A first-order inquisitive witness semantics
Student version

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Abstract

This paper develops a first-order inquisitive semantics whose main feature is that an existentially quantified sentence like $\exists x.Px$ is only supported in a state s if a specific *witness* had been introduced in s whose value is known to have the property P. Thus, in this system, an inquisitive sentence may not only request a response that provides certain information; it may also request a response such that the information it provides gives rise to the introduction of a specific *witness* with certain properties. Thus, the notion of inquisitiveness is richer than in previous work on first-order inquisitive semantics (Ciardelli, 2009, 2010; Roelofsen, 2011).

Our conjecture is that because the notion of inquisitiveness is enriched in this way, the problems that were encountered earlier by Ciardelli (2009, 2010) in developing a first-order system no longer arise.

Overnight problems

After the presentation yesterday evening, we discovered problems. Starting from Section 1, we have left our notes as they were at the time of presentation, but we will start out here with an indication of the issues we found.

The problems are due to the fact that in the clauses for implication and universal quantification in the definition of support, we use the notion of witness-extensions of states.

0.1 Vacuous implication

Let us abbreviate $\top = \neg\bot$. Then we have for all s and g: $s \models_g \top$. Now consider a sentence $\top \to \varphi$, which we might call a 'vacuous implication'. Under the definition of support in Section 1 (assuming that Persistence holds) we get:

- $s \models_g \top \to \varphi$ iff $\exists t. t \geq_{\Delta} s \colon t \models_g \varphi$

Clearly, then it need not hold for every φ that $\varphi \equiv \top \to \varphi$. We get, e.g., that:

- $\exists x.Px \not\equiv \top \to \exists x.Px$

The reason is that whereas $s \models_g \exists x.Px$ requires the presence of a witness in s, this witness requirement does not apply to $s \models_g \top \to \exists x.Px$.

This may not seem very dramatic at first sight, but it is. It immediately leads to counterexamples to the central claim in Section 2.6 that there are minimal supporting states for any formula.

- Let $\exists x.Bx$ be the boundedness formula from Ciardelli (2009, 2010).

  There are no minimal supporting states for $\top \to \exists x.Bx$.

It is a good excercise to convince yourself that this, unfortunately, is the case.

0.2 Vacuous universal quantification

In the fact on Binding it is claimed that vacuous universal quantification has no semantic effect. A footnote already warns that this may not be the case. And indeed it isn't.

- If x does not occur free in φ, then $s \models_g \forall x.\varphi$ iff $\exists t.t \geq_\Delta s \colon t \models_g \varphi$

So the effects of vacuous universal quantification are essentially the same as those of vacuous implication. And we get a second case of a counterexample to the claim that there are minimal supporting states for every formula.

- Let $\exists x.Bx$ be the boundedness formula from Ciardelli (2009, 2010).

  There are no minimal supporting states for $\forall y \exists x.Bx$.

0.3 Avoiding witness extensions?

What causes the problems with vacuous implication and vacuous universal quantification is the same: the fact that in the support clauses for implication and universal quantification the notion of witness extension plays a role. So, if we think of ways to repair the definition of support, we should see whether we can do without 'uncontrolled' witness extensions.

As is discussed to some extent in Section 2.2, we could in the clause for universal quantification not consider witnesses extensions of the state of evaluation s at large, but just add for each $d \in D$ that we have to consider in evaluating $\forall x.\varphi$ only d hypothetically as a witness to s. That would be the minimal, and easy solution to the 'vacuous binding problem'.

In case of implication, we can decide that we get back to the standard interpretation of implication, not allowing for witness extensions at all. This had repercussions for the definition of entailment, which then should be as it standardly is in inquisitive semantics as well. But if we then wish to stick to the fact that $Pa \models \exists x.Px$, we have to make sure that atomic sentences require the denotation of the constants occurring in them to be present as witnesses in states that support them.

0.4 Exercise

Try to implement the changes in the support definition as suggested above, and compare the two. In doing so, there are several remarks in Section 2 that pertain to the difference between these two notions. There are more and less forcefull arguments there, why we do need witness extensions in the clauses for implication and universal quantification. One can evaluate these arguments, and to the extent that they are convincing, try to come up with alternative ways to meet them that do not involve the arbitrary addition of witnesses in the clauses for implication and universal quantification.

1 An inquisitive witness semantics

In this section we will present a notion of support for a first-order language $\mathcal{L}$. We denote by $\mathcal{L}_C$ the set of individual constants in $\mathcal{L}$, and by $\mathcal{L}_P$ the set of predicate symbols in $\mathcal{L}$. As usual, we assume a countably infinite set of variables, which we denote by $x, y, z, \ldots$. We refer to the variables and the constants in $\mathcal{L}_C$ as the *terms* in $\mathcal{L}$.

We take as primitives the connectives $\bot, \wedge, \vee, \rightarrow$, both quantifiers $\exists$ and $\forall$, and the identity symbol $=$. We use $\neg\varphi$ as an abbreviation for $\varphi \rightarrow \bot$, $?\varphi$ for $\varphi \vee \neg\varphi$ and $!\varphi$ for $\neg\neg\varphi$.

We will only consider models for $\mathcal{L}$ over a fixed domain D. We will refer to the models for $\mathcal{L}$ as $\mathcal{L}$-*worlds*, and denote the set of all suitable models for $\mathcal{L}$ with domain D by $W_{\mathcal{L}}$. We will usually suppress explicit reference to $\mathcal{L}$.

Given that the domain is fixed for all $w \in W$, we write $w(c)$ for the denotation in w of a constant $c \in \mathcal{L}_C$, where $w(c) \in D$; and we write $w(P^n)$ for the denotation in w of an n-place predicate $P^n \in \mathcal{L}_P$, where $w(P^n) \subseteq D^n$.

Next to a fixed domain, we also assume a fixed interpretation for the individual constants in $\mathcal{L}_C$. For all $c \in \mathcal{L}_C$ and all $w, v \in W\colon w(c) = v(c)$.

As usual, a state contains a set of worlds, those worlds that have not been excluded by the information that the state embodies. Additionally, a state contains a set of objects from the domain, which have been introduced in the state as witnesses.

The assumptions that there is a fixed domain for all worlds, and that the denotation of all individual constants is fixed as well, mean that there is no uncertainty in states as to what constitutes the domain of discourse, nor as to what the objects are that the names in the language refer to.

Definition 1 (States and witnesses)
Let W be the set of worlds with as domain the set of objects D.

- *A* ***state*** *s is a pair $\langle V, \Delta\rangle$, where V is a subset of W and Δ a subset of D.*[1]
    - *If $d \in \Delta$, we say that d is a* ***witness in*** *s.*

[1] It would be better if Δ is required to be finite, but this comes unhandy for the notion of maximal non-absurd states characterized below.

  - *If $w \in V$, we say that w is a* ***world in*** *s.*

- *A state $s = \langle V, \Delta \rangle$ is an* ***absurd state*** *iff $V = \emptyset$.*
- *A state $s = \langle V, \Delta \rangle$ is an* ***ignorant state*** *iff $V = W$.*
- *The* ***initial state*** *s_0 is the state $\langle W, \emptyset \rangle$.*

An absurd state is a state that excludes every world. An ignorant state is a state where no world is excluded. The initial state is the ignorant state in which no witnesses have been introduced.

Definition 2 (Extension of states)
Let $s = \langle V, \Delta \rangle$ and $s' = \langle V', \Delta' \rangle$ be two states.

- *s' is an* ***extension*** *of s, $s' \geq s$, iff $V' \subseteq V$ and $\Delta \subseteq \Delta'$.*
- *s' is a* ***witness extension*** *of s, $s' \geq_\Delta s$, iff $V' = V$ and $\Delta \subseteq \Delta'$.*

Both notions are used in the semantics. We can extend states by eliminating worlds, or by introducing new witnesses. Witness extension is restricted to the latter aspect of extending a state.

Fact 1 *Every state is an extension of the initial state.*

Fact 2 *Every absurd state is an extension of $\langle \emptyset, \emptyset \rangle$.*

The latter fact guarantees that $\langle \emptyset, \emptyset \rangle$ is the unique minimal state that supports $\perp$.

Fact 3 (Maximal (non-absurd) state(s))

1. *There is no proper extension of $\langle \emptyset, D \rangle$. So $\langle \emptyset, D \rangle$ is the maximal state.*
2. *A state $\langle V, \Delta \rangle$ is a maximal non-absurd state iff $V = \{w\}$ for some w, and $\Delta = D$.*

The maximal non-absurd states, we might call them *points*, will be used to relate the semantics to classical first order semantics.

Definition 3 (Assignments)

- *An assignment g is a function that maps every variable to an object in D*
- *If g is an assignment and d an object in D, then $g[x/d]$ is the assignment that is just like g except that $g = d$.*

Definition 4 (The denotation of terms)

- *Let w be a world, g an assignment, x a variable, and c an individual constant. Then:*

- $[x]_{w,g} \coloneqq g(x)$
- $[c]_{w,g} \coloneqq w(c)$

Definition 5 (Support)
Let $s = \langle V, \Delta \rangle$ be a state. Then:

1. $s \models_g Rt_1 \ldots t_n$ *iff* *for all* $w \in V : \langle [t_1]_{w,g}, \ldots, [t_n]_{w,g} \rangle \in w(R)$, *and if* t_i *is a variable* x*, then* $g(x)$ *is a witness in* s
2. $s \models_g t = t'$ *iff* *for all* $w \in V : [t]_{w,g} = [t']_{w,g}$, *and if* t *or* t' *is a variable* x*, then* $g(x)$ *is a witness in* s
3. $s \models_g \bot$ *iff* s *is an absurd state*
4. $s \models_g \varphi \wedge \psi$ *iff* $s \models_g \varphi$ *and* $s \models_g \psi$
5. $s \models_g \varphi \vee \psi$ *iff* $s \models_g \varphi$ *or* $s \models_g \psi$
6. $s \models_g \varphi \to \psi$ *iff* $\forall t \geq s$: *if* $t \models_g \varphi$ *then* $\exists t' \geq_\Delta t \colon t' \models_g \psi$
7. $s \models_g \forall x.\varphi$ *iff* *all* $d \in D$ *are such that* $\exists t \geq_\Delta s \colon t \models_{g[x/d]} \varphi$
8. $s \models_g \exists x.\varphi$ *iff* *some* $d \in D$ *is such that* $s \models_{g[x/d]} \varphi$

Given the way implication is defined, the natural notion of entailment that the semantics gives rise to is:

Definition 6 (Entailment) $\varphi \models \psi$ *iff for every state* s *and assignment* g *such that* $s \models_g \varphi$*, there is a state* $t \geq_\Delta s$ *such that* $t \models_g \psi$*.*

We first need to explain the semantics a bit, but what the notion of entailment amounts to is this:

- φ entails ψ just in case for any state s that supports φ, we can introduce witnesses in s such that the resulting state t supports ψ.

The important part of this informal reformulation, which is not immediately clear from the formal definition, is that the fact that s supports φ must give rise to witnesses that ψ needs in order to be supported.

That ψ needs witnesses in order to be supported is specifically the case when ψ is an existentially quantified sentence, the simplest case is $\exists x.Px$. What is a witness for this sentence relative to a state? A specific object d such that in every world in that state d has the property P. Assuming individual constants to be rigid, as we do, if a sentence Pc is supported in a state, then there cannot fail to be a specific object in that state that can serve as a witness for $\exists x.Px$, viz., the denotation of c. So, in every state that supports Pc, we are indeed capable to add an object to the witness set of that state such that thus extended, the state supports $\exists x.Px$.

Definition 7 (Validity)
$\models \varphi$ *iff for every state* s *and assignment* g*:* $s \models_g \varphi$

Fact 4 *$s \models \varphi$ iff for every assignment g: $s_0 \models_g \varphi$*

Definition 8 (Equivalence)
$\varphi \equiv \psi$ iff for every state s and assignment g: $s \models_g \varphi$ iff $s \models_g \psi$.

2 Examples, motivation, issues

2.1 Existential quantification and the atomic clause

The basic idea behind inquisitive witness semantics is that support of $\exists x.\varphi$ by a state s requires the presence of a witness in s. However, the clause for existential quantification looks completely standard and does not mention the 'witness requirement'. True, but the atomic clause and the clause for identity statements are non-standard, and force the witness requirement to be satisfied for existential quantification.

Consider the simple case $\exists x.Px$. According to the support definition, $\exists x.Px$ is supported by a state s relative to an assignment g iff for some object d, s supports Px relative to $g[x/d]$. This brings us to the atomic clause, which now first of all requires quite standardly that for all worlds in s the object d assigned to x it holds that $d \in w(P)$. I.e., the state contains the information that d has the property P. But, furthermore, the atomic clause also requires in order for s to support Px relative to $g[x/d]$ that $g[x/d](x) = d$ is a witness in s.

So, all in all, for $s \models_g \exists x.Px$, it is required that there is some object d such that $d \in w(P)$ for all worlds w in s, and that d is a witness in s. Note that, of course, there can be several such objects that satisfy these requirements in s.

A bit about motivation for implementing the witness requirement in this roundabout way. One effect of putting the witness requirement in the atomic clause is that we deal properly with cases of vacuous quantification, as in $\exists x.\exists y.Py$. As is standardly the case, this comes out equivalent with $\exists y.Py$. This would be difficult to achieve if we put the witness requirement in the clause for existential quantification.

2.2 Universal and existential quantification

Against expectations, the clause for universal quantification looks less standard than the one for existential quantification. This has directly to do with the fact that the witness requirement for existential quantification is forced by the atomic clause. That would mean that if we more standardly would require: $s \models_g \forall x.\varphi$ iff for all $d \in D$: $s \models_{[g/d]} \varphi$, universal quantification would make a heavy witness requirement as well. There may be something to say for this, but this is not our intention.

What we want to require for s to support $\forall x.Px$ is simply that for all $d \in D$ and w in s: $d \in w(P)$. Given the way the atomic clause is defined, for which we have good reasons, as we saw above, in order to achieve this, we have to make sure that when we consider $s \models_{g[x/d]} Px$, we can add d on the fly, hypothetically so to speak, to the witnesses in s.

The non-standard feature of the clause for universal quantification is that we do not consider s as such but allow to consider a state t instead, which results from s by adding an arbitrary number of witnesses to it. This may seem overkill, and perhaps in the end this will turn out to be so, why not just consider for each $d \in D$ the extension of s with a witness d in evaluating $\forall x.\varphi$? That would indeed suffice for a case like $\forall x.Px$. However, there are independent (though perhaps also less compelling) reasons for allowing the addition of an arbitrary number of witnesses in evaluating universal quantification.

Consider the sentence $\forall x.\exists y.Rxy$. First note that under the clause for universal quantification as it is now we also liberate the existential quantifier embedded under the universal one from the requirement that there be a witness for it in the state of evaluation. A state s can support $\forall x.\exists y.Rxy$ if there are no witnesses at all in s.

Consider what would happen in case we adopt the stronger, and admittedly more intuitive clause for universal quantification, where we may only add on the fly one by one hypothetical witnesses for the universal quantifier. Then, in evaluating $\forall x.\exists y.Rxy$, for every choice of an object d as a value for x, there should be a witness d' in s such that $\langle d, d' \rangle \in w(R)$ for every world w in s.

This is problematic. The point is that for different choices of values for x, we may need different choices for witnesses for the existential quantifier. We may need the presence of many, in fact in some cases infinitely many, witnesses in s in order for s to support $\forall x.\exists y.Rxy$.

As a more concrete case, consider $\forall x.\exists y.y = x$. You would need the whole domain D as witness set in the state of evaluation. We would like it to be the case that every state supports $\forall x.\exists y.y = x$. That would not be so if it is required that every object in the domain is present as a witness.

Anyway, the current clause for universal quantification is designed to circumvent such problems. You could also put it like this: if an existential quantifier occurs in a sentence, we do not want to require for more than one witness for that quantifier in a state that supports that sentence.[2]

We have not necessarily reached the end of the story here. The deliberations above very much depend on the decision that witnesses are just objects in the domain. But this is not a necessary feature. (And we did things differently in other versions of the semantics that we considered, and moved away from for the time being.) Consider the situation where our domain consists of the natural numbers. And take the sentence: $\forall x.\exists y.y > x$. You can easily think here of a (single) witness for the existential quantifier, not as a specific object, a number, but as a specific function from objects to objects, from numbers to numbers, the function f such that for all $d \in D$: $f(d) = d + 1$. But there are other options as well, of course. It may not be straightforward to implement this, but it certainly would make sense to add witnesses of this kind.

A more down to earth example. Read $\forall x.\exists y.Rxy$ as "Everyone loves someone." If a witness for $\exists y$ should be required, one way or the other, to be a single

[2]Note that this would motivate to require the witness set in a state to be finite, since there can never be infinitely many occurences of existential quantifiers in a sentence.

object in the state of evaluation, then the 'intended object' as a witness could be, e.g., Mary. The sentence would be inquisitive as to which object is 'intended' to serve as a witness. Note that in such a case, the order of the quantifiers does not seem to matter. Now, if we may consider functions from individuals to individuals as well as potential witnesses, then we could also take the function f, where for every $d \in D$: $f(d) =$ the mother of d. Where of course, again, there is more than one option for such a function, just as there is more than one option if we consider single objects.

This is not something new. E.g., in the semantics of questions it has been proposed that on one reading of the question: "Whom does every man love?", it can be answered with: "His mother." To implement this technically we would have to use so-called Skolem-functions.

2.3 Implication

In the clause for implication, we do not just require that every extension of s that supports φ, supports ψ as well. Here we also allow the addition of any number of witnesses before evaluating ψ. The motivation for this is more or less comparable to what we saw in the case of universal quantification.[3]

But first, let us point at the fact that as far as the antecedent of an implication is concerned, where we have to consider all extensions of the state of evaluation that support the antecedent, whether or not the antecedent as such requires the presence of witnesses plays no role. Consider the case of a formula of the form $\exists x.Px \to \varphi$ evaluated in a state s. Any state s' that is an extension of s such that s' supports $\exists x.Px$ will be such that it contains a witness for the existential quantifier. In other words, no such witness needs to be present in s as such to support an implication of which $\exists xPx$ is the antecedent.

One way to motivate this (because it is not impossible to arrange things otherwise), is to point at: $\exists x\varphi \to \bot$, which corresponds, as usual, to $\neg\exists x.\varphi$. It would be rather strange to require for support of $\neg\exists Px$ that there is a witness for the quantifier in the state of evaluation. So, generally, the witness requirement for existential quantification does not apply to any occurrence of the quantifier somewhere in a sentence, it depends on its 'environment'.

Note that, irrespective of whether we allow for the addition of further witnesses or not, when we move to the consequent, it cannot fail to be the case that any state, with or without witnesses, supports $\exists x.Px \to \exists x.Px$. Or, for that matter, $\exists x.Px \to \exists y.Py$.

Things are slightly different for $\forall x.Px \to \exists x.Px$, which is classically valid under the usual assumption that the domain of discourse $D \neq \emptyset$. Whether or not this sentence is supported by any state does depend on whether we allow to add witnesses on the fly when we move to the consequent. Remember, a state with no witnesses can support $\forall x.Px$. So if we move from a state s to a state s' that is an extension of s that supports the antecedent $\forall x.Px$, no witness has

[3] Side remark: the type of clause we find here, and that holds also for universal quantification, bears striking similarities to the corresponding clauses in dynamic logic. But there you would see similar features for the other clauses as well, which are lacking here.

to be added. We only have to eliminate worlds from s where not every object has property P. So, if we then turn to the consequent $\exists x.Px$ and see whether it is supported by s', it is required that there is a witness for $\exists x.Px$. There is no guarantee that this is the case for s' as such, but it is guaranteed if we allow for arbitrary additions of witnesses to s'. So, under the choice we made for the interpretation of implication $\forall x.Px \to \exists x.Px$ is supported in any state. If we use the standard interpretation of implication — i.e., every extension of the state of evaluation that supports the antecedent, supports the consequent as well — only states that contain a witness d such that d has the property P when everyone else has that property as well, we get that $\forall x.Px \to \exists x.Px$ is supported.

This may be open to some discussion, but it seems reasonable to think that what we get under the clause for implication as it is now, is the preferable option. Note that the sentence $\exists x.(\forall y.Py \to Px)$, or perhaps more sensibly, $\exists x.(\forall y.(y \neq x \to Py) \to Px)$, does require the presence of a witness in the state of evaluation, sort of 'the last person who has the property P'.

2.4 Individual constants

After writing this we decided to make the general assumption that individual constants have a fixed, rigid interpretation. This is not taken for granted here, but the conclusion in this section is that that better be the case.

In the current version of the semantics, unlike in practically all previous versions that we considered, we do nothing special concerning individual constants. Consider a simple case like the atomic sentence Pa, the only thing that the atomic clause requires for a state s to support Pa is that in every world w in s: $w(a) \in w(P)$, which is as standard as you can get.

In general there are two options for the interpretation of individual constants. One is to require globally that an individual constant c, a name, is a *rigid designator*, i.e., for $w, w' \in W$: $w(c) = w'(c)$. Note that if we do that, then it cannot fail to be the case that in every (information) state s, constants are rigid as well, i.e., we assume that we are fully informed about the denotation of every name. This is rather unrealistic, but nevertheless quite a standard practice.

Note that this is independent of whether or not we adopt Kripke's view that names are rigid designators. As Kripke stresses himself, this is a *metaphysical* point of view, which as such at most implies *epistemically* that we know that names are rigid, not that we know what the (rigid) denotation of every name is.

So, the idea that names are rigid is quite compatible with the situation that in an *information* state s there are worlds w, w' such that $w(c) \neq w'(c)$. So this is the second option for the interpretation of individual constants in our epistemic, information oriented setting: not requiring rigidity.

It is a different ball game to implement both the knowledge that names are rigid, and at the same time allow for not knowing the rigid denotation of certain names. Neither of the two options we have do anything like that. From that perspective both options are viable options.

This much for an introduction. As things are, we did not make the rigidity assumption for individual constants.[4] And given the way the atomic clause and the clause for identity statements (and the clause for universal quantification and implication) are formulated in the current support definition, we get that: $Pc \equiv \forall x.(x = c \rightarrow Px)$, i.e., "Whoever c is, he has the property P." Note that this is a classical equivalence. But there is another one that doesn't hold here: $Pc \not\equiv \exists x.(x = c \wedge Px)$. Support of $\exists x.(x = c \wedge Px)$ in a state s requires the presence of a witness d in s, a specific object d, such that for all worlds w in s: $w(c) = d$ and $d \in w(P)$. I.e., Unlike for $\forall x.(x = c \rightarrow Px)$, it is required by $\exists x.(x = c \wedge Px)$ that c is rigid in a state s if it is to support $\exists x.(x = c \wedge Px)$.

There is another way to make the difference. As was the case in the semantics for propositional logic, we get under the current semantics that for atomic sentences φ, including identity statements, $\varphi \equiv \neg\neg\varphi \equiv\ !\varphi$. But like this does not hold for disjunctions in the propositional case, it does not hold for existentially quantified sentences (nor for disjunctions) in the first order case, whence $Pc \not\equiv \exists x.(x = c \wedge Px)$.

However, this has a somewhat nasty consequence. Under the notion of entailment defined above, we get that $Pc \models \exists x.(x = c \wedge Px)$. If a state s supports Pc, then just add any individual d to the witness of s such that in some world w in s: $w(d) = w(c)$, and you cannot fail to arrive in a state s' that supports $\exists x.(x = c \wedge Px)$.

Why is this "somewhat nasty"? Well, the usual way to look upon $\varphi \models \psi$, is that if φ is an assertion (which is the case for Pc given the way we interpret it), then φ provides at least as much information as ψ (which is indeed the case for Pc in relation to $\exists x.(x = c \wedge Px)$), and φ also fully resolves any issue that ψ may raise. And the latter does not hold. $\exists x.(x = c \wedge Px)$ requires establishing for some specific object d that it has the property P. That is not what Pc does, it just tells us that whoever c is, that object has the property P.

Now, if we move from the entailment definition as given above, to the more standard alternative that for φ to entail ψ, any state that supports φ is to support ψ as well, then we get rid of the somewhat nasty result that Pc entails $\exists x.(x = c \wedge Px)$. But the price we have to pay is that we loose the connection between the way entailment is defined and the interpretation of implication. If Pc does not entail $\exists x.(x = c \wedge Px)$, then we shouldn't get either that every state supports $Pc \rightarrow \exists x.(x = c \wedge Px)$, but that requires the standard interpretation of implication, rather than the non-standard one in the current definition of support.

We can avoid the dilemma by, unlike what we did sofar, go for the option that constants are rigid designators. Then under the current semantics, and the current definition of entailment, we get that (i) Pc and $\exists x.(x = c \wedge Px)$ are equivalent; (ii) we still have that $\neg\neg Pc$ and Pc and $\forall x.(x = c \rightarrow Px)$ are equivalent as well; (iii) we still get that $Pc \models \exists x.(x = c \wedge Px)$, but now Pc also resolves the issue embodied by $\exists x.(x = c \wedge Px)$; and (iv) every state supports $Pc \rightarrow \exists x.(x = c \wedge Px)$.

[4]In the meantime we did.

Conclusion to draw: the current semantics most naturally goes with the assumption that (i) names are rigid, and (ii) we know what their rigid denotation is. As said before, it is quite common to make this assumption in an epistemic setting, as ours is, but it would be better if we were not obliged to make this assumption.

This is not really put to use here yet, but in the meantime we have a decent informal characterization of the entailment relation that comes with the semantics. See the remarks after the definition of entailment. That may make things easier to grasp.

2.5 Relation with classical first order semantics

The following fact gives us the relation between the support notion defined above and the classical support notion $w \models_g \varphi$.

Claim 1 $\langle \{w\}, D\rangle \models_g \varphi$ *iff* $w \models_g \varphi$

Likewise we can make a relation between our support notion and Ivano's first order support notion, where states are sets of worlds.

Claim 2 $\langle V, D\rangle \models_g \varphi$ *iff* $V \models_g \varphi$

Note that the first fact follows from the second, since under Ivano's support notion it holds that $\{w\} \models_g \varphi$ iff $w \models_g \varphi$.

It may be good to observe that the standard notions concerning binding are not affected by the semantics. The introduction of witnesses does not as such introduce effects of dynamic binding. This means in particular that classical facts like the ones below stay in force.[5,6]

Fact 5 (Binding)

1. *If* x *does not occur free in* φ*, then* $\exists x\varphi \equiv \varphi$ *and* $\forall x\varphi \equiv \varphi$
2. *If* y *does not occur free in* φ *and* y *is free for* x *in* φ*, then* $\exists x.\varphi \equiv \exists y.[y/x]\varphi$

2.6 Persistence and the existence of minimal extensions

There are two important facts about the semantics (that still need formal proof). The first one is familiar from general inquisitive semantics.

Claim 3 (Persistence) *If* $s \models_g \varphi$*, then for all* $s' \geq s$: $s' \models_g \varphi$.

In words, if a state supports a sentence, then so do all of its extensions.

[5] The syntactic operation $[t/x]\varphi$ results in the formula φ' where the term t is substituted for every free occurrence of x in φ. And *y is free for x in φ* means that x does not occur as a free variable within the scope of any quantifier $\exists y$ or $\forall y$ in φ. We use the definitions from Gamut.

[6] CHECK: is it really true that $\forall x\varphi \equiv \varphi$? What about $\forall x.\exists y.Py$ and $\exists y.Py$?

Claim 4 (Minimal extensions supporting a formula)
For any state s, any assignment g, any formula φ, and any extension t of s such that $t \models_g \varphi$, there is a minimal extension t_{min} of s such that $t_{min} \models_g \varphi$ and $t \geq t_{min}$.

A simplified version of this claim is the following:

Claim 5 (Minimal supporting states)
If $s \models_g \varphi$, then there exists a state s' such that $s \geq s'$ and $s' \models_g \varphi$, and for no $s'' \neq s'$ such that $s' \geq s''$ it holds that $s'' \models_g \varphi$.

2.7 Propositions

Definition 9 (Propositions and possibilities for a formula in a state)

- *The **proposition** expressed by a formula φ **in a state** s relative to an assignment g, $s[\varphi]_g$, is the set of $\geq$-minimal extensions of s that support φ relative to g.*
- *The elements of $s[\varphi]_g$ are called the **possibilities** for φ **in** s relative to g.*
- *If for every two assignments g and g' we have that $s[\varphi]_g = s[\varphi]_{g'}$, we simply refer to $s[\varphi]_g$ as the proposition expressed by φ in s (not relative to g), and abbreviate it as $s[\varphi]$. In this case we also simply refer to the elements of $s[\varphi]$ as the possibilities for φ in s (not relative to g).*

Definition 10 (Propositions and possibilities simpliciter)

- *The proposition expressed by a formula φ in the initial state s_0 relative to an assignment g, $s_0[\varphi]_g$, is referred to as the **proposition** expressed by φ relative to g, and is abbreviated as $[\varphi]_g$.*
- *The elements of $[\varphi]_g$ are called the **possibilities** for φ relative to g.*
- *If for every two assignments g and g' we have that $[\varphi]_g = [\varphi]_{g'}$, we simply refer to $[\varphi]_g$ as the proposition expressed by φ (not relative to g), and abbreviate it as $[\varphi]$. In this case we also simply refer to the elements of $[\varphi]$ as the possibilities for φ (not relative to g).*

Fact 6 (Proposition expressed by sentences do not depend on g)
For any sentence φ (a formula without free variables), any state s, and any two assignments g and g', we have that $s[\varphi]_g = s[\varphi]_{g'}$.

Definition 11 (The discourse effect of uttering a sentence)

- *In uttering a sentence φ in a state s, a speaker proposes to extend s to a state that supports φ.*
- *A proposal may be both informative and inquisitive:*

– *On the one hand, any world in s that does not persist in any extension of s that supports φ is proposed to be eliminated. Thus, if there are such worlds, the proposal is* ***informative****.*

– *On the other hand, in order to extend s to a state that supports φ it may be necessary for other participants to introduce new witnesses and/or provide new information. If this is the case, the proposal is* ***inquisitive****.*

Remark: the system does not really offer the option to participants in a conversation to "introduce new witnesses" in the sense that they can utter a sentence that has the semantic effect of *adding* a witness to the witness set in a state. What can be done by a participant is to provide information that makes it possible for every participant to introduce a witness to the common ground such that an inquisitive proposal is resolved. This has to be further explained, but see the remarks made concerning the entailment relation. The typical example is that the information provided by Pc makes it possible to extend a state with a witness d, i.e., the denotation of c, such that the thus extended state, with the witness d and the information that d has the property P, supports $\exists x.Px$.

Example 1 (Boundedness formulas) *In this system it is indeed possible to capture the meaning distinctions pointed out in Ciardelli (2009, 2010). Consider the boundedness formula and its variants.*

- $[\exists x.Bx] = \left\{ \begin{array}{l} \langle \{w \in W \mid \textit{all elements of } w(P) \textit{ are } \leq 0\}, \{0\} \rangle \\ \langle \{w \in W \mid \textit{all elements of } w(P) \textit{ are } \leq 1\}, \{1\} \rangle \\ \langle \{w \in W \mid \textit{all elements of } w(P) \textit{ are } \leq 2\}, \{2\} \rangle \\ \langle \{w \in W \mid \textit{all elements of } w(P) \textit{ are } \leq 3\}, \{3\} \rangle \\ \ldots \end{array} \right\}$

- $[\exists x.B_+\, x] = \left\{ \begin{array}{l} \langle \{w \in W \mid \textit{all elements of } w(P) \textit{ are } \leq 1\}, \{1\} \rangle \\ \langle \{w \in W \mid \textit{all elements of } w(P) \textit{ are } \leq 2\}, \{2\} \rangle \\ \langle \{w \in W \mid \textit{all elements of } w(P) \textit{ are } \leq 3\}, \{3\} \rangle \\ \ldots \end{array} \right\}$

- $[\exists x.B_e\, x] = \left\{ \begin{array}{l} \langle \{w \in W \mid \textit{all elements of } w(P) \textit{ are } \leq 0\}, \{0\} \rangle \\ \langle \{w \in W \mid \textit{all elements of } w(P) \textit{ are } \leq 2\}, \{2\} \rangle \\ \langle \{w \in W \mid \textit{all elements of } w(P) \textit{ are } \leq 4\}, \{4\} \rangle \\ \ldots \end{array} \right\}$

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