A generalized inquisitive semantics.*

Ivano A. Ciardelli

In Inquisitive Semantics, formulas are evaluated on ordered pairs of indices; actually, the order of the pair is irrelevant, so these pairs could just just as well be taken to be non-empty sets of indices of cardinality at most two. This last restriction, however, sounds particularly unnatural, especially considering that the definition of inquisitive semantics can be easily reformulated in such a way that it is meaningful for any non-empty set of indices.

In this little paper I investigate the consequences of undertaking this generalized approach. In section 1 I introduce the generalized inquisitive semantics, I reformulate the notions of standard inquisitive semantic for this extended setting, and I prove many basic properties of the system, most of which are the analogue of properties of inquisitive logic.

In section 2 I prove some results concerning the expressive completeness of the language with respect to the given semantics. It turns out that all the "meanings" are expressible by a formula which uses only negation and disjunction, while "classical meanings" - those which correspond to meanings of assertions - can be expressed using only negation and conjunction. These results also yield two normal form results.

In section 3 I analyse the logic GIL arising from our semantics. I show that this strictly contains intuitionistic logic and is strictly contained in standard inquisitive logic. I give examples of formulas which are valid in inquisitive logic but not in our generalized setting. Additionally, I also prove a result expressing a strong form of adequacy of GIL as a logic to reason about sets of possibilities.

Finally, in section 4 I compare the way standard inquisitive semantics and this generalized inquisitive semantics deal with the inquisitive component of the meaning of disjunction, that is, the issue of formulas specifying possibilities. I point out a rather severe shortcoming of standard inquisitive logic, which in many case does not render all the possibilities that a formula should - intuitively speaking - specify. I show that the generalized setting is immune from this shortcoming and thus provides a more faithful account of the meaning of disjunction. In the final part of section 4 I give an explanation of what exactly

*This paper was written as a term paper for the course *Semantics and pragmatics* taught at the University of Amsterdam in autumn 2008. It builds on Groenendijk (2008b), which is an extended version of Groenendijk (2008a); it is the semantics presented in those papers that I refer to as *standard* or *restricted* Inquisitive Semantics. Another reference for this semantics is Mascarenhas (2008). Please note that Floris Roelofsen and I are currently in the process of writing up an article which improves and extends the present paper and will make the new version available as soon as possible.

goes wrong in standard inquisitive semantics and why the generalized semantics is a more natural environment for the specification of possibilities.

1 Definitions and basic facts

First let me briefly recall the syntax of the language we are going to use. Given a finite set P of propositional letters, the language L_P is nothing but the propositional language on P where the operators that we take as primitive are $\bot$, $\vee$, $\wedge$ and $\rightarrow$; in other words, L_P contains exactly those formulas which are built up from propositional letters in P and the symbol $\bot$ using the binary connectives $\wedge, \vee$ and $\rightarrow$.

Note that I am not going to take the operators $\neg$, ! and ? as primitive, since also in this generalized setting they would turn out to be definable in terms of the other operators. This allows shorter inductive proofs and definitions. Rather, for any formula $\phi \in L_P$, we will use the following notations:

- $\neg\phi$ is the formula $\phi \rightarrow \bot$;
- $?\phi$ is the formula $\phi \vee \neg\phi$;
- $!\phi$ is the formula $\neg\neg\phi$.

Recall that a P-index (or a P-valuation) is a map from P to $\{0, 1\}$, and we denote by I_P the set of P-indices. For any set X, I denote by $\mathcal{P}_0(X)$ the set $\mathcal{P}(X) - \{\emptyset\}$ of non-empty subsets of X.

Definition (Generalized Inquisitive Semantics). [1] The satisfaction relation $\models$ between $\mathcal{P}_0(I_P)$ and L_P is defined inductively by the following clauses. For any non-empty[2] subset $S \subseteq I_p$:

1. $S \models p$ iff $v(p) = 1$ for all $v \in S$;
2. $S \not\models \bot$;
3. $S \models \phi \wedge \psi$ iff $S \models \phi$ and $S \models \psi$;
4. $S \models \phi \vee \psi$ iff $S \models \phi$ or $S \models \psi$;
5. $S \models \phi \rightarrow \psi$ iff for any non-empty $T \subseteq S$, if $T \models \phi$ then $T \models \psi$.

[1] Note that this semantic, while resembling standard Inquisitive Semantics, differs significantly from Cresswell's possibility semantics (Cresswell, 2004). For, when it comes to the meaning of disjunction, Cresswell refuses the rule that is used here as it is not "classically respectable". This means that in general, it is not the case that $S \models \phi \vee \psi$ iff for any $v \in S$, $v \models \phi \vee \psi$.

But this fact is precisely the strength of our system, since it is exactly this key property of disjunction which gives rise to inquisitiveness: without it, as we shall see, all formulas would admit only (at most) one possibility, and so every formula would behave classically.

[2] This non-emptiness restriction can and should be lifted, and indeed it will be lifted in the forthcoming article; this leads to *much* nicer formal properties of the system as we no longer have to treat contradictions as a special case.

Fact (Persistency). Let S, T be non-empty sets of P-indices. If $T \subseteq S$ and $S \models \phi$, then $T \models \phi$.

Proof. By induction on the complexity of ϕ.

1. First consider the case that ϕ is a propositional letter p. For any $v \in T$, $v \in S$ and since $S \models p$, by definition $v(p) = 1$. So $T \models p$.

2. Consider $\phi = \chi \wedge \xi$. Then $S \models \phi$ amounts to $S \models \chi$ and $S \models \xi$; then by induction hypothesis we have $T \models \chi$ and $T \models \xi$, whence $T \models \phi$.

3. Consider $\phi = \chi \vee \xi$. Then $S \models \phi$ amounts to $S \models \chi$ or $S \models \xi$; in the former case by induction hypothesis on χ we have $T \models \chi$; in the latter case, by induction hypothesis on ξ we have $T \models \xi$; so in any case we know that $T \models \chi$ or $T \models \xi$, whence $T \models \phi$.

4. Consider $\phi = \chi \to \xi$. Take any non-empty $U \subseteq T$: since $T \subseteq S$ we have $U \subseteq S$, and by the semantics of implication, $S \models \chi \to \xi$ implies that if $U \models \chi$, then $U \models \xi$. So we conclude that for any non-empty $U \subseteq T$, if $U \models \chi$ then $U \models \xi$: by definition this means $T \models \phi$.

Fact (Classical behaviour of singletons/endpoints). For any P-index v and any formula $\phi \in L_P$, $\{v\} \models \phi$ iff $v \models \phi$ in the classical sense (i.e. ϕ is true classically under the valuation v).

In particular, for any ϕ we have either $\{v\} \models \phi$ or $\{v\} \models \neg\phi$.

Proof. We will prove the claim by induction on ϕ. Let v be a P-index and proceed by induction on ϕ. For propositional letters we have: $\{v\} \models p$ iff $u(p) = 1$ for all indices $u \in \{v\}$, iff $v(p) = 1$.

The case for $\perp$ is trivial and the inductive steps for conjunction and disjunction are utterly straightforward.

Finally we come to the inductive step for implication; suppose our claim holds for χ, ξ, we have: $\{v\} \models \chi \to \xi$ iff for all non-empty $T \subseteq \{v\}$, if $T \models \chi$ then $T \models \xi$; since the only non-empty subset of $\{v\}$ is $\{v\}$, this condition boils down to: if $\{v\} \models \chi$ then $\{v\} \models \xi$; then by induction hypothesis this is equivalent to: if $v \models \chi$, then $v \models \xi$, and this is in turn equivalent to $v \models \chi \to \xi$. Our inductive proof is thus complete.

Having established this result, in the following I will not distinguish the statements $v \models \phi$ (in the classical sense) and $\{v\} \models \phi$, since we now know that they both amount to the very same thing. Note that as a particular case of persistency we have the following: if $S \models \phi$ and $v \in S$, then $v \models \phi$.

Fact (Truth conditions for $\neg$, ! and ?). For any non-empty set S of P-indices and any formula $\phi \in L_P$ we have:

1. $S \models \neg\phi$ iff for all $v \in S$, $v \models \neg\phi$.
2. $S \models !\phi$ iff for all $v \in S$, $v \models \phi$.
3. $S \models ?\phi$ iff either $S \models \phi$, or all $v \in S$ validate $\neg\phi$.

Proof. We exploit heavily the classical behaviour of singletons.

1. The left-to-right direction is immediate by persistency. For the converse, reasoning by contraposition suppose that $S \not\models \neg\phi$: by the meaning of implication, this means that there is a non-empty $T \subseteq S$ such that $T \models \phi$ (and trivially $T \not\models \bot$); but then take a $v \in T$: by persistency, $v \models \phi$; and since $v \in S$ (because $v \in T \subseteq S$) it is *not* the case that for all $v \in S$, $v \models \neg\phi$.

2. Since $!\phi$ is $\neg\neg\phi$, we use the previous point which gives us: $S \models !\phi$ iff for all $v \in S$, $v \models \neg\neg\phi$; but this amounts exactly to: for all $v \in S$, $v \models \phi$.

3. Suppose $S \models ?\phi$: by the meaning of disjunction, either $S \models \phi$, or else $S \models \neg\phi$, in which case by persistency all $v \in S$ validate $\neg\phi$. Conversely, if $S \models \phi$ obviously $S \models ?\phi$; and if all $v \in S$ validate $\neg\phi$, then by point 1 we have $S \models \neg\phi$, so $S \models ?\phi$.

Fact (Generalized Inquisitive Semantics extends Inquisitive Semantics). Let v, w be two P-indices. For any formula $\phi \in L_p$ we have the following: $\langle v, w\rangle \models \phi$ (in the sense of standard Inquisitive Semantics) iff $\{v, w\} \models \phi$. We omit the explicit proof of this fact, which is utterly tedious and straightforward (by induction on ϕ): basically, this is true because spelling out the definition we see that $\langle i, j\rangle \models \phi$ and $\{i, j\} \models \phi$ are simply *defined* in exactly the same way!

This fact allows us to recover immediately results about the classical version of Inquisitive Semantics from this generalized setting. For instance, by persistency we see that if $\langle v, w\rangle \models \phi$, then both $\langle v, v\rangle \models \phi$ and $\langle w, w\rangle \models \phi$; also, we have $\langle v, w\rangle \models \phi$ iff $\{v, w\} \models \phi$ iff $\langle w, v\rangle \models \phi$. So we see immediately that the meaning of a formula (in the sense of standard Inquisitive Semantics) is a reflexive symmetric relation. Or, again, if $v = w$, then $\{v, w\} = \{v\}$ is a singleton and behaves classically: so all the points $\langle v, v\rangle$ behave classically in standard Inquisitive Semantic.

Definition (Meaning). For $\phi \in L_P$, the meaning of ϕ is the set containing all non-empty sets of P-indices which validate ϕ, namely: $[\![\phi]\!] := \{S \in \mathcal{P}_0(I_P) \mid S \models \phi\}$.

We also define the *classical extension* of ϕ, in symbols $\lfloor\phi\rfloor$, to be the set of valuations satisfying ϕ (in the classical sense): $\lfloor\phi\rfloor = \{v \in I_P \mid v \models \phi\}$.

Note that for any ϕ, $S \models \phi$ implies $S \subseteq \lfloor\phi\rfloor$. For, let $S \in [\![\phi]\!]$: then for any $v \in S$ we have $v \models \phi$ and so $v \in \lfloor\phi\rfloor$; thus $S \subseteq \lfloor\phi\rfloor$.

In terms of meanings, this fact can be rewritten as: for any formula ϕ, $[\![\phi]\!] \subseteq \mathcal{P}_0(\lfloor\phi\rfloor)$.

Fact (Generalized Inquisitive Semantics formulated in terms of meaning). The following facts are immediate consequences of the definition of the Generalized Inquisitive Semantics and the above remark on the semantics of $\neg, !$ and $?$.

1. $[\![p]\!] = \mathcal{P}_0(\lfloor p\rfloor)$
2. $[\![\phi \vee \psi]\!] = [\![\phi]\!] \cup [\![\psi]\!]$
3. $[\![\phi \wedge \psi]\!] = [\![\phi]\!] \cap [\![\psi]\!]$
4. $[\![\phi \to \psi]\!] = \{S \mid \mathcal{P}_0(S) \subseteq [(\mathcal{P}_0(I_P) - [\![\phi]\!]) \cup [\![\psi]\!]]\}$
5. $[\![\neg\phi]\!] = \mathcal{P}_0(\lfloor\neg\phi\rfloor)$
6. $[\![!\phi]\!] = \mathcal{P}_0(\lfloor\phi\rfloor)$
7. $[\![?\phi]\!] = [\![\phi]\!] \cup \mathcal{P}_0(\lfloor\neg\phi\rfloor)$

Definition (Possibilities). A non-empty set of indices S is a possibility for ϕ in case $S \models \phi$ and there is no $T \supsetneq S$ such that $T \models \phi$: in other words, a possibility for ϕ is a maximal set validating ϕ. In terms of meanings, this definition can be rewritten nicely: a possibility for ϕ is a maximal element of the partial order $\langle [\![\phi]\!], \subseteq\rangle$. The set of possibilities for ϕ is denoted by $\pi[\phi]$.

Fact. Let $S \in \mathcal{P}_0(I_P)$. $S \models \phi$ iff S is included in a possibility for ϕ (possibly itself).

Proof. Suppose that S can be extended to a possibility for ϕ, i.e. suppose that there is $T \supseteq S$ such that T is a possibility for ϕ. By persistency, since $T \models \phi$, also $S \models \phi$. Conversely, suppose $S \models \phi$, and let $T_0 = S$. If there is no strictly bigger $T_1 \supset T_0$ which validates ϕ, then S itself is a possibility, by definition; otherwise, consider a $T_1 \supset T_0$ which validates ϕ: either T_1 is a possibility, or there is $T_2 \supset T_1$ validating ϕ; then we look whether T_2 is a possibility, and so on. For some n, T_n must be a possibility: for, otherwise the sets $T_0 \subset T_1 \subset \dots$ would constitute a chain of strictly increasing subsets of I_P, which is impossible since I_P is finite. In conclusion, for some natural n we will have that $S \subseteq T_n$ and T_n is a possibility.

Corollary. For any formula ϕ, $\bigcup \pi(\phi) = \lfloor\phi\rfloor$.

Proof. If $v \in \bigcup\pi(\phi)$, then $v \in S$ for some possibility S for ϕ; then, since $S \models \phi$, by persistency $v \models \phi$, so $v \in \lfloor\phi\rfloor$. Conversely, if $v \in \lfloor\phi\rfloor$ then $v \models \phi$ and therefore $v \subseteq S$ for some possibility $S \in \pi(\phi)$, whence $v \in \bigcup\pi(\phi)$.

Definition (Inquisitiveness, Informativeness). Let $\phi \in L_P$:

1. we say that ϕ is informative in case $\bigcup \pi[\phi] \neq I_P$.

2. we say that ϕ is inquisitive in case $|\pi[\phi]| \geq 2$.

Remark (ϕ is informative iff it is not a classical tautology). Note that according to the previous corollary, item 1 of this definition is equivalent to: ϕ is informative iff $\lfloor\phi\rfloor \neq I_P$, that is to say, iff it is not a classical tautology.

We say that ϕ is a *contradiction* in case $[\![\phi]\!] = \emptyset$, that is, if ϕ is true on no $S \in \mathcal{P}_0(I_P)$. As we will see later on, ϕ is a contradiction iff it is classical contradiction. Also, note that ϕ is a contradiction iff $\pi(\phi) = \emptyset$.

Note that the notion of informativeness (and thus the notion of question that we will see in a moment) given here differs from that in Groenendijk (2008b), allowing a contradiction to be informative. This difference only depends on my choice of the definitions, and it has nothing to do with the relations between the generalized and the restricted inquisitive semantics. I have chosen this definition exclusively because it gives slightly more elegant properties.

Definition (Assertions, Questions) . Let $\phi \in L_P$.

1. ϕ is a question iff it is not informative.

2. ϕ is an assertion iff it is not inquisitive. In other words, ϕ is an assertion iff $|\pi(\phi)| = 1$ or ϕ is a contradiction. We say that an assertion ϕ is *consistent* in case it not a contradiction; so, ϕ is a consistent assertion in case $|\pi(\phi)| = 1$.

Fact (Alternative characterizations of questions and assertions.)

1. ϕ is a question iff it is a classical tautology.

2. ϕ is a consistent assertion iff $\lfloor\phi\rfloor \models \phi$. In fact, we will prove this nice, stronger fact: if ϕ is a consistent assertion, then the unique possibility for ϕ is $\lfloor\phi\rfloor$; conversely, if $\lfloor\phi\rfloor \models \phi$, then $\lfloor\phi\rfloor$ is a possibility for ϕ, and it is the *unique* one.

Proof.

1. It is immediate from the definition of question and the above remark that ϕ is informative iff it is not a classical tautology.

2. Suppose that $\lfloor\phi\rfloor \models \phi$. Consider any other set S such that $S \models \phi$: by persistency, for any $v \in S$ we have $v \models \phi$ and therefore $v \in \lfloor\phi\rfloor$; so $S \subseteq \lfloor\phi\rfloor$. In particular, if S is any possibility, then S is a *maximal* subset validating ϕ, so (since $\lfloor\phi\rfloor \models \phi$) S cannot be *strictly* contained in $\lfloor\phi\rfloor$: thus $S = \lfloor\phi\rfloor$. This proves that every possibility for ϕ coincides with $\lfloor\phi\rfloor$,

i.e. that $\lfloor \phi \rfloor$ is the *unique* possibility for ϕ. So in particular $|\pi(\phi)| = 1$ and ϕ is a consistent assertion.

Conversely, suppose ϕ is a consistent assertion, and let S be the unique possibility for ϕ. Consider any $v \in \lfloor \phi \rfloor$: then $v \models \phi$, so $\{v\}$ can be extended to a possibility for ϕ, and since the only possibility for ϕ is S, $v \in S$. So $\lfloor \phi \rfloor \subseteq S$. Conversely, suppose $v \in S$: since $S \models \phi$, by persistency we have $v \models \phi$ and so $v \in \lfloor \phi \rfloor$; this proves $S \subseteq \lfloor \phi \rfloor$ and thus $S = \lfloor \phi \rfloor$. So $\lfloor \phi \rfloor$ *is* the unique possibility for ϕ. In particular $\lfloor \phi \rfloor \models \phi$.

One immediate consequence of the second fact is that if ϕ is an assertion, then the following property holds:

$$\text{(Hrd)} \qquad \text{for any } S \in \mathcal{P}_0(I_P)\ (\ S \models \phi \ \Leftrightarrow\ \ v \models \phi \text{ for any } v \in S\ )$$

But in fact, we shall be able to say something stronger in a moment, namely that the converse implication holds as well: if ϕ has the property (Hrd), then ϕ is an assertion.

Remark. For all consistent [3] ϕ, $\lfloor \phi \rfloor \models \phi$ iff $[\![\phi]\!] = \mathcal{P}_0(\lfloor \phi \rfloor)$.

Proof. If $[\![\phi]\!] = \mathcal{P}_0(\lfloor \phi \rfloor)$, then since $\lfloor \phi \rfloor \in \mathcal{P}_0(\lfloor \phi \rfloor)$ we have $\lfloor \phi \rfloor \models \phi$. Conversely, if $\lfloor \phi \rfloor \models \phi$, then for any $S \subseteq \lfloor \phi \rfloor$, $S \models \phi$, so $\mathcal{P}_0(\lfloor \phi \rfloor) \subseteq [\![\phi]\!]$, and since the converse inclusion holds for any formula, $\mathcal{P}_0(\lfloor \phi \rfloor) = [\![\phi]\!]$.

This remark shows that ϕ is an assertion *if and only if* the property Hrd holds for ϕ. In other terms, we can think of assertions as precisely those formulas whose meaning is "hereditary": whenever all the elements of a set satisfy an assertion, the set satisfies it as well. Conversely, any formula displaying this kind of behaviour is an assertion.

Fact. For all $p \in P$ and all $\phi \in L_P$:

1. p is an assertion;
2. $\neg\phi$ is an assertion;
3. $!\phi$ is an assertion;
4. $?\phi$ is a question;
5. if ϕ, ψ are assertions, then $\phi \wedge \psi$ is an assertion.
6. if ψ is an assertion, then $\phi \to \psi$ is an assertion.

[3] The assumption of consistency here is only necessary because in the formulation of the semantics we do not allow the empty state, and thus for contradictions the statement would be meaningless; let me remark once again that I do not agree with this choice anymore.

Proof . Items 1 to 3 follow from the previous remark together with the fact proved above, that $[\![\psi]\!] = \mathcal{P}_0(\lfloor\psi\rfloor)$ for $\psi = p, \neg\phi$ and $!\phi$. As for item 4, $?\phi$ is a question because it is a propositional tautology.

For item 5, suppose ϕ, ψ are assertions: then for any $S \in \mathcal{P}_0(I_P)$, we have $S \models \phi \wedge \psi$ iff $S \models \phi$ and $S \models \psi$; since ϕ, ψ are assertions, this happens iff all $v \in S$ validate ϕ and all $v \in S$ validate ψ, that is to say, iff all $v \in S$ validate both ϕ and ψ; this, in turn, is the case iff all $v \in S$ validate $\phi \wedge \psi$. So $\phi \wedge \psi$ has the property Hrd and therefore it is an assertion.

Finally, suppose ψ is an assertion. Let $S \in \mathcal{P}_0(I_P)$: if $S \models \phi \to \psi$, then by persistency any $v \in S$ validates $\phi \to \psi$. Conversely, suppose $v \models \phi \to \psi$ for all $v \in S$ and consider any subset $T \subseteq S$; suppose $T \models \phi$: then any $v \in T$ validates ϕ; but any such v is in S and so it validates $\phi \to \psi$: thus for any $v \in T$, $v \models \psi$, and since ψ is an assertion this implies $T \models \psi$. This shows that $S \models \phi \to \psi$. So $\phi \to \psi$ satisfies (Hrd) and therefore it is an assertion.

Note that items 1,2, 5 and 6 immediately imply the following fact, which shows that disjunction is the "source" of inquisitiveness in our language.

Fact (All disjunction-free formulas are assertions). Call a formula *disjunction-free* in case no disjunction occurs in it, and *conjunctive* in case it is built up from propositional letters using only negations and conjunctions.

Then we have the following: if ϕ is disjunction-free, then ϕ is an assertion. In particular, every conjunctive formula is an assertion. (We shall see later on why the conjunctive fragment of the language is relevant).

Definition (Equivalence). We say that two formulas $\phi, \psi \in L_P$ are equivalent, in symbols $\phi \equiv \psi$, in case $[\![\phi]\!] = [\![\psi]\!]$.

Fact (Iteration of ? and !) . For any formula ϕ we have $!!\phi \equiv !\phi$ and $??\phi \equiv ?\phi$.

Proof. For any $S \in \mathcal{P}_0(I_P)$, $S \models !!\phi$ iff for any $v \in S$, $v \models \neg\neg\phi$, iff for any $v \in S$, $S \models \phi$, iff $S \models !\phi$.

As for the second claim, obviously if $S \models ?\phi$ we have $S \models ??\phi$. Conversely, suppose $S \models ??\phi$: then $S \models ?\phi$ or $S \models \neg ?\phi$. But $\neg ?\phi$ is a contradiction (because $[\![\neg ?\phi]\!] = \mathcal{P}_0(I_P - \lfloor ?\phi \rfloor) = \mathcal{P}_0(\emptyset) = \emptyset$), so $S \not\models \neg ?\phi$; therefore $S \models ?\phi$ and our claim is proved.

Fact (Division in Theme and Rheme) For any formula ϕ we have $\phi \equiv !\phi \wedge ?\phi$

Proof. Take any $S \in \mathcal{P}_0(I_P)$. Suppose $S \models \phi$: then immediately we have $S \models \phi \vee \neg\phi$, that is $S \models ?\phi$; moreover, by persistency, for any $v \in S$, $v \models \phi$ and therefore $S \models !\phi$; so we have $S \models !\phi \wedge ?\phi$. Conversely, suppose $S \models !\phi \wedge ?\phi$; since $S \models ?\phi$, either $S \models \phi$ or for all $v \in S$, $v \models \neg\phi$; but the latter cannot be the case, since $S \models !\phi$ implies that any $v \in S$ validates ϕ and S is non-empty; therefore, $S \models \phi$ must hold.

Fact (Weak Distribution Laws). For any formulas $\phi_1, \ldots, \phi_n$ we have:

1. $\neg\neg(\phi_1 \vee \ldots \vee \phi_n) \equiv \neg(\neg\phi_1 \wedge \ldots \wedge \neg\phi_n)$
2. $\neg\neg(\phi_1 \wedge \ldots \wedge \phi_n) \equiv \neg(\neg\phi_1 \vee \ldots \vee \neg\phi_n)$

Proof. Let $S \in \mathcal{P}_0(I_P)$. We have $S \models \neg\neg(\phi_1 \vee \ldots \vee \phi_n)$ iff for any $v \in S$ it is $v \models \phi_1 \vee \ldots \vee \phi_n$; this holds iff for any $v \in S$, $v \models \neg(\neg\phi_1 \wedge \ldots \wedge \neg\phi_n)$, which in turn holds iff $S \models \neg(\neg\phi_1 \wedge \ldots \wedge \neg\phi_n)$. The second weak distribution law is proved analogously.

2 Expressive completeness

Definition (Picture). A (possibly empty) collection $\Pi \subseteq \mathcal{P}_0(I_P)$ of non-empty P-indices is called a *picture* in case there are no two distinct elements P, Q in Π such that $P \subseteq Q$. In other words, Π is a picture iff its elements are pairwise incomparable with respect to $\subseteq$.

For any formula ϕ, its set of possibilities $\pi(\phi)$ is a picture: for, by definition, a possibility is a *maximal* subset satisfying ϕ, and therefore it cannot be contained in another, different possibility. So the possibilities for ϕ are pairwise incomparable, and thus $\pi(\phi)$ is a picture, which we call the picture of ϕ. By extension, if Π is any picture we refer to the elements of Π as the *possibilities* of Π.

Note that the picture of a formula ϕ can be effectively computed from ϕ: clearly, by the finiteness of our semantics, for any set S we can effectively decide whether $S \models \phi$; then, once we know which sets satisfy ϕ, with an easy algorithm we can select all the maximal ones, which are precisely the elements of $\pi(\phi)$.

The following fact says that the picture of ϕ characterizes ϕ up to logical equivalence.

Fact (Pictures characterize formulas up to logical equivalence). For any formulas ϕ, ψ in L_P, $\pi(\phi) = \pi(\psi)$ iff $\phi \equiv \psi$.

Proof. If $\phi \equiv \psi$, then ϕ and ψ are true on exactly the same sets, so clearly $\pi(\phi) = \pi(\psi)$. Conversely, suppose $\pi(\phi) = \pi(\psi)$ and consider *any* set S: we have $S \models \phi$ iff there is $T \in \pi(\phi)$ such that $S \subseteq T$, iff there is $T \in \pi(\psi)$ such that $S \subseteq T$, iff $S \models \psi$. So ϕ and ψ are equivalent.

We know that any formula describes a picture. Is every picture described by a formula? If so, our language is rich enough to express all "possible meanings". The following proposition gives a positive answer to this question.

Theorem (Expressive Completeness). For any picture Π there is a "characteristic" formula χ_Π such that $\Pi = \pi(\chi_\Pi)$. Moreover, χ_Π can be constructed using only negation and disjunction as connectives, so already the fragment of the language using only $\neg, \vee$ is expressively complete.

Proof. Let Π be any picture. First, for any P-index v we want to define a sentence ν_v such that for any P-index w, $w \models \nu_v \Leftrightarrow w = v$. For any propositional letter p, let $\delta_v(p)$ be p if $v(p) = 1$, and $\neg p$ if $v(p) = 0$. Then consider $\nu_v = \bigwedge_{p \in P} \delta_v(p)$: for any P-index w, $w \models \nu_v$ iff for any letter p, $w \models \delta_v(p)$, iff for any p it is $w(p) = v(p)$, that is iff $w = v$.

Now consider the formula $\mu_v := \neg \bigvee_{p \in P} \neg\delta_v(p)$, which uses only $\neg$ and $\vee$: since μ_v and ν_v are classically equivalent, for any P-index w, $w \models \mu_v$ iff $w = v$.

Now for any possibility $P \in \Pi$, define $\xi_P := !\bigvee_{v \in P} \mu_v$. We will now show that ξ_P characterizes the possibility P in the following sense: for any $S \in \mathcal{P}_0(I_P)$, $S \models \xi_P$ iff $S \subseteq P$. In fact, for any set S we have $S \models \xi_P$ iff for any $w \in S$, $w \models \bigvee_{v \in P} \mu_v$; but $w \models \bigvee_{v \in P} \mu_v$ iff there is some $v \in P$ such that $w \models \mu_v$, and so such that $w = v$; so the above condition amounts to the fact that any $w \in S$ is in P, i.e. that $S \subseteq P$.

Finally, define $\chi_\Pi := \bigvee_{P \in \Pi} \xi_P$. Note that χ_Π is built up from propositional letters with the use of the only connectives $\neg$ and $\vee$. For any $S \in \mathcal{P}_0(I_P)$, $S \models \chi_\Pi$ iff there is $P \in \Pi$ such that $S \models \xi_P$, and so iff there is $P \in \Pi$ such that $S \subseteq P$, that is to say, iff S is included in a possibility of Π.

Now it is easy to check that $\Pi = \pi(\chi_\Pi)$. If $P \in \Pi$, then $P \models \chi_\Pi$ and so it is included in some $S \in \pi(\chi_\Pi)$; but since $S \in \pi(\chi_\Pi)$, $S \models \chi_\Pi$ and so S is included in some $P' \in \Pi$: so $P \subseteq P'$ and since two elements of a picture are incomparable it must be $P = P'$, whence also $P = S \in \pi(\chi_\Pi)$. This shows that $\Pi \subseteq \pi(\chi_\Pi)$.

As for the converse inclusion, take $S \in \pi(\chi_\Pi)$: since $S \models \chi_\Pi$, by the above property of χ_Π there must be $P \in \Pi$ such that $S \subseteq P$; but since $\Pi \subseteq \pi(\chi_\Pi)$, P is a possibility for χ_Π and therefore by maximality it must be $S = P$, whence $S \in \Pi$. This shows $\pi(\chi_\Pi) \subseteq \Pi$. The equality $\pi(\chi_\Pi) = \Pi$ is thus proved.

Note that if $\Pi = \emptyset$, then $\chi_\Pi \bigvee_{P \in \emptyset} \xi_P = \bot$ and our proof still works, since $\pi(\bot) = \emptyset$.

Note that the expressive completeness of the language immediately gives us a Normal Form Theorem. Call a formula ξ *normal* in case $\xi = \chi_\Pi$ for some picture Π. Note that, by construction, $\pi(\chi_\Pi) = \Pi$ (this is precisely what the theorem amounts to!).

Corollary (Normal Form Theorem). Any formula is equivalent to exactly one normal formula. Moreover, this formula can be effectively computed from the original one.

Proof. For any formula ϕ, $\chi_{\pi(\phi)}$ is a normal formula and $\phi \equiv \chi_{\pi(\phi)}$ because $\pi(\phi) = \chi_{\pi(\phi)}$ and we have seen above that pictures characterize formulas up to logical equivalence.

For uniqueness, suppose $\phi = \chi_\Pi$ for some picture Π: then $\pi(\phi) = \pi(\chi_\Pi) = \Pi$, so $\chi_\Pi = \chi_{\pi(\phi)}$. Thus, $\chi_{\pi(\phi)}$ is the *unique* normal formula equivalent to ϕ.

That $\chi_{\pi(\phi)}$ can be effectively computed from ϕ is fairly obvious: we know that $\pi(\phi)$ can be computed from ϕ; then the above theorem tells us how to construct $\chi_{\pi(\phi)}$ out of $\pi(\phi)$.

Definition (Classical Picture). We say that a picture Π is *classical* iff $|\Pi| \leq 1$. So a classical picture is either empty, or a singleton.

By definition, the picture of an assertion is classical. Conversely, any classical picture is the picture of an assertion: for, given a classical picture Π, the formula χ_Π is an assertion, since it has Π as its picture.

But we can say more: recall that we have shown above that any conjunctive formula (i.e. containing only negation and conjunction) is an assertion. Conversely, we shall now show any classical picture can be described by a conjunctive formula.

Fact (Expressive completeness of the conjunctive fragment with respect to classical pictures). If $\Pi = \{P\}$ is a classical picture, there is a conjunctive formula ξ_Π such that $\Pi = \pi(\xi_\Pi)$.

Proof. This fact is more or less a byproduct of the previous proof of expressive completeness with respect to arbitrary pictures. If $\Pi = \emptyset$, then $\Pi = \pi(\bot)$ and $\bot$ is a conjunctive formula.

Otherwise, Π is a singleton, say $\Pi = \{P\}$. For any P-index v, we have shown above how to define a formula ν_v which uses only the connectives $\neg, \wedge$ and such that for any P-index w, $w \models \nu_v \Leftrightarrow w = v$.

Then define ξ_Π as: $\xi_\Pi = \neg \bigwedge_{v \in P} \neg \nu_v$. Note that ξ_Π is a formula which only uses the connectives $\neg, \wedge$; this is just a syntactic variant of what we were doing for each possibility in the above proof. We claim that for any set S, $S \models \xi_\Pi \Leftrightarrow S \subseteq P$.

Recall that by the weak distributive laws, ξ_Π is equivalent to $!\bigvee_{v \in P} \nu_v$. So for any $S \in \mathcal{P}_0(I_P)$, we have: $S \models \xi_\Pi$ iff for any $w \in S$, $w \models \bigvee_{v \in P} \nu_v$. Now for any w, $w \models \bigvee_{v \in P} \nu_v$ iff there is $v \in P$ such that $w \models \nu_v$, which happens iff there is $v \in P$ such that $w = v$, that is to say iff $w \in P$. So we conclude: $S \models \xi_\Pi \Leftrightarrow S \subseteq P$.

Now, since $P \subseteq P$, $P \models \xi_\Pi$ and therefore it is included in a possibility $S \in \pi(\xi_\Pi)$; but then $S \models \xi_\Pi$ and so $S \subseteq P$, whence $P = S \in \pi(\xi_\Pi)$. Conversely, for any possibility $S \in \pi(\xi_\Pi)$, S validates ξ_Π and therefore $S \subseteq P$, and since $P \in \pi(\xi_\Pi)$, $S = P$. This proves that P is the unique possibility for ξ_Π, i.e. that $\Pi = \pi(\xi_\Pi)$. This completes the proof.

Note that also in this case this expressive completeness result comes with an associated normal form result. Call a formula ξ *normal assertive* in case

$\xi = \xi_\Pi$ for some classical picture Π. Note that normal assertive formulas are conjunctive formulas.

Fact (Normal form for assertions). Any assertion is equivalent to exactly one normal assertive formula.

Proof. If ϕ is an assertion then by construction of $\xi_{\pi(\phi)}$ we have $\pi(\phi) = \pi(\xi_{\pi(\phi)})$, and so, by a previous result, $\phi \equiv \xi_{\pi(\phi)}$.

For uniqueness, suppose ϕ is equivalent to ξ_Π: then $\pi(\phi) = \pi(\xi_\Pi) = \Pi$, so actually $\xi_\Pi = \xi_{\pi(\phi)}$: this shows that $\xi_{\pi(\phi)}$ is the *unique* normal assertive formula equivalent to ϕ.

As a particular case of this result we have that ϕ is an assertion if and only if it is equivalent to a conjunctive formula.

3 Generalized Inquisitive Logic

Definition (GIL-Entailment, GIL-Validity). Let $\phi_1, \ldots, \phi_n, \psi \in L_P$. We say that ψ *is entailed by* $\phi_1, \ldots, \phi_n$ - in symbols $\phi_1, \ldots, \phi_n \models \psi$ - in case for any $S \in \mathcal{P}_0(I_P)$, if $S \models \phi_i$ for $i = 1, \ldots, n$ then $S \models \psi$.

We say that ψ is *valid* in case $\models \psi$, that is to say, in case $S \models \psi$ for any $S \in \mathcal{P}_0(I_P)$.

We call *Generalized Inquisitive Logic* (GIL for short) the set of formulas which are valid in the sense just defined.

Fact (Validity reduces to truth on I_P). For any formula ϕ, $\models \phi$ iff $I_P \models \phi$.

Proof. The left-to-right direction is trivial: $\models \phi$ implies $S \models \phi$ for any non-empty set of indices. Conversely, suppose $I_P \models \phi$: then for *any* non-empty $S \subseteq I_P$ we have $S \models \phi$ by persistency; so $\models \phi$.

Fact. For any formula ϕ, ϕ is a contradiction in generalized Inquisitive Semantics iff it is a classical contradiction. (Note that the same holds for standard Inquisitive Logic).

Proof. If ϕ is a contradiction in inquisitive semantics, then for no index v we have $\{v\} \models \phi$, which is the same as $v \models \phi$: so ϕ is a classical contradiction. Conversely, suppose ϕ is not a contradiction in inquisitive semantics: then there is some non-empty $S \models \phi$, and because it is non-empty, there is some $v \in S$ for which, by persistency, we have $v \models \phi$: so ϕ is not a classical contradiction.

This fact also shows that in Inquisitive Logic (generalized or standard), it is not the case that ϕ is valid iff $\neg\phi$ is a contradiction (although this is true if ϕ is an assertion); instead, it is the case that ϕ is a *question* iff $\neg\phi$ is a contradiction: for, ϕ is a question iff it is a classical tautology, iff $\neg\phi$ is a classical contradiction, iff $\neg\phi$ is a contradiction.

Fact. All the inquisitive entailments and validities listed in Fact 14 of the student paper hold in our generalized setting. The proof of any of those facts is more or less trivial, and so I omit it. In particular, note that we have the deduction theorem: for any $\phi_1, \ldots, \phi_n, \psi$, it is $\phi_1, \ldots, \phi_n \models \psi \iff \models \phi_1 \wedge \ldots \wedge \phi_n \rightarrow \psi$.

Definition (Entailment for pictures). Let Π, Π' be two pictures. We say that Π entails Π', and we write $\Pi \leq \Pi'$, in case any possibility $P \in \Pi$ is included in some possibility $P' \in \Pi'$.

Fact (Entailment is a partial order on pictures) . Let *Pic* be the set of pictures for a fixed finite set P of propositional letters. Then $(Pic, \leq)$ is a partial order.

Proof.

- (Reflexivity). For any $\Pi \in Pic$, $\Pi \leq \Pi$ because trivially, any $P \in \Pi$ is included in itself.
- (Antisymmetry). For any $\Pi, \Pi' \in Pic$, suppose $\Pi \leq \Pi'$ and $\Pi' \leq \Pi$. Then for any $P \in \Pi$ there is a $P' \in \Pi'$ such that $P \subseteq P'$; in turn, it must be $P' \subseteq Q$ for some $Q \in \Pi$; but then we have $P \subseteq Q$, and since distinct elements of a picture are incomparable, this implies $P = Q$ and thus $P = P'$.

  This shows that $\Pi \subseteq \Pi'$. Analogously we prove the converse inclusion and thus the equality $\Pi = \Pi'$.
- (Transitivity). For any $\Pi, \Pi', \Pi'' \in Pic$, suppose $\Pi \leq \Pi' \leq \Pi''$: then for any $P \in \Pi$ there is $P' \in \Pi'$ such that $P \subseteq P'$; but in turn, this P' must be contained in P'' for some $P'' \in \Pi''$, whence $P \subseteq P''$ as well. This shows that $\Pi \leq \Pi''$.

Fact (Entailment between formulas amounts to entailment between their pictures). For any formulas ϕ, ψ, we have $\phi \models \psi$ iff $\pi(\phi) \leq \pi(\psi)$.

Proof. Fix any ϕ, ψ and suppose $\phi \models \psi$. Then, consider a possibility $P \in \pi(\phi)$: $P \models \phi$ and so $P \models \psi$, whence P must be contained in a possibility $S \in \pi(\psi)$. So $\pi(\phi) \leq \pi(\psi)$.

Conversely, suppose $\pi(\phi) \leq \pi(\psi)$. Then if $S \models \phi$, $S \subseteq P$ for some possibility $P \in \pi(\phi)$; then $P \subseteq P'$ for some possibility $P' \in \pi(\psi)$, so $S \subseteq P'$ and therefore $S \models \psi$. So $\phi \models \psi$.

Let me now indulge in a short mathematical aside. It is easy to see that the relation of equivalence between formulas is indeed an equivalence relation on L_P, whence we can form the quotient $\mathbb{L}_P$. We denote the equivalence class of ϕ by $[\phi]$.

If $\phi \equiv \phi'$ and $\psi \equiv \psi'$, then $\phi \models \psi$ iff $\phi' \models \psi'$: this guarantees that we can consistently define a relation of entailment on the quotient $\mathbb{L}_P$ by putting: $[\phi] \models [\psi]$ iff $\phi \models \psi$.

It is in fact very easy to show that $\models$ is a partial order on the quotient $\mathbb{L}_P$. Now, the crucial fact is the following.

Proposition. The posets $(\mathbb{L}_P, \models)$ and $(Pic, \leq)$ are isomorphic, and the map $\pi^* : \mathbb{L}_P \to Pic$ defined by $\pi^*([\phi]) := \pi(\phi)$ is an isomorphism between them.

Proof. This is a consequence of the results we have been showing so far. First, we have shown that $\phi \equiv \psi$ iff $\pi(\phi) = \pi(\psi)$: this translates to $[\phi] = [\psi]$ iff $\pi^*([\phi]) = \pi^*([\psi])$, which shows that the map π^* is well-defined and injective. The expressive completeness result above shows that π^* is a surjection, because for any $\Pi \in Pic$ there is $\phi \in L_P$ such that $\Pi = \pi(\phi) = \pi^*([\phi])$. So π^* is a bijection.

Finally, we have proved that for any ϕ, ψ, $\phi \models \psi$ iff $\pi(\phi) \leq \pi(\psi)$; this translates to the fact that $[\phi] \models [\psi]$ iff $\pi^*([\phi]) \leq \pi^*([\psi])$. This shows that π^* preserves the order in both directions. So π^* is an isomorphism and our claim is proved.

This result can be read a statement of adequacy (in the strongest possible sense) of GIL as a logic for possibilities [4]: in fact, GIL soundly and completely represents the notion of entailment between possibilities.

We now turn to a different issue; we are going to compare the logic GIL with other familiar logics: Intuitionistic Propositional Logic (IPL), Inquisitive Logic (IL) and Classical Propositional Logic (CPL). In order to distinguish between validity in these different settings, we will use the following notational convention. For $\lambda =$ GIL, IPL, IL, CPL, we write $\models_\lambda \phi$ for $\phi \in \lambda$ ("ϕ is valid in the logic λ"); in particular, $\models_{\text{GIL}} \phi$ is the same as $\models \phi$ as defined above.

It will turn out that we have the following scenario: $IPC \subset GIL \subset IL \subset CPL$, and each of these inclusions is strict. In particular, GIL is a superintuitionistic logic, and it is *not* the same as IL: we shall see that there are formulas which are not valid in GIL but which cannot be falsified on a pair of indices.

In order to prove the first inclusion $IPC \subseteq GIL$ we first recall the completeness of the Kripke semantics for Intuitionistic Logic. Recall that a Kripke model for Intuitionistic Logic is a triple $K = \langle W, R, V \rangle$ where R is a reflexive transitive relation on W and $V : P \to \mathcal{P}(W)$ is a map which is *persistent*, namely: for any propositional letter $p \in P$, if wRv and $w \in V(p)$, then $v \in V(p)$.

[4] here the term *possibilities* is of course used in our technical sense!

Moreover, recall that if $K = \langle W, R, V \rangle$, the satisfaction relation $\Vdash$ is defined inductively as follows: $K, w \Vdash p$ iff $w \in V(p)$; $K, w \nVdash \bot$; the clauses for $\vee$ and $\wedge$ are the obvious ones; finally, $K, w \Vdash \phi \to \psi$ iff for any R-successor v of w, if $K, v \Vdash \phi$ then $K, v \Vdash \psi$.

Finally, recall the completeness result for this semantics: for any ϕ it is $\models_{\mathrm{IPL}} \phi$ iff for any Kripke model K for intuitionistic logic and any point w in K, $K, w \Vdash \phi$.

We will now show how the satisfaction relation of our Generalized Inquisitive Semantics amounts to satisfaction on a suitable Kripke model K_C for Intuitionistic Logic based on the set $\mathcal{P}_0(I_P)$.

Definition (Canonical Kripke model for GIL). Denote $\mathcal{P}_0(I_P)$ by W_C. Define $V_C : P \to \mathcal{P}(W_C)$ as follows: $V_C(p) = \{S \mid S \models p\}$. The canonical Kripke frame for GIL is the Kripke frame $K_C = \langle W_C, \supseteq, V_C \rangle$.

Observe that K_C is a Kripke model for intuitionistic logic. The relation $\supseteq$ is clearly reflexive and transitive. Moreover, suppose $S \supseteq T$ and $S \in V_C(p)$: this means that $S \models p$, and so by persistency $T \models p$, which means that $T \in V_C(p)$. So the valuation V_C is persistent.

Proposition (GIL-satisfaction coincides with Kripke satisfaction on K_C). For any $\phi \in L_P$ and $S \in \mathcal{P}_0(I_P)$ we have $S \models \phi \iff K_C, S \Vdash \phi$.

Proof. Fix S and proceed by induction on the complexity of ϕ.

- If ϕ is a propositional letter p, then $K_C, S \Vdash \phi \iff S \in V_C(p) \iff S \models p$.
- Obviously, $K_C, S \nVdash \bot$ and $S \not\models \bot$.
- Suppose the claim holds for χ, ψ and consider $\phi = \chi \vee \psi$. $S \models \chi \vee \psi$ iff $S \models \chi$ or $S \models \psi$; by inductive hypothesis, this happens iff $K_C, S \Vdash \chi$ or $K_C, S \Vdash \psi$, which by definition amounts to $K_C, S \Vdash \chi \vee \psi$.
- The inductive step for conjunction is similar to that for disjunction and totally straightforward.
- Suppose the claim holds for χ, ψ and consider $\phi = \chi \to \psi$. $S \models \chi \to \psi$ iff for any $T \in W_C$, if $T \subseteq S$ and $T \models \chi$ then $T \models \psi$; by inductive hypothesis, this happens iff for any $T \in W_C$, if $S \supseteq T$ and $K_C, T \Vdash \chi$ then $K_C, T \Vdash \psi$, which by definition amounts to $K_C, S \Vdash \chi \to \psi$.

Corollary. IPL $\subseteq$ GIL.

Proof. Suppose $\not\models_{\mathrm{GIL}} \phi$. Then $I_P \not\models \phi$ and so by the previous result $K_C, I_P \nVdash \phi$. Thus, by soundness of the Kripke semantics for Intuitionistic Logic, $\not\models_{IPL} \phi$.

We will now give two witnesses of the fact that the inclusion IPL $\subset$ GIL is strict, i.e. that the logic GIL is different from IPL.

Fact.

1. For any propositional letter $p \in P$, $\models_{\mathrm{GIL}} \neg\neg p \to p$.

2. For any formulas ϕ, χ, ψ, $\models_{\mathrm{GIL}} (\neg\phi \to \chi \vee \psi) \to (\neg\phi \to \chi) \vee (\neg\phi \to \psi)$.

Proof.

1. Let $p \in P$. For any $S \in \mathcal{P}_0(I_P)$, suppose $S \models \neg\neg p$, that is, $S \models !p$; by a previous result, this implies for all $v \in S$, $v \models p$; thus by definition $S \models p$. This proves that $I_P \models \neg\neg p \to p$, whence $\models_{\mathrm{GIL}} \neg\neg p \to p$.

2. Fix any ϕ, χ, ψ and fix any $S \in \mathcal{P}_0(I_P)$. Suppose towards a contradiction that $\not\models_{\mathrm{GIL}} (\neg\phi \to \chi \vee \psi) \to (\neg\phi \to \chi) \vee (\neg\phi \to \psi)$. This implies that there is $S \in \mathcal{P}_0(I_P)$ such that $S \models \neg\phi \to \chi \vee \psi$ but $S \not\models \neg\phi \to \chi$ and $S \not\models \neg\phi \to \psi$.

   Now, $S \not\models \neg\phi \to \chi$ implies that there is $T_1 \subseteq S$ such that $T_1 \models \neg\phi$, but $T_1 \not\models \chi$; similarly, $S \not\models \neg\phi \to \psi$ implies that there is $T_2 \subseteq S$ such that $T_2 \models \neg\phi$, but $T_2 \not\models \chi$; but then consider $T := T_1 \cup T_2$:

   - obviously, $T_1 \cup T_2 \subseteq S$;
   - moreover, $T_1 \cup T_2 \not\models \chi \vee \psi$: for, if it were $T_1 \cup T_2 \models \chi$ we would have $T_1 \models \chi$ by persistency, and if it were $T_1 \cup T_2 \models \psi$ we would have $T_2 \models \psi$;
   - finally, we have $T_1 \cup T_2 \models \neg\phi$: for, consider any $v \in T_1 \cup T_2$; then at least one of $v \in T_1$ or $v \in T_2$ holds, and since $T_i \models \neg\phi$ for $i = 1, 2$, by persistency $v \models \neg\phi$; so for all $v \in T_1 \cup T_2$, $v \models \neg\phi$, and by a previous result this suffices to establish $T_1 \cup T_2 \models \neg\phi$.

   But then what we have shown is that $T \subseteq S$, $T \models \neg\phi$ but $T \not\models \chi \vee \psi$, so $S \not\models \neg\phi \to \chi \vee \psi$, contrarily to assumption. Thus we have the contradiction we were heading to.

This completes the proof of the first, strict inclusion IPL $\subsetneq$ GIL. The above fact that the Inquisitive Semantics is a particular case of the Generalized Inquisitive Semantics immediately show the inclusion GIL $\subseteq$ IL: for, suppose $\not\models_{\mathrm{IL}} \phi$ for some $\phi \in L_P$; then there are P-indices v, w such that $\langle v, w \rangle \not\models \phi$, and by a previous result this is equivalent to $\{v, w\} \not\models \phi$; so $\not\models_{\mathrm{GIL}} \phi$.

One may still wonder whether GIL and IL might be the same. I will now show that this is not the case by giving a concrete example of a formula which is valid in IL but not in GIL.

Fact. The formula $\xi := (p \to q) \vee (p \to \neg q) \vee (q \to p)$ is valid in IL but not in GIL.

Proof. A simple way to see that ξ is valid in IL would be to simply check that all pairs $\langle v, w \rangle$ of $\{p, q\}$-indices validate ξ. For a less brute-force argument, let v, w be $\{0,1\}-$indices and suppose $\{v, w\} \not\models p \to q$ and $\{v, w\} \not\models p \to \neg q$.

Certainly it cannot be $\{v, w\} \models q$, otherwise all the subsets of $\{w, v\}$ would validate q and therefore $\{w, v\}$ would validate $p \to q$; thus at least one of v, w does not validate q. Analogously, it cannot be $\{v, w\} \models \neg q$, otherwise all the subsets would validate $\neg q$ and therefore $\{w, v\}$ would validate $p \to \neg q$; so at least one of v, w validates $\neg q$. So we are in this situation: one of v, w makes q true, while the other makes it false; without loss of generality, we may assume that $v \models q$ and $w \models \neg q$. So $\{v\}$ is the only subset of $\{v, w\}$ which validates q.

Now, to have $p \to \neg q$ falsified at $\{v, w\}$ requires a subset $S \subseteq \{v, w\}$ which validates p and not $\neg q$; this S cannot be $\{w\}$ because $w \models \neg q$: so $v \in S$, whence by persistency $v \models p$. This shows that for any $T \subseteq \{v, w\}$, if $T \models q$ then $T \models p$, because the only subset T validating q is $\{v\}$, which in fact also validates p. Hence, $\{v, w\} \models q \to p$.

This shows that for any v, w, $\{v, w\}$ validates at least one of $p \to q$, $p \to \neg q$ and $q \to p$, and so by definition $\{v, w\} \models (p \to q) \vee (p \to \neg q) \vee (q \to p)$, which is the same as $\langle v, w \rangle \models \xi$. So ξ is valid in IL.

Now consider the set $I := I_{\{p,q\}}$. Denote by ij the index mapping p to i and q to j, so that, for instance, 10 is the index mapping p to 1 and q to 0. We have: $I \not\models p \to q$, because $\{10\} \subseteq I$ and $10 \models p$, $10 \not\models q$; $I \not\models p \to \neg q$, because $\{11\} \subseteq I$ and $11 \models p$, $11 \not\models \neg q$; finally, $I \not\models q \to p$, because $\{01\} \subseteq I$ and $01 \models q$, $01 \not\models p$. Hence, by definition, $I \not\models (p \to q) \vee (p \to \neg q) \vee (q \to p)$, and thus ξ is *not* valid in GIL.

Observe that a minor modification in the above proof shows that the formula $\xi' = (p \to q) \vee (p \to \neg q) \vee (q \to p) \vee (q \to \neg p)$ - which is equivalent to $(p \to ?q) \vee (q \to ?p)$ - is another example of a formula valid in IL but not in GIL. That ξ' is valid in IL is obvious, because $\xi \models \xi'$ and we have just seen that ξ is IL-valid. To see that ξ' is *not* valid in the generalized setting we reason exactly as above, showing that for each disjunct in ξ' we can find a subset $\{v\} \subseteq I$ which falsifies it, and therefore $I \not\models \xi'$.

Finally, we have IL $\subsetneq$ CPL: for, suppose $\not\models_{\mathrm{CPL}} \phi$; then there is a valuation $v \not\models \phi$, and as we already know, this is the same as $\{v\} \not\models \phi$, which in turn, by a previous result, is the same as $\langle v, v \rangle \not\models \phi$. This shows the inclusion IL $\subseteq$ CPL; that this inclusion is strict is obvious, for instance we have $\not\models_{\mathrm{IL}} ?\phi$.

This completes the proof of the above claim that: IPL $\subsetneq$ GIL $\subsetneq$ IL $\subsetneq$ CPL.

4 Accounting for the behaviour of disjunction.

The purpose, or at least one of the main purposes, of Inquisitive Semantics is to account for the behaviour of disjunction, starting from the intuitive assumption that the meaning of a disjunction $\phi \vee \psi$ consists of two components:

- an informative component, which consists in excluding the case that neither ϕ nor ψ;

- an inquisitive component, which consists in specifying two possibilities, namely the possibility that ϕ holds and the possibility that ψ does.

Now, the generalized and the restricted version of inquisitive semantics both amount to the same thing as for the first issue, since they both represent informative content in the classical way.

As for the description of the inquisitive component of the meaning of a disjunction, however, the two semantics behave quite differently. Indeed, we are now going to see that the generalized version of Inquisitive Semantic is considerably more well-behaved than its restricted counterpart when it comes to specifying possibilities. We begin by considering a few examples of the shortcomings of the restricted setting. We will then explain what the general problem is, and show how the present, generalized version of inquisitive logic is immune from this problem.

First, consider again the formula that we encountered above: $\xi = (p \to q) \vee (p \to \neg q) \vee (q \to p)$. We have seen that this formula is valid in IL; this means that $\xi \equiv \top$, so $\pi(\xi) = \{I_{p,q}\}$. So according to the restricted inquisitive semantics there is only a possibility for ξ, namely the whole universe. The reason why this is a problem is that the whole universe is not a possibility for *any* of the disjuncts in ξ. So, in the restricted setting we may have a set which is a possibility for a disjunction while not being a possibility for *any* of the disjuncts!

On the contrary, our linguistic intuition suggests that ξ should specify the three possibilities $p \to q$, $p \to \neg q$, and $q \to p$. Note that none of these formulas implies any other, so these should actually be three *distinct* possibilities for ξ ! This is *not* what the restricted version of inquisitive logic gives, but as we shall now see, it *is* what comes out according to generalized semantics.

Before discussing the general problem, let us give another couple of examples. The first one is the choice question $\xi' = (p \to ?q) \vee (q \to ?p)$: we have seen above that also this formula is IL-valid, so according to the standard semantics it admits only one possibility, namely the whole universe. Again, note that the whole universe is a possibility for neither of the disjuncts! Moreover, the intuitive meaning that we are trying to capture with $(p \to ?q) \vee (q \to ?p)$ is: "answer one of the following two questions: if p, then is it the case that q? if q, is it the case that p?"; intuitively, this should have the effect of specifying four possible answers: "if p then q", "if p then $\neg q$", "if q then p", "if q then $\neg p$". So we would like ξ' to determine four possibility: but in the restricted setting, it does not. On the contrary, we will see in a moment that in the generalized setting, ξ' specifies *exactly* the four possibilities corresponding to the four mentioned answers.

A third example shows that this problem is not limited to the misrepresentation of the possibilities of formulas which are IL-valid. Consider $\xi'' := p \vee q \vee \neg(p \leftrightarrow q)$, where $P = \{q, p\}$. We see that $[\![p]\!] = \{10, 11\}$, $[\![q]\!] = \{01, 11\}$, $[\![\neg(p \leftrightarrow q)]\!] = \{01, 10\}$. So $\{10, 01, 11\}$ is not a possibility for any of the disjuncts of ξ''. However, it is easy to check that $\{10, 01, 11\}$ *is* a possibility for ξ''. Note that in this case ξ'' is not an IL-valid formula: for instance, $\langle 00, 00\rangle \not\models \xi''$

because $\langle 00, 00 \rangle$ validates none of the disjuncts.

All of these examples show that the definition of possibility which arises from the restricted Inquisitive Semantics is not fully satisfactory: in particular, it leads to pictures which are poorer than we would expect.

We would like our notion of possibility for a formula ϕ to capture the intuitive idea of "way in which ϕ can be realized"; since a disjunction is realized exactly in case one of its disjuncts is, we would expect that any possibility for a disjunction is also a possibility for at least one of its disjuncts. The following fact says that, unlike in the standard setting, in the generalized version of Inquisitive Semantics this expectation is always met.

Fact. For any $\phi_1, \ldots, \phi_n \in L_P$, $\pi(\phi_1 \vee \ldots \vee \phi_n) \subseteq \pi(\phi_1) \cup \ldots \cup \pi(\phi_n)$. In words, if a set is a possibility for a disjunction, then it is a possibility for at least one of its disjuncts.

Proof. For any $P \in \mathcal{P}_0(I_P)$, if $P \in \pi(\phi_1 \vee \ldots \vee \phi_n)$, then $P \models \phi_1 \vee \ldots \vee \phi_n$, so there is $i \leq n$ such that $P \models \phi_i$; so P is included in some possibility $P' \in \pi(\phi_i)$; but now $P' \models \phi_i$, so $P' \models \phi_1 \vee \ldots \vee \phi_n$. But then since P is a *maximal* set validating the disjunction, it must be $P = P'$, so $P \in \pi(\phi_i)$.

This fact guarantees that the unwanted situation in which a set is a possibility for a disjunction without being a possibility for any disjunct does not occur when possibilities are defined in generalized semantics. As a consequence, the pictures that we get in our generalized semantics are richer, and I believe they are rich enough to specify all the "intuitive" possibilities to which a disjunction gives rise.

This is not to say that any disjunction gives rise to at least as many possibilities as its disjuncts: for instance, if $\phi \models \psi$, then the disjunction $\phi \vee \psi$ is equivalent to ψ, so the possibilities for $\phi \vee \psi$ coincide with the possibilities of ψ. But this should be so, intuitively: if ϕ entails ψ, then of course $\phi \vee \psi$ can only be realized when ψ is realized, and the disjunction does not really specify two alternatives, since one of the disjuncts is superfluous. As an example, think of disjunctions like $p \vee (q \to p)$, or $p \vee (p \wedge q)$.

There are more complicated cases of ϕ, ψ for which $\pi(\phi \vee \psi) \subsetneq \pi(\phi) \cup \pi(\psi)$, that is, for which there is a possibility P for a disjunct - say ϕ - which is not a possibility for the disjunction. But as the following fact shows, this can only be the case if $P \subsetneq Q$ for some possibility Q for ψ; this, of course, generalizes immediately to disjunctions of an arbitrary number of formulas.

Fact. Let $\phi, \psi \in L_P$. If $P \in \pi(\phi)$, then $P \subseteq Q$ for some $Q \in \pi(\phi \vee \psi)$; in particular, if $P \in \pi(\phi)$ but $P \notin \pi(\phi \vee \psi)$, then $P \subsetneq Q$ for some $Q \in \pi(\psi)$.

Proof. If $P \in \pi(\phi)$, then $P \models \phi$, so $P \models \phi \vee \psi$ and thus $P \subseteq Q$ for some $Q \in \pi(\phi \vee \psi)$. Then, suppose $P \in \pi(\phi)$ but $P \notin \pi(\phi \vee \psi)$: take $Q \in \pi(\phi \vee \psi)$ such that $P \subseteq Q$; since $P \notin \pi(\phi \vee \psi)$, $P \subsetneq Q$, and therefore by the maximality of P, $Q \notin \pi(\phi)$. But we know that $Q \in \pi(\phi) \cup \pi(\psi)$: therefore, $Q \in \pi(\psi)$ and we are done.

This fact has the following consequence: if the possibilities of ϕ and the possibilities of ψ are pairwise incomparable (i.e. there are no $P \in \pi(\phi)$, $P' \in \pi(\psi)$ such that $P \subset P'$ or $P' \subset P$) then $\pi(\phi \vee \psi) = \pi(\phi) \cup \pi(\psi)$. This generalizes to disjunctions of an arbitrary number of formulas.

Now, let us return to check that the notion of possibility defined in our generalized semantics behaves according to our expectations in the critical examples seen above.

Consider our first example, $\xi = (p \to q) \vee (p \to \neg q) \vee (q \to p)$ where $P = \{p, q\}$. It is immediate to check that: $p \to q$ admits the only possibility $P_1 = \{00, 01, 11\}$, $p \to \neg q$ admits the only possibility $P_2 = \{00, 01, 10\}$, and $q \to p$ admits the only possibility $P_3 = \{00, 10, 11\}$. Note that none of these possibilities is included in any other, so by the previous result, $\pi(\xi) = \pi(p \to q) \cup \pi(p \to \neg q) \cup \pi(q \to p) = \{P_1, P_2, P_3\}$, which is precisely what we expected.

Now consider our second example, $\xi' = (p \to ?q) \vee (q \to ?p)$. We recycle the notation of the previous example: put $P_1 = \{00, 01, 11\}$, $P_2 = \{00, 01, 10\}$, $P_3 = \{00, 10, 11\}$, $P_4 = \{10, 01, 00\}$. As we noted before, P_1 is the only possibility for $p \to q$, P_2 for $p \to \neg q$, P_3 for $q \to p$, and P_4 for $q \to \neg p$. So, P_1 and P_2 correspond to the possible answers to $p \to ?q$, while P_3 and P_4 correspond to the possible answers to $q \to ?p$.

It easy to verify that indeed $\pi(p \to ?q) = \{P_1, P_2\}$ and $\pi(q \to ?p) = \{P_3, P_4\}$. The possibilities in these two sets are pairwise incomparable, and thus again by the previous result: $\pi(\xi') = \pi(p \to ?q) \vee \pi(q \to ?p) = \{P_1, P_2, P_3, P_4\}$: so the choice question ξ' specifies precisely the four possibilities corresponding to its four possible answers, as we would expect.

Analogously it is easy to check that in the case of our third sentence $\xi'' = p \vee q \vee \neg(p \leftrightarrow q)$, ξ'' turns out to specify three possibilities, corresponding to p, q and $\neg(p \leftrightarrow q)$ respectively.

What goes wrong in standard inquisitive semantics? So, what exactly is wrong with the notion of possibility that we get from the restricted semantics? We are going to try to give an intuitively clear explanation of the problem.

The problem can be perfectly illustrated with an analogy. Using classical logic, we could define a set S to be a possibility for ϕ iff it is a maximal set such for all $v \in S$, $v \models \phi$ (in the classical sense). In other words, a possibility is a maximal set such that all of its subsets of size 1 validate ϕ. Clearly, it turns out that according to this definition the unique possibility for ϕ is its classical extension $\lfloor \phi \rfloor = \{v \mid v \models \phi\}$.

Now, consider the formula $p \vee q$. In Inquisitive Semantics we have two possibilities for it, namely $\{10, 11\}$ and $\{01, 11\}$; these are maximal only because $\{10, 01, 00\}$ is *not* a possibility in Inquisitive Semantics, and this is because it contains the subset $\{01, 10\}$ (of cardinality 2) which does not validate $p \vee q$.

But when we are in the classical setting, we only look at subsets of cardinality 1: and since all the singletons in $\{01, 10, 11\}$ validate $p \vee q$, $\{01, 10, 11\}$ will be a possibility. We can look at it this way: the classical setting is unable to "distinguish" the two possibilities specified by $p \vee q$, which are sort of "hidden inside $\{01, 10, 11\}$", because it can only look at subsets of cardinality 1.

In this respect, the restricted inquisitive semantics is just one step further than this classical setting, in that it can only look at subsets of cardinality at most 2.

Suppose for instance that ϕ has three possibilities of the form $\{v, w\}, \{w, u\}, \{u, v\}$ (note that whatever indices v, w, u are, the existence of such a ϕ is guaranteed by the expressive completeness theorem). Consider the set $S := \{v, w, u\}$. For any $\{i, j\} \subseteq S$, $\{i, j\} \models \phi$; moreover, it is easily seen that S is maximal with respect to this property; so in the restricted setting, S is a (the) possibility for ϕ.

The restricted perspective is unable to distinguish the three possibilities specified by ϕ, which are "hidden inside S", because they are interconnected too tightly to be detected by a set of size 2; but if we could just have a look to subsets of size 3...

The above examples, for instance, show that there are formulas (like ξ and ξ' above) which "look tautological" to sets of size 2 - and thus to standard inquisitive logic - but which are *not* tautological to the generalized semantics, nor they are in natural language!

Recapitulating, if we look at subsets of size one, we may "mistake two possibilities for one"; if we look at subsets of size at most 2, we may "mistake three possibilities for one". The general fact - the easy proof of which I will omit here - is that restricting our attention to subsets of any fixed size n is not sufficient if we want an accurate representation of the notion of "possibility" for a formula, since in the present, generalized setting we could easily construct a formula ϕ having $n+1$ possibilities which will be different but will "look like one" to an observer who only looks at subsets of size n.

References

Cresswell, M. J. (2004). Possibility semantics for intuitionistic logic. *Australasian Journal of Logic*, **2**, 11–29.

Groenendijk, J. (2008a). Inquisitive semantics and dialogue management. ESSLLI course notes, `http://www.illc.uva.nl/inquisitive-semantics`.

Groenendijk, J. (2008b). Inquisitive semantics: Student version. Amsterdam lecture notes, `http://www.illc.uva.nl/inquisitive-semantics`.

Mascarenhas, S. (2008). Inquisitive semantics and logic. Manuscript, University of Amsterdam.