Stochastic Process Ch10

2024 Fall

Chapter 10.

10-1 Random walk and Brownian Motion / Wiener Process

$\{X_n\}_{n=1}^{\infty}$ is a sequence of IID random variables such that

$P(X_n=1)=P(X_n=-1)=\frac{1}{2}$, $\forall n \in N \rightarrow E[X_n]=0,\ E[X_n^2]= 1^2\cdot\frac{1}{2}+(-1)^2\cdot\frac{1}{2}=1$

$\Delta X>0$, $\Delta t>0$ $\quad X(0)\triangleq 0$

$$X(t)\triangleq \Delta X\sum_{k=1}^{\lfloor \frac{t}{\Delta t}\rfloor} X_k\ ,\ \forall t\in(0,\infty)$$

(1) $E[X(t)] = \Delta X \sum E[X_k] = 0$

(2) $var[X(t)] = var\left[\Delta X\sum_{k=1}^{\lfloor \frac{t}{\Delta t}\rfloor} X_k\right] = (\Delta X)^2 var\left[\sum_{k=1} X_k\right] = (\Delta X)^2\sum_{k=1}^{\lfloor \frac{t}{\Delta t}\rfloor} var(X_k)$

$= (\Delta X)\lfloor \frac{t}{\Delta t}\rfloor$

X(t) is NOT SSS, since var(X(t)) is not constant

$$X(t)\triangleq \Delta X\sum_{k=1}^{\lfloor \frac{t}{\Delta t}\rfloor} X_k$$

Consider $\Delta X = c\sqrt{\Delta t}$, where $c>0$

Then , $(\Delta X)^2=c^2\Delta t \Rightarrow \frac{(\Delta X)^2}{\Delta t} = c^2$

Thus, $(\Delta X)^2\cdot\frac{t}{\Delta t} = c^2 t$

Definition :

$$B_c(t)\triangleq \lim_{\Delta t\to 0} c\sqrt{\Delta t}\sum_{k=1}^{\lfloor \frac{t}{\Delta t}\rfloor} X_k\ ,\ \forall t\in[0,\infty)$$

$X(t)$ with $\Delta X = c\sqrt{\Delta t}$

Brownian motion / Wiener process with parameter c

1. $E[B_c(t)]=0$
2. $Var[B_c(t)]=c^2 t$
3. $B_c(t)\sim N(0,c^2 t)$ by CLT

Definition (version 2)

Let $c>0$, $\{B_c(t), t \geq 0\}$ is a Wiener process with parameter c if

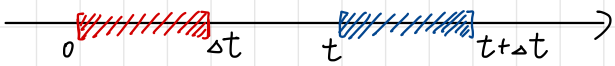
(1) $X(0) = 0$

(2) $X(t) \sim N(0, c^2 t)$

(3) $\{X(t), t \geq 0\}$ has stationary and independent increments

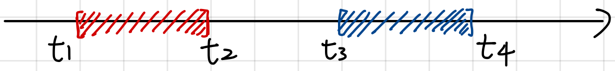
If $B_c(t)$ is seen as the reward you get till time t

(3.1) $B_c(t+\Delta t) - B_c(t)$ and $B_c(\Delta t)$ have the same pdf



(3.2) $B_c(t_4) - B_c(t_3)$ is independent to $B_c(t_2) - B_c(t_1)$

$\forall\ t_4 > t_3 \geq t_2 > t_1 \geq 0$



Theorem 1. $R_{B_c}(t_1, t_2) = c^2 \cdot \min(t_1, t_2), \forall\ t_1, t_2 \geq 0$

proof.

① Consider $0 < t_1 < t_2$

$R_{B_c}(t_1, t_2)$

$= E[B_c(t_1) \cdot B_c(t_2)]$ 因為 $t_2 > t_1$，才可以這樣減

$= E[B_c(t_1) \cdot \boxed{(B_c(t_2) - B_c(t_1))} + B_c^2(t_1)]$

$= E[B_c(t_1)] \cdot E[B_c(t_2) - B_c(t_1)] + \overbrace{E[B_c^2(t_1)]}^{c^2 t_1}$

$= c^2 t_1$

Then, when $0 \leq t_1 \leq t_2$

$R_{B_c}(t_1, t_2) = c^2 t_1$

② when $0 \leq t_2 \leq t_1$, it can be similary prove that

$R_{B_c}(t_1, t_2) = c^2 t_2$

③ Based on ① and ②, $R_{B_c}(t_1, t_2) = c^2 \min(t_1, t_2)$ ※

$B_c(t, \xi)$ is a continuous but NOT differentiable with respect to t

Theorem 2 $\{B_c(t), t \geq 0\}$ is a Gaussian process

To prove it, we have to show that for each $n \in N$ and $0 \leq t_1 < t_2 \cdots < t_n$ $(B_c(t_1), B_c(t_2) \cdots B_c(t_n))$ is a Gaussian random vector

Lemma 1.

A random vector $(X_1 X_2 \cdots X_n)$ is Gaussian, if $\forall (a_1 a_2 \cdots a_n) \in \mathbb{R}^n$ $\sum_{k=1}^{n} a_k X_k$ is a Gaussian random variable

$a_1 X_1 + a_2 X_2 + \cdots a_n X_n$

$B_c(t_1)$

$B_c(t_2) = [B_c(t_2) - B_c(t_1)] + B_c(t_1) \sim N(0, c^2 t_2)$

$B_c(t_3) = [B_c(t_3) - B_c(t_2)] + B_c(t_2)$

$\vdots$

$B_c(t_n) = [B_c(t_n) - B_c(t_{n-1})] + B_c(t_{n-1}) \sim N(0, c^2 t_n)$

$(B_c(t_1), B_c(t_2) \cdots B_c(t_n))$

$\sum a_k B_c(t_n) = \sum_{k=1}^{n} a_k (B_c(t_k) - B_c(t_{k-1})) + a_k B_c(t_{k-1})$

<u>$n=2$</u>

$a_1 B_c(t_1) + a_2 B_c(t_2)$ — linear combination of indep Gaussian RVs

$= a_1 B_c(t_1) + a_2 [B_c(t_2) - B_c(t_1)] + a_2 B_c(t_1)$

Let $B(t)$ be a Brownian motion (with parameter 1)

Then, (1) $B(t) \sim N(0,t)$, $\forall t \geq 0$

(2) $R_{BB}(t_1,t_2) = \min(t_1,t_2)$, $\forall t_1,t_2 \geq 0$

Theorem 1.

Let $\{B(t), t \geq 0\}$ be a Brownian motion with parameter 1

Consider $0 \leq s \leq t$, Then $\forall y \in \mathbb{R}$

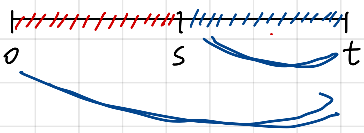
$$\boxed{f_{B(s)|B(t)}(x|y) = K \exp\left\{\frac{-t(x-\frac{ys}{t})^2}{2s(t-s)}\right\}}, \quad \forall x \in \mathbb{R}\ K \in \mathbb{R}_+$$

Namely, $B(s)|B(t)=y \sim N(\frac{ys}{t}, s(t-s))$; $B(t) \sim N(0,t)$, $B(s) \sim N(0,s)$

Proof.

① $$f_{B(s)|B(t)}(x|y) = \frac{f_{B(s),B(t)}(x,y)}{f_{B(t)}(y)} = \frac{f_{B(s)}(x)\, f_{B(t)-B(s)}(y-x)}{f_{B(t)}(y)}$$

Since $\begin{cases} B_c(t+\Delta t) - B_c(t) \text{ and } B_c(\Delta t) \text{ have the same pdf} \\ B_c(t) - B_c(s) \text{ is independent to } B(s) \end{cases}$



②

$$\begin{cases} f_{B(t)}(y) = K_t \cdot \exp\left(\frac{-y^2}{2t}\right) & B(t) \sim N(0,t) \\ f_{B(s)}(x) = K_s \cdot \exp\left(\frac{-x^2}{2s}\right) & B(s) \sim N(0,s) \end{cases}$$

$$\rightarrow f_{B(t)-B(s)}(y-x) = K_{t-s} \exp\left(\frac{-(y-x)^2}{2(t-s)}\right) \quad \because B(t)-B(s) \sim N(0,t-s)$$

③ Based on ① and ②

$$f_{B(s)|B(t)}(x|y) = K_i \exp\left\{\frac{-x^2}{2s} + \frac{-(y-x)^2}{2(t-s)} - \frac{-y^2}{2t}\right\}$$

$\exp\{\frac{-y^2}{2t}\}$ is independent to x

$$= K_2 \exp\left\{\frac{-x^2}{2s} + \frac{-(y-x)^2}{2(t-s)}\right\}$$

$$= K \cdot \exp\left\{\frac{-t(x-\frac{ys}{t})^2}{2s(t-s)}\right\}$$