Wireless Communication Ch12

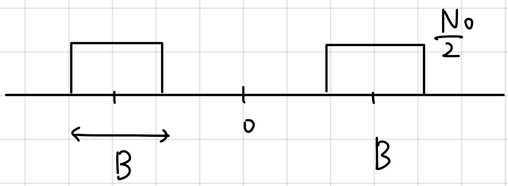
2024 Fall

Demodulation.

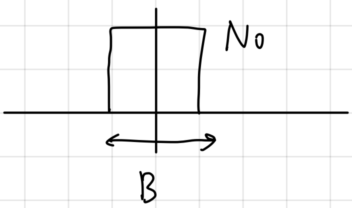
- Noise channel:

$$r_{cp}(t) = \alpha s_{cp}(t) + n_{cp}(t)$$

  - where α is the channel $= |\alpha| \exp(j\theta)$ which is know by receiver
  - $n_{cp}(t)$ is complex with Gaussian noise.
  - In passband, we accually have white noise with $S_N(f) = \frac{N_0}{2}$ and $BW = B$



  - The equivalent baseband noise is:



Derive the Optimal receiver.

- Assume the channel and modulation have no memory
- Assume α is perfectly know
  - Assume without loss of generality that α is real number.
- The optimal receiver is the " maximum a posterior " receiver (MAP)
  → If a signal $r(t)$ is received, then which symbol $s(t)$ is the mostly likely transmitted.
  → we want to find the $S_m(t)$ such that $Pr(S \text{ sent} \mid r \text{ received})$ max.

$$\rightarrow \hat{m} = \arg\max_m Pr(S_m \mid r)$$
$$= \arg\max_m Pr(r \mid S_m) \cdot P(S_m)$$
$$= \arg\max Pr(n(t) = r(t) - \alpha S_m(t))\, Pr(S_m)$$

- If we assume that all symbol are $P(S_m) = \frac{1}{M}$.

$$\rightarrow \hat{m} = \arg\max_m Pr(n(t) = r(t) - \alpha S_m(t)) \quad \text{ML rule}$$

- with signal space. we have:
  - $S_m(t) = \sum_{n=1}^{N} S_{m,n} \phi_n(t)$, $S_{m,n} = \int_0^T S_m(t) \cdot \phi_n^*(t)\, dt$
  - $r(t) = \sum_{n=1}^{N} r_n \cdot \phi_n(t)$, $r_n = \int_0^T r(t)\, \phi_n^*(t)\, dt$
- Since the transmit signal only span for N-dimension. thus those dimension are not related to the signal can be ignore

$$\rightarrow r_N(t) = \sum_{n=1}^{N} \alpha S_{m,n} \cdot \phi_n^*(t) + \sum_{n=1}^{N} n_n \phi_n^*(t) + \cancel{\sum_{n=N+1}^{\infty} n_n \phi_n^*(t)}$$

- using the above expression. we have

$$\hat{m} = \arg\max_{m} P_r(n = r - \alpha s_m)$$

  - $n = [n_1\ n_2 \cdots n_N]^T$
  - $r = [r_1\ r_2 \cdots r_N]^T$
  - $s_m = [s_{m,1}\ s_{m,2} \cdots s_{m,N}]^T$

- Since noise power spectrum density $= N_0$, we have

$$P_r(r \mid \alpha s_m) \propto \exp\left(-\frac{1}{2N_0}\|r - \alpha s_m\|^2\right)$$

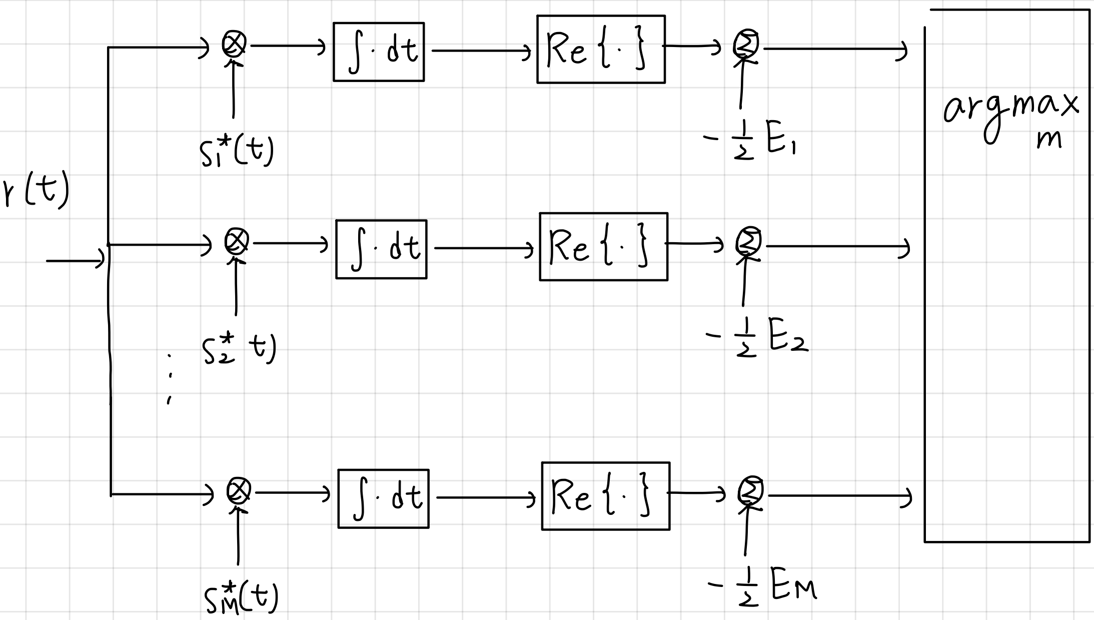
$$= \prod_{n=1}^{N} \exp\left(-\frac{1}{2N_0}(r - \alpha s_m)^2\right)$$

$$\Rightarrow \underline{\hat{m} = \arg\min_{m} \|r - \alpha s_m\|^2} \leftarrow \text{MD dicision rule.}$$

- let $\mu(s_m) = \|r - \alpha s_m\|^2 = \|r\|^2 + \|\alpha s_m\|^2 - 2\alpha \mathrm{Re}\{r \cdot s_m^*\}$

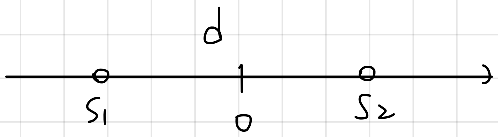
$$\Rightarrow \hat{m} = \arg\min_{m} \left(\alpha^2 E_m - 2\alpha \mathrm{Re}\{r \cdot s_m^*\}\right)$$

$$= \arg\max_{n} \left(\underline{\mathrm{Re}\{r \cdot s_m^*\}} - \frac{1}{2}E_m\right) \leftarrow \text{correlation receiver.}$$



BER analysis with noise and fading

- What is the BER if we have the following signal space diagram



- For the ML, we know if $\|r-S_1\|^2 < \|r-S_2\|^2$, we say S_1 is transmit
- Then, the error happen if we transmit S_2 but detector say S_1.
- Suppose $n \sim (0, \frac{N_0}{2})$, the error happen if $n < -\frac{d}{2}$

Remark:

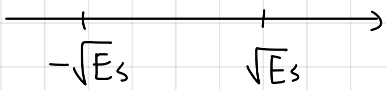
the noise power density is $\frac{N_0}{2}$ because we only one-dimension.

Thus,

$$BER = \int_{-\infty}^{-\frac{d}{2}} \frac{1}{\sqrt{\pi N_0}} e^{-\frac{x^2}{N_0}} dx = \int_{-\infty}^{\frac{-d}{\sqrt{2N_0}}} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt = \boxed{Q\left(\sqrt{\frac{d^2}{2N_0}}\right)}$$

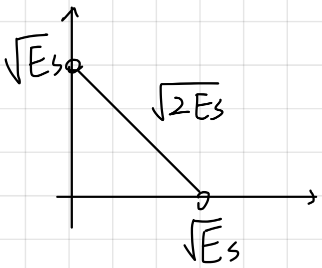
BER of BPSK (antipodal)

$$SNR = \gamma_s = \frac{E_s}{N_0}$$



$$\Rightarrow BER = Q\left(\sqrt{\frac{4E_s}{2N_0}}\right) = Q\left(\sqrt{2\frac{E_s}{N_0}}\right) = Q(\sqrt{2\gamma_s}) = Q(\sqrt{2\gamma_B})$$

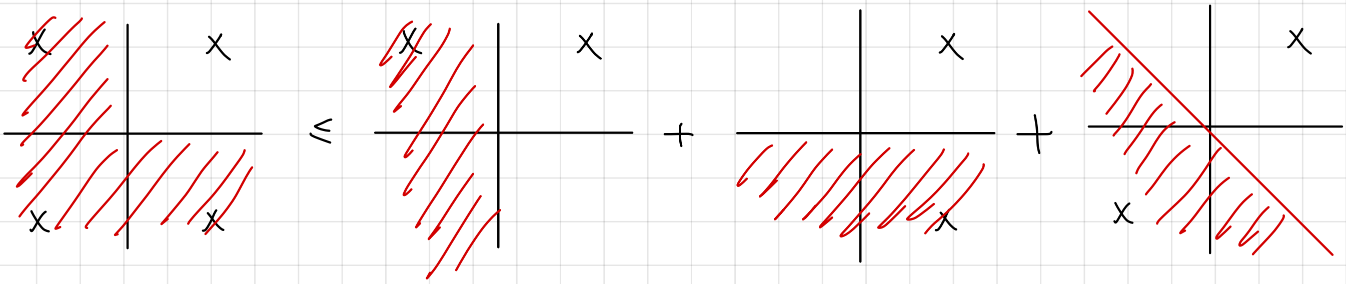
BER of BFSK



$$BER = Q\left(\sqrt{\frac{2E_s}{2N_0}}\right) = Q(\sqrt{\gamma_s}) = Q(\sqrt{\gamma_B})$$

BER of QPSK

- we have multiple symbol.
- Union bound.
- Union bound consider the sum of pair-wise error rate which is the upper bound of the exact error rate



- $SER_{s_i} \le \sum_{j \ne i} SER_{i \to j} = \sum_{j \ne i} Q\left(\sqrt{\frac{d_{ij}^2}{2N_0}}\right)$
- $BER_{s_i} \le \sum_{i \ne j} b_{ij}\, Q\left(\sqrt{\frac{d_{ij}^2}{2N_0}}\right) \Big/ \bar{b}$

- b_{ij} : bit in error if $S_i \to S_j$
- $\bar{b}$: total of bit in symbol $= \log_2 M$

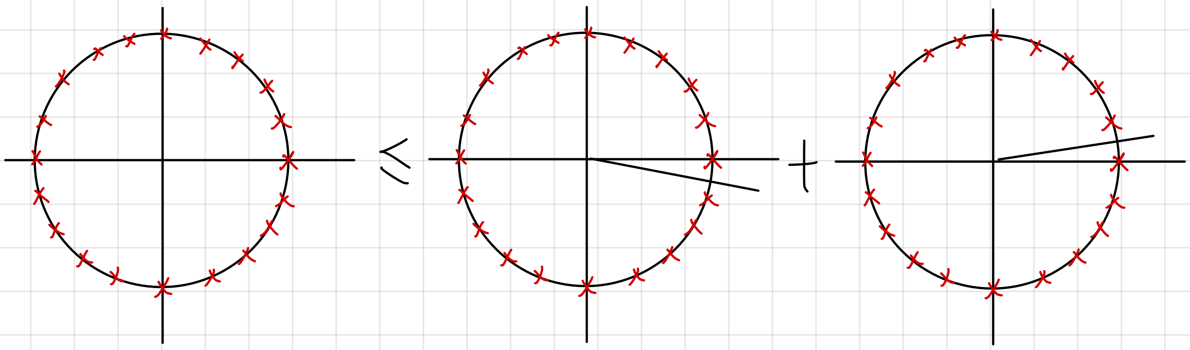
$$SER \le \frac{1}{M} \sum_{i=1}^{M} \sum_{j \ne i} Q\left(\sqrt{\frac{d_{ij}^2}{2N_0}}\right)$$

$$BER \le \frac{1}{M\bar{b}} \sum_{i=1}^{M} \sum_{j \ne i} b_{ij}\, Q\left(\sqrt{\frac{d_{ij}^2}{2N_0}}\right)$$

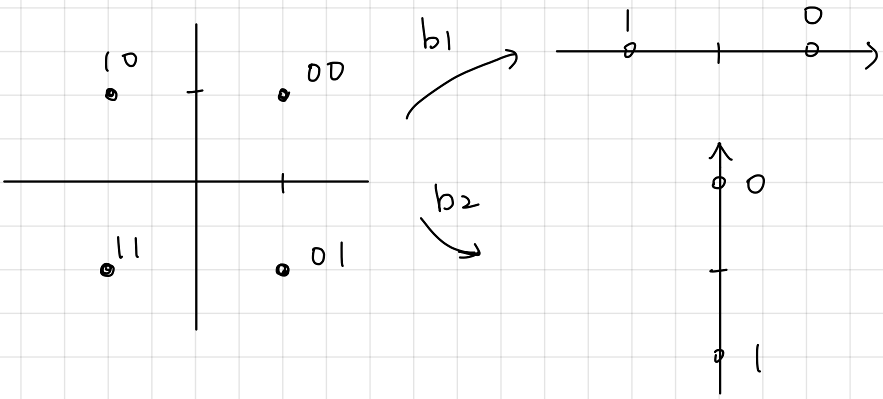
續.

If all symbol are 2D-plan, we only need to consider the polar-wise error rate of neighboring pairs.

EX.



Gray-coding.



→ This is equivalent to independent BPSK.

→ Bit Error Rate $= Q\left(\sqrt{\frac{2E_s}{2N_0}}\right) = Q(\sqrt{\gamma_s})$

→ $E_s = 2E_b$. If we express in terms of E_b, we have

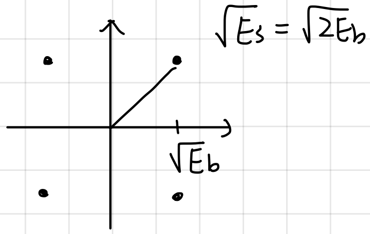
$$BER = Q\left(\sqrt{\frac{4E_b}{2N_0}}\right) = Q(\sqrt{2\gamma_b}) = \text{BER of BPSK}$$

→ When use QPSK with gray-coding, the QPSK has the same BER as BPSK. If they have same E_b.

- But QPSK has a larger transmission rate.
- QPSK need twice of the energy per symbol to achieve.

Example 12-1

QPSK, look at BER, SER.



pair-wise error rate, we have $Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$

- $SER \leq 2Q\left(\sqrt{\frac{2E_b}{N_0}}\right) = 2Q\left(\sqrt{2\gamma_b}\right)$

- If we consider the gray-coding QPSK
  we know that the $BER = Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$

- By BER, we can derive exact SER.

$$SER = 1-(1-P_e)^2 = 2P_e - P_e^2 \lesssim 2P_e$$

- BER of DPFSK $= \frac{1}{2}\exp\left(-\frac{E_b}{N_0}\right) = \frac{1}{2}\exp(\gamma_b)$
- BER of differential orthogonal binary shift keying $= \frac{1}{2}\exp\left(\frac{E_b}{2N_0}\right) = \frac{1}{2}\exp\left(-\frac{1}{2}\gamma_b\right)$
- Noncoherent BFSK, $BER = \frac{1}{2}\exp\left(-\frac{E_b}{2N_0}\right)$

Ex 12.2

Given SNR = 5dB. What are the BER of BPSK, BFSK, DBPSK under noise.

$\gamma_s = 5dB = \gamma_B$

$\Rightarrow BER_{BPSK} = Q(\sqrt{2\gamma_B}) = 0.006$

$BER_{BFSK} = Q(\sqrt{\gamma_B}) = 0.038$

$BER_{DBPSK} = \frac{1}{2}e^{-\gamma_B} = 0.021$

EX.

$10\log_{10} 20\times10^7 = 73$

$f_c = 5GHz$, $B = 20MHz$, $G_{TX} = 2dB$ $G_{RX} = 2dB$ $P_{TX} = 20dBm$ $L = 3dB$.

$F = 5dB$, $N_0 = -173\,dBm/Hz$, $d_{break} = 10m$, $d = 100m$, $n = 3.4$.

(a) SNR at the RX = ?

(b) BER at BPSK, BFSK, QPSK.

SNR

- $P_n = -173 + 73 + 5 = -95\,dBm$ F
- $PL = 20\log_{10}(\frac{4\pi}{\lambda_c}) + 20\log_{10}(d_{break}) + 10n\log_{10}(\frac{d}{d_{break}})$

$= 32 + 2\times7 + 20 + 34 = 100\,dB$

- $P_{RX} = 20 + 2 + 2 - 100 = -79\,dBm$

$SNR = \gamma_s = -79 - (-95) = 16dB = 40$ 次

不是dB

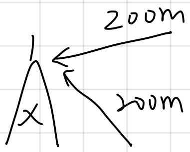
BER

$BER_{BPSK} = Q(\sqrt{2\gamma_b}) = Q(\sqrt{2\gamma_s}) = Q(\sqrt{80}) = 1.87\times10^{-19}$

$BER_{BFSK} = Q(\sqrt{\gamma_b}) = Q(\sqrt{\gamma_s}) = Q(\sqrt{40}) = 1.26\times10^{-10}$

$BER_{QPSK} = Q(\sqrt{2\gamma_b}) = Q(\sqrt{\gamma_s}) = Q(\sqrt{40}) = 1.26\times10^{-10}$

Ex : 2-user system



$P_{MS} = 20\,dBm$, $f_c = 900\,MHz$, $G_{MS} = 0\,dB$, $G_{BS} = 5\,dB$

$N_0 = -174\,dBm/Hz$, $B = 20\,MHz$. $d_{break} = 10\,m$

$u = 4$. $F = 5\,dB$.

(a) $SNR = 17\,dB = 50.12$

(b) Suppose $SIR = 17\,dB$, Then what is our BER if we can treat interference as additional noise. i.e. $\hat{n} = n + I$

Q. BER of BPSK = ?

$$SINR = \frac{E}{I+N} = \frac{1}{\frac{1}{SNR} + \frac{1}{SIR}} = \frac{1}{\frac{1}{50.12} + \frac{1}{50.12}} = 25.06 = 14\,dB$$

$$\Rightarrow BER_{BPSK} = Q(\sqrt{2\gamma_s}) = Q(\sqrt{50.12}) = 7.23 \times 10^{-23}$$

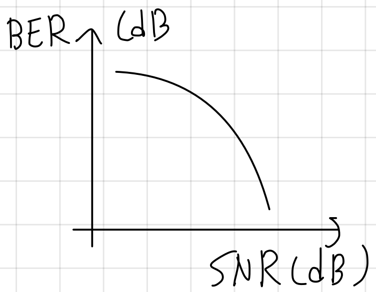
BER under flat fading channel.

- With fading, the channel gain is a random variable.

  Thus, BER is NOT only determined by noise but also the distribution of fading.

- In this case, we compute the average BER, using the following procedure
  1. Determine the BER for an arbitrary given SNR
     e.g. BPSK $\Rightarrow$ BER $= Q(\sqrt{2\gamma_B}) = Q(\sqrt{2\gamma_s})$
  2. Determine the pdf of the channel power gain.
  3. Find the average BER using the pdf.
     e.g. BPSK, $\overline{BER} = \int_0^\infty Q(\sqrt{2\gamma_s})\, pdf(\gamma_s)\, d\gamma_s$

- Remark: average BER is dominated by the low SNR case.
  - BER with respect to SNR is exponential function



之後縮寫成 w.p.

e.g. BPSK at 10dB, BER $\approx 10^{-4}$ with probility 0.5

15dB, BER $\approx 10^{-8}$ with probility 0.5

$\rightarrow BER = \frac{1}{2}(10^{-4} + 10^{-8}) \approx \frac{1}{2} \times 10^{-4}$

EX 12.3

Suppose we have a fading channel with average SNR = 10dB

where $SNR = \begin{cases} 13\text{dB}, & \text{w.p } 0.5 \longrightarrow \text{given } BER = 10^{-9} \\ -\infty \text{ dB}, & \text{w.p } 0.5 \longrightarrow \text{given } BER = 0.5 \end{cases}$

13dB = 20

- $\overline{SNR} = \frac{1}{2} \cdot (20 + 0) = 10\text{ dB}$
- $\overline{BER} = \frac{1}{2}(0.5 + 10^{-9}) = 0.25$

If No fading and SNR = 10dB $\Rightarrow$ BER $= 2 \cdot 10^{-5}$

Average BER at Rayleigh fading.

$$pdf_R(r) = \frac{r}{\sigma^2}\exp\left(-\frac{r^2}{2\sigma^2}\right)$$

$$\Rightarrow pdf_P(P) = \frac{1}{\bar{P}}\exp\left(-\frac{P}{\bar{P}}\right); \quad \bar{P}: \text{average received power.}$$

$$\begin{cases} SNR = \frac{P}{\sigma_n} = \gamma_s \longrightarrow \text{instantaneous SNR} \\ \overline{SNR} = \frac{\bar{P}}{\sigma_n} = \bar{\gamma}_s \longrightarrow \text{average SNR} \end{cases}$$

$$\rightarrow pdf_{\gamma_s}(\gamma_s) = \frac{1}{\bar{\gamma}_s}\exp\left(-\frac{\gamma_s}{\bar{\gamma}_s}\right)$$

$$\boxed{\text{Formula: } 2\int_0^\infty Q(\sqrt{2x})\cdot a\exp(-ax)\,dx = 1-\sqrt{\frac{1}{1+a}}}$$

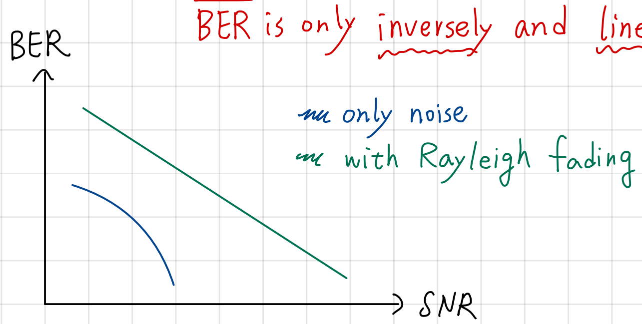
- $$\overline{BER}_{BPSK} = \int_0^\infty \frac{1}{\bar{\gamma}_s}\exp\left(-\frac{\gamma_s}{\bar{\gamma}_s}\right)Q(\sqrt{2\gamma_s})\,d\gamma_s \qquad \text{If let } a = \frac{1}{\bar{\gamma}_s}$$

$$= \frac{1}{2}\left[2\int_0^\infty a\exp\left(-\frac{\gamma_s}{\bar{\gamma}_s}\right)Q(\sqrt{2\gamma_s})\,d\gamma_s\right]$$

$$= \frac{1}{2}\left(1-\sqrt{\frac{1}{1+\frac{1}{\bar{\gamma}_s}}}\right) \qquad \boxed{\bar{\gamma}_s \gg 1}$$

$$\approx \frac{1}{4\bar{\gamma}_s} \qquad \sqrt{\frac{\bar{\gamma}_s}{\bar{\gamma}_s+1}} = \left(1-\frac{1}{1+\bar{\gamma}_s}\right)^{\frac{1}{2}} \approx 1-\frac{1}{2}\frac{1}{1+\bar{\gamma}_s}$$

$\overline{BER}$ is only inversely and linearly proportional to $\bar{\gamma}_s$



- $\overline{BER}$ of binary orthogonal signal (eg. BFSK)

$$\Rightarrow \overline{BER} = \frac{1}{2}\left[1-\sqrt{\frac{1}{2+\bar{\gamma}_s}}\right] \approx \frac{1}{2\bar{\gamma}_s}$$

$$\Rightarrow \overline{BER}_{DBPSK} = \frac{1}{2(1+\bar{\gamma}_s)} \approx \frac{1}{2\bar{\gamma}_s}$$

- $\overline{BER}$ of differential binary orthogonal signal.

$$\Rightarrow \overline{BER} = \frac{1}{2(1+\bar{\gamma}_s)} \approx \frac{1}{2\bar{\gamma}_s}$$

BER of Rician fading channel.

- Usually no closed form.
- $pdf_{\gamma_s}(\gamma_s) = \frac{1+k}{\bar{\gamma}_s} \exp\left(-\frac{\gamma_s(1+k)k\bar{\gamma}_s}{\bar{\gamma}_s}\right) I_0\left(\sqrt{\frac{4(1+k)k\gamma_s}{\bar{\gamma}_s}}\right)$
- $\overline{BER} = \int_0^\infty pdf_{\gamma_s}(\gamma_s)\, BER(\gamma_s)\, d\gamma_s$

- For differential BPSK: (can find close form)

$$\overline{BER} = \frac{1+K}{2(1+K+\bar{\gamma}_s)} \exp\left(-\frac{k\bar{\gamma}_s}{1+K+\bar{\gamma}_s}\right)$$

- differential binary orthogonal signal.

$$\overline{BER} = \frac{1+K}{2+2K+\bar{\gamma}_s} \exp\left(-\frac{k\bar{\gamma}_s}{1+K+\bar{\gamma}_s}\right)$$

- Alternative $\overline{BER}$ compute. (NOT general)

if our $SER(\gamma_s)$ or $BER(\gamma_s)$ can be express as:

- $SER(\gamma_s) = \int_{\theta_1}^{\theta_2} f_1(\theta) \exp(-\gamma_s f_2(\theta))\, d\theta$
  - $f_1(\theta)$, $f_2(\theta)$ are some function e.g. BER of DBPSK $= \frac{1}{2}\exp(-\gamma_s)$
- We have formula:

$$\overline{SER} = \int_{\theta_1}^{\theta_2} f_1(\theta) M_{\gamma_s}(-f_2(\theta))\, d\theta$$

where $M_{\gamma_s}(x)$ is moment generating function of γ_s at x:

$$M_{\gamma_s}(x) = \int_0^\infty pdf_{\gamma_s}(\gamma_s) \exp(\gamma_s \cdot x)\, dx.$$

NOTE:

Rayleigh: $M_{\gamma_s}(s) = \frac{1}{1-s\bar{\gamma}_s}$

Rician: $M_{\gamma_s}(s) = \frac{1+k}{1+k+s\bar{\gamma}_s} \exp\left(\frac{ks\bar{\gamma}_s}{1+k-s\bar{\gamma}_s}\right)$

Nakagami: $M_{\gamma_s}(s) = \left(1-\frac{s\bar{\gamma}_s}{m}\right)^{-m}$

where m is Nakagami fading parameter

Outage prob. for a BER requirement

- With fading and BER/SER as a function of SNR.
  we can define the $\boxed{\text{outage prob.}}$ for a given BER/SER requirement.

  $\Rightarrow$ $P_{out} = Pr(BER \geq BER_{req})$

- Note that we have: $BER(\gamma_s) = f(\gamma_s)$ ← as a function of SNR

→ we obtain : $\gamma_{req} = f^{-1}(BER_{req})$ ↙ 反函數

→ $P_{out} = Pr(\gamma_s \leq \gamma_{req}) = Pr(\gamma_s \leq f^{-1}(BER_{req}))$

→ we get another way for outage prob. computation and fading margine computation.

Ex:
Suppose use BPSK

$P_{TX} = 20\,dBm$, $f_c = 1\,GHz$, $G_{TX} = 0$ $G_{RX} = 6\,dB$. $L = 5\,dB$ $d = 200\,m$

$B = 20\,MHz$. $N_0 = -174\,dBm/Hz$, $d_{break} = 10\,m$, $n = 4$, $F = 5\,dB$

(a) Find $\overline{BER}$ under Rayleigh fading. $\overline{\gamma}_s = 13\,dB = 20$

(b) Suppose we want $BER \leq 10^{-2}$. find P_{out} $[Q(\sqrt{4.8}) \approx 0.01]$

a. $\overline{BER} = \frac{1}{2}\left[1 - \sqrt{\frac{\overline{\gamma}_s}{1+\overline{\gamma}_s}}\right] = \frac{1}{2}\left[1 - \sqrt{\frac{20}{21}}\right] = 0.012$

$\approx \frac{1}{4\overline{\gamma}_s} = \frac{1}{80} = 0.013$ ← 不是dB

b. P_{out} if $BER \leq 10^{-2}$

$BER = Q(\sqrt{2\gamma_s}) \leq 0.01 \Rightarrow 2\gamma_s \geq 4.8$, $\gamma_s \geq 2.4$. $\rightarrow \gamma_{min}^2 = 2.4$

<法1> $P_{out}(\gamma_{min}) = 1 - \exp\left(-\frac{\gamma_{min}^2}{2\sigma^2}\right) = 1 - \exp\left(-\frac{2.4}{20}\right) = 0.11$ (γ_s, $\overline{\gamma}_s$)

<法2> $Pr(\gamma_s \leq 2.4) = \int_0^{2.4} pdf_{\gamma_s}(\gamma_s)\,d\gamma_s = \int_0^{2.4} \frac{1}{\overline{\gamma}_s} \exp\left(-\frac{\gamma_s}{\overline{\gamma}_s}\right) d\gamma_s$

$= \int_0^{2.4} \frac{1}{20} \exp\left(-\frac{\gamma_s}{20}\right) d\gamma_s = 0.11$

EX:

Suppose $\overline{SNR} = 12\,dB$, consider DBPSK under Rician fading.

- $\overline{\gamma_s} = 12\,dB = 15.8$

<u>(a) if $k=0$, $\overline{BER} = ?$</u> Rayleigh

- $\overline{BER} = \dfrac{1+k}{2(1+k+\overline{\gamma_s})} \exp\left(\dfrac{-k\overline{\gamma_s}}{1+k+\overline{\gamma_s}}\right) = \dfrac{1}{2(1+\overline{\gamma_s})} = 0.03$

<u>(b) if $k=1$, $\overline{BER} = ?$</u>

- $\overline{BER} = \dfrac{2}{2(15.8+2)} \exp\left(-\dfrac{15.8}{15.8+2}\right) = 0.023$

<u>(c) if $k=10$, $\overline{BER} = ?$</u>

- $\overline{BER} = \dfrac{11}{2 \cdot 26.8} \exp\left(-\dfrac{15.8 \times 10}{26.8}\right) = 5.6 \times 10^{-4}$

<u>(d) if $k=100$, $\overline{BER} = ?$</u>

- $\overline{BER} = 5.8 \times 10^{-7}$

<u>(e) if we only have noise.</u>

- $\overline{BER} = BER(\gamma_s) = \frac{1}{2}\exp(-\overline{\gamma_s}) = 6.87 \times 10^{-8}$

<u>(g) if $k \to \infty$</u>

- $\overline{BER} = \dfrac{\cancel{1+k}^{\,\frac{1}{2}}}{\cancel{2}(1+k+\overline{\gamma_s})} \exp\left(-\dfrac{\cancel{k\overline{\gamma_s}}^{\,-\overline{\gamma_s}}}{\cancel{1}+k+\overline{\gamma_s}}\right) = \underline{\frac{1}{2}\exp(-\overline{\gamma_s})}$

> **Remark:**
>
> - $k = 0 \to$ Rayleigh
> - $k \nearrow$ $BER \searrow$
> - $k \to \infty \to$ noise only

Suppose $\overline{SNR} = 7dB = 5$

$$\gamma_B = \begin{cases} 7.5 & , \text{w.p. } 0.5 \\ 2.5 & , \text{w.p. } 0.5 \end{cases}$$

(a) What is the average SER when BPSK used ?

(b) Define required $SER = Q(\sqrt{10})$, what is the outage prob.

a.

$BER = SER = Q(\sqrt{2\gamma_B})$

$\overline{BER} = 0.5\left(Q(\sqrt{15}) + Q(\sqrt{5})\right)$ #

b.

Require $SER = Q(\sqrt{10})$

w.p $0.5 \rightarrow SER = Q(\sqrt{\underline{15}})$ ✓

$0.5 \rightarrow SER = Q(\sqrt{5})$

$\Rightarrow P_{out} = 0.5$