Digital Signal Processing

Ch3 Z-transform

2024 Fall

Z-transform : $\boxed{X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}}$

等比級數 : $\frac{a_1(1-r^n)}{1-r}$

- Right-side sequence : $x[n]=0$ for all $n<N$ 不一定是 0
- Left-side sequence : $x[n]=0$ for all $n>N$

EX 1.

$$x[n] = a^n u[n] = \begin{cases} a^n ; & n \geq 0 \\ 0 ; & n<0 \end{cases}$$ Right-Side seq.

- $X(z) = \frac{z}{z-a} = \frac{1}{1-az^{-1}}$

x(n)為 Right-side
X(z) ROC為外側

- ROC : $|az^{-1}| < 1 \rightarrow |a| < |z|$
- zero : $z=0$
  pole : $z=a$

EX2.

$$x[n] = -a^n u[-n-1] = \begin{cases} -a^n ; & n \leq -1 \\ 0 ; & n > -1 \end{cases}$$ left-side seq

- $X(z) = \frac{z}{z-a} = \frac{1}{1-az^{-1}}$

x(n)為 left-side
X(z) ROC為內側

- ROC : $|az^{-1}| > 1 \rightarrow |a| > |z|$
- zero : $z=0$
  pole : $z=a$

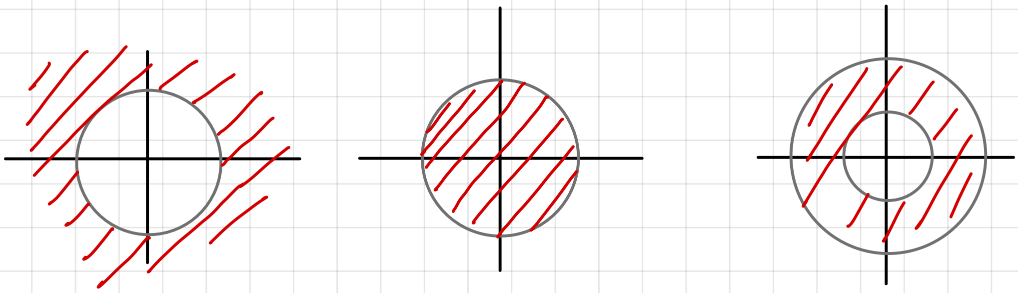
不同 x(n) 可能有相同 X(z)

$\rightarrow$ x(n) and ROC determined X(z)

Properties of ROC

- Convergence depend only on $|z|$
- ROC can NOT contain any pole
- Fourier transform exist $\longleftrightarrow$ ROC include "unit circle"

- ROC of right-side seq: $|z| > r_{max}$ ← outmost finite pole
- ROC of left-side seq: $|z| < r_{min}$ ← innermost finite pole



- Stability LTI system
  → ROC of $H(z)$ Contain "unit circle"
- Causal LTI system
  → ROC of $H(z)$ are "outmost finite pole"

EX3 right-side left-side

- $x[n] = \sum_{k=1}^{M} a_k^n u[n] - \sum_{\ell=1}^{N} b_\ell^n u[-n-1]$

- $X(z) = \sum_{k=1}^{M} \boxed{\frac{1}{1-a_k z^{-1}}} + \boxed{\sum_{\ell=1}^{N} \frac{1}{1-b_\ell z^{-1}}}$

$|a_k z^{-1}| < 1$ $|b_\ell z^{-1}| > 1$

$$ROC = \left(\bigcap_{k=1}^{M} |a_k| < |z| \right) \cap \left(\bigcap_{\ell=1}^{N} |b_k| > |z| \right)$$

zero and pole of rational system

$$H(z) = \frac{B(z)}{A(z)} = \frac{c\prod_{i=1}^{M}(z-z_i)}{\prod_{i=1}^{N}(z-z_i)}$$

- if $N>M$, there are $N-M$ zero at $z=\infty$
- If $N<M$, there are $M-N$ pole at $z=\infty$
- Number of zero = Number of pole (if $z=\infty$ counted)

Propertie of z-transform

- Time shift: $\boxed{x[n-n_0] \leftrightarrow z^{-n_0}X(z)}$

$$\sum x[n-n_0]z^{-n} = \sum x[n']z^{-(n'+n_0)} = X(z)z^{-n_0}$$

- z-domain scale: $\boxed{z_0^n x[n] \longleftrightarrow X\left(\frac{z}{z_0}\right)}$

$$\sum z_0^n x(n) z^{-n} = \sum x[n]\left(\frac{z}{z_0}\right)^{n} = X\left(\frac{z}{z_0}\right)$$

- Differential of $X(z)$: $\boxed{nx(n) \longleftrightarrow -z\frac{d}{dz}X(z)}$

$$-z\frac{d}{dz}X(z) = -z\frac{d}{dz}\sum x(n)z^{-n} = -n\sum x(n)z^{-n-1}(-z)$$

- Conjugation of $x(n)$: $\boxed{x^*(n) \longleftrightarrow X^*(z^*)}$

$$\sum x^*(n)z^{-n} = \left(\sum x(n)(z^*)^{-n}\right)^* = X^*(z^*)$$