

Inquisitive semantics with compliance^{*}

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1 Introduction

This note follows up on three earlier notes on first-order inquisitive semantics:

- *Maximality and support in the first-order setting*, December 2, 2010
- *First-order inquisitive semantics revisited*, February 18, 2011
- *Algebraic foundations for inquisitive semantics*, June 15, 2011
(to appear as Roelofsen, 2011)

The main result obtained in these earlier notes was a first-order system in which:

- Propositions capture informative and inquisitive content, and nothing more than that.
- There is a natural notion of entailment, which forms a partial order on the set of all propositions; in particular, it is anti-symmetric, which means that propositions are really uniquely determined by the informative and inquisitive content that they embody.
- This notion of entailment, seen as a partial order on propositions, gives rise to *meet*, *join*, and (*relative*) *complementation* operators, which makes it possible to formulate a semantics for the language of first-order logic that is motivated by general algebraic considerations, rather than specific linguistic examples (just like classical first-order logic).

Let me refer to this basic system as Inq_B , and let me refer to the system developed by Ivano in his AC paper (Ciardelli, 2010) as Inq_A . I argued in the earlier notes that, as long as we are interested in capturing only informative and inquisitive content, Inq_B is a more appropriate system than Inq_A , precisely because it has the features listed above. Of course, to make such an argument,

*The idea presented here has its roots in many many discussions with Ivano Ciardelli, Jeroen Groenendijk, and Matthijs Westera, for which I am very grateful. This is a very first version of the note, changes are bound to be made over the next few weeks/months. Feedback would of course be more than welcome.

it is important to say precisely what is meant by informative and inquisitive content, especially the latter. In the above notes, the assumption was that the inquisitive content of a sentence φ is determined by the range of responses that satisfy the request for information that is made in uttering φ . In short, the inquisitive content of a sentence is determined by the range of *issue-resolving responses* to that sentence.

Now, it must be noted that Ivano's system was in fact developed to do *more* than just capturing informative and inquisitive content (in the above sense). For instance, it was intended to be able to capture the differences in meaning between different boundedness formulas, even though in terms of issue-resolving responses these formulas are all equivalent.

What Ivano aimed for was a notion of propositions that would not just allow us to distinguish issue-resolving responses from non-issue-resolving responses, but rather a notion that would allow us to distinguish *compliant* responses from non-compliant ones, where, intuitively, compliant responses are responses that provide 'exactly the information that is requested,' and not more than that. Issue-resolving responses on the other hand may provide, besides enough information to satisfy the given request, additional (possibly irrelevant) information as well.

Inq_B only allows us to characterize compliant responses to a limited extent. For instance, we could say that ψ is a compliant response to φ if $[\psi]$ consists of a single possibility which coincides with a *maximal* possibility for φ . If this requirement is fulfilled, ψ is certainly an issue-resolving response—it provides enough information to establish one of the possibilities for φ —and moreover, it does not provide any more information than necessary—any response that provides *less* information than ψ is not issue-resolving.

However, this is only a partial characterization of compliant responses. And indeed, it becomes clear from Ivano's considerations that in Inq_B it is impossible to obtain a fully satisfactory characterization of compliant responses. This is essentially because there are formulas which are assigned exactly the same proposition in Inq_B , while they differ in their range of compliant responses. The even and odd boundedness formulas are a case in point. It should perhaps be emphasized that, in my view, there is nothing wrong with the fact that these formulas are assigned exactly the same proposition in Inq_B , since in terms of issue-resolving responses they are indeed indistinguishable. However, in terms of compliant responses they do differ, and it is impossible to capture this difference in Inq_B .

The purpose of this note is to develop a system Inq_C that allows us to distinguish compliant responses from non-compliant ones, like Inq_A , but whose notions of propositions and possibilities are better understood, and whose clauses are, as a consequence, better understood and better behaved as well.

2 Inquisitive semantics with compliance

2.1 Propositions

As in Inq_B , and all other recent implementations of inquisitive semantics, we will define propositions as sets of possibilities. However, we will think of propositions and possibilities slightly differently.

In Inq_B , we thought of the proposition expressed by a sentence φ as capturing the range of issue-resolving responses to φ . That is, we thought of every possibility in $[\varphi]$ as corresponding to an issue-resolving response, and vice versa, of every issue-resolving response as corresponding to some possibility in $[\varphi]$.

If we think about propositions in this particular way, then we have to define them as *persistent* sets of possibilities. To see this, suppose that we have a proposition $[\varphi]$, a possibility $\alpha \in [\varphi]$, and another possibility $\beta \subset \alpha$. Then, since $\alpha \in [\varphi]$, α must correspond to an issue-resolving response to φ . But then, since $\beta \subset \alpha$, β must also correspond to an issue-resolving response to φ . But this means that β must also be in $[\varphi]$. Thus, propositions in Inq_B must be defined as persistent sets of possibilities.

In Inq_C , we will not think of the proposition expressed by φ as capturing the range of issue-resolving responses to φ , but rather as capturing the range of compliant responses to φ . More precisely, the system will be based on the following ideas:

- A sentence φ expresses a *proposal* to update the common ground of the conversation in one or more ways.
- It *provides* the information that the actual world is not one of the worlds that do not survive any of the proposed updates.
- Moreover, it *requests* a response that provides enough information to establish one or more of the proposed updates.
- We call a sentence φ *inquisitive* iff in order to satisfy the requests for information issued by φ a response needs to provide more information than the information that φ itself already provides.
- We call a sentence φ an *assertion* iff it proposes exactly one update.
- A response ψ to an initiative φ is a *compliant* response iff:
 - There is a non-empty set Z of updates proposed by φ , such that ψ provides exactly the information that is needed to establish all the updates in Z , and
 - ψ does not raise any further issues, i.e., it is an assertion.¹

¹We may want to refer to these responses as *basic* compliant responses, and broaden the general notion of compliant responses to also include inquisitive responses.

- The *proposition* expressed by φ , $[\varphi]$, characterizes the range of compliant responses to φ . That is, $[\varphi]$ contains a possibility α iff any sentence that expresses the proposition $\{\alpha\}$ is a compliant response to φ .

A consequence of this way of thinking about propositions is that they should be *closed under intersection*. That is, whenever we have a proposition $[\varphi]$ and two possibilities α and β in $[\varphi]$, then $\alpha \cap \beta$ must also be in $[\varphi]$. After all, if α and β are in $[\varphi]$, then they both correspond to a compliant response. This means that there are two non-empty sets of updates proposed by φ , Z_1 and Z_2 , such that any response expressing the proposition $\{\alpha\}$ provides exactly the information that is needed to establish all the updates in Z_1 , and any response expressing the proposition $\{\beta\}$ provides exactly the information that is needed to establish all the updates in Z_2 . But then any response expressing the proposition $\{\alpha \cap \beta\}$ provides exactly the information that is needed to establish all the updates in $Z_1 \cup Z_2$. Therefore, $\alpha \cap \beta$ must also be a possibility in $[\varphi]$, and more generally, propositions have to be closed under intersection.

It also follows that propositions must be *non-empty* sets of possibilities. This is because sentences are taken to express proposals to update the common ground in *one or more* ways. Thus, there is always at least one proposed update, and consequently also always at least one possibility.

Definition 1 (Propositions)

A proposition is a non-empty set of possibilities, closed under intersection.

Notice that the way we have set things up, it is not always true that every possibility in a proposition $[\varphi]$ corresponds with a *single* update proposed by φ (this is how we often think about possibilities, so we should be particularly careful not to do so in the setting of Inq_C). Rather, possibilities correspond to compliant responses, and every compliant response in turn corresponds with one or more proposed updates. It is true that every *maximal* possibility corresponds with a single proposed update. But this does not necessarily hold for non-maximal possibilities.

Now that we have fixed a particular way of thinking about propositions, it is relatively straightforward to state a recursive semantics for the language of propositional logic and for the language of first-order logic. Each clause can be read as a specification of the range of compliant responses for a complex formula in terms of the range of compliant responses for its simpler constituents. For instance, the clause for conjunction specifies the range of compliant responses for $\varphi \wedge \psi$, given the range of compliant responses for φ and for ψ ; and similarly for other connectives and quantifiers.

2.2 A semantics for the language of propositional logic

Let us first consider a propositional language L_P . The language is defined as usual. In order to define the semantics we will use the following operations on propositions and possibilities.

Definition 2 (Operations on propositions and possibilities)

Let A and B be two propositions, and let α and β be two possibilities. Then:

1. $A \sqcap B = \{\alpha \cap \beta \mid \alpha \in A \text{ and } \beta \in B\}$
2. $A \sqcup B = \{\bigcap D \mid D \subseteq (A \cup B) \text{ and } D \neq \emptyset\}$
3. $\chi(A) = \{\chi(\alpha, A) \mid \alpha \in A\}$ where:

$$\chi(\alpha, A) = \alpha - \bigcup\{\beta \in A \mid \alpha \not\subseteq \beta\}$$
4. $\alpha \Rightarrow \beta = \{w \mid \text{if } w \in \alpha \text{ then } w \in \beta\}$

$A \sqcap B$ is the *pointwise intersection* of A and B . $A \sqcup B$ is the proposition obtained by taking the union of A and B , and then closing off under intersection. $\chi(A)$ is the *exclusive strengthening* of A (see Roelofsen and van Gool, 2010). And finally, \Rightarrow is the classical *material implication* operator, which in the present setting does not apply at the level of propositions, but at the level of possibilities (sets of worlds).

Definition 3 (Semantics for L_P)

1. $[p] = \{\{w \mid w(p) = 1\}\}$
2. $[\perp] = \{\emptyset\}$
3. $[\varphi \wedge \psi] = [\varphi] \sqcap [\psi]$
4. $[\varphi \vee \psi] = [\varphi] \sqcup [\psi]$
5. $[\varphi \rightarrow \psi] = \{\bigcap_{\alpha \in \chi[\varphi]} \alpha \Rightarrow f(\alpha) \mid f \text{ is a function from } \chi[\varphi] \text{ to } [\psi]\}$

Let us go through the clauses one by one.

Atoms. The clause for atomic sentences is simple: the proposition expressed by p contains a single possibility, which consists of all worlds where p is true. Thus, p proposes a single update—the update that eliminates every world where p is false. In terms of compliant responses, this clause says that a response to p is compliant iff it provides the information that p is true, and nothing more than that. Note that atomic sentences are never inquisitive.

Contradictions. The proposition expressed by \perp consists of a single possibility, namely the empty possibility. Thus, \perp proposes a single update, which eliminates all worlds from the common ground. That is, \perp proposes an unacceptable update, leading to an inconsistent common ground. We will call sentences with this property *contradictions*. The only way to respond compliantly to a contradiction is with another contradiction. Contradictions are never inquisitive.

Conjunction. The proposition expressed by $\varphi \wedge \psi$ is obtained from $[\varphi]$ and $[\psi]$ by pointwise intersection. This means that ξ is a compliant response to $\varphi \wedge \psi$ iff (i) there is a non-empty set Z_1 of updates proposed by φ and a non-empty set Z_2 of updates proposed by ψ such that ξ provides exactly the information that is needed to establish all the updates in $Z_1 \cup Z_2$, and (ii) ξ does not raise any further issues, i.e., it is an assertion.

Notice that conjunction is *idempotent*: for every φ , $[\varphi \wedge \varphi] = [\varphi]$ (recall that this desirable property does not hold in Inq_A). To see that conjunction is idempotent, suppose that α is in $[\varphi \wedge \varphi]$. Then $\alpha = \beta \cap \gamma$ for some β in $[\varphi]$ and some γ in $[\varphi]$. But then, since $[\varphi]$ is closed under intersection, α must also be in $[\varphi]$. So $[\varphi \wedge \varphi] \subseteq [\varphi]$. Clearly the opposite inclusion also holds.

Disjunction. The proposition expressed by $\varphi \vee \psi$ is obtained by first taking the union of $[\varphi]$ and $[\psi]$, and then closing the resulting set of possibilities off under intersection. This means that ξ is a compliant response to $\varphi \vee \psi$ iff (i) there is a non-empty set Z of updates proposed either by φ or by ψ such that ξ provides exactly the information that is needed to establish all the updates in Z , and (ii) ξ does not raise any further issues, i.e., it is an assertion.

Implication. The clause for implication is the most involved. The proposition expressed by $\varphi \rightarrow \psi$ is obtained as follows. First, we apply the exclusive strengthening operator χ to $[\varphi]$. This gives us the proposition $\chi[\varphi]$, which is a set of mutually exclusive possibilities. More precisely, the possibilities in $\chi[\varphi]$ correspond to *exhaustive* responses to φ . That is, for every possibility $\alpha \in \chi[\varphi]$ there is a non-empty set Z of updates proposed by φ such that α corresponds to a response that provides exactly enough information to establish all the updates in Z and reject all the other updates proposed by φ . And vice versa, for every non-empty set Z of updates proposed by φ there is a possibility $\alpha \in \chi[\varphi]$ such that α corresponds to a response that provides exactly enough information to establish all the updates in Z and reject all the other updates proposed by φ .

Now let's return to the computation of $[\varphi \rightarrow \psi]$. After we applied χ to $[\varphi]$, we consider all the functions from $\chi[\varphi]$ to $[\psi]$, and for each of these functions we construct the possibility $\bigcap_{\alpha \in \chi[\varphi]} \alpha \Rightarrow f(\alpha)$. All these possibilities together form $[\varphi \rightarrow \psi]$.

Thus, every possibility β in $[\varphi \rightarrow \psi]$ has a ‘characteristic function’ f which maps possibilities in $\chi[\varphi]$ to possibilities in ψ . Another way to think about this function f is as a mapping from sets of updates proposed by φ to sets of updates proposed by ψ . The possibility β , then, consists of all worlds w such that for every non-empty set of updates Z proposed by φ , if w survives all the updates in Z and none of the other updates proposed by φ , then it also survives all the updates in $f(Z)$.

The idea here is very similar to the idea behind the clause for implication that Ivano introduced (see, for instance, Ciardelli, 2009). The crucial difference is that we now apply the exclusive strengthening operator to $[\varphi]$ before doing anything else. As a result, a conditional question like (1-a) is now taken to

license a response like (1-b) (which was not the case in earlier systems).^{2,3}

- (1) a. If John or Bill goes, will Mary go as well?
- b. Well, if only John or only Bill goes, then Mary will go as well, but if John and Bill both go, then Mary won't go.

It should be verified that $[\varphi \rightarrow \psi]$, as defined above, is indeed always a proposition, especially that it is always closed under intersection. To see this, suppose that β and γ are two possibilities in $[\varphi \rightarrow \psi]$. Then there are functions f_β and f_γ from $\chi[\varphi]$ to $[\psi]$ such that:

- $\beta = \bigcap_{\alpha \in \chi[\varphi]} \alpha \Rightarrow f_\beta(\alpha)$
- $\gamma = \bigcap_{\alpha \in \chi[\varphi]} \alpha \Rightarrow f_\gamma(\alpha)$

This means that:

$$\begin{aligned} \beta \cap \gamma &= (\bigcap_{\alpha \in \chi[\varphi]} \alpha \Rightarrow f_\beta(\alpha)) \cap (\bigcap_{\alpha \in \chi[\varphi]} \alpha \Rightarrow f_\gamma(\alpha)) \\ &= \bigcap_{\alpha \in \chi[\varphi]} ((\alpha \Rightarrow f_\beta(\alpha)) \cap (\alpha \Rightarrow f_\gamma(\alpha))) \\ &= \bigcap_{\alpha \in \chi[\varphi]} \alpha \Rightarrow (f_\beta(\alpha) \cap f_\gamma(\alpha)) \end{aligned}$$

Now let $f_{\beta\gamma}$ be the function from $\chi[\varphi]$ to $[\psi]$ such that for all $\alpha \in \chi[\varphi]$:

$$f_{\beta\gamma}(\alpha) = f_\beta(\alpha) \cap f_\gamma(\alpha)$$

This function is guaranteed to exist since $[\psi]$ is closed under intersection. Now:

$$\beta \cap \gamma = \bigcap_{\alpha \in \chi[\varphi]} \alpha \Rightarrow f_{\beta\gamma}(\alpha)$$

which means that $\beta \cap \gamma$ is a possibility in $[\varphi \rightarrow \psi]$. This establishes that $[\varphi \rightarrow \psi]$ is indeed closed under intersection.

Negation. Negation is defined in terms of implication and bot: $\neg\varphi := \varphi \rightarrow \perp$. This means that $[\neg\varphi]$ always consists of a single possibility, which is the intersection of the complements of all the possibilities in $\chi[\varphi]$, $\bigcap_{\alpha \in \chi[\varphi]} \overline{\alpha}$, or equivalently, the complement of the union of all the possibilities in $\chi[\varphi]$, $\overline{\bigcup \chi[\varphi]}$, which

²It is not *necessary* to make this change in the clause for implication—we could also stick to the old definition. However, independently of the main issue addressed here, I think that the old definition often does not give us enough compliant responses, in particular in the case of a conditional question with a disjunctive antecedent. By applying the exclusive strengthening operator to the antecedent before doing anything else we get a wider range of compliant responses for the conditional as a whole. Note that this solution could not be implemented in Inq_B , because it is not always possible to determine the range of exhaustive responses based on the range of issue-resolving responses. On the other hand, it *is* always possible to determine the range of exhaustive responses based on the range of compliant responses.

³It would be interesting to look at recent analyses of conditionals that treat the antecedent as a definite description (Schlenker) or a free relative (Schulz) picking out a certain set of worlds. In both cases, a notion of maximality/exhaustivity plays a crucial role. Perhaps the present account can be seen as a generalization of these analyses to the inquisitive setting.

is the same as $\overline{\bigcup[\varphi]}$. Notice that negated sentences are never inquisitive. In terms of compliance, the (derived) clause for negation says that a response ξ to $\neg\varphi$ is compliant iff (i) it provides exactly the information that is needed to reject all the updates proposed by φ , and (ii) it does not raise any further issues.

2.3 A semantics for the language of first-order logic

Now let us consider a first-order language L_{FO} .

Definition 4 (Semantics for L_{FO})

1. $[Rt_1 \dots t_n]_g = \{\{w \mid \langle [t_1]_{w,g}, \dots, [t]_{w,g} \rangle \in w(R)\}\}$
2. $[\perp]_g = \{\emptyset\}$
3. $[\varphi \wedge \psi]_g = [\varphi]_g \sqcap [\psi]_g$
4. $[\varphi \vee \psi]_g = [\varphi]_g \sqcup [\psi]_g$
5. $[\varphi \rightarrow \psi]_g = \{\bigcap_{\alpha \in \chi[\varphi]_g} \alpha \Rightarrow f(\alpha) \mid f \text{ is a function from } \chi[\varphi]_g \text{ to } [\psi]_g\}$
6. $[\exists x.\varphi]_g = \bigsqcup_{d \in D} [\varphi]_{g[x/d]}$
7. $[\forall x.\varphi]_g = \prod_{d \in D} [\varphi]_{g[x/d]}$

The clauses for existential and universal quantification are straightforward generalizations of the clauses for disjunction and conjunction respectively.

Existential quantification. The proposition expressed by $\exists x.\varphi$ relative to an assignment g is obtained by looking at the propositions expressed by φ relative to all the assignments that differ from g at most in the object that they assign to x . We take the union of all these propositions, and then close off under intersection. This means that a response ξ to $\exists x.\varphi$ is compliant just in case there is a non-empty set Z of updates, each proposed by φ relative to some assignment $g[x/d]$, $d \in D$ (different updates in Z may be proposed by φ relative to different assignments), such that ξ provides exactly the information that is needed to establish all the updates in Z , and does not raise any further issues.

Universal quantification. The proposition expressed by $\forall x.\varphi$ relative to an assignment g is also obtained by looking at all the propositions expressed by φ relative to all the assignments that differ from g at most in the object that they assign to x . But now we take the pointwise intersection of all these propositions. This means that a response to $\forall x.\varphi$ is compliant just in case for every $d \in D$ there is a non-empty set Z_d of updates proposed by φ relative to $g[x/d]$, such that the response provides exactly the information that is needed to establish all the updates in $\bigcup_{d \in D} Z_d$, and does not raise any further issues.

2.4 The boundedness formulas

The boundedness examples are straightforwardly dealt with in Inq_C .

2.5 Entailment

Propositions in Inq_C can be ordered in terms of their informative content, in terms of their inquisitive content, and in terms of their range of compliant responses. In defining these entailment orderings we will use $\text{info}(\varphi)$ to denote the informative content of a sentence φ :

- $\text{info}(\varphi) = \bigcup[\varphi]$

Definition 5 (Entailment)

1. $\varphi \models_{\text{info}} \psi$ iff $\text{info}(\varphi) \subseteq \text{info}(\psi)$
2. $\varphi \models_{\text{inq}} \psi$ iff every $\alpha \in [\varphi]$ is contained in some $\beta \in [\psi]$
3. $\varphi \models_{\text{com}} \psi$ iff for every $\alpha \in [\varphi]$ and every $\beta \in [\psi]$, $\alpha \cap \beta \in [\varphi]$
4. $\varphi \models \psi$ iff $\varphi \models_{\text{info}} \psi$, $\varphi \models_{\text{inq}} \psi$, and $\varphi \models_{\text{com}} \psi$

The definition of \models_{info} speaks for itself. The definition of \models_{inq} says that φ is at least as inquisitive as ψ iff every compliant response to φ is at least as informative as some compliant response to ψ . This means that every issue-resolving response to φ is also an issue-resolving response to ψ . So, conceptually, \models_{inq} amounts to the familiar notion of inquisitive entailment in Inq_B .⁴ Finally, $\varphi \models_{\text{com}} \psi$ holds iff for every $\alpha \in [\varphi]$ and every $\beta \in [\psi]$, $\alpha \cap \beta \in [\varphi]$. The idea behind this definition is that for every set Z_φ of updates proposed by φ and every set Z_ψ of updates proposed by ψ , a response that provides exactly the information that is needed to establish all the updates in both Z_φ and Z_ψ is a compliant response to φ . In other words, whenever we have a compliant response to φ and we strengthen it in order to establish one or more updates proposed by ψ as well, we still have a compliant response to φ .⁵

Fact 1 (Partial order) \models forms a partial order.

Proof. It is clear that \models is reflexive. In fact, \models_{info} , \models_{inq} , and \models_{com} are all reflexive. Now consider transitivity. Suppose that $\varphi \models \psi$ and $\psi \models \xi$. It is clear, then, that $\varphi \models_{\text{info}} \xi$ and $\varphi \models_{\text{inq}} \xi$. What remains to be shown is that $\varphi \models_{\text{com}} \xi$. To see this, suppose that $\alpha \in [\varphi]$ and $\gamma \in [\xi]$. Then, since $\varphi \models_{\text{inq}} \psi$, α must be contained in some $\beta \in [\psi]$. But then, since $\psi \models_{\text{com}} \xi$, we have that $\beta \cap \gamma \in [\psi]$, and since $\varphi \models_{\text{com}} \psi$, $\alpha \cap \beta \cap \gamma \in [\varphi]$. Now recall that $\alpha \subseteq \beta$, which means that $(\alpha \cap \beta \cap \gamma) = (\alpha \cap \gamma)$. So $\alpha \cap \gamma \in [\varphi]$, which is what we set

⁴This is only true at the conceptual level. The two notions are not extensionally equivalent, due to the change we made here in the clause for implication.

⁵This seems to be a particularly natural notion of entailment, but of course it is not the only natural notion. In the appendix, we mention one alternative.

out to show. Finally consider *anti-symmetry*. Suppose that $\varphi \models \psi$ and $\psi \models \varphi$, and suppose that $\alpha \in [\varphi]$. Then, since $\varphi \models_{inq} \psi$, α must be contained in some $\beta \in [\psi]$. But then, since $\psi \models_{com} \varphi$, $\alpha \cap \beta$ must be in $[\psi]$ as well, and since $\alpha \subseteq \beta$, we have that $\alpha \cap \beta = \alpha$. So $\alpha \in [\psi]$, which means that $[\varphi] \subseteq [\psi]$. The opposite inclusion can be established in exactly the same way. So $[\varphi] = [\psi]$, which is what we set out to show. \square

2.6 Joins, meets, and complements

As in classical logic, and as in Inq_B , conjunction behaves like a *meet* operator in Inq_C w.r.t. the above notion of entailment. This means that $[\varphi \wedge \psi]$ is always the weakest proposition among all those propositions that are stronger than both $[\varphi]$ and $[\psi]$.

Fact 2 (Conjunction behaves as a meet operator w.r.t. entailment)
For every φ and ψ , $\varphi \wedge \psi$ is the greatest lower bound, i.e. the meet, of φ and ψ w.r.t. \models .

Proof. First notice that $\varphi \wedge \psi$ is indeed stronger than both φ and ψ . That is, $\varphi \wedge \psi \models \varphi$ and $\varphi \wedge \psi \models \psi$. What remains to be shown is that $\varphi \wedge \psi$ is entailed by any other sentence that is stronger than both φ and ψ . Let ξ be such a sentence, i.e., let $\xi \models \varphi$ and $\xi \models \psi$. It is clear that $\xi \models_{info} \varphi \wedge \psi$ in this case. It remains to be shown that $\xi \models_{inq} \varphi \wedge \psi$ and $\xi \models_{com} \varphi \wedge \psi$. First, let's show that $\xi \models_{inq} \varphi \wedge \psi$. Let γ be a possibility in $[\xi]$. Then, since $\gamma \models_{inq} \varphi$ and $\gamma \models_{inq} \psi$, there are possibilities $\alpha \in [\varphi]$ and $\beta \in [\psi]$ such that $\gamma \subseteq \alpha$ and $\gamma \subseteq \beta$. But this means that $\gamma \subseteq \alpha \cap \beta$, and $\alpha \cap \beta$ is a possibility in $[\varphi \wedge \psi]$. So, indeed, $\xi \models_{inq} \varphi \wedge \psi$.

Now let's show that $\xi \models_{com} \varphi \wedge \psi$. Let $\gamma \in [\xi]$ and $\mu \in [\varphi \wedge \psi]$. We have to show that $\gamma \cap \mu \in [\xi]$. We know that $\mu = \alpha \cap \beta$ for some $\alpha \in [\varphi]$ and $\beta \in [\psi]$. Moreover, since $\xi \models_{com} \varphi$ and $\xi \models_{com} \psi$, we know that $\gamma \cap \alpha \in [\xi]$ and $\gamma \cap \beta \in [\xi]$. But then, since ξ is closed under intersection, $(\gamma \cap \alpha) \cap (\gamma \cap \beta)$ must also be in $[\xi]$, and $(\gamma \cap \alpha) \cap (\gamma \cap \beta)$ amounts exactly to $\gamma \cap \mu$. \square

Thus, the conjunction of two sentences $\varphi \wedge \psi$ can be characterized as follows:

1. $\varphi \wedge \psi$ provides at least as much information as φ and as ψ
2. $\varphi \wedge \psi$ requests at least as much information as φ and as ψ
3. Whenever we have a compliant response to $\varphi \wedge \psi$ and we strengthen it to establish one or more updates proposed by φ or by ψ , we still have a compliant response to $\varphi \wedge \psi$
4. Any formula ξ that has the above three characteristics entails $\varphi \wedge \psi$

Now let us turn to *disjunction*. In classical logic, and in Inq_B , disjunction behaves like a *join* operator w.r.t. entailment. However, in the setting of Inq_C , with the entailment relation as defined above, it would not make sense if disjunction behaved like a join operator. This is because, if it did, $\varphi \vee \psi$ would

always have to be entailed by φ and by ψ , which would mean in particular that whenever we would have a compliant response to φ and we would strengthen it to establish one or more updates proposed by $\varphi \vee \psi$, we would still have a compliant response to φ . It is clear that this is not what we want.

Rather, in the setting of Inq_C , we want $\varphi \vee \psi$ to have the following characteristics:

1. $\varphi \wedge \psi$ provides at most as much information as φ and as ψ

That is, $\varphi \models_{\text{info}} \varphi \vee \psi$ and $\psi \models_{\text{info}} \varphi \vee \psi$

2. $\varphi \wedge \psi$ requests at most as much information as φ and as ψ

That is, $\varphi \models_{\text{inq}} \varphi \vee \psi$ and $\psi \models_{\text{inq}} \varphi \vee \psi$

3. Whenever we have a compliant response to $\varphi \vee \psi$ and we strengthen it to establish one or more updates proposed by φ or by ψ , we still have a compliant response to $\varphi \vee \psi$

That is, $\varphi \vee \psi \models_{\text{com}} \varphi$ and $\varphi \vee \psi \models_{\text{com}} \psi$

4. Any formula ξ that has the above three characteristics is such that:

- $\varphi \vee \psi \models_{\text{info}} \xi$
- $\varphi \vee \psi \models_{\text{inq}} \xi$
- $\xi \models_{\text{com}} \varphi \vee \psi$

Another way of putting this is that we want disjunction to behave as a join operator with respect to the following entailment order:

Definition 6 (*Entailment)

$\varphi \models^* \psi$ if and only if:

- $\varphi \models_{\text{info}} \psi$
- $\varphi \models_{\text{inq}} \psi$
- $\psi \models_{\text{com}} \varphi$

Fact 3 (Partial order) \models^* forms a partial order.

Proof. Reflexivity is clear, and anti-symmetry can be established exactly as for \models . It remains to be shown that \models^* is *transitive*. Suppose that $\varphi \models^* \psi$ and $\psi \models^* \xi$. It is clear that $\varphi \models_{\text{info}} \xi$ and $\varphi \models_{\text{inq}} \xi$ in this case. We need to show that $\xi \models_{\text{com}} \varphi$. Let $\gamma \in [\xi]$ and $\alpha \in [\varphi]$. To show: $\gamma \cap \alpha \in [\xi]$. Since $\varphi \models_{\text{info}} \psi$, $\alpha \subseteq \beta$ for some $\beta \in [\psi]$. But then, since, $\psi \models_{\text{com}} \varphi$, $\alpha \cap \beta = \alpha$ must also be in $[\psi]$. In the same way it follows from $\psi \models_{\text{info}} \xi$ and $\xi \models_{\text{com}} \psi$ that $\alpha \in [\xi]$. But then, since $[\xi]$ is closed under intersection, $\gamma \cap \alpha$ must also be in $[\xi]$, which is what we set out to show. \square

Given the way we formulated the clauses of Inq_C , disjunction indeed behaves like a join operator w.r.t. \models^* .

Fact 4 (Disjunction behaves as a join operator w.r.t. \models^*)

For every φ and ψ , $\varphi \vee \psi$ is the join of φ and ψ w.r.t. \models^* .

Proof. It is clear that $\varphi \vee \psi$ is indeed weaker than both φ and ψ w.r.t. \models^* . That is, $\varphi \models^* \varphi \vee \psi$ and $\psi \models^* \varphi \vee \psi$. Now suppose that ξ is also weaker than φ and ψ . Then we have to show that $\varphi \vee \psi \models \xi$. First, since $\varphi \models_{info} \xi$ and $\psi \models_{info} \xi$, it must be the case that $info(\varphi) \subseteq info(\xi)$ and $info(\psi) \subseteq info(\xi)$, which means that $info(\varphi) \cup info(\psi) \subseteq info(\xi)$. Thus, $info(\varphi \vee \psi) \subseteq info(\xi)$, which means that $\varphi \vee \psi \models_{info} \xi$. Second, since $\varphi \models_{inq} \xi$ and $\psi \models_{inq} \xi$, it must be the case that every possibility for φ and every possibility for ψ is contained in some possibility for ξ . But this means that every possibility for $\varphi \vee \psi$ is contained in some possibility for ξ as well. So $\varphi \vee \psi \models_{inq} \xi$. Finally, we have to show that $\xi \models_{com} \varphi \vee \psi$. Let $\gamma \in [\xi]$ and $\delta \in [\varphi \vee \psi]$. Then $\delta = \alpha \cap \beta$ for some $\alpha \in [\varphi]$ and some $\beta \in [\psi]$. Since $\xi \models_{com} \varphi$ and $\xi \models_{com} \psi$, it must be the case that $\gamma \cap \alpha$ and $\gamma \cap \beta$ are in $[\xi]$. But then, since $[\xi]$ is closed under intersection, $(\gamma \cap \alpha) \cap (\gamma \cap \beta)$ must also be in $[\xi]$, and $(\gamma \cap \alpha) \cap (\gamma \cap \beta)$ amounts exactly to $\gamma \cap \delta$. So $\gamma \cap \delta \in [\xi]$, which is what we set out to show. \square

Next, let us consider *negation*. As in Inq_B , negation behaves here as a *pseudo-complement* operator w.r.t. entailment. This means that $[\neg\varphi]$ is the weakest proposition such that its meet with $[\varphi]$ amounts to $\{\emptyset\}$.

Fact 5 (Negation behaves as a pseudo-complement operator w.r.t. \models)

For every φ , $\neg\varphi$ is the pseudo-complement of φ w.r.t. \models .

Proof. It is clear that the meet of φ and $\neg\varphi$, i.e., $\varphi \wedge \neg\varphi$ is always a contradiction. What remains to be shown is that every formula ξ such that $[\varphi \wedge \xi] = \{\emptyset\}$ entails $\neg\varphi$. Let ξ be a formula such that $[\varphi \wedge \xi] = \{\emptyset\}$. It follows that every possibility in ξ must be contained in $info(\varphi)$. It is clear, then, that $info(\xi) \subseteq info(\neg\varphi)$, which means that $\xi \models_{info} \neg\varphi$. Moreover, since $\overline{info(\varphi)}$ is the unique possibility for $\neg\varphi$ it follows immediately that every possibility in $[\xi]$ is contained in some possibility for $\neg\varphi$, which means that $\xi \models_{inq} \neg\varphi$. And finally, it follows that for every possibility $\gamma \in [\xi]$ and every possibility $\alpha \in [\varphi]$ (of which there is only one, namely $\overline{info(\varphi)}$), $\gamma \cap \alpha$ coincides with γ and is therefore in $[\xi]$. This means that $\xi \models_{com} \varphi$. We conclude that $\xi \models \varphi$, which is what we set out to show. \square

Finally, let us consider *implication*. In Inq_B , implication behaves semantically as a *relative pseudo-complement* operator w.r.t. entailment. That is, for every φ and every ψ , $\varphi \rightarrow \psi$ expresses the weakest proposition whose *meet* with $[\varphi]$ entails $[\psi]$.

This result does *not* carry over to Inq_C . In fact, in Inq_C the pseudo-complement of one proposition relative to another is not guaranteed to exist. That is, there are certain propositions $[\varphi]$ and $[\psi]$ such that there is no unique weakest proposition whose *meet* with $[\varphi]$ entails $[\psi]$. Two such propositions are depicted in figure 1.

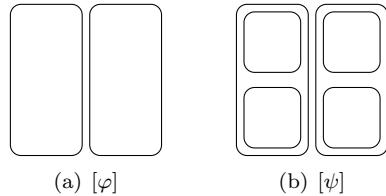


Figure 1: Two propositions that do not have a relative pseudo-complement.

To see that there is no unique weakest proposition whose *meet* with $[\varphi]$ entails $[\psi]$ consider the two propositions depicted in figure 2. Both of these propositions are such that their *meet* with $[\varphi]$ entails $[\psi]$. However, neither of them is weaker than the other. So, there is no unique weakest proposition whose *meet* with $[\varphi]$ entails $[\psi]$.

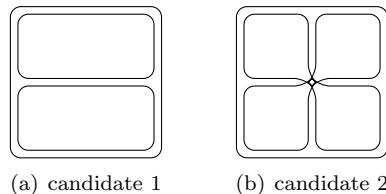


Figure 2: Two incomparable propositions whose *meet* with $[\varphi]$ entails $[\psi]$.

The significance of this result is that, unlike in the classical setting and in the setting of Inq_B , in Inq_C we cannot associate implication with the operation of relative pseudo-complementation. Of course, this does not mean that there is no suitable treatment of implication in Inq_C at all. Indeed, we have proposed one particular treatment, which seems quite attractive from a linguistic point of view—it improves on earlier treatments of conditionals, within inquisitive semantics and beyond, in that it captures a broader range of compliant responses. It may be good to clarify in this regard that in Inq_B , we did not treat implication as a relative pseudo-complement operator because we think that this yields the most appropriate predictions from a linguistic point of view. Rather, we chose to do so in order to obtain a system whose underlying algebra is closest to that of classical logic. Thus, we think of Inq_B as the counterpart of classical logic in the inquisitive setting: a system that serves as a logically well-understood point of departure, but one that certainly will not have the last word on the semantics of connectives (in particular conditionals) in natural language. Thus, it is not really a disaster that in Inq_C , implication can no longer be associated with relative pseudo-complementation. It just means that in moving to a setting in which propositions are taken to capture the range of compliant responses to a sentence, the classical way of thinking about implication is no longer applicable.⁶

⁶In non-classical logics, there are other ways of thinking about implication, and some of

3 Conclusion and outlook

The proposition expressed by a sentence φ in Inq_C allows us to distinguish compliant responses to φ from non-compliant ones. It improves on Inq_A , which was developed for the same purpose, in that its clauses are better behaved (for instance, conjunction is idempotent) and better understood, in the sense that we can read every clause as a specification of the range of compliant responses to a complex formula, given the range of compliant responses to its simpler constituents.

This is possible because we specified more precisely from the outset what we mean by compliant responses. Namely, a response to a sentence φ is compliant just in case there is a non-empty set Z of updates proposed by φ such that the response provides exactly the information that is needed to establish all the updates in φ (and does not raise any further issues).

Notice that this notion of compliant responses is slightly different from the one we (or at least I) usually had in mind. That is, we used to think of a compliant response as one that provides exactly enough information to establish one or more of the updates proposed by φ . The difference with the notion I have adopted here is one of scope. In Inq_C , a response is compliant iff there is a specific set of updates proposed by φ such that the response provides exactly enough information to establish all the updates in that specific set. Indeed, this switch in conception of compliant responses is, I think, the main reason that everything seems to work out so well in Inq_C . According to the old notion, compliant responses can only correspond with *maximal* possibilities—after all, if a response picks out a non-maximal possibility then it provides more information than necessary to establish at least one of the proposed updates—and if there are no maximal possibilities for a formula, we are forced to conclude that there are no compliant responses either.

Of course, the notion of compliant responses adopted here is still an arbitrary notion; we could in principle also decide to think of compliant responses in some other way, and then the system would turn out differently. For instance, if we are less strict about what it takes to be a compliant response, and identify, say, compliant responses with issue-resolving responses, then we get Inq_B . Similarly, we could imagine being even more strict than in Inq_C about what it takes to be a compliant response, and we would obtain yet another system.

This underlines the point that in developing a semantic framework, it is absolutely crucial to specify very precisely how the basic semantic objects in the system, in particular propositions, are to be conceived of. There are many different ways of thinking about propositions, and there is not one ‘right’ way to do so. Different conceptions may be appropriate for different purposes. But once we fix a certain way of thinking about propositions, everything else should basically follow from it.

This leads me to one final remark, which is that Inq_C does not allow us to give an analysis of attentive *might*. This is perfectly acceptable, since Inq_C was

these may indeed carry over to the setting of Inq_C as well. This would be an interesting direction to explore.

designed to capture compliance, and not attentive content. For instance, the pragmatic principle of *attentive sincerity*, which requires that a speaker who utters a sentence φ considers every possibility for φ a live possibility, does not make sense in Inq_C . Related to this, propositions in Inq_C are not fine-grained enough to distinguish sentences like $p \vee q \vee (p \wedge q)$ and $p \vee q$. This is as it should be, since in terms of compliant responses (as conceived of here) these two sentence are indeed indistinguishable. But this does mean that we are not done yet. In order to capture attentive content as well, Inq_C needs to be further enriched.

A An alternative notion of entailment

It was mentioned above that the adopted notion of entailment is not necessarily the only natural notion of entailment in Inq_C . Here we consider one alternative. In defining this alternative entailment order we will write $B\langle\text{info}(A)\rangle$ for the *restriction* of B to the informative content of A :

- $B\langle\text{info}(A)\rangle = \{\beta \cap \text{info}(A) \mid \beta \in B\}$

Definition 7 (An alternative notion of entailment)

1. $A \models_{\text{info}} B$ iff $\text{info}(A) \subseteq \text{info}(B)$
2. $A \models_{\text{inq}} B$ iff every $\alpha \in A$ is contained in some $\beta \in B$
3. $A \models_{\text{com}} B$ iff $A \subseteq B\langle\text{info}(A)\rangle$
4. $A \models B$ iff $A \models_{\text{info}} B$, $A \models_{\text{inq}} B$, and $A \models_{\text{com}} B$

The definition of \models_{info} and \models_{inq} is as before, but \models_{com} is now defined differently. $A \models_{\text{com}} B$ can now be read as “ A is at least as *restrictive* as B in its range of compliant responses.” That is, the range of compliant responses to A is a subset of the range of compliant responses to B , relativized to the information provided by A .

Thus, all three entailment notions now require, in one way or another, that A offers *less options* than B . In the case of \models_{info} , A offers less candidates for the actual world than B . In the case of \models_{inq} , A admits a smaller range of issue-resolving responses than B , and in the case of \models_{com} , A admits a smaller range of compliant responses than B .⁷

Fact 6 (Partial order) \models forms a partial order.

⁷Of course, this is not necessarily the *only* sensible way of ordering propositions.

Proof. It is clear that \models is reflexive and transitive. In fact, \models_{info} , \models_{inq} , and \models_{com} are all reflexive and transitive. It remains to be shown that \models is *anti-symmetric*, which certainly does not hold for \models_{info} , \models_{inq} , and \models_{com} individually. Suppose that $A \models B$ and $B \models A$. Then, since $B \models_{info} A$, $B\langle\text{info}(A)\rangle = B$. But then it follows, since $A \models_{com} B$, that $A \subseteq B$. In the same way we can show that $B \subseteq A$. So $A = B$, and \models is indeed anti-symmetric. \square We would expect that relative to this notion of entailment, disjunction should behave as a *join* operator. But this is not the case. In fact, the *join* of two propositions does not always exist. In order to see this, consider:

- $\varphi := p \vee (p \wedge q)$
- $\psi := \neg p \vee (\neg p \wedge q)$
- $\xi := \top \vee q$

Then we have that:

- $\varphi \models \xi$
- $\psi \models \xi$
- $\varphi \models \varphi \vee \psi$
- $\psi \models \varphi \vee \psi$

So both ξ and $\varphi \vee \psi$ are weaker than φ and ψ . However:

- $\xi \not\models \varphi \vee \psi$
- $\varphi \vee \psi \not\models \xi$

So $[\varphi]$ and $[\psi]$ do not have a join.

I have not looked at *meets* and *pseudo-complements* in detail yet.

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