# Stochastic Process Ch13

2024 Fall

# Ch13. Mean Square Estimation.

- $\underbrace{\hat{s}(t)}_{\text{Estimation of } s(t)} \triangleq \hat{E}\{\underbrace{s(t)}_{\text{target data}} \mid \underbrace{X(\xi),\ a \le \xi \le b}_{\text{Collected data}}\} = \int_a^b h(\alpha) X(\alpha)\, d\alpha \quad (13\text{-}1)$

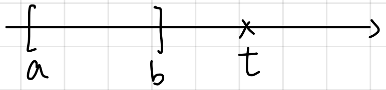
- we aim to find an optimal continuous function $h(\alpha)$

  such that

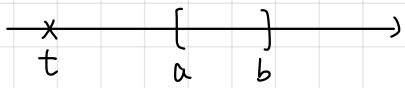
  $$P \triangleq E\{[s(t) - \hat{s}(t)]^2\} = E\{[s(t) - \int_a^b X(\alpha) h(\alpha)\, d\alpha]^2\} \text{ is minimized}$$

case

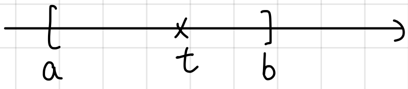
(1) Prediction




(2) backward prediction




(3) filtering



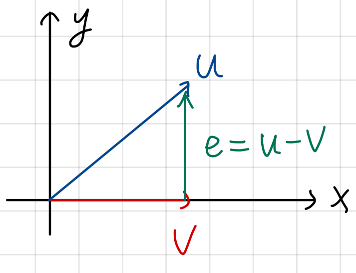

Theorem 13-1 Orthogonal principle

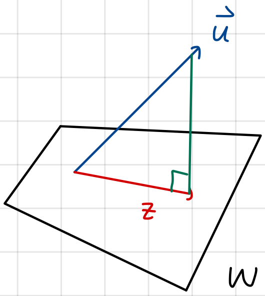
The mean square (MS) error $P$ of the estimation of a process $s(t)$ by the integral in (13-1) is minimum, if $\boxed{s(t) - \hat{s}(t)}$ are orthogonal to $\boxed{X(\xi)}$, i.e.

$$\begin{cases} E\{\underbrace{[s(t) - \int_a^b X(\alpha) h(\alpha)\, d\alpha]}_{[s(t) - \hat{s}(t)]} \cdot X(\xi)\} = 0, \quad \forall\, \xi \in [a, b] \quad (13\text{-}3) \\ R_{sx}(t, \xi) = \int_a^b h(\alpha) R_{xx}(\alpha, \xi)\, d\alpha \quad \forall\, \xi \in [a, b] \quad (13\text{-}4) \end{cases}$$

$e(t)$ $\quad e(t) \perp X(\xi)$

or equivalently.




Q: Find a $z \in W$ s.t. $\|u - z\|$ is minimized

$W = \text{span}\{v_1, v_2 \cdots v_m\}$

Ex 1. Prediction : we want to estimate the future value $s(t+\lambda)$

$\lambda > 0$. $s(t)$ is WSS

$$\hat{s}(t+\lambda) = E\{s(t+\lambda) \mid \underbrace{s(t)}_{X(\xi)}\} = a\,s(t).\quad \text{Find } a \text{ and } P$$

① Based on Theorem 13-1.

$E\{[s(t)-\hat{s}(t)]\cdot X(\xi)\}=0$

$$E\{[s(t+\lambda)-a\,s(t)]\cdot s(t)\} = 0$$

$$\Rightarrow R_{ss}(\lambda) - a R_{ss}(0) = 0$$

$$\Rightarrow a = \frac{R_{ss}(\lambda)}{R_{ss}(0)} \in [-1, 1]$$

※

② $P = E\{[s(t)-\hat{s}(t)]^2\}$

$$= E\{[s(t+\lambda) - a\,s(t)]^2\}$$

$$= E\{[s(t+\lambda)-a\,s(t)]\cdot s(t+\lambda)\} - E\{[s(t+\lambda)-a\,s(t)]\cdot \underbrace{a\,s(t)}_{aX(\xi)}\}$$

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$$= R_{ss}(0) - a R_{ss}(\lambda) - 0$$

Then,

$$P = R_{ss}(0) - \frac{R_{ss}(\lambda)}{R_{ss}(0)} R_{ss}(\lambda) = \frac{R_{ss}^2(0) - R_{ss}^2(\lambda)}{R_{ss}^2(0)} \geq 0$$

※

## EX 2. Prediction : we want to estimate the future value $S(t+\lambda)$

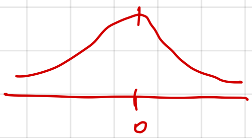
$\hat{S}(t+\lambda) = a_1 S(t) + a_2 S'(t)$ , Find $(a_1, a_2)$ , $P$

① Based on Theorem 13-1.

$$
\begin{cases}
E\{[S(t+\lambda) - a_1 S(t) - a_2 S'(t)] \cdot S(t)\} = 0 \\
\Rightarrow R_{ss}(\lambda) - a_1 R_{ss}(0) - a_2 R_{ss'}(0) = 0 \\
E\{[S(t+\lambda) - a_1 S(t) - a_2 S'(t)] \cdot S'(t)\} = 0 \\
\Rightarrow R_{ss'}(\lambda) - a_1 R_{ss'}(0) - a_2 R_{s's'}(0) = 0
\end{cases}
$$

$$\boxed{\begin{array}{l} (9\text{-}106) \\ R_{ss'}(\lambda) = -R'_{ss}(\lambda) \\ R_{s's'}(\lambda) = -R''_{ss}(\lambda) \end{array}}$$

Based on 9-106



$$
\begin{cases}
R_{ss}(\lambda) - a_1 R_{ss}(0) + a_2 \cancel{R'_{ss}(0)}^{\,0} = 0 \\
-R'_{ss}(\lambda) + a_1 \cancel{R'_{ss}(0)}^{\,0} + a_2 R''_{ss}(0) = 0
\end{cases}
$$

$$\Rightarrow a_1 = \frac{R_{ss}(\lambda)}{R_{ss}(0)} \qquad a_2 = \frac{R'_{ss}(\lambda)}{R''_{ss}(0)} \quad \#$$

$$\hat{S}(t+\lambda) = \frac{R_{ss}(\lambda)}{R_{ss}(0)} S(t) + \frac{R'_{ss}(\lambda)}{R''_{ss}(0)} S'(t)$$

If $\lambda$ is small,

$R_{ss}(\lambda) \cong R_{ss}(0)$

$R'_{ss}(\lambda) \cong R'_{ss}(0) + \lambda R''_{ss}(0)$

$$\Rightarrow \hat{S}(t+\lambda) \simeq S(t) + \lambda S'(t)$$

EX3.

$X(t) = S(t) + V(t)$

$\hat{S}(t) = \hat{E}\{S(t) \mid X(\xi)\ ; -\infty < \xi < \infty\} = \int_{-\infty}^{\infty} h(\alpha) X(t-\alpha)\, d\alpha \quad (13\text{-}12)$

$= \int_{-\infty}^{\infty} \hat{h}(\alpha) X(\alpha)\, d\alpha$ , Find $H(\omega)$ and $P$

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Theorem 1.

- $H(\omega) = \dfrac{S_{sx}(\omega)}{S_{xx}(\omega)} \qquad (13\text{-}14)$
- $P = \dfrac{1}{2\pi}\int_{-\infty}^{\infty} S_{ss}(\omega) - H^*(\omega) S_{sx}(\omega)\, d\omega \qquad (13\text{-}15)$

Proof.

① Based on Theorem 13-1. $\forall \tau \in R$

$E\{[S(t) - \int_{-\infty}^{\infty} h(\alpha) X(t-\alpha) d\alpha] \cdot X(t-\tau)\} = 0$

$\Rightarrow R_{sx}(\tau) - \boxed{\int_{-\infty}^{\infty} h(\alpha) \cdot R_{xx}(\tau-\alpha)\, d\alpha} = 0$

$h(\tau) * R_{xx}(\tau)$

$\Rightarrow S_{sx}(\omega) - H(\omega) \cdot S_{xx}(\omega) = 0$ ※

② $P$

$E\{e(t) \cdot \int_{-\infty}^{\infty} h(\beta) X(t-\beta) d\beta\}$

$= \int_{-\infty}^{\infty} h(\beta)\, \underbrace{E[e(t) X(t-\beta)]}_{=0}\, d\beta = 0$

$= E\{[S(t) - \int_{-\infty}^{\infty} h(\alpha) X(t-\alpha) d\alpha]^2\}$

$= E\{[S(t) - \int_{-\infty}^{\infty} h(\alpha) X(t-\alpha) d\alpha] \cdot S(t)\} - \cancel{E\{[S(t) - \int_{-\infty}^{\infty} h(\alpha) X(t-\alpha) d\alpha] \cdot \int_{-\infty}^{\infty} h(\beta) X(t-\beta) d\beta\}}$

$= R_{ss}(0) - \int_{-\infty}^{\infty} h(\alpha) R_{sx}(\alpha)\, d\alpha$

$= \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{ss}(\omega) e^{j\omega \cdot 0} d\omega - \int_{-\infty}^{\infty} h(\alpha) R_{sx}(\alpha)\, d\alpha \quad \because \int_{-\infty}^{\infty} h(\alpha) R_{sx}(\alpha)\, d\alpha = h(-\tau) * R_{sx}(\tau) \Big|_{\tau=0}$

$= \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{ss}(\omega) - H^*(\omega) S_{sx}(\omega) \cancel{e^{j\omega \cdot 0}}\, d\omega$ ※

- Important special case in which $\boxed{S(t) \perp V(t) \longleftrightarrow R_{sv}(\tau) = 0}$

- In the case. Theorem 1. become

- if $S(t)$ and $V(t)$ are indep. zero mean RPs.

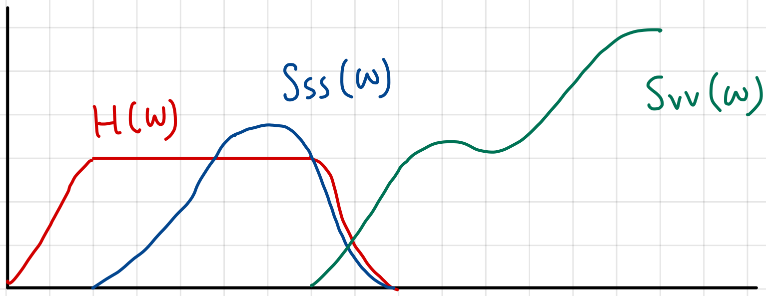
$$R_{sv}(\tau) = E[S(t+\tau)\, v(t)]$$
$$= E[S(t+\tau)] \cdot E[v(t)]$$
$$= 0$$

$$\boxed{\begin{aligned} H(\omega) &= \frac{S_{ss}(\omega)}{S_{ss}(\omega) + S_{vv}(\omega)} \\ P &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{S_{ss}(\omega) \cdot S_{vv}(\omega)}{S_{ss}(\omega) + S_{vv}(\omega)} \, d\omega \end{aligned}} \quad (13\text{-}16)$$

in this case

$$S_{sx}(\omega) = S_{ss}(\omega)$$

$$S_{xx}(\omega) = S_{ss}(\omega) + S_{vv}(\omega) \longleftrightarrow R_{xx}(\tau) = R_{ss}(\tau) + R_{vv}(\tau) + \overset{0}{\cancel{2R_{sv}}}(\tau)$$




→ If $S_{ss}(\omega)$ and $S_{vv}(\omega)$ NOT overlap.

$$\begin{cases} H(\omega) = 1 & \text{in the band of signal} \\ H(\omega) = 0 & \text{in the band of noise.} \end{cases}$$

# 13-2 Prediction.

$$\hat{s}[n] = \hat{E}\{s[n] \mid \overbrace{s[n-k],\ k \geq 1}^{\text{post}}\} = \sum_{k=1}^{\infty} h[k]\, s[n-k]$$

- $s[n]$ is a regular process
- $P \triangleq E\{(s[n] - \hat{s}[n])^2\}$
- $s[n] = \sum_{k=0}^{\infty} l[n]\, i[n-k]$ (13-27)
- $i[n] = \sum_{k=0}^{\infty} r[k]\, s[n-k]$ (13-28)

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Theorem 1.

If $s[n]$ is a regular process,

$$\hat{s}[n] = \sum_{k=1}^{\infty} l[k]\, i_s[n-k] \quad (13\text{-}31)$$

where $i_s[n] = \sum_{k=0}^{\infty} r[k]\, s[n-k]$

$$s[n] \longrightarrow \boxed{r[n]} \xrightarrow{i_s[n]} \boxed{h_i[n]} \xrightarrow{\hat{s}[n]}$$

<proof>

Consider $m \geq 1$.

- $E\{(\overbrace{s[n] - \sum_{k=1}^{\infty} l[k]\, i_s[n-k]}^{\hat{s}[n]})\, \overbrace{i_s[n-m]}^{\text{data}}\}$

$$= R_{si_s}[m] - \sum_{k=1}^{\infty} l[k]\, R_{i_s}[m-k]$$

$$= R_{si_s}[m] - \overset{\text{Lamma 1}}{R_{si_s}[m]} = 0$$

- Based on theorem 13-1

$$\hat{E}\{s[n] \mid s[n-k], k \geq 1\} = \sum_{k=1}^{\infty} l[k]\, i_s[n-k]$$

$$E\{s[n] \mid i_s[n-k], k \geq 1\} = \sum_{k=1}^{\infty} h_i[k] \cdot i_s[n-k]$$

$$\Rightarrow l[k] = h_i[k]$$

$$s[n] \longrightarrow \underbrace{\boxed{\frac{1}{L(z)}} \xrightarrow{i_s[n]} \boxed{H_i(z)}}_{H(z)} \longrightarrow \hat{s}[n]$$

Lamma 1.

$$R_{si_s}[m] = l[m] \quad (13\text{-}30)$$

<proof>

(1) $R_{si_s}[n, n-m]$

$$= E\{\underbrace{s[n]}_{\downarrow 13\text{-}27} \cdot i_s[n-m]\}$$

$$= E\{\sum_{k=0}^{\infty} l[k]\, i_s[n-k] \cdot i_s[n-m]\}$$

$$= \sum_{k=0}^{\infty} l[k] \cdot R_{i_s}[m-k]$$

$$= \sum_{k=0}^{\infty} l[k] \cdot \delta[m-k]$$

$$= l[m] \quad \#$$

$$i_s[n] \longrightarrow \boxed{l[n]} \longrightarrow s[n]$$

$$H_i(z) = \sum_{k=1}^{\infty} h_i[k]\, z^{-k} = \sum_{k=1}^{\infty} l[k]\, z^{-k} = \sum_{k=0}^{\infty} l[k]\, z^{-k} - l[0] = L(z) - l[0]$$

$$\Rightarrow H(z) = \frac{1}{L(z)} H_i(z) = 1 - \frac{l[0]}{L(z)} \quad (13\text{-}33)$$

EX 13-3

$$S(w) = \frac{5-4\cos w}{10-6\cos w} \; ; \; L(z) = \frac{2z-1}{3z-1} \Rightarrow \ell[0] = \frac{2}{3}$$

$$H(z) = 1 - \frac{\ell[0]}{L(z)} = 1 - \frac{2}{3}\,\frac{3z-1}{2z-1} = \frac{-1}{6z-3} = -\frac{1}{6}\,\frac{z^{-1}}{1-0.5z^{-1}} = \frac{\hat{S}(z)}{S(z)}$$

$$\Rightarrow (1-0.5z^{-1})\,\hat{S}(z) = -\frac{1}{6} z^{-1} S(z)$$

$$\Rightarrow \hat{s}[n] - \frac{1}{2}\hat{s}[n-1] = -\frac{1}{6} s[n-1] \;\#$$

Lamma 2.

If $s[n]$ is a regular process

$$P \triangleq E\{(\hat{s}[n]-s[n])^2\} = \ell[0]^2$$

$$\begin{cases} \hat{s}[n] = \sum_{k=1}^{\infty} \ell[k]\, i_s[n-k] = s[n] - \ell[0]\, i[n] \\ E\{(\ell[0]\, i[n])^2\} = \ell[0]^2 \cdot E\{i[n]^2\} \end{cases}$$

# Predicting Continuous-time Regular Process

$$\hat{s}(t+\lambda) = E\{s(t+\lambda) \mid s(t-\tau);\ \tau \geq 0\} \quad \lambda > 0$$

$$= \int_0^\infty \underbrace{h(\alpha)}_{\text{continuous function}} s(t-\alpha)\, d\alpha \qquad (13\text{-}39)$$

- $h(\alpha)$ is optimal $\longleftrightarrow$ $E\{[s(t-\lambda) - \int_0^\infty h(\alpha) s(t-\alpha)\, d\alpha] \cdot s(t-\tau)\} = 0 \ , \forall \tau \geq 0$

$$\Rightarrow \underline{R_{ss}(\tau+\lambda)} - \int_0^\infty h(\alpha) R_{ss}(\tau-\alpha)\, d\alpha = 0, \ \forall \tau \geq 0 \quad (13\text{-}40)$$

Wiener-Hopf equation.

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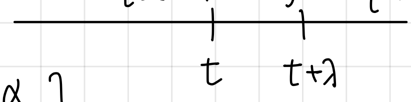
Theorem 2.

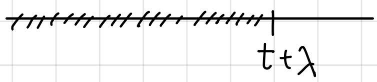
$\{s(\alpha) \mid \alpha \leq t\} \sim \{i_s(\alpha) \mid \alpha < t\}$




- $$\left.\begin{aligned} \hat{s}(t+\lambda) &= \int_\lambda^\infty \ell(\alpha)\, i_s(t+\lambda-\alpha)\, d\alpha \\ &= \int_0^\infty \ell(\beta+\lambda) \cdot i_s(t-\beta)\, d\beta \end{aligned}\right\} (13.43)$$

- $$s(t+\lambda) = \int_0^\infty \ell(\alpha)\, i_s(t+\lambda-\alpha)\, d\alpha \quad (13.42)$$
$$= (\ell * i_s)\big|_{t+\lambda}$$

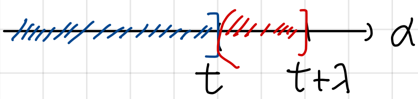
$s(t+\lambda)$ depend on white noise $i_s(t)$




<proof>

$$E\left\{\left[s(t+\lambda) - \int_\lambda^\infty \ell(\alpha)\, i_s(t+\lambda-\alpha)\, d\alpha\right]^2 i_s(t-\tau)\right\} \overset{?}{=} 0$$

- $s(t+\lambda) - \int_\lambda^\infty \ell(\alpha)\, i_s(t+\lambda-\alpha)\, d\alpha$




$$= \int_0^\infty \ell(\alpha)\, i_s(t+\lambda-\alpha) d\alpha - \int_\lambda^\infty \ell(\alpha)\, i_s(t+\lambda-\alpha)\, d\alpha = \int_0^\lambda \ell(\alpha) i_s(t+\lambda-\alpha)\, d\alpha$$

- $X$ depend only on $\{i_s(\alpha) \mid \alpha \in (t, t+\lambda)\}$

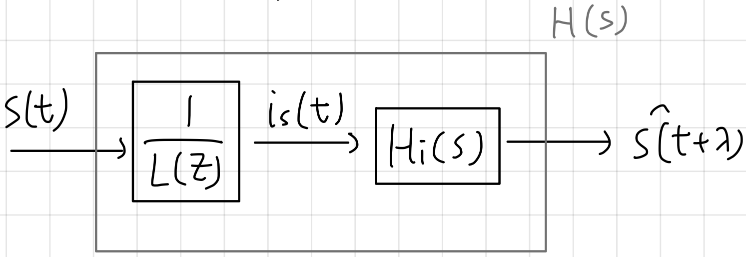
  on the other hand, $t-\tau \in (-\infty, t]$

  $(-\infty, t] \cap (t, t+\lambda] = \phi$

- In addition, $R_{i_s i_s}(\tau) = \delta(\tau)$. Thus $E\{X \cdot i_s(t-\tau)\} = E[X] \cdot E[i_s(t-\tau)] = 0$

- $h_i(t) \triangleq l(t+\lambda)\, u(t)$
- $H_i(s) = \int_0^\infty h_i(t) e^{-st} dt$

$$H(s) = \frac{H_i(s)}{L(s)}$$

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<u>Anolog process</u>.

$\lambda > 0$, $\hat{s}(t+\lambda) = E\{s(t+\lambda) \mid s(t-\tau), \tau \geq 0\} = \int_0^\infty h(\alpha) s(t-\alpha) d\alpha$ (13-39)

Then, based on Theorem 13-1,

$R(\tau+\lambda) = \int_0^\infty h(\alpha) R(\tau-\alpha) d\alpha$, $\forall \tau \geq 0$ (13-40)

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$s(t+\lambda) = \int_0^\infty l(\alpha) i_s(t+\lambda-\alpha) d\alpha$ ; $s(t)$ is regular process (13-42)

Theorem 1.

Let $s(t)$ be a regular process,

$$\hat{s}(t+\lambda) = \int_{\lambda}^{\infty} \ell(\alpha)\, i_s(t+\lambda-\alpha)\, d\alpha$$

$$= \int_{0}^{\infty} \ell(\beta+\lambda)\, i_s(t-\beta)\, d\beta \quad (13-43)$$

<pf>

① $\hat{e}(t+\lambda)$

$\triangleq s(t+\lambda) - \hat{s}(t+\lambda)$ $\quad s(t)$ is regular process

$= \int_{0}^{\infty} \ell(\alpha)\, i_s(t+\lambda-\alpha)\, d\alpha - \int_{\lambda}^{\infty} \ell(\alpha)\, i_s(t+\lambda-\alpha)\, d\alpha.$

$= \int_{0}^{\lambda^-} \ell(\alpha)\, i_s(t+\lambda-\alpha)\, d\alpha$ (improper integral)

$\left( \int_{0}^{\lambda^-} f(\alpha)\, d\alpha \triangleq \lim_{\delta\to 0} \int_{0}^{\lambda-\delta} f(\alpha)\, d\alpha \right)$

② Consider $\tau \geq 0$

$E\{\hat{e}(t+\lambda)\cdot i_s(t-\tau)\}$

$= E\{\int_{0}^{\lambda^-} h(\alpha)\, i_s(t+\lambda-\alpha)\, d\alpha\; i_s(t-\tau)\}$

$= \int_{0}^{\lambda^-} h(\alpha)\, E\{i_s(t+\lambda-\alpha)\, i_s(t-\tau)\}\, d\alpha$

$= \int_{0}^{\lambda^-} h(\alpha)\, \delta(\tau-\lambda+\alpha)\, d\alpha = 0$

$\because \tau-\lambda+\alpha \in [\tau, \tau-\lambda]$

but $\tau > 0$, thus $\tau-\lambda+\alpha \neq 0$

③ Since $s(t)$ is regular

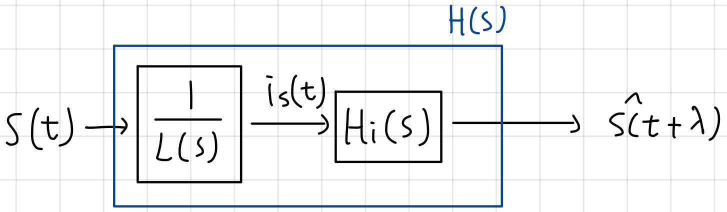
$E\{\hat{e}(t+\lambda)\, s(t-\tau)\} = 0$

④ Based on Theorem 13-1,

$\int_{\lambda}^{\infty} \ell(\alpha)\, i_s(t+\lambda-\alpha)\, d\alpha$ is optimal linear MSE for $s(t+\lambda)$

**Lemma 1.** Let $h_i(t) = \ell(t+\lambda)u(t)$ (13.45)

$$\int_0^\infty \ell(\beta+\lambda)\, i_s(t-\beta)\, d\beta = (h_i * i_s)(t+\lambda)$$




$$\begin{cases} H_i(s) \triangleq \int_0^\infty h_i(t) e^{-st} dt & (13\text{-}45) \\ H(s) = \dfrac{H_s(s)}{L(s)} & (13\text{-}46) \end{cases}$$

**Algorithm for $\hat{s}(t+\lambda)$**

Input : $S(\omega)$ , $\lambda$

Output : $H(\omega)$

(1) Factorization : $S(s) = L(s)L(-s)$

(2) $\ell(t) = \mathcal{L}^{-1}\{L(s)\}$; $h_i(t) = \ell(t+\lambda)u(t)$

(3) $H_i(s) = \mathcal{L}\{h_i(t)\}$ ; $H(s) = \dfrac{H_i(s)}{L(s)}$

**EX 13-5** $R_{ss}(\tau) = (2\alpha)^{-1} e^{-\alpha|\tau|}$ , $\alpha > 0$ $H(s) = ?$

⓪ $R_{ss}(\tau) = \frac{1}{2\alpha} e^{-\alpha|\tau|}$

$$S_{ss}(\omega) = \frac{1}{2\alpha}\frac{2\alpha}{\omega^2+\alpha^2} = \frac{1}{\omega^2+\alpha^2}$$

① Define $s = j\omega$ , $\omega^2 = -s^2$

$$S_s(s) = \frac{1}{\alpha^2 - s^2} \; ; \; \forall s \in C$$

$$S_s(s) = \frac{1}{\alpha+s}\,\frac{1}{\alpha-s}$$

$$\Rightarrow L(s) = \frac{1}{\alpha+s}$$

② $\ell(t) = \mathcal{L}^{-1}\left\{\frac{1}{s+\alpha}\right\} = e^{-\alpha t}u(t)$

$$h_i(t) = \ell(t+\lambda)u(t)$$
$$= e^{-\alpha(t+\lambda)} u(t+\lambda)u(t)$$
$$= e^{-\alpha t} e^{-\alpha\lambda} u(t)$$

③ $H_i(s) = \mathcal{L}\{h_i(t)\} = e^{-\alpha\lambda}\frac{1}{s+\alpha}$

$$\to H(s) = \frac{H_i(s)}{L(s)} = e^{-\alpha\lambda} \;\text{※}$$

$$\to h(t) = e^{-\alpha\lambda}\delta(t)$$

$$\to \hat{s}(t+\lambda) = s(t)\, e^{-\alpha\lambda} \;\text{※}$$

Lemma 2.

If $s(t)$ is regular and $R_{ss}(\tau) = c \cdot e^{-\alpha|\tau|}$

Then

$$\hat{s}(t+\lambda) \triangleq \hat{E}\{s(t+\lambda) \mid s(t-\lambda), \tau \geq 0\}$$
$$= \hat{E}\{s(t+\lambda) \mid s(t)\}$$

Definition :

A RP $s[n]$ is said to be predictable if

$$s[n] = \sum_{k=1}^{\infty} h[n]\, s[n-k] \quad (13\text{-}49)$$

where $\hat{s}[n] = \hat{E}\{s[n] \mid s[n-k]\, ; k \geq 1\} = \sum_{k=1}^{\infty} h[k]\, s[n-k]$

$$\underset{\text{WSS}}{s[n]} = \underset{\text{regular}}{s_r[n]} + \underset{\text{predictable}}{s_p[n]}$$

In this case

$$E\{(s[n] - \hat{s}[n])^2\} = 0$$

Theorem 13-3 (Wold decomposition)

Let $s[n]$ be a WSS RP, Then there exist two RP $s_1[n]$, $s_2[n]$, such that

(1) $s[n] = \underbrace{s_1[n]}_{\text{regular}} + \underbrace{s_2[n]}_{\text{Predictable}}$

(2) $s_1[n] = \hat{E}\{s[n] \mid \varepsilon[n-k], k \geq 0\} = \sum_{k=0}^{\infty} w_k \varepsilon[n-k]$ (13-55)

where $E[z] \triangleq 1 - H(z) = 1 - \sum_{k=1}^{\infty} h[k] z^{-k}$

error filter

$$s[n] \longrightarrow \boxed{E[z]} \longrightarrow \varepsilon[n] = s[n] - \hat{s}[n]$$

$s_2[n] = s[n] - s_1[n]$

(3) $s_1[n] \perp s_2[n-k] \quad \forall k \in N$

$\longleftrightarrow E\{s_1[n] \cdot s_2[n-k]\} = 0 \quad \forall k \in N$

(4) $s_1[n]$ is a regular process.

(5) $s_2[n]$ is a predictable process

$s_2[n] = \sum_{k=1}^{\infty} h[k] s_2[n-k]$ (13-58)

13-3 $\hat{s}[n] = \hat{E}\{s[n] \mid \underbrace{x[n-k]}_{\text{another RP}} ; k \geq 0\}$

(1) $x[n] = s[n] + w[n]$

(2) $x[n] = h[n] s[n] + w[n]$

(3) $x[n] = s^2[n] w[n]$

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## 13-3 Filtering and ~~Prediction~~ 13-2

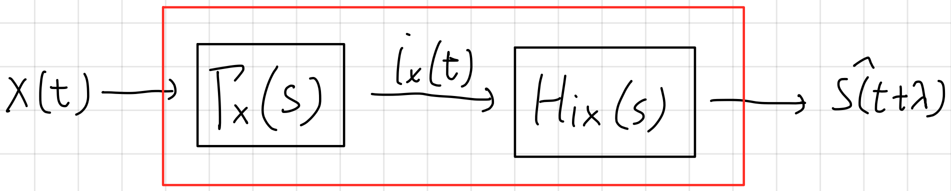
- $\hat{S}(t+\lambda) = \hat{E}\{\underbrace{S(t+\lambda)}_{\text{signal process}} \mid \underbrace{X(t+\tau)}_{\text{data process}}, \tau \geq 0\}$

$$= \int_0^\infty h_x(\alpha) X(t-\alpha)\, d\alpha \qquad (13\text{-}87)$$

- Based on Theorem 13-1

$$E\{[S(t+\lambda) - \int_0^\infty h_x(\alpha) X(t-\alpha)\, d\alpha]\, X(t-\tau)\} = 0, \ \forall t \geq 0$$

$$\rightarrow R_{SX}(t+\lambda) = \int_0^\infty h_x(\alpha) R_{XX}(\tau-\alpha)\, d\alpha, \ \forall \tau \geq 0 \qquad (13\text{-}88)$$




$H_x(s) = \Gamma_x(s) \cdot H_{ix}(s)$

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- To find $h_{ix}(t)$, we have to solve the following equation,

$$E\{[S(t+\lambda) - \int_0^\infty h_{ix}(\alpha)\, i_x(t-\alpha)\, d\alpha]\, i_x(t-\tau)\} = 0, \ \forall \tau \geq 0$$

$$\rightarrow R_{Si_x}(\tau+\lambda) = \int_0^\infty h_{ix}(\alpha) \underbrace{R_{i_x i_x}(\tau-\alpha)}_{\delta(\tau-\alpha)}\, d\alpha$$

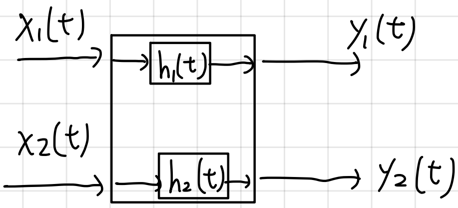
$$= h_{ix}(\tau) \qquad (13\text{-}92)$$

- Then, since $h_{ix}(\tau)$ is causal, $h_{ix}(\tau) = R_{Si_x}(\tau+\lambda)\, u(\tau)$ (13-93)
- We usually know $S_{SX}(S)$ or $R_{SX}(\tau)$

- Lemma 1:

$$S_{Si_x}(S) = S_{SX}(S) \cdot \Gamma_x(-S) \qquad (13\text{-}94)$$

The proof is based on (9-130), (9-170)




$y_2(t)$ statically depend on $X_1(t)$ (and $X_2(t)$

$$R_{X_1X_2}(t_1, t_2) = \int_{-\infty}^{\infty} R_{X_1X_2}(t_1, t_2-\alpha)\, h_2^*(\alpha)\, d\alpha$$

$$R_{Y_1Y_2}(t_1, t_2) = \int_{-\infty}^{\infty} h_1(\alpha)\, R_{X_1Y_2}(t_1-\alpha, t_2)\, d\alpha$$