} - (\rho_{23} + \rho_{32}) & 2(\rho_{34} - \rho_{24}) \\ 0 & 2(\rho_{42} - \rho_{43}) & 2(\rho_{43} - \rho_{42}) & 2(\rho_{23} + \rho_{32}) + 2(1 + \rho_{11} + \rho_{44}) \end{pmatrix}.$$

Following the procedure to get the concurrence (see Section [2]), we calculate the matrix $R_{st} = \rho_{st}(\sigma_y \otimes \sigma_y)\rho_{st}^*(\sigma_y \otimes \sigma_y)$. For real matrix elements, R_{st} has only one nonzero eigenvalue:

$$\lambda = \frac{1}{4}(\rho_{22} + \rho_{33} - 2\,\mathrm{Re}(\rho_{23}))^2. \qquad (135)$$

Thus, the condition to have a disentangled steady state is

$$\rho_{22} + \rho_{33} = 2\rho_{23}. \qquad (136)$$

Now, we will consider some examples of different initial separable states.

(a) Let us first have two atoms in excited state $|\Psi(0)\rangle = |11\rangle$. The concurrence is

$$C(t) = \max\{0, 2te^{-2t} - 2\sqrt{e^{-2t}(1 - (1 + 2t)e^{-2t})}\} = 0. \qquad (137)$$

Thus, this state is separable for all times. In the stationary state, both atoms go to the ground state. Thus its concurrence is $C(t) = 0$.

(b) For the case when one atom is in an excited state $|1\rangle$ and the other one in the ground state $|0\rangle$, the concurrence is

$$C(t) = \max\left\{ 0, \frac{1}{2}\left|e^{-2t} - 1\right| \right\}, \tag{138}$$

this result being the same for $|10\rangle$ and $|01\rangle$. The initial value of the concurrence increases to its steady-state value of $\frac{1}{2}$.

(c) When both atoms are initially in the ground state, $|\Psi(0)\rangle = |00\rangle$, the system remains in its ground state for all times, and the concurrence is $C(t) = 0$. As mentioned above, the only way to generate entanglement (for the vacuum reservoir) is when one atom is in the ground state and the other one in the excited state, since this combination makes photon exchanges between the atoms possible. In the cases when there is no photon exchange, the entanglement generation never occurs. To study this effect we consider linear combinations of the $|0\rangle$ and $|1\rangle$ states, for one of the atoms, and the ground or the excited state for the other one, respectively.

(d) Consider first

$$|\Psi(0)\rangle = |0\rangle \otimes (\alpha|1\rangle + \beta|0\rangle) = \alpha|01\rangle + \beta|00\rangle, \tag{139}$$

with $\alpha^2 + \beta^2 = 1$. Figure 19(a) shows that the entanglement reaches a higher value when α is greater.

(e) The second possible initial condition is

$$|\Psi(0)\rangle = (\alpha|0\rangle + \beta|1\rangle) \otimes |1\rangle = \alpha|01\rangle + \beta|11\rangle, \tag{140}$$

with $\alpha^2 + \beta^2 = 1$. Figure 19(b) shows the time evolution of the concurrence for different values of α. As in the previous case, when α decreases, the maximum value of the entanglement decreases, but unlike in the previous case, the entanglement creation has a time delay.

(f) Consider a more general initial condition

$$(\alpha_1|0\rangle + \beta_1|1\rangle) \otimes (\alpha_2|0\rangle + \beta_2|1\rangle). \tag{141}$$

According to Eq. (135), the condition to get a completely disentangled stationary state is

Figure 19
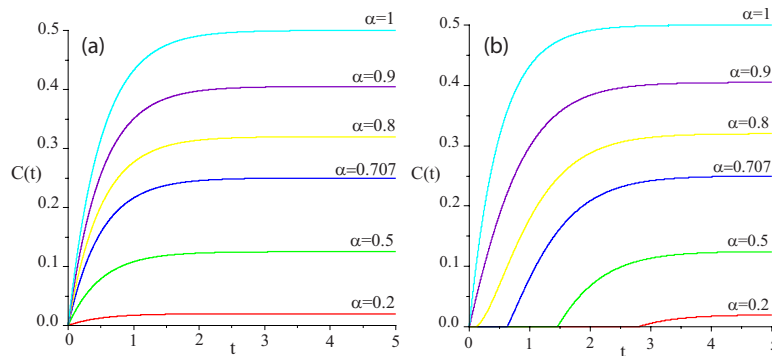
(a) Time evolution of concurrence for initial $|0\rangle \otimes (\alpha|1\rangle + \beta|0\rangle)$ with different values of α. (b) Time evolution of concurrence for initial $(\alpha|0\rangle + \beta|1\rangle) \otimes |1\rangle$ with different values of α. The reservoir is at $T=0$.

$$\frac{\alpha_1}{\beta_1} = \frac{\alpha_2}{\beta_2}, \tag{142}$$

implying that when both qubits are in the same state, the created entanglement is eventually destroyed, at large times. We show this effect in the Fig. 20, where we plotted the evolution of the concurrence for the various initial conditions in Eq. (141) when $\alpha_1 = \alpha_2 \equiv \alpha$.

Finally, we analyze the effects of having a reservoir with a finite (nonzero) temperature. In a vacuum reservoir the initial states $|\Psi(0)\rangle = |11\rangle$ and $|\Psi(0)\rangle = |00\rangle$ are kept separable at all times. However it is noted that the concurrence increases with the temperature, since thermal excitation may help in the generation of entanglement; see Fig. 21. We observe in both cases an entanglement that is actually enhanced by increasing the reservoir temperature, and even the minimum temperature rise is enough to generate entanglement.

The opposite is true for the cases $|\Psi(0)\rangle = |10\rangle$ and $|\Psi(0)\rangle = |01\rangle$ (see Fig. 22), where the one photon exchange, necessary for the entanglement in the case of the vacuum reservoir, is disturbed by the presence of thermal photons.

Figure 20
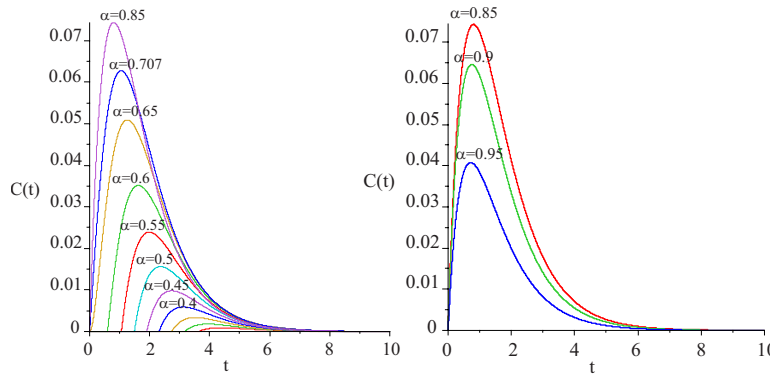
Concurrence versus time for different initial conditions $\alpha_1 = \alpha_2 \equiv \alpha$. The reservoir is at $T=0$.

Figure 21
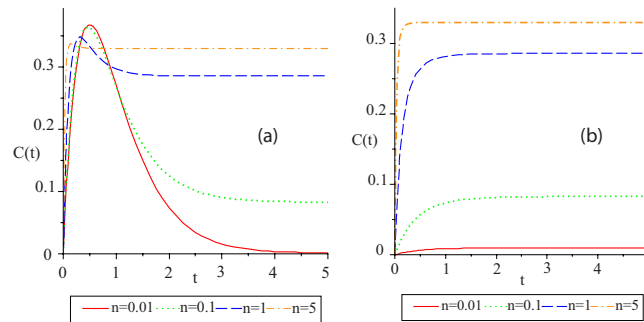
Evolution of the concurrence for the initial conditions (a) $|11\rangle$, and (b) $\|00\rangle$, and different values of n.

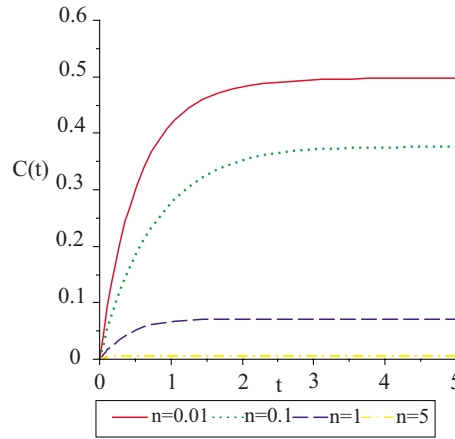Figure 22

Concurrence versus time for the initial conditions $|\Psi(0)\rangle=|10\rangle$ and $|\Psi(0)\rangle=|01\rangle$. The parameter of each curve is the reservoir temperature (average thermal photon number).

7. Other Models

A somewhat more general model was studied by Ficek and Tanás [69], where the Lehmberg–Agarwal master equation was used [94,95], which includes both collective damping and dipole–dipole interaction, namely,

$$\frac{\partial\rho}{\partial t}=-i\omega_0\sum_{i=1}^{2}[\sigma_i^z,\rho]-i\sum_{i=1}^{2}\Omega_{ij}[\sigma_i^+\sigma_i^+,\rho]-\frac{1}{2}\sum_{i=1}^{2}\gamma_{ij}([\rho\sigma_i^+,\sigma_j^-]+[\sigma_i^+,\sigma_j^-\rho]),$$

where $\gamma_{ii}\equiv\gamma$ are the spontaneous rates of the atoms, caused by their direct coupling to the vacuum field, and both γ_{ij} and Ω_{ij} depend on the distance between the atoms. The collective damping γ_{ij} is given by

$$\gamma_{ij}=\frac{3}{2}\gamma\left[\frac{\sin(kr_{ij})}{(kr_{ij})}+\frac{\cos(kr_{ij})}{(kr_{ij})^2}-\frac{\sin(kr_{ij})}{(kr_{ij})^3}\right],$$

and the dipole–dipole interaction Ω_{ij} is

$$\Omega_{ij}=\frac{3}{4}\gamma\left[-\frac{\cos(kr_{ij})}{(kr_{ij})}+\frac{\sin(kr_{ij})}{(kr_{ij})^2}+\frac{\cos(kr_{ij})}{(kr_{ij})^3}\right],$$

with $k=\omega_0/c$ and $r_{ij}=|\mathbf{r}_i-\mathbf{r}_j|$ being the relative distance between the atoms.

For an initial state $(\sqrt{p}|11\rangle+\sqrt{1-p}|00\rangle)$, as discussed in the previous section, under certain initial conditions, sudden death occurs. It is also observed that the evolution of the system depends crucially on the initial conditions and, in the present example, is independent of the dipole–dipole interaction between the atoms, but is strongly affected by the collective damping γ_{12}. In the particular case when the atoms radiate independently, that is, $\gamma_{12}=0$, then the initial entanglement dies after some finite time

$$t_d = \frac{1}{\gamma} \ln\left(\frac{p + \sqrt{p(1-p)}}{2p - 1} \right).$$

As we can see, this death time is a sensitive function of the initial condition, and sudden death for independent atoms is possible only if $p > \frac{1}{2}$, which agrees with the results of Section 5. A revival is also observed, after a short disentanglement time, for the case $r_{12}/\lambda = \frac{1}{20}$, which agrees with the results shown in Section 6, corresponding to the common reservoir.

Finally, Ficek and Tanás [75] find that for some separable nonsymmetrical initial state, and depending on (r_{12}/λ) and the angle between the excitation direction and $\mathbf{r}_{12}$, sudden birth of entanglement is created after some time delay.

Qualitatively, these results agree with the previous examples presented in this review, particularly, in Subsection 6.6, related to the common reservoir $(r_{12}/\lambda \ll 1)$, where for a separable initial state that is not symmetrical a delayed generation of entanglement was found, the delay in this case being a very sensitive function of the initial condition .

8. Conclusion

In this review, we gave an overall picture of decoherence and disentanglement, first, in the context of a general discussion, and later, giving particular examples related to two qubits in contact with different types of reservoirs. We included Dicke-type models, without a direct interaction between atoms, and also other models that include dipole–dipole interaction and collective damping effects.

Entanglement is in the heart of quantum mechanics. Also, besides its conceptual relevance, it is also crucial in many applications in quantum communications and quantum computing. On the other hand, it is a subtle and delicate effect that can be easily altered or destroyed. In the various cases studying the dynamics of entanglement, we encountered different situations, described in the Fig. 23,

Figure 23
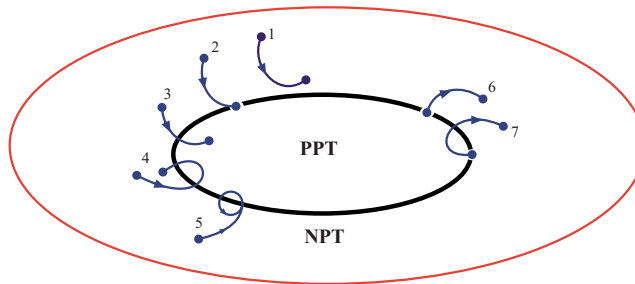
Trajectories between the PPT (separable) and the NPT (entangled) areas: 1, Initial and final states are both entangled; 2, Initial entanglement going asymptotically to zero; 3, Initial entanglement with sudden death; 4, Initial entanglement with sudden death and revival; 5, Periodic death and revival; 6, Entanglement generation starting from a separable state; 7, Time delayed generation of entanglement starting from a separable state.

where we show trajectories between the PPT and the NPT areas. The first case corresponds to a system that has both the initial and final states entangled. In the second case, the initial entanglement goes asymptotically to zero. In the third and fourth cases, we observe sudden death and death with revival, respectively. Case number 5 corresponds to a periodic Hamiltonian system (for example, the Jaynes–Cummings model), where death and revival appear periodically. Death and revival are intriguing effects that basically depend on the initial conditions as well as the nature of the reservoir. Since, for many applications, sudden death is an undesired effect, one can try to protect these states, using, for example, the distillation procedures.

Finally, in cases 6 and 7, we have generation of entanglement without and with time delay, respectively, starting from a separable state. These are very interesting effects where, in spite of the fact that there is no coupling between the atoms, the common reservoir acts as an effective coupling, with particularly good results, in the case of a vacuum reservoir, when the initial atomic state combines an excited atom with the other one in the ground state, thus allowing, via a common reservoir, a photon exchange. When the reservoir is at a finite temperature, this effect is reduced.

Acknowledgments

References

1. E. Schrödinger, "Discussion of probability relations between separated systems," Proc. Cambridge Philos. Soc. **31**, 555–563 (1935).
2. A. Einstein, B. Podolosky, and N. Rosen, "Can quantum-mechanical description of physical reality be considered complete?," Phys. Rev. **47**, 777–780 (1935).
3. M. Born, *The Born–Einstein Letters 1916–1955* (Macmillan, 2005).
4. J. S. Bell, "On the Einstein–Podolsky–Rosen paradox," Physics **1**(3), 195–200 (1964).
5. A. Aspect, J. Dalibard, and G. Roger, "Experimental test of Bell's inequalities using time-varying analyzers," Phys. Rev. Lett. **49**, 1804–1807 (1982).
6. C. H. Thompson and H. Holstein, "The chaotic ball model: local realism and Bell test "detection loophole," arXiv.org, arXiv:quant-ph/0210150 (2002).
7. E. Santos, "Optical tests of Bell's inequalities not resting upon the absurd fair sampling assumption," arXiv.org, arXiv:quant-ph/0401003 (2004).
8. C. H. Bennett, G. Brassard, C. Crepeau, R. Jozsa, A. Peres, and W. K. Wootters, "Teleporting an unknown quantum state via dual classical and Einstein–Podolsky–Rosen channels," Phys. Rev. Lett. **70**, 1895–1899 (1993).
9. A. K. Ekert, "Quantum cryptography based on Bell's theorem," Phys. Rev. Lett. **67**, 661–663 (1991);
10. D. Deutsch, A. K. Ekert, R. Jozsa, C. Macchiavello, S. Popescu, and A. San-

pera, "Quantum privacy amplification and the security of quantum cryptography over noisy channels," Phys. Rev. Lett. **77**, 2818–2821 (1996).

11. C. H. Bennett and S. J. Wiesner, "Communication via one- and two-particle operators on Einstein–Podolsky–Rosen states," Phys. Rev. Lett. **69**, 2881–2884 (1992).

12. J. H. Eberly, "Schmidt analysis of pure-state entanglement," arXiv.org, arXiv:quant-ph/0508019 (2005).

13. S. J. Freedman and J. F. Clauser, "Experimental test of local hidden-variable theories," Phys. Rev. Lett. **28**, 938–941 (1972).

14. J. H. Eberly, "Bell inequalities in quantum mechanics," Am. J. Phys. **70**, 276–279 (2002).

15. M. Orszag, *Quantum Optics* (Springer, 2000).

16. R. Grobe, K. Rzazewski, and J. H. Eberly, "Measure of electron–electron correlation in atomic physics," J. Phys. B **27**, L503–L508 (1994).

17. A. Barenco, C. H. Bennett, R. Cleve, D. P. DiVicenzo, N. Margolus, P. Shor, T. Sleator, J. A. Smolin, and H. Weinfurter, "Elementary gates for quantum computation," Phys. Rev. A **52**, 3457–3467 (1995).

18. D. Bruss, "Characterizing entanglement," J. Math. Phys. **43**, 4237–4251 (2002).

19. R. Werner, "Quantum states with Einstein–Podolsky–Rosen correlations admitting a hidden-variable model," Phys. Rev. A **40**, 4277–4281 (1989).

20. A. Peres, "Separability criterion for density matrices," Phys. Rev. Lett. **77**, 1413–1415 (1996).

21. M. Horodecki, P. Horodecki, and R. Horodecki, "Separability of mixed states: necessary and sufficient conditions," Phys. Lett. A **223**, 1–8 (1996).

22. P. Horodecki, "Separability criterion and inseparable mixed states with positive partial transposition," Phys. Lett. A **232**, 333–339 (1997).

23. A. Sen, U. Sen, M. Lewestein, and A. Sampera, "The separability versus entanglement problem," arXiv.org, arXiv:quant-phys/0508032 (2005).

24. C. Bennett, D. DiVicenzo, J. Smolin, and W. Wootters, "Mixed-state entanglement and quantum error correction," Phys. Rev. A **54**, 3824–3851 (1996).

25. S. Hill and W. Wootters, "Entanglement of a pair of quantum bits," Phys. Rev. Lett. **78**, 5022–5025 (1997).

26. W. Wootters, "Entanglement of formation of an arbitrary state of two qubits," Phys. Rev. Lett. **80**, 2245–2248 (1998).

27. C. H. Bennett, G. Brassard, S. Popescu, B. Schumacher, J. A. Smolin, and W. K. Wootters, "Purification of noisy entanglement and faithful teleportation via noisy channels," Phys. Rev. Lett. **76**, 722–725 (1996).

28. D. M. Mundarain and M. Orszag, "Entanglement distillation with local common reservoirs," Phys. Rev. A **79**, 022306 (2009).

29. D. M. Mundarain and M. Orszag, "Entanglement preservation by continuous distillation," Phys. Rev. A **79**, 052333 (2009).

30. W. H. Zurek, "Decoherence and the transition from quantum to classical," Phys. Today **44**(10), 36–44 (1991).

31. W. H. Zurek, "Pointer basis of quantum apparatus: into what mixture does the wave packet collapse?," Phys. Rev. D **24**, 1516–1525 (1981).

32. W. H. Zurek, "Environment-induced superselection rules," Phys. Rev. D **26**, 1862–1880 (1982).

33. N. Bohr, "The quantum postulate and the recent development of atomic theory," Nature (London) **121**, 580–590 (1928).

34. J. von Neumann, *Mathematische Grundlagen der Quanten mechanik* (Springer-Verlag, 1932).

35. For generalized measurements and quantum state discrimination, see, for example, J. A. Bergou, U. Herzog, and M. Hillery, in *Quantum State Discrimination in Quantum State Estimation*: M. Paris and J. Rehacek, eds., Vol. 649 of Lecture Notes in Physics (Springer-Verlag, 2004), pp. 417–456, and [36,37].

36. J. A. Bergou, "Discrimination of quantum states," to be published in J. Mod. Opt..

37. J. Preskill, *Quantum Information and Computation* Vol. 229 of Lecture Notes in Physics (Springer-Verlag, 1998).

38. E. Joos, H. D. Zeh, C. Kiefer, D. Giulini, J. Kupsch, and I.-O. Stamatescu, *Decoherence and the Appearance of Classical World in Quantum Theory* (Springer, 1997).

39. E. Joos and H. D. Zeh, "The emergence of classical properties through interaction with the environment," Z. Phys. B **59**, 223–243 (1985).

40. A. Caldeira and A. J. Leggett, "Path integral approach to quantum Brownian motion," Physica A **121**, 587–616 (1983).

41. D. F. Walls and G. J. Milburn, "Effect of dissipation on quantum coherence," Phys. Rev. A **31**, 2403–2408 (1985).

42. D. F. Walls and G. J. Milburn, *Quantum Optics* (Springer, 1994).

43. V. Buzek and P. L. Knight, in *Progress in Optics*, E. Wolf, ed. (Elsevier, A1995), Vol. XXXIV, p. 1.

44. M. S. Kim and V. Buzek, "Schrödinger-cat states at finite temperature: influence of a finite-temperature heat bath on quantum interferences," Phys. Rev. A **46**, 4239–4251 (1992).

45. M. S. Kim and V. Buzek, "Decay of quantum coherences under the influence of a thermal heatbath: Schroedinger cat states at finite temperature," J. Mod. Opt. **39**, 1609–1614 (1992).

46. M. S. Kim and V. Buzek, "Photon statistics of superposition states in phase-sensitive reservoirs," Phys. Rev. A **47**, 610–619 (1993).

47. S. Schneider and G. J. Milburn, "Decoherence in ion traps due to laser intensity and phase fluctuations," Phys. Rev. A **57**, 3748–3752 (1998).

48. M. Murao and P. L. Knight, "Decoherence in nonclassical motional states of a trapped ion," Phys. Rev. A **58**, 663–669 (1998).

49. S. Schneider and G. J. Milburn, "Decoherence and fidelity in ion traps with fluctuating trap parameters," Phys. Rev. A **59**, 3766–3774 (1999).

50. D. A. Lidar, I. L. Chuang, and K. B. Whaley, "Decoherence-free subspaces for quantum computation," Phys. Rev. Lett. **81**, 2594–2597 (1998).

51. D. A. Lidar and K. B. Whaley, "Decoherence-free subspaces and subsystems," arXiv.org, arXiv:quant-phys/0301032 (2003).

52. D. Kielpinski, V. Meyer, M. A. Rowe, C. A. Sackett, W. M. Itano, C. Monroe, and D. J. Wineland, "A decoherence-free quantum memory using trapped ions," Science **291**, 1013–1015 (2001).

53. P. Zanardi and M. Rasetti, "Error avoiding quantum codes," Mod. Phys. Lett. B **11**, 1085–1093 (1997).

54. P. Zanardi and M. Rasetti, "Noiseless quantum codes," Phys. Rev. Lett. **79**, 3306–3309 (1997).

55. W. H. Zurek, "Decoherence, einselection, and the quantum origins of the classical," Rev. Mod. Phys. **75**, 715 (2003).

56. T. Yu and J. H. Eberly, "Finite-time disentanglement via spontaneous emission," Phys. Rev. Lett. **93**, 140404 (2004).

57. L. Diósi, "Progressive decoherence and total environmental disentanglement," Lect. Notes Phys. **622**, 157–163 (2003).

58. P. Marek, J. Lee, and M. S. Kim, "Vacuum as a less hostile environment to entanglement," Phys. Rev. A **77**, 032302 (2008).

59. Y. X. Gong, Y.-S. Zhang, Y.-L. Dong, X.-L. Niu, Y.-F. Huang, and G.-C. Guo, "Dependence of the decoherence of polarization states in phase-damping channels on the frequency spectrum envelope of photons," Phys. Rev. A. **78**, 042103 (2008).

60. D. Tolkunov, V. Privman, and P. K. Aravind, "Decoherence of a measure of entanglement," Phys. Rev. A. **71**, 060308(R) (2005).

61. J. P. Paz and A. J. Roncaglia, "Dynamical phases for the evolution of the entanglement between two oscillators coupled to the same environment," Phys. Rev. A. **79**, 032102 (2009).

62. C. Cormick and J. P. Paz, "Decoherence of Bell states by local interactions with a dynamic spin environment," Phys. Rev. A. **78**, 012357 (2008).

63. T. Gorin, C. Pineda, and T. H. Seligman, "Decoherence of an *n*-qubit quantum memory," Phys. Rev. Lett. **99**, 240405 (2007).

64. C. Pineda, T. Gorin, and T. H. Seligman, "Decoherence of two-qubit systems: a random matrix description," New J. Phys. **9**, 106 (2007).

65. M. Ikram, F.-l. Li, and M. Zubairy, "Disentanglement in a two-qubit system subjected to dissipation environments," Phys. Rev. A. **75**, 062336 (2007).

66. K.-L. Liu and H.-S. Goan, "Non-Markovian entanglement dynamics of quantum continuous variable systems in thermal environments," Phys. Rev. A. **76**, 022312 (2007).

67. A. Al-Qasimi and D. F. V. James, "Sudden death of entanglement at finite temperature," Phys. Rev. A. **77**, 012117 (2008).

68. M. Hernandez and M. Orszag, "Decoherence and disentanglement for two qubits in a common squeezed reservoir," Phys. Rev. A. **78**, 042114 (2008).

69. Z. Ficek and R. Tanás, "Dark periods and revivals of entanglement in a two-qubit system," Phys. Rev. A. **74**, 024304 (2006).

70. M. P. Almeida, F. de Melo, M. Hor-Meyll, A. Salles, S. P. Walborn, P. H. Souto-Ribeiro, and L. Davidovich, "Environment-induced death of entanglement," Science **316**, 579–582 (2007);

71. A. Salles, F. de Melo, M. P. Almeida, M. Hor-Meyll, S. P. Walborn, P. H. Souto-Ribeiro, and L. Davidovich, "Experimental investigation of the dynamics of entanglement: sudden death, complementarity and continuous monitoring of the environment," Phys. Rev. A. **78**, 022322 (2008).

72. D. Braun, "Creation of entanglement by interaction with a common heat bath," Phys. Rev. Lett. **89**, 277901 (2002);

73. Z. Ficek and R. Tanas, "Entanglement induced by spontaneous emission in spatially extended two-atom systems," J. Mod. Opt. **50**, 2765–2779 (2003).

74. F. Benatti, R. Floreanini, and M. Piani, "Environment induced entanglement in Markovian dissipative dynamics," Phys. Rev. Lett. **91**, 070402 (2003).

75. Z. Ficek and R. Tanás, "Delayed sudden birth of entanglement," Phys. Rev. A. **77**, 054301 (2008).

76. B. Bellomo, R. Lo Franco, and G. Compagno, "Non-Markovian effects on the dynamics of entanglement," Phys. Rev. Lett. **99**, 160502 (2007).

77. S. Maniscalco, S. Olivares, and M. G. A. Paris, "Entanglement oscillations in non-Markovian quantum channels," Phys. Rev. A **75**, 062119 (2007); see also the related work in [78,79] and, for multipartite systems, [80].

78. L. Mazzola, S. Maniscalco, J. Piilo, K.-A. Suominen, and B. M. Garraway,

"Sudden death and sudden birth of entanglement in common structured reservoirs," Phys. Rev. A **79**, 042302 (2009).

79. B. Bellomo, R. Lo Franco, S. Maniscalco, and G. Compagno, "Entanglement trapping in structured environments," Phys. Rev. A **78**, 060302 (2008).

80. C. E. Lopez, G. Romero, F. Lastra, E. Solano, and J. C. Retamal, "Sudden birth versus sudden death of entanglement in multipartite systems," Phys. Rev. Lett. **101**, 080503 (2008).

81. M. Franca-Santos, P. Milman, L. Davidovich, and N. Zagury, "Direct measurement of finite-time disentanglement induced by a reservoir," Phys. Rev. A **73**, 040305 (2006).

82. T. Yu and J. H. Eberly, "Evolution from entanglement to decoherence of standard bipartite mixed states," arXiv.org, arXiv:quant-ph/0503089 (2006).

83. F. Lastra, S. Wallentowitz, M. Orszag, and M. Hernandez, "Quantum recoil effects in finite-time disentanglement of two distinguishable atoms," J. Phys. B **42**, 065504 (2009).

84. D. Mundarain and M. Orszag, "Decoherence-free subspace and entanglement by interaction with a common squeezed bath," Phys. Rev. A **75**, 040303(R) (2007).

85. D. Mundarain, M. Orszag, and J. Stephany, "Total quantum Zeno effect and intelligent states for a two-level system in a squeezed bath," Phys. Rev. A **74**, 052107 (2006).

86. T. Yu and J. H. Eberly, "Negative entanglement measure and what it implies," arXiv.org, arXiv:quant-ph/0703083 (2007).

87. R. Tanas and Z. Ficek, "Stationary two-atom entanglement induced by nonclassical two-photon correlations," J. Opt. B **6**, S610–S617 (2004).

88. A. Beige, S. Bose, D. Braun, S. F. Huelga, P. L. Knight, M. B. Plenio, and V. Vedral, "Entangling atoms and ions in dissipative environments," J. Mod. Opt **47**, 2583—2598 (2000)..

89. A. M. Basharov, "Entanglement of atomic states upon collective radiative decay," JETP Lett. **75**, 123–126 (2002).

90. L. Jakobczyk, "Entangling two qubits by dissipation," J. Phys. A **35**, 6383–6392 (2002).

91. M. Paternostro, S. M. Tame, G. M. Palma, and M. S. Kim, "Entanglement generation and protection by detuning modulation," Phys. Rev. A **74**, 052317 (2006).

92. S. Natali and Z. Ficek, "Temporal and diffraction effects in entanglement creation in an optical cavity," Phys. Rev. A **75**, 042307 (2007).

93. L. Derkacz and L. Jakobczyk, "Vacuum-induced stationary entanglement in radiatively coupled three-level atoms," arXiv.org, arXiv:0710.5048 (2007).

94. R. H. Lehmberg, "Radiation from *n*-atom system. I. General formulation," Phys. Rev. A **2**, 883–888 (1970).

95. G. S. Agarwal, in *Quantum Statistical Theories of Spontaneous Emission and Their Relation to Other Approaches*, edited by G. Hohler, Vol. 70 of Springer Tracts in Modern Physics (Springer-Verlag, 1974).

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