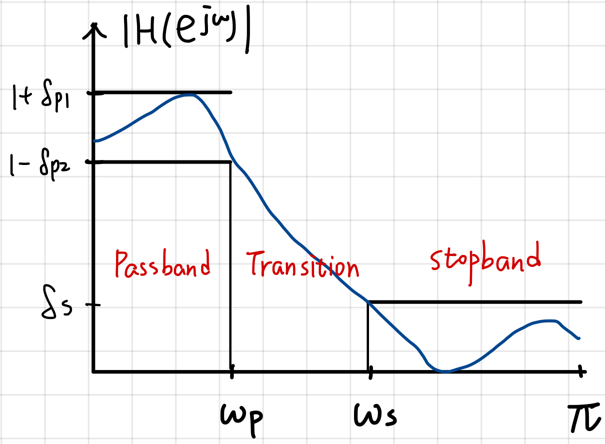
Digital Signal Processing

Ch7 Filter Design

2024 Fall

Filter Specification.



- passband edge ω_p
- stopband edge ω_s
- Passband ripple δ_{p1} δ_{p2}
- Stopband ripple δ_s

- passband : $1-\delta_{p2} \leq |H(e^{j\omega})| \leq 1+\delta_{p1}$
- stopband : $|H(e^{j\omega})| \leq \delta_s$
- transition bandwidth : $\omega_s - \omega_p$
- stopband attenuation : $A_s = -20\log_{10} \delta_s$

Outline

IIR Filter design

$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1-\sum_{k=1}^{N} a_k z^{-k}}$$

1.
   - Butterworth Filter
   - Chebyshev Filter
   - Elliptic Filter

2. Bilinear transformation. $\Omega = \frac{2}{T_d}\tan\left(\frac{\omega}{2}\right)$

3. Impulse invariance transformation $h_d[n] = T_d h_c(nT_d)$

FIR Filter design

$$H(z) = \sum_{n=0}^{N} h[n] z^{-n}$$

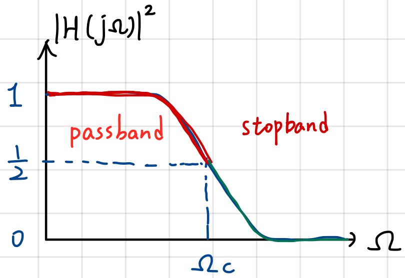
1. Window design : hamming window, Kaiser window
2. Optimum approximate : Parks-McClellan algorithm.

Butterworth Filter

parameter N

- $|H_c(j\Omega)|^2 = \dfrac{1}{1+(\Omega/\Omega_c)^{2N}} = H(s)\cdot H(-s) = \dfrac{1}{1+(s/j\Omega)^{2N}}$

1. $|H_c(j0)|^2 = 1$
2. $|H_c(j\infty)|^2 = 0$
3. $|H_c(j\Omega_c)|^2 = \frac{1}{2}$
4. $N\uparrow$, 中間陡



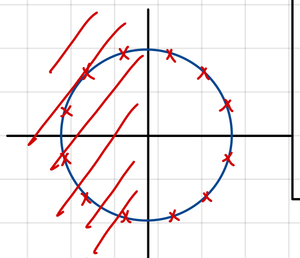
EX $N=6$,

- $H_c(s)\cdot H_c(-s) = \dfrac{1}{1+(s/j\Omega_c)^{12}}$
- pole : $(s/j\Omega_c)^{12} = -1 = e^{j(k2\pi+\pi)}$

$\rightarrow s/j\Omega_c = e^{j(k\frac{\pi}{6}+\frac{\pi}{12})}$

$\rightarrow s = \underline{\Omega_c}\, e^{j(\frac{\pi}{2}+k\frac{\pi}{6}+\frac{\pi}{12})}$

半徑為 Ω_c 的 unit circle.



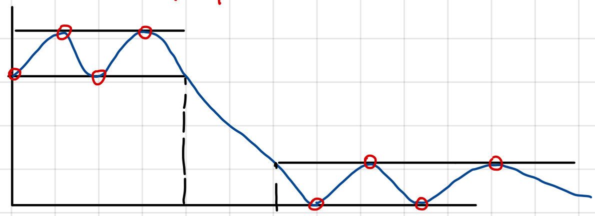
左半邊的 pole
構成 causal & stable 的 $H_c(s)$

Elliptic Filter

$|H_c(j\Omega)|^2 = \dfrac{1}{1+\epsilon^2 R_N^2(\Omega/\Omega_c)}$

- equiripple in passband and stopband

$N=4$



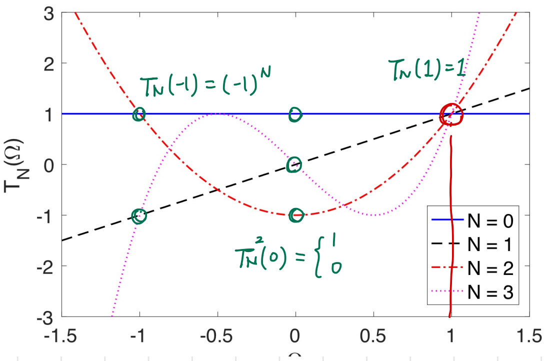
Chebyshev Filter.

parameter N, ϵ

- Chebyshev polynomial

$$T_N(\Omega) = \begin{cases} \cos(N\cos^{-1}\Omega), & |\Omega|\le 1 \\ \cosh(N\cosh^{-1}\Omega), & |\Omega|>1 \end{cases}$$

1. $T_N(\Omega)$ is even/odd function
2. $T_N(1) = 1$ if N is even/odd
3. $T_N(-1) = (-1)^N$
4. $T_N^2(0) = \begin{cases} 0 ; N \text{ is odd} \\ 1 ; N \text{ is even} \end{cases}$



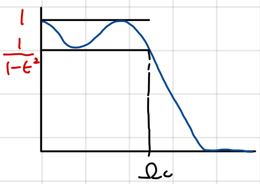
- Type I Chebyshev filter.

$|H_c(j\Omega)|^2 = \dfrac{1}{1+\epsilon^2 T_N^2(\Omega/\Omega_c)}$

- equiripple in passband.

if N odd

- $T_N^2(0) = 0$
- $|H(j0)|^2 = 1$



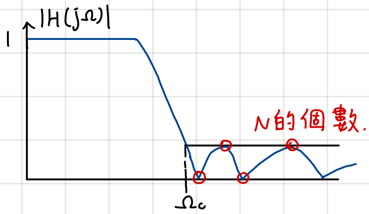
if N even

- $T_N^2(0) = 1$
- $|H(j0)|^2 = \frac{1}{1-\epsilon^2}$

- Type II Chebyshev filter.

$|H_c(j\Omega)|^2 = \dfrac{1}{1+1/\epsilon^2 T_N^2(\Omega/\Omega_c)}$

- equiripple in stopband.



Bilinear Transformation

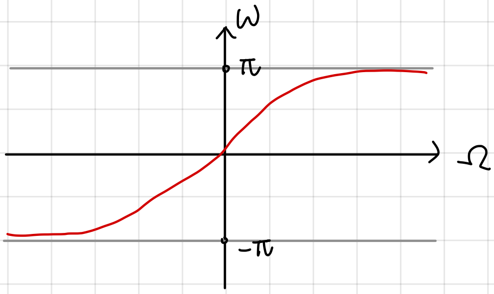
使用 $\Omega = \frac{2}{T_d}\tan(\frac{\omega}{2})$，使 $H(e^{j\omega}) = H_c(j\Omega)\Big|_{\Omega=\frac{2}{T_d}\tan(\frac{\omega}{2})}$
。振幅不變，橫軸尺度改變。

- $S = \frac{2}{T_d}\left[\frac{1-z^{-1}}{1+z^{-1}}\right]$ ← S和Z之間的關係

$$= \frac{2}{T_d}\left[\frac{1-e^{-j\omega}}{1+e^{-j\omega}}\right] = \frac{2}{T_d}\left[\frac{e^{j\omega/2}-e^{-j\omega/2}}{e^{j\omega/2}+e^{-j\omega/2}}\right] = \frac{2}{T_d}\left[\frac{2j\sin(\frac{\omega}{2})}{2\cos(\frac{\omega}{2})}\right] = \frac{2j}{T_d}\tan(\frac{\omega}{2})$$

- Thus

$$\begin{cases} \Omega = \frac{2}{T_d}\tan(\frac{\omega}{2}) \ ; \ -\infty < \Omega < \infty \\ \omega = 2\tan^{-1}(\frac{\Omega T_d}{2}) \ ; \ -\pi < \omega < \pi \end{cases}$$



Impulse Invariance Transformation.

- Given a continuous-time filter $h_c(t)$, then
  $h[n] = T_d h_c(nT_d)$

Ch4. Sampling $\omega = \Omega T_d$

- $\boxed{H(e^{j\omega}) = \sum H_c(j(\frac{\omega}{T_d} - k\frac{2\pi}{T_d}))}$

- Consider

$$\begin{cases} h_c(t) = \sum_{k=1}^{N} A_k e^{S_k t} \longleftrightarrow H_c(s) = \sum_{k=1}^{N} \frac{A_k}{S - S_k} \\ h_d[n] = T_d \sum_{k=1}^{N} A_k e^{S_k n T_d} \longleftrightarrow H(z) = \sum_{k=1}^{N} \frac{T_d A_k}{1 - e^{S_k T_d} \cdot z^{-1}} \end{cases}$$

1. Pole : $S_k \rightarrow e^{S_k T_d}$
2. if $Re\{S_k\} < 0$ then $|e^{S_k T_d}| < 1$

FIR Filter : Design by windowing.

◦ FIR Filter : $H(z) = \sum_{n=0}^{M} h[n] z^{-n}$ 有限個 h[0], h[1] ⋯ h[M] 組成的 H(z), 即為 FIR.

1. $h_d'[n] = \frac{\sin \omega_c n}{\pi n} \longleftrightarrow H_d'(e^{j\omega}) = \begin{cases} 1 ; & |\omega| < \omega_c \\ 0 ; & \text{else} \end{cases}$

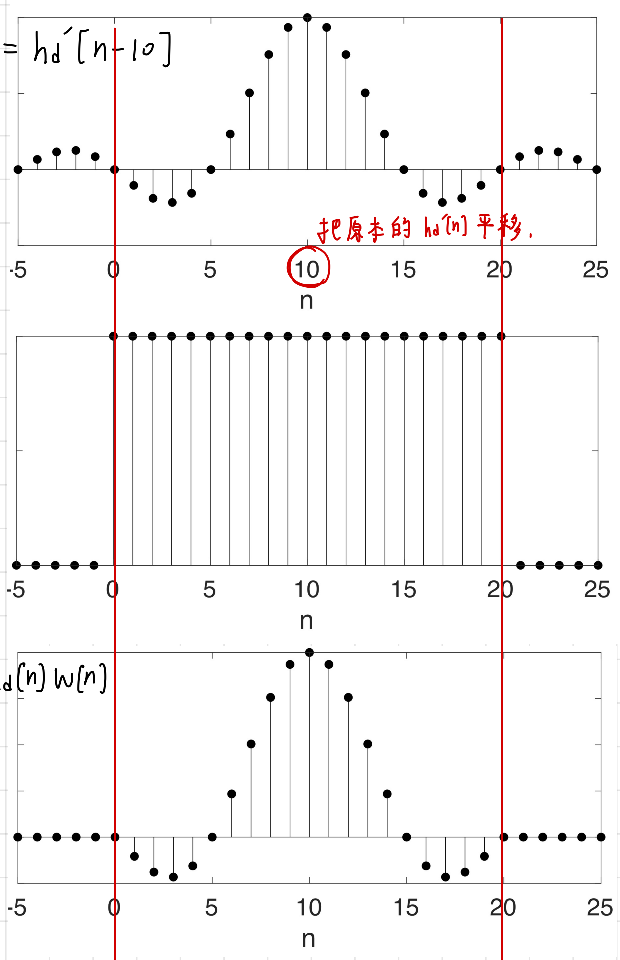
2. $h_d[n] = h_d'[n-L] = \frac{\sin\omega(n-L)}{\pi(n-L)}$ → 平移，才能在 windowing 之後為 causal

3. $h[n] = h_d[n]\,w[n]$; $\underline{w[n]} = \begin{cases} 1 ; & 0 < n < 2L \\ 0 ; & \text{else} \end{cases}$ 用這個 window w[n], 使 IIR $h_d[n]$ ⟶ FIR h[n]

◦ $h_d[n] = h_d'[n-10]$

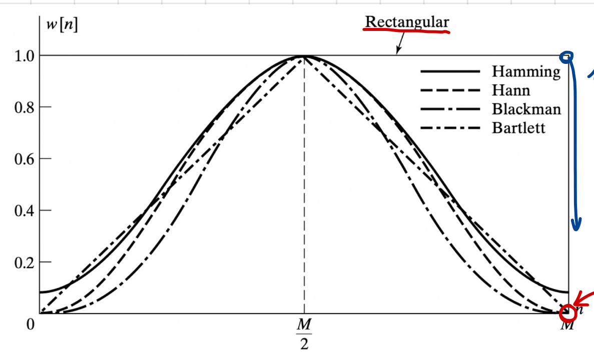
◦ w[n]

◦ $h[n] = h_d[n]\,w[n]$



變成有限的序列

Commonly used window



會有高頻變化出現在 M

以免出現高頻變化 因此切到 |w[n]| = 0

Generalized Linear Phase

- Any symmetric filter $h_d[n] = \pm h_d[M-n]$
- Any symmetric window $w[n] = w[M-n]$

$H(e^{j\omega})$ has linear phase iff. $h[n] = e^{j\theta} h^*[N-n]$
又 $w[n] = w[M-n]$，故 $W(e^{j\omega})$ 為 linear phase.

$\rightarrow h[n] = w[n]\, h_d[n] = \pm h[M-n]$ 。 real coefficient FIR linear phase filter 必定滿足此條件.

〈說明〉

- if $h_d[n] = h_d[M-n]$ (Type I or II)

$$H(e^{j\omega}) = \underbrace{A_e(e^{j\omega})}_{\text{real even}} e^{-j\omega \cdot \frac{M}{2}}$$

- if $h_d[n] = -h_d[M-n]$ (Type III or IV)

$$H(e^{j\omega}) = j\, \underbrace{A_o(e^{j\omega})}_{\text{real odd}} e^{-j\omega \frac{M}{2}}$$

- $W(e^{j\omega}) = W_e(e^{j\omega}) e^{-j\omega \frac{M}{2}}$

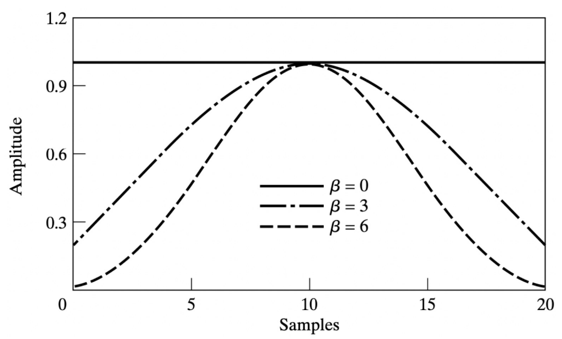
$$\Rightarrow H(e^{j\omega}) = \frac{1}{2\pi}\int H_d(e^{j\theta}) \cdot W(e^{j(\omega-\theta)})\, d\theta$$

$$= \underbrace{\frac{1}{2\pi}\int H_e(e^{j\theta})\, W_e(e^{j(\omega-\theta)})}_{A_e(e^{j\omega})} \cdot \cancel{e^{-j\theta\frac{M}{2}}} \cdot e^{-j(\omega \cancel{-\theta})\frac{M}{2}}\, d\theta$$

$$= A_e(e^{j\omega})\, e^{-j\omega\frac{M}{2}}$$ 亦為 linear phase.

Kaiser Window.

- $w[n] = \dfrac{I_0\left(\beta\sqrt{1-\left(\frac{n-M/2}{M/2}\right)^2}\right)}{I_0(\beta)}$



- We can use Kaiser window to approximate other window. e.g rectangular, Hamming window, and so on.

- Design procedure.

- step 0. Given discrete-time filter δ_p, δ_s, ω_p, ω_s
- step 1. $\omega_c = (\omega_p + \omega_s)/2$
- step 2. Determine β from $\begin{cases} \delta = \min(\delta_p, \delta_s) \\ A_s = -20\log_{10}\delta \end{cases}$

$$\beta = \begin{cases} 0.1102(A_s - 8.7), A_s > 50 \\ 0.5842(A_s - 21)^{0.4} + 0.07886(A_s - 21), 21 \le A_s \le 50 \\ 0, A_s < 21 \end{cases}$$

- Step 3. Estimate order M from $\Delta\omega = \omega_s - \omega_p$

$$M \approx \frac{A_s - 8}{2.285\Delta\omega}$$