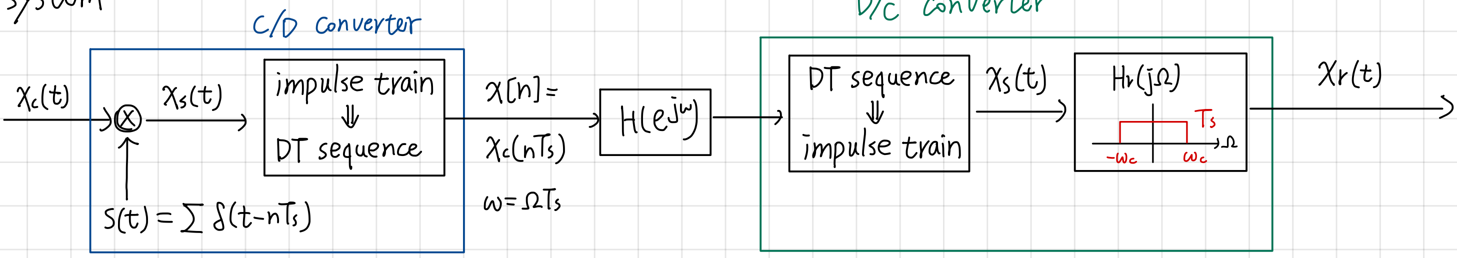
Digital Signal Processing

Ch4 Sampling

2024 Fall

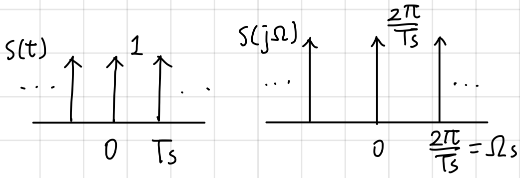
system



C/D converter

- ideal sampling

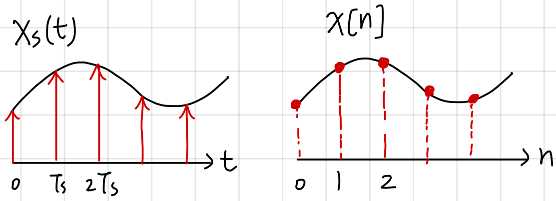
$$S(t)=\sum\delta(t-nT_s)\longleftrightarrow S(j\Omega)=\frac{2\pi}{T_s}\sum\delta\left(\Omega-k\frac{2\pi}{T_s}\right)$$



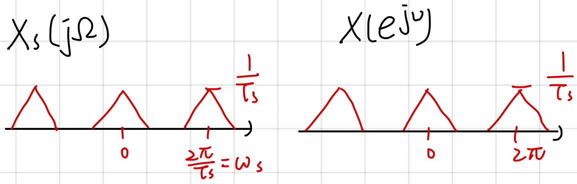
$$x_s(t)=\sum x_c(nT_s)\,\delta(t-nT_s)\longleftrightarrow$$

$$X_s(j\Omega)=\frac{1}{2\pi}X_c(j\Omega)*S(j\Omega)=\boxed{\frac{1}{T_s}\sum X_c\left(j\left(\Omega-k\frac{2\pi}{T_s}\right)\right)}$$

橫軸尺度改變



$$X(e^{j\omega})=X_s(j\omega/T_s)$$



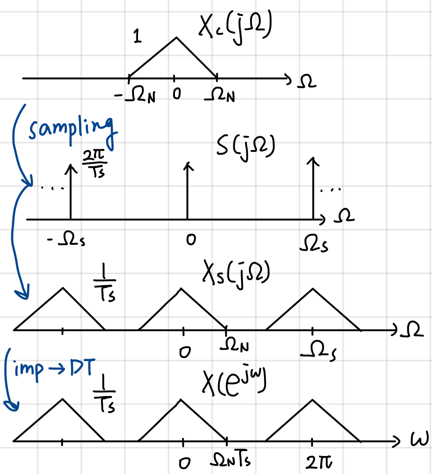
- impulse train ⟶ DT sequence

$$\begin{cases} X_s(j\Omega)=\sum x_c(nT_s)\mathcal{F}\{\delta(t-nT_s)\}=\sum x_c(nT_s)e^{-jn\Omega T_s} \\ X(e^{j\omega})=\sum x[n]e^{-j\omega n}=\sum x_c(nT_s)e^{-j\omega n} \end{cases}$$

- When $\omega=\Omega T_s$; $X(e^{j\omega})=X_s(j\Omega)\big|_{\omega=\Omega T_s}$

$$\begin{cases} X_s(j\Omega)=\frac{1}{T_s}\sum X_c\left(j\left(\Omega-k\frac{2\pi}{T_s}\right)\right) \\ X(e^{j\omega})=\frac{1}{T_s}\sum X_c\left(j\left(\frac{\omega-2\pi k}{T_s}\right)\right) \end{cases}\longrightarrow x[n]=x_c(nT_s)$$

以2π為週期

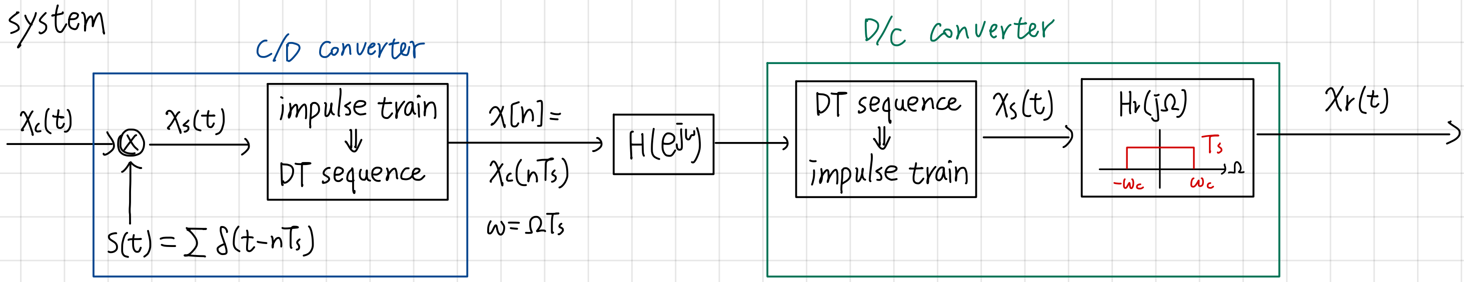


Nyquest-Shanno Sampling theorem

When $\Omega_s\geq 2\Omega_N$

$x_c(t)$ is uniquely determined by its sample $x[n]=x_c(nT_s)$

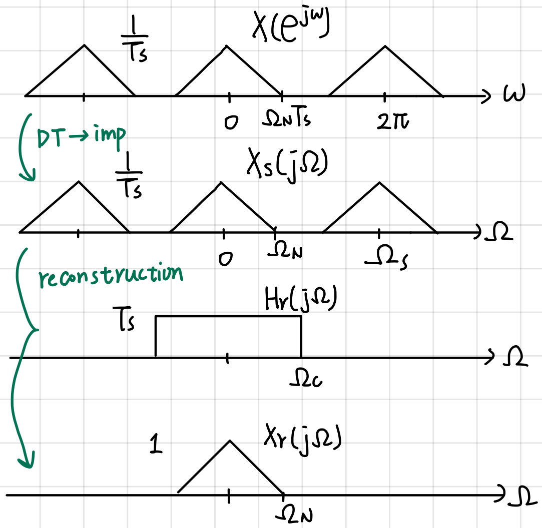
system



D/C converter

- DT sequence ⟶ impulse train

$$\begin{cases} x_s(t) = \sum x[n]\,\delta(t-nT_s) \\ X_s(j\Omega) = \sum x[n]\, e^{-j\Omega n T_s} = X(e^{j\omega})\big|_{\omega=\Omega T_s} \end{cases}$$

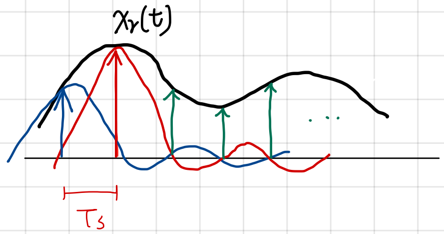


- Signal reconstruction

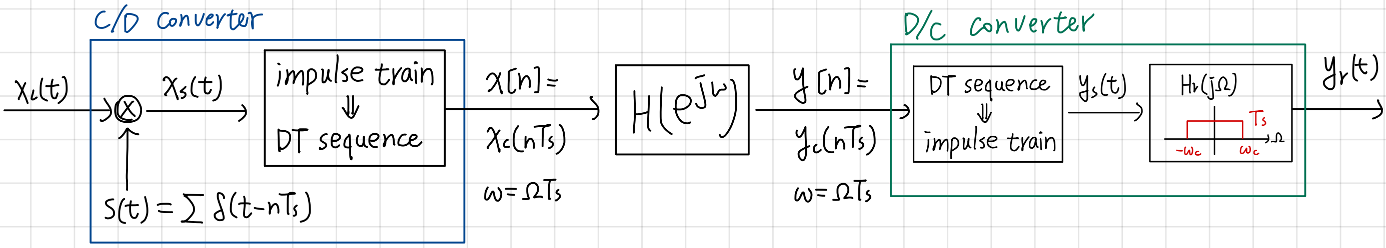
$$\begin{cases} H_r(j\Omega) = T_s\,\Pi\left(\frac{\Omega}{2\omega_c}\right) \\ h_r(t) = \frac{T_s \omega_c}{\pi}\,\mathrm{sinc}\left(\frac{\omega_c t}{\pi}\right) \xrightarrow{\substack{\omega_s = 2\omega_c \\ \omega_c = \frac{\pi}{T_s}}} \mathrm{sinc}\left(\frac{t}{T_s}\right) \end{cases}$$

$$X_r(j\Omega) = X(e^{j\Omega T_s})H_r(j\Omega)$$
$$= T_s X(e^{j\Omega T_s}) \ ; \ |\Omega| < \Omega_c$$

$$x_r(t) = \sum x_c(nT_s)\,\mathrm{sinc}\left(\frac{t-nT_s}{T_s}\right)$$

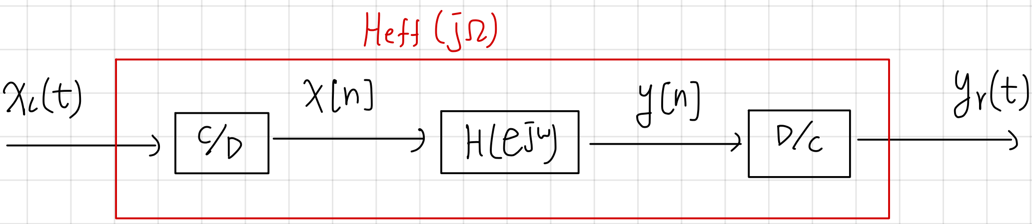


Conclusion



- $x_s(t) = \sum x_c(nT_s)\,\delta(t - nT_s) \longleftrightarrow X_s(j\Omega) = \frac{1}{T_s}\sum X_c(j(\Omega - k\Omega_s))$
- $x_d[n] = x_c(nT_s) \xleftrightarrow[\omega = \Omega T_s]{} X(e^{j\Omega T_s}) = X_s(j\Omega)$
- $y_s(t) = \sum y[n]\,\delta(t - nT_s) \longleftrightarrow Y_s(j\Omega) = \sum y[n] e^{-jn\Omega T_s} = Y(e^{j\Omega T_s})$
- $y_r(t) = y_s(t) * h_r(t) \longleftrightarrow Y_r(j\Omega) = Y_s(j\Omega) H_r(j\Omega)$

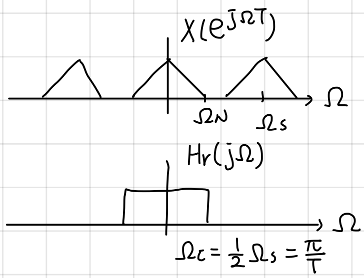
- $X_s(j\Omega) = \frac{1}{2\pi} X_c(j\Omega) * S(j\Omega) = \frac{1}{T_s}\sum X_c(j(\Omega - k\Omega_s))$
- $X(e^{j\omega}) = X_s(j\frac{\omega}{T_s}) = \frac{1}{T_s}\sum X_c\left(j\left(\frac{\omega - 2\pi k}{T_s}\right)\right)$
- $Y(e^{j\omega}) = X(e^{j\omega}) H(e^{j\omega})$ $\quad \Omega = \omega T_s$
- $Y_s(j\omega) = Y(e^{j\Omega T_s})$
- $Y_r(j\omega) = Y_s(j\Omega) H_r(j\Omega)$

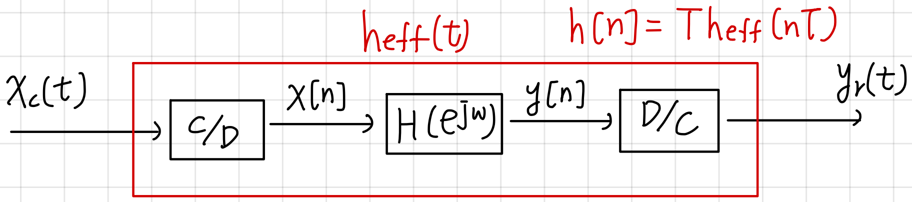


$$Y_r(j\Omega) = X_c(j\Omega) H_{eff}(j\Omega) \quad : \quad H_{eff}(j\Omega) = \begin{cases} H(e^{j\Omega T}) & ; |\Omega| < \frac{\pi}{T} \\ 0 & ; \text{else} \end{cases}$$

<pf>

$$Y_r(j\Omega) = X(e^{j\Omega T}) H(e^{j\Omega T}) H_r(j\Omega)$$
$$= H(e^{j\Omega T}) X_c(j\Omega)$$





Differentiator : $y_r(t) = \frac{d}{dt} x_c(t)$

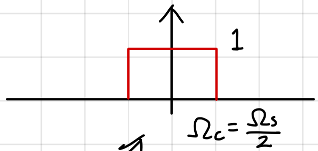
- Find impulse response $h[n]$: let $x[n] = \frac{1}{T_s}\delta[n]$

$x[n] = \frac{1}{T_s}\delta[n]$, $y[n] = \frac{1}{T_s} h[n]$

$X_c(t) = \frac{\sin(\Omega_c t)}{\pi t}$

$X(e^{j\omega}) = \frac{1}{T_s}\sum X_c(j(\Omega - k\Omega_s))$

$X_c(j\Omega)$



$$x[n] = x_c(nT) = \frac{1}{T}\delta[n]$$
$$y[n] = y_r(nT) = \frac{1}{T}h[n]$$
$$\Rightarrow h[n] = T\, y_r(nT)$$

$$\begin{cases} h[n] = T h_{eff}(nT) = T\,\frac{\sin(\Omega_c nT)}{\pi nT} = \frac{\sin(\omega_c n)}{\pi n} \ \# \\ H(e^{j\omega}) = \begin{cases} 1 ; & |\omega| < \omega_c \\ 0 ; & \text{else} \end{cases} \end{cases}$$