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Abstract

This document is intended to be a self-contained introduction to the Erdos-Szekeres conjecture. By introducing Ramsey Theory in the beginning, we hope to motivate the cups-caps theorem of Erdos and the structure of the conjecture itself. All tools necessary to understand the proof of the best possible known asymptotic upper bound (by Suk) are introduced in the document. It is conceivable that a complete proof of the Erdos-Szekeres conjecture will be found in the next few years, so this document will hopefully serve as a useful tool for explaining the current status of the conjecture to the uninitiated and interested reader.

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Chapter 1

Introduction

The problem we consider dates back to 1934, in a paper titled "A Combinatorial Problem in Geometry" [1], co-authored by Paul Erdős and George Szekeres. In [1], the authors first consider a simple case of a general question.

Problem 1. *Show that given 5 points in the Euclidean plane $\mathbb{R}^2$, such that no three lie on the same straight line, it is always possible to select a subset of 4 points which form a convex quadrilateral.*

Proof. Suppose the least convex polygon that encloses all the points is a quadrilateral or a pentagon. In this case, the problem becomes trivial. If, on the other hand, the least such convex polygon is a triangle ABC, then the points D, E lie inside the triangle. Two of the points, say A and C lie on the same side of line $\overline{DE}$. In this case, $AEDC$ is a convex quadrilateral, as required. ■

1.1 The Erdős-Szekeres Conjecture

Having considered the problem above for five points, it is only natural to wonder if we can generalize the result for an arbitrary number of points n. The original problem that Erdős and Szekeres were faced with was stated as follows:

Problem 2. *Can we find for $n \in \mathbb{N}$ a number $N(n)$ such that from any set containing at least N points in "general position", it is possible to find n points which form a convex polygon?*

By "general position" in $\mathbb{R}^2$, we refer to points in the plane such that no three are collinear. The problem above can be divided into two separate parts. We will consider both questions in this document.

Problem 3. *Does there exist finite N for every $n \in \mathbb{N}$?*

Problem 4. *What are the best possible upper and lower bounds on $N(n)$?*

It turns out that **Problem 3** has an affirmative answer and this result is now referred to as the Erdős-Szekeres theorem. We will discuss a proof of the "cups-caps" theorem in the next section which gives a recurrence for bounding $ES(n)$ above. This will give better intuition for **Problem 4** as well. In fact, we attempt to look at the Erdős-Szekeres conjecture, which is a definitive statement regarding **Problem 4**. We will denote $N(n)$ as $ES(n)$ from here on out:

Conjecture 1. *The minimum number of points in $\mathbb{R}^2$ required to ensure there exists a subset of n points which form a convex polygon is $2^{n-2}+1$, i.e. $ES(n) = 2^{n-2}+1$.*

We will now introduce the necessary tools to tackle this statement. The problems above are some of the most famous in a broad area of Mathematics known as Ramsey-Theory, which deals with the question of *"how many instances are required before a certain property on these instances begins to hold true?"*. We will deal with this topic on a stronger footing below.

1.2 An Introduction to Ramsey Theory

It is not entirely clear when Ramsey Theory emerged as a separate subdiscipline of combinatorial analysis. Frank P. Ramsey, a close friend of the philosopher Ludwig Wittgenstein, was a British philosopher and logician who passed away at the early age of 26 in 1930. In his paper *On a problem of formal logic*, Ramsey proved his now-famous theorem as a lemma towards proving the ***Entscheidungsproblem***, originally posed by David Hilbert in 1928. Church's theorem, proved by Alonzo Church in 1936, showed that the ***Entscheidungsproblem***, the decision problem in first-order logic, was actually unsolvable.
Despite not having attained his final goal, Ramsey left behind a theorem that led to the emergence of an entire field of mathematics. Ramsey's theorem was one of the first indications of a general notion that in sufficiently large systems, there must exist some order, regardless of how disordered the system is. We will now develop a basic understanding of Ramsey Theory by proving Ramsey's theorem.
The exposition in this subsection makes heavy use of the development and notation used in the book "Ramsey Theory" by Ronald L.Graham and co. [1].

Definition 1. *An r-coloring of a set S is a map*

$$\chi : S \to [r] \tag{1.1}$$

where $s \in S$ and $\chi(s)$ is the color of s.

We say a set T is monochromatic if χ is constant on T. We also need arrow notation, commonly used in Ramsey Theory:

Definition 2. *We have:*

$$n \to (l) \tag{1.2}$$

if, given a 2-coloring of $[n]^2$, $\exists T \subseteq [n]$, $|T| = l$, *such that* $[T]^2$ *is monochromatic. In this case,* T *is called a monochromatic* K_l.

We generalize this notation further:

Definition 3. *We have:*

$$n \to (l_1, .., l_r) \tag{1.3}$$

if for every r-coloring of $[n]^2, \exists i, 1 \le i \le r$, *and a set* $T \subseteq [n]$, $|T| = l_i$ *such that* $[T]^2$ *is monochromatic in color* i.

We can now define the Ramsey function and proceed to state Ramsey's theorem with the help of the notation we have developed thus far.

Definition 4. *The Ramsey Function* $R(l_1, ..., l_r)$ *is the minimal* $n \in \mathbb{N}$ *such that*

$$n \to (l_1, .., l_r) \tag{1.4}$$

We now state Ramsey's Theorem in its simplified form. Understanding the proof of the simple version of Ramsey's Theorem will allow us to grasp the general version of the theorem better. Furthermore, we use induction to prove the general version of Ramsey's Theorem and proving the statement for the simple base case below is further indication that the general version is also true.

Theorem 1. *Ramsey's Theorem (Simplified)*
The function R *is well defined. In other words,* $\forall\ l_1, ..., l_r$, $\exists\ n$ *such that*

$$n \to (l_1, .., l_r) \tag{1.5}$$

Proof. ■

We must now generalize the theorem for colorations of $[n]^k$, where $k \in \mathbb{N}$. So far, we have only considered $k = 2$ in the simplified version of Ramsey's theorem.

Definition 5. *We write*

$$n \to (l_1, .., l_r)^k \tag{1.6}$$

if for every r-coloring of $[n]^k$, $\exists\ i$ *such that* $1 \le i \le r$ *and a set* T, $|T| = l$ *such that* $[T]^k$ *is colored* i.

When $l_1 = l_2 ... = l_r = l$ we use the shorthand

$$n \to (l)_r^k \tag{1.7}$$

In this case, every r-coloring of $[n]^k$ gives a monochromatic $[l]^k$. When we do not specify the number of colors r, we assume $r = 2$.
We are now in a position to state the complete Ramsey's Theorem:

Theorem 2. ***Ramsey's Theorem***
The function R is well-defined. In other words, $\forall\ k, l_1, ..., l_r\ \exists\ n_0$ *such that* $\forall\ n \geq n_0$,

$$n \to (l_1, ..., l_r)^k \tag{1.8}$$

Proof. The idea is to induct on k. When $k = 1$, we have:

$$n = 1 + \sum_{i=1}^{r} (l_i - 1) \to (l_1, ..., l_r)^1 \tag{1.9}$$

Now, if we color n points using r colors where $n \geq 1 + \sum_{i=1}^{r} (l_i - 1)$, then we can find some color i used at least l_i times by a simple pigeonhole argument.
Now assume that the result holds for $k - 1$. We then need to find n such that

$$n \to (l)_r^k \tag{1.10}$$

The remaining induction is tedious (although important) and the interested reader is encouraged to refer to [2] for the complete proof. However, we will be focusing on making our way towards the best known upper bounds on $ES(n)$ for the rest of this document, so a general version of Ramsey's theorem is not necessary at this stage. ■

1.2.1 Properties of Ramsey Numbers

It is important to obtain intuition for the Ramsey Numbers by computing actual values. This is not an easy problem by any means. Paul Erdős once said,

> "Suppose an evil alien would tell mankind, 'Either you tell me [the value of R(5,5)] or I will exterminate the human race'... It would be best in this case to try to compute it, both by mathematics and with a computer. If he would ask [for the value of R(6,6)], the best thing would be to destroy him before he destroys us, because we couldn't."

Our first result is fairly easy to prove but gives us some insight into the properties of Ramsey numbers.

Theorem 3. *R(2,k)* $= k\ \forall\ k \geq 2$

Proof. Consider a complete graph with k vertices. Let the two colors we consider be red and blue. If not all of the edges are blue, there must exist two vertices connected by a red edge. Hence, we have $R(2,k) \le k$. Consider any complete graph with $k-1$ vertices. If all the edges are blue, clearly we do not have a complete subgraph K_2 of red edges or a complete subgraph K_k with only blue edges. Hence, $R(2,k) = k$, as required. ■

Several such properties of Ramsey Numbers can be proven using simple combinatorial arguments. We list a few below.

- $R(m,n) = R(n,m)$
- $R(m,n) \le R(m-1,n) + R(m,n-1)$
- $R(m,n) \le \binom{m+n-2}{m-1}$ for $m,n \ge 2$

Let us prove the last property above, which gives us a fairly good upper bound on the Ramsey Numbers.

Proof. Ramsey's theorem shows that $R(m,n)$ exists. Now let us use double induction along with the second property above. The proof of the second property is left as an exercise to the reader. We induct on the value of $m+n$. For $n=2$, we have $R(m,2) = m = \binom{m+2-2}{m-1}$, as required.
Now suppose the property holds $\forall s,t \ge 2$ and $s+t=n$. Now let $k,l \ge 3$ with $k+l = n+1$. We then have:

$$R(k,l) \le R(k-1,l) + R(k,l-1) \tag{1.11}$$

By the induction hypothesis, we then have:

$$R(k,l) \le \binom{k+l-3}{k-2} + \binom{k+l-3}{l-2} \tag{1.12}$$

Note that:

$$\binom{k+l-3}{l-2} = \binom{k+l-3}{k-1} \tag{1.13}$$

Then using Pascal's identity, we finally have:

$$R(k,l) \le \binom{k+l-2}{k-1} \tag{1.14}$$

With this, our induction is complete and we have the required result. ■

Erdős is also famous for his "probabilistic-method", which he used to derive an upper bound on the Ramsey Numbers.

Theorem 4. *If* $\binom{n}{k}\, 2^{1-\binom{k}{2}} < 1$, *then* $R(k,k) > n$

Proof. Look at the complete graph K_n. Randomly color each edge red or blue, each with probability $\frac{1}{2}$. For a subset S of k vertices, let E_S be the event that the induced subgraph on S is monochromatic. Then,

$$Prob[E_S] = 2^{1-\binom{k}{2}}. \tag{1.15}$$

We have $\binom{n}{k}$ possible induced subgraphs S, so the probability of finding a monochromatic induced subgraph is at most $\binom{n}{k}2^{1-\binom{k}{2}}$. Therefore if $\binom{n}{k}2^{1-\binom{k}{2}} < 1$, there exists at least one two-coloring on K_n in which there does not exist a monochromatic K_k.
In particular, when $n = \lfloor 2^{k/2} \rfloor$ and $k \geq 3$ notice that the above condition is satisfied. Therefore for $k \geq 3$,

$$R(k,k) > \lfloor 2^{k/2} \rfloor \tag{1.16}$$

∎

1.3 Proof of The Erdős-Szekeres Theorem

It may interest the reader to look at [1] for a proof of the existence of an upper bound and the calculation of a (very loose) explicit upper bound using Ramsey's theorem and forming recurrences. However, we postpone our discussion of upper bounds till we introduce all the tools necessary to gain intuition for Suk's proof.

Chapter 2

Background and Results

Before diving into the proof of the best known bound today on $ES(n)$, we need to introduce some tools and theorems that will give us intuition and background for the methods used.

2.1 Posets and Dilworth's Theorem

Definition 6. *A Poset P is a set with a partial order defined on it. The partial order ($\leq$) is a binary relation that has the following three properties $\forall a, b, c \in P$:*

- *$a \leq a$ (reflexivity)*
- *$a \leq b$ and $b \leq a \implies a = b$ (anti-symmetry)*
- *$a \leq b$ and $b \leq c \implies a \leq c$ (transitivity)*

A poset P such that any two elements $a, b \in P$ are *comparable* (i.e. $a \leq b$ or $b \leq a$) is called a total order. Not all elements have to be comparable.

Example 1. *One standard example of a partial order is the division operator "|" defined on the set of natural numbers. Note that division | satisfies all three properties above and hence induces a poset structure on the natural numbers.*

Definition 7. *A Chain is a sequence of elements $\{x_i\} \subseteq P$ such that $x_1 < x_2 < \dots < x_n$. Here $x_i < x_j$ means that $x_i \leq x_j$ and $x_i \neq x_j$.*
We define an AntiChain as a set of elements $\{x_i\} \subseteq P$ such that no two distinct elements in the set are comparable.

The one main theorem in the theory of posets is Dilworth's theorem. Its so called "dual" theorem is also referred to as Mirsky's theorem. We look at the standard proofs of both below.

Theorem 5. *Dilworth's Theorem - Let P be a finite poset. Then, if $d_1 = min\{m : \exists$ chains $C_1,...,C_m\}$ with $P = \cup_{i=1}^{m} C_i$ and $d_2 = max\{|A| : A$ is an antichain$\}$, we have $d_1 = d_2$.*

Proof. Given Chains $\{A_i\}$ with distinct elements, note that any Anti-Chain can contain at most one element from each Chain. Hence, $d_1 \geq d_2$.
We now induct on $|P|$ to show that $d_1 \leq d_2$.
Define a *minimal* element $p_0 \in P$ to be such that $\forall p \in P$, $p_0 \leq p$ or p_0, p are incomparable. Define a maximal element $p_1 \in P$ analogously. We impose the condition that $p_1 \geq p_0$ as well. Some two minimal/maximal elements must satisfy this because otherwise, all elements must be incomparable. If all elements are incomparable, then we have $|P|$ chains and can find one antichain P with size $|P|$, as required.
Now, let $C' = \{p_0, p_1\}$ be a chain in P. If every antichain in $P - C'$ has length at most $d'_2 \leq d_2 - 1$, then by the induction assumption, the minimum number of chains for any chain decomposition in the new poset is $d'_1 \leq d_2 - 1$ such that $d'_1 = d'_2$. Adding the chain C to the decomposition of P' increases both d'_1 and d'_2 by 1. Hence, we are done.
So, we only need to look at the case of some antichain A in $P - C'$ having size d_2. We consider the sets:

$$P_1 = \{x \in P : \exists a \in A, x \geq a\}, P_2 = \{x \in P : \exists a \in A, x \leq a\}$$

- $P_1 \cup P_2$ is simply P. This is obvious because if $x' \notin P_1 \cup P_2$, then x' is incomparable with every element of A. Hence, $A \cup x'$ would result in a larger antichain.

- $P_1 \cap P_2 = A$. This also follows easily because every element $a \in A$ is such that $a \leq a$ and $a \geq a$, so $A \subseteq P_1 \cap P_2$. Since every element of $P_1 \cap P_2$ is both greater and lesser than some element in A,if we construct chains (using the transitive property), they imply that every chain collapses to a single element. Therefore, $P_1 \cap P_2 \subseteq A$ as well.

$p_0 \notin A$ and p_0 is minimal, so $p_0 \notin P_1$. Similarly, $p_1 \notin P_2$. So, $|P_1|, |P_2| < |P|$. Both P_1, P_2 can be partitioned into d_2 disjoint chains. These sets of disjoint chains can be combined (note that transitivity is necessary for this combining process) to give d_2 disjoint chains for $P_1 \cup P_2 = P$. Therefore, $d_1 \leq d_2$. This proves Dilworth's theorem! ■

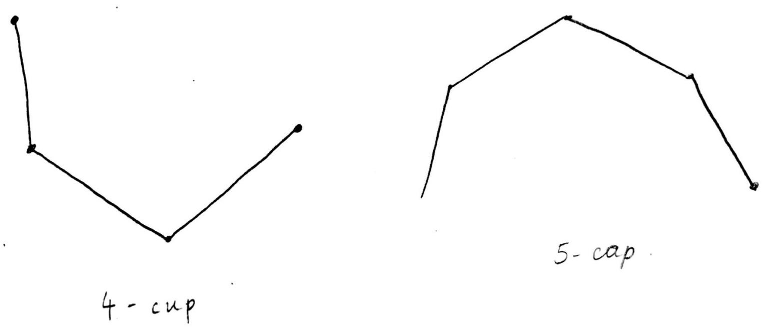
Theorem 6. *Dual Dilworth - Let P be a finite poset. Then, if $m_1 = min\{m : \exists$ AntiChains $A_1,...,A_m\}$ with $P = \cup_{i=1}^{m} A_i$ and $m_2 = max\{|C| : C$ is a Chain$\}$, we have $m_1 = m_2$.*

Proof. Given Antichains $\{A_i\}$ with distinct elements, note that any chain can contain at most one element from each Antichain. Hence, $m_1 \geq m_2$.
The inequality in the other direction follows directly from a simple induction on $|P|$, as in the proof of Dilworth's theorem. ■

2.2 Supports, Cups and Caps

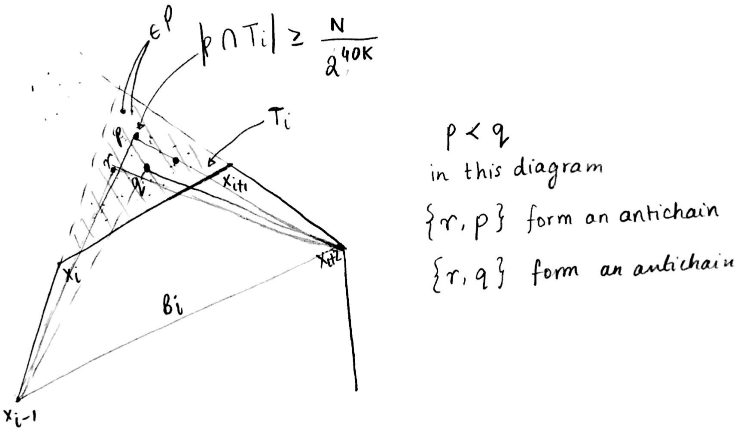
We begin by defining a few terms in similar fashion to [6].

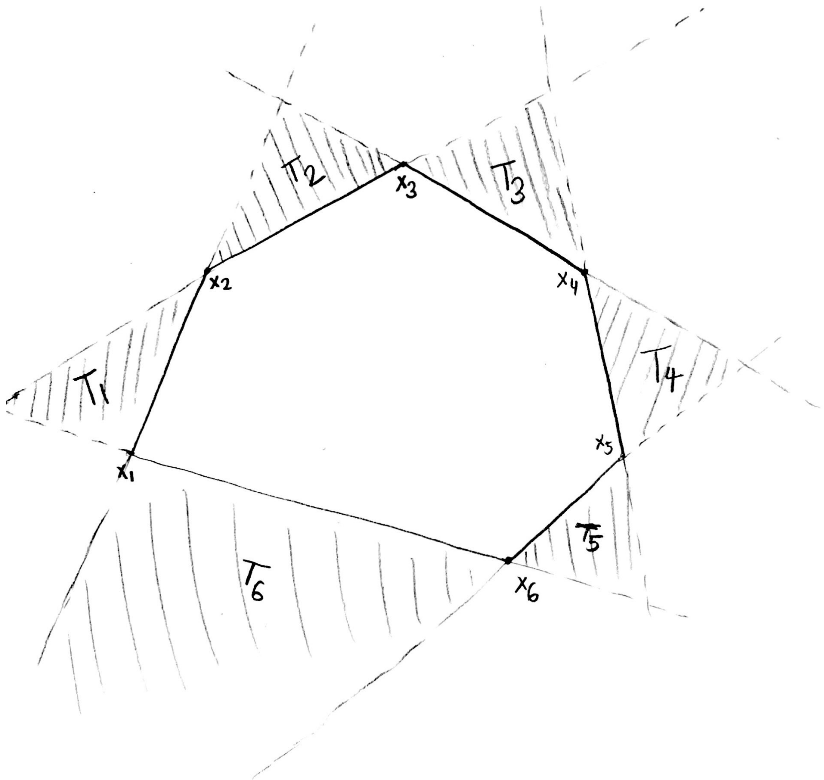
Definition 8. *A k-element point set X in the plane (in general position) is said to form a k-cup (k-cap) if X is in convex position and its convex hull is bounded above (below) by a single edge. In other words, $\forall p \in X$, there exists a line L passing through p such that all other points of X lie above or below the line L.*



Note that if we remove an edge of a convex n-gon, we get a cup or a cap. The concept of a support is natural, when we consider the possible points we can add to a convex set (in general position) and retain convexity.

Definition 9. *Given a k-cap (k-cup) $X = \{x_1, \ldots, x_k\}$ where the points appear in order from left to right, the support of X is the collection of open regions $C = \{T_1, \ldots, T_k\}$, with T_i being the region outside conv(X) bounded by the segment $\overline{x_i x_{i+1}}$ and by the lines $x_{i-1}x_i$, x_{i+1}, x_{i+2}.*





Theorem 7. *If every four points of a set $P \in \mathbb{R}^2$ (in general position such that no three points are collinear) are in a convex configuration, then the entire set is in convex configuration.*

Proof. First, for $|P| = 4$ it is trivial. Suppose the claim holds inductively for $|P| = 4,5,...,k$. We show it holds for $|P| = k+1$.
Consider any set $P' = P - \{p\}$ of size k, where p is some arbitrary point. Then, P' forms a convex k-set by the inductive hypothesis. It is obvious to see that if we add a point s belonging to the support of P' (shaded region in the diagram), then the convex property is retained.
Also notice that if we add a point s to the regions "in between" the supports, say in the open region enclosed by rays $\overline{x_i x_{i+1}}$ and $\overline{x_{i+1} x_{i+2}}$, then the points s, x_i, x_{i+1}, x_{i+2} violate the convexity condition for 4-points, since x_{i+1} is inside the triangle formed by s, x_i and x_{i+2}. Similarly, if we add point s to the open region inside set $conv(P')$ and excluding the boundary, then the convexity condition for four points is violated.
Therefore, the point p must belong to the support of P' and hence $P' \cup \{p\} = P$ forms a convex set. ■

Note that by combining cups and caps with shared end vertices, we form a convex polygon. This makes these objects worth exploring, and so we are naturally led to the Erdos cups-caps theorem. We refer to the exposition in [4] for a proof.

Theorem 8. *Erdos cups-caps theorem - Let $f(k,l)$ be the smallest positive integer for which a set of points X contains a k-cup or an l-cap when $|X| \geq f(k,l)$. Then,*

$$f(k,l) = \binom{k+l-4}{k-2} + 1$$

Proof. We begin by showing an inequality. Note that $f(k,3) = f(3,k) = k$. This is obvious since if we have three points in succession such that the slopes of the line segments between them are decreasing, then we have a 3-cap. Otherwise, the slopes must keep increasing and after k points, we have a k-cup. So, $f(k,3) = k$. Similarly, $f(3,k) = k$.
Let us consider a set of points X, with $|X| = f(k-1,l) + f(k,l-1) - 1$. Now, let us just look at the set of leftmost endpoints of $k-1$ cups of X, denoted by Y. If $|X - Y| \geq f(k-1,l)$, then it has an l cap. Otherwise, $|Y| \geq f(k,l-1)$. Suppose Y has an $l-1$ cap $\{(x_{i_1}, y_{i_1}), (x_{i_2}, y_{i_2}), ..., (x_{i_{l-1}}, y_{i_{l-1}})\}$. Then, there exists some $k-1$ cup $\{(x_{j_1}, y_{j_1}), (x_{j_2}, y_{j_2}), ..., (x_{j_{k-1}}, y_{j_{k-1}})\}$ such that $i_{l-1} = j_1$. Then, it is immediately obvious that we can add either $(x_{i_{l-2}}, y_{i_{l-2}})$ or (x_{j_2}, y_{j_2}) to create a k cup or an l cap. This proves the recurrence. ■

Expanding the recurrence out gives us the inequality:

$$f(k,l) \leq \binom{k+l-4}{k-2} + 1$$

We now want to obtain equality. For the case of $k = 3$ or $l = 3$, this reduces to the base cases, which we know are true.

- *Let there exist set A with $\binom{k+l-5}{k-3}$ points. A has no $k-1$-cup and no l-cap.*
- *Similarly, we have set B with $\binom{k+l-5}{k-2}$. It has no k-cup and no $l-1$-cap.*
- *Note that if either of these sets did not exist, we would have a lower bound on $f(k,l)$ corresponding to the upper bound and we would be done.*
- *Fix a coordinate system on the plane.*
- *Translate sets A and B such that all points of B has a greater $x-$coordinate than any point in A. Also, by moving B far enough up in the y direction, we ensure that the slope of a line connecting any two points $a \in A, b \in B$ is greater than the slope of any line connecting two points of A or two points of B.*
- *Let us denote $A \cup B$ by X.*

Now, note that any cup in X containing points of both A and B can contain at most one point of B. This is obvious from the slope condition. This implies that X has no k-cup. In the exact same way,

we can see that X has no l-cap.
Therefore, we have the result:

$$f(k,l) \geq \binom{k+l-5}{k-3} + \binom{k+l-5}{k-2} + 1 = \binom{k+l-4}{k-2} + 1$$

by Pascal's identity (trivial). Combining the upper and lower bounds, we finally obtain the result of the cups-caps theorem.

Now, $ES(n) \leq f(n,n)$, so we get a non-trivial upper bound:

$$ES(n) \leq \binom{2n-4}{n-2} + 1$$

As described in the survey [4], successive results from Chung, Graham, and Toth,Valtr improved the upper bound to:

$$ES(n) \leq \binom{2n-5}{n-3} + 2$$

Note that asymptotically, the upper bound grows faster than $(4-\epsilon)^n$ for any constant $\epsilon > 0$, but slower than 4^n Suk uses the following theorem at a critical juncture in his proof to ensure the existence of both a convex k-gon and supports with a non-negligible number of points in them.

Theorem 9. *Partitioned-Version of Erdos-Szekeres([5]) - It implies that if we have a finite point set P in the plane in general position, with $|P| \geq 2^{32k}$, then $\exists$ a k-element subset $X \subset P$, with X being a k-cup or k-cap, and $T_1,...,T_{k-1}$, in the support of X satisfying $|T_i \cap P| \geq \frac{|P|}{2^{32k}}$.*

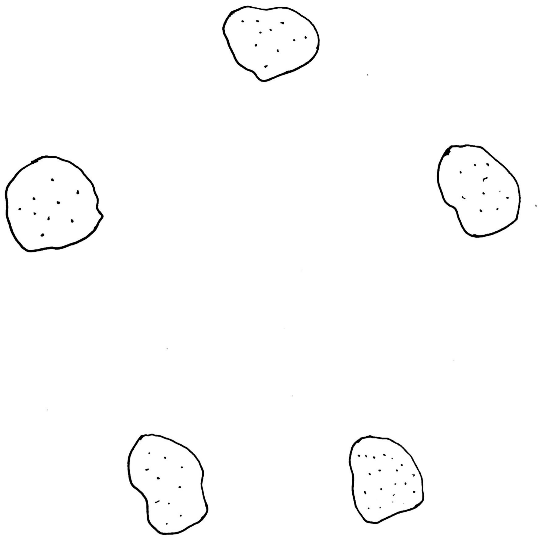
Proved by Por and Valtr, the above theorem is actually stated in a different way in the original paper, in the language of "convex clusterings".

Definition 10. *A finite planar point set X is called a convex k-clustering if it is a disjoint union of sets $\{X_i\}$, $X = \cup_{i=1}^{k} X_i$, with each $|X_i|$ the same and $x_1,...,x_k$ a convex k-gon for each $x_i \in X_i$.*

The theorem they prove states:

Theorem 10. $\forall k \geq 3$, $\exists \epsilon_k > 0$ *such that if X is a set of points in general position, then $|X| \geq f(k)$ implies the existence of a convex k-clustering of size $\geq \epsilon_k |X|$.*

And finally, the authors show that the theorem holds for $\epsilon_k = \frac{k}{2^{32k}}$.



Convex 5-clustering

CHAPTER 3

RECENT DEVELOPMENTS

In 2006, Szekeres and Peters [7] found a computer proof of the Erdos-Szekeres conjecture for $n = 6$, showing that exactly 17 points in general position were needed to guarantee a convex polygon in $\mathbb{R}^2$. Yet, even with their methods, they could not extend their proof for $n = 7$. This should give an indication for how difficult bounding $ES(n)$ would be using computational methods, unless we find new methods to enumerate configurations of points in the plane.
In early 2016, Andrew Suk [6] made a major breakthrough, proving an asymptotic This year, Andrew Suk published a paper proving the following:

Theorem 11. *$\forall\ n \geq n_0$, where n_0 is a large absolute constant, $ES(n) \leq 2^{n+6n^{2/3}log(n)}$*

Note that Suk's result does not give exact results for a general n. Rather, it is an asymptotic result which illuminates the rate of growth of $ES(n)$ as $n \to \infty$. The best previous bounds were of the form:

$$ES(n) \leq 2^{n+\Theta(n)}$$

The new bound implies

$$ES(n) \leq 2^{n+o(n)}$$

As this has been the first major result on the conjecture in several decades, we give a detailed explanation of the proof below.

3.1 Suk's Proof

Proof. Let us begin by defining the objects and main terms under consideration.

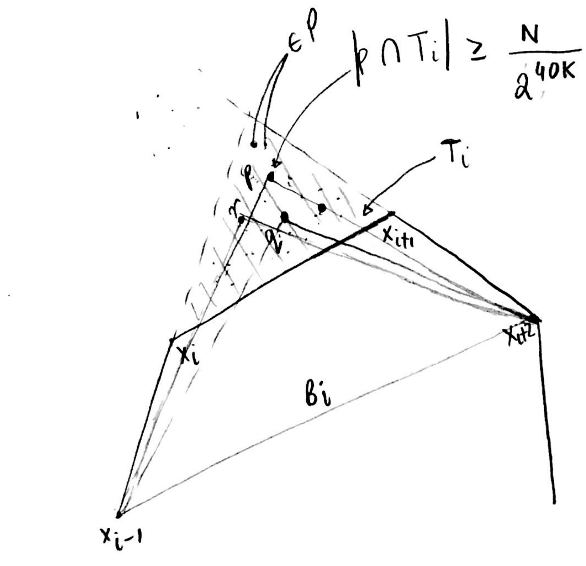
- P a point set (general position), $|P| = N$. N is chosen to be $\lfloor 2^{n+6n^{2/3}logn} \rfloor$.

- $n \geq n_0$, some large constant.
- $k = \lfloor n^{2/3} \rfloor$.
- $\alpha = 3n^{-1/3} log(n)$.
- T_i refers to the region over segment $\overline{x_i x_{i+1}}$ in the support of a point set X.
- $\prec$ refers to a partial order.

Recall the partitioned version of Erdos-Szekeres from [5]. Applying it to set P, with $k+3$ (instead of k), we get a subset of P denoted by $X = \{x_1, ..., x_{k+3}\}$. By [5], X can be chosen to be a cup or cap. We let it be a cap W.L.O.G.
We also take the points to be appearing from left to right. From [5]:

$$|T_i \cap P| \geq \frac{N}{2^{32(k+3)}} \geq \frac{N}{2^{40k}}$$

- Let $P_i = T_i \cap P$, $i \in [k+2]$.
- T_i and T_j are considered adjacent in the usual sense (i.e. if i and j are consecutive indices).
- Let B_i denote segment $\overline{x_{i-1} x_{i+2}}$, $i \in \{2, ..., k+1\}$.
- Define a partial order $\prec$ on P_i, $i \in \{2, ..., k+1\}$ as follows:
  If $p, q \in P$ then $p \prec q$ if $p \neq q$ and $q \in \text{conv}(B_i \cup p)$.

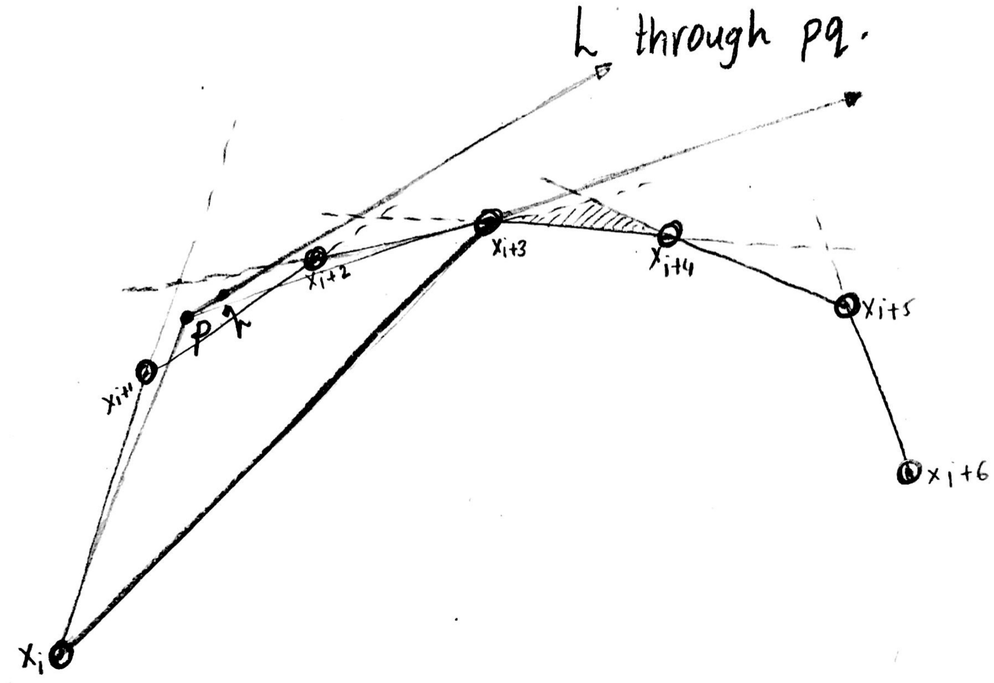


$p \prec q$
in this diagram
$\{r, p\}$ form an antichain
$\{r, q\}$ form an antichain

Now, we apply Dilworth's theorem on $(P_i, <)$. So, $\exists$ either a chain of size $\geq |P_i|^{1-\alpha}$ or an antichain of size $\geq |P_i|^{\alpha}$. Suk divides the proof into two cases as follows.

Case 1. *Let $t = \lceil \frac{n^{1/3}}{2} \rceil$. Assume there are t non-adjacent parts $P_i \in \{P_2, ..., P_{k+1}\}$ such that each part has a corresponding antichain $Q_i \subseteq P_i$, $|Q_i| \geq |P_i|^{\alpha}$.*

- *Let the selected antichains be $Q_{j_1}, ..., Q_{j_t}$.*
- *Note that the line through any two points in Q_{jr} with $r \in [t]$ cannot intersect T_{jw} for $w \neq r$ (Refer to the Figure).*



- *We let $40k < n^{2/3} log(n)$, since n is arbitrarily large and we are looking at the asymptotic behavior.*

Then,

$$|Q_{jr}| \geq |P_i|^{\alpha} \geq (\frac{N}{2^{40k}})^{\alpha} \geq 2^{3n^{2/3}log(n)+15n^{1/3}log^2(n)} \geq \binom{n+\lceil 2n^{2/3}\rceil-4}{n-2}+1 = f(n,\lceil 2n^{2/3}\rceil)$$

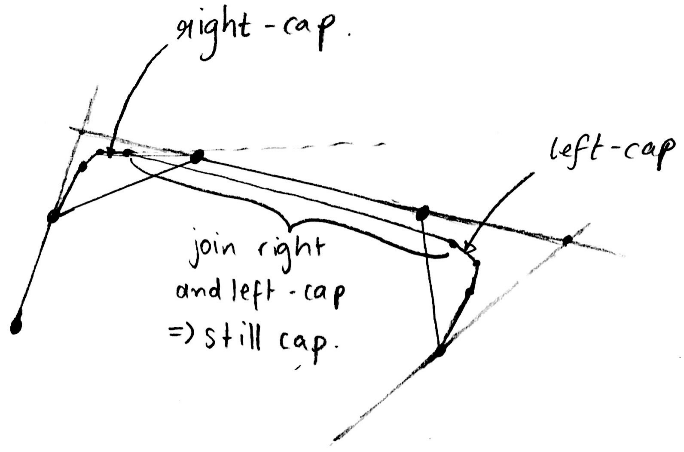
Hence, the cups-caps theorem guarantees that there exists an n cup in Q_{jr} or a $\lceil 2n^{2/3}\rceil$ cap. If we have an n-cup, we also have a corresponding convex n-gon directly. Hence we are done.
Otherwise, suppose each Q_{j_r} contains a $\lceil 2n^{2/3}\rceil$ cap denoted by C_{j_r}.
Consider:

$$C^* = \cup_{r=1}^{t} C_{j_r}$$

The insight here is that each C_{j_r} is a cap, implying that there exists some point $q \in C_{j_r}$ for every $p \in C_{j_r}$ such that all other points of C_{j_r} lie below the line L through segment $\overline{pq}$. L also does not intersect B_{j_r}, so all points in $C^* - \{p,q\}$ lie below L.
The above argument is a little difficult to visualize, but it is important to note that we are considering antichains C_{j_r} which imposes restrictions on which lines L can be drawn.
We conclude that C^* is a cap, so:

$$|C^*| \geq \frac{n^{1/3}}{2}(2n^{2/3}) = n$$

as required.



This concludes Case 1. We must now see what happens if there do not exist non-consecutive parts as described Case 1. This implies that there must exist some consecutive block of P_i's each containing a chain of length $|P_i|^{1-\alpha}$. If it did not, then we would be guaranteed some P_i with an antichain of length $\geq |P_i|^{\alpha}$ in every consecutive $\lceil n^{1/3}\rceil$ indices. This simply corresponds to Case 1.

Case 2. *There are $\lceil n^{1/3}\rceil$ consecutive indices, with each index corresponding to a part P_i and each part containing a chain of size $\geq |P_i|^{1-\alpha}$.*

- *Let* $t = \lceil n^{1/3} \rceil$.
- *Let* $Q_1, ...Q_t$ *denote the chains corresponding to some* $P_{j+1}, P_{j+2}, ..., P_{j+t}$ *or simply* $P_1, ..., P_t$.
- *We refer to the corresponding supports as* $T_1, ..., T_t$.
- *We order each* $Q_i = \{p_1, ..., p_{|Q_i|}\}$ *according to* $\prec$ *such that* $p_a \prec p_b$ *if* $a < b$.

Definition 11. *A subset* Y *of* Q_i *is a right-cap if* $x_i \cup Y$ *is in convex position. A left-cap corresponds to* $x_{i+1} \cup Y$ *in convex position.*

We now look closely at how right-caps and left-caps can be used to produce cups and caps.

- Every triple in Q_i is a left-cap or right-cap because Q_i is a chain. However, it cannot be both.
- Let $i_1 < i_2 < i_3 < i_4$. If $(p_{i_1}, p_{i_2}, p_{i_3})$ and $(p_{i_2}, p_{i_3}, p_{i_4})$ are right-caps, say, then $(p_{i_1}, p_{i_2}, p_{i_4})$ and $(p_{i_1}, p_{i_3}, p_{i_4})$ are both right-caps. The same holds true for left-caps.
- The above observations show that if $|Q_i| \geq f(k,l)$, then Q_i contains either a $k-$left-cap or an $l-$right-cap.
- Note that the above argument very closely resembles the Erdos-Szekeres cups-caps theorem and it is essentially the same, if we use a combinatorial reformulation of the cups-caps theorem in terms of transitive-2-colorings and cliques of the same color ([3]).

Lemma 1. *If we look at adjacent chains* Q_{i-1} *and* Q_i *as defined above, and if* Q_{i-1} *contains a* k*-left-cap* Y_{i-1} *and* Q_i *contains an* l*-right-cap* Y_i*, then* $Y_{i-1} \cup Y_i$ *is a set of* $k + l$ *points in convex position.*

Proof. First off, the lemma makes complete intuitive sense. A few diagrams clearly show why joining a left-cap and a right-cap give you a complete (cup or cap) convex figure.

- If we show every four points of Y are in convex configuration, we're done.
- If all four points for some randomly chosen set of four points lie exclusively in Y_{i-1} or Y_i, then they are in convex position and we are done.
- Otherwise, We either have two points in each Y_{i-1}, Y_i, or one point in Y_{i-1} and three points in Y_i, or three points in Y_{i-1} and one point in Y_i.
- In the 2-2 point distribution, the conclusion is immediate since the supports T_{i-1} and T_i do not intersect and the lines joining points in the same support do not intersect the adjacent support. Hence, all four points are in convex configuration.

- In the 3-1 configuration, simply notice that there is no way to choose a point p_4 in T_i such that p_1, p_2, p_3, p_4 form a triangle with one point in the interior. You can see this rigorously by drawing lines through the three points in T_{i-1} and then noticing the position of p_4 with respect to these lines.
- The last case (1-3) is the same as above. In fact, we we change the orientation of the plane, it becomes the same question.

Since every set of four points is in convex configuration, the entire set Y is in convex configuration. □

Therefore, we can add left-caps and right-caps together to get a convex set. Proceeding with the calculations of the proof, we have:
For $i \in [\lceil n^{1/3} \rceil]$,

$$|Q_i| \geq |P_i|^{1-\alpha} \geq (\frac{N}{2^{40k}})^{1-\alpha} \geq f(k,n)$$

as before for Case 1. So, each Q_i either contains an n-right-cap or a k-left-cap. We'd like to use the lemma above, so we start with set Q_1.
If Q_1 has an n-right-cap, we are done. Otherwise, we know it has a k-left-cap. We can rewrite $f(n,k)$ as $f(2k, n-k)$. So now, if Q_2 has an $n-k$- right cap, we are done by the previous lemma. Hence, we assume Q_2 contains a $2k$-left-cap. Continuing this process, Q_i has an ik-left-cap.
We then know that $Q_{\lceil n^{1/3} \rceil}$ contains a $(\lceil n^{1/3} \rceil)k$-left-cap. Since we chose $k = \lceil n^{2/3} \rceil$ before, we can conclude that $Q_{\lceil n^{1/3} \rceil}$ has an n-left-cap, as required.
Hence we are done! ■

CHAPTER 4

Beyond Erdos-Szekeres in $\mathbb{R}^2$ and Future Work

We can generalize the Erdos-Szekeres Problem to higher dimensions. The idea is to reinterpret the notion of convex position, and replace it with the notion of a convex hull.

Definition 12. *A set of points in Euclidean space $\mathbb{R}^d$ is in general position if no $d+1$ points of X lie on a hyperplane.*

Problem 5. *Let $ES_d(n)$ be the smallest number of points in Euclidean space $\mathbb{R}^d$ such that $\exists n$ points in convex position (no point belongs to the convex hull of the remaining $n-1$ points). Does $ES_d(n)$ always exist $\forall d, n \geq 1$? Are there upper and lower bounds?*

It turns out that a simple application of Caratheodory's theorem and the same general strategy as Erdos-Szekeres' initial proof for the existence of $ES(n)$ proves the existence of $ES_d(n)$. As far as I know, the only known lower bound is:

$$ES_d(n) = \Omega(c^{(d-1)\sqrt{n}})$$

where c_d is a constant associated with each d. One known upper bound is:

$$ES_d(n) \leq \binom{2n-2d-1}{n-d} + d$$

As we can see, the situation is not any better or significantly worse. It seems that the proof of the lower bound for $ES(n)$ does not easily generalize for $ES_d(n)$. However, I am particularly interested in applying Suk's methods and clever application of Dilworth's theorem to higher dimensions and seeing if better results can be obtained.

For more known results, open problems and current lines of research, the interested reader is encouraged to refer to [4].

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