Digital Signal Processing

Ch2 Signal system

2024 Fall

- **Discrete-time signal**

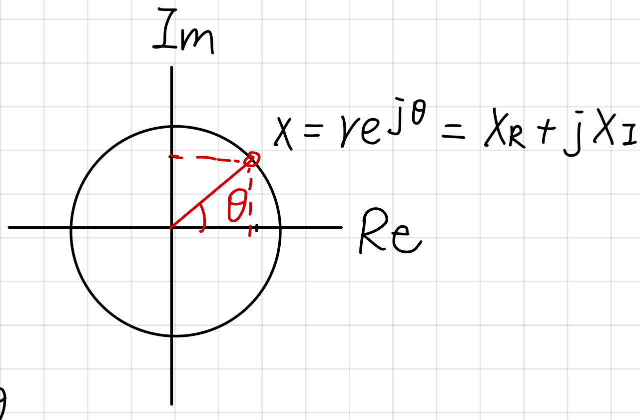
Time discrete, but amplitude continuous

- **Digital signal**

Both time and amplitude discrete

Polar form of complex number

- $X = X_R + jX_I$
- Polar form: $x = re^{j\theta}$
  - $r = \sqrt{X_R^2 + X_I^2}$
  - $\theta = \tan^{-1}\left(\frac{X_I}{X_R}\right)$
  - $X_R = r\cos\theta$, $X_I = r\sin\theta$



Is $x[n] = e^{j\omega_0 n}$ always periodic?

$x[n] = x[n+N]$

$e^{j\omega_0 n} = e^{j\omega_0 (n+N)}$

$\rightarrow \omega_0 N = 2\pi k$ for $k \in \mathbb{Z}$

$\rightarrow \omega_0 = 2\pi \frac{k}{N}$

Ans: No、當 ω_0 為 2π 的有理數倍時，才是 periodic

Properties of system

Memoryless

- A system is memoryless iff
$y[n]$ depend only on input $x[n]$ at same value of n

EX. $y(n)=5$ memoryless
, 因為過去或未來的輸入不影響現在的輸出

Causal

- A system is causal iff

$$\begin{cases} y[n_0] \text{ depend only on input } x[n],\ \text{for } n<n_0 \\ h(n)=0,\ \text{for } n<0 \end{cases}$$

EX. $y(n]=x[n+2]$: Non causal
$y(n]=5$: causal
$y(n]=x(n)\cdot\cos[w(n+3)]$: causal
因為未來的輸入不影響現在的輸出

Linear

- $T\{aX_1[n]+bX_2[n]\}=aT\{x_1(n]\}+bT\{y_1(n]\}$

Time invariant

- $T\{x(n-n_0]\} = y[n-n_0]$
(i) (ii)

Stable

- $|x[n]|\le B_x<\infty \rightarrow |y(n)|\le B_y<\infty$
- $\sum|h(k]|<\infty$

LTI system

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k] = \cdots\ x[-1]\,\delta[n+1] + x[0]\,\delta[n] + x[1]\,\delta[n-1]\ \cdots$$

$$T\{x[n]\}$$

$$= T\left\{\sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k]\right\} = \sum_{k=-\infty}^{\infty} x[k]\,T\{\delta[n-k]\}$$

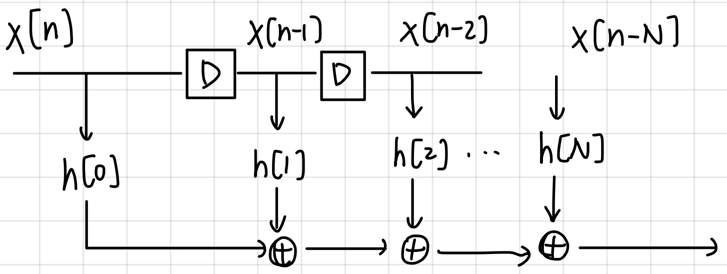
$$= \sum_{k=-\infty}^{\infty} x[k]\cdot h[n-k]$$

$$= x[n] * h[n]$$ ※

FIR system (Finite Impulse Response)

- impulse response $h[n]$ has finite duration

  i.e $h[n] \neq 0$ only for $0 < n < N$



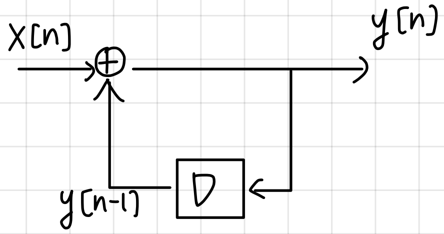
$$y[n] = x[n]h[0] + x[n-1]h[1] \cdots$$

$$= \sum_{k=0}^{N} h[k]\,x[n-k]$$

IIR system (Infinite Impulse Response)

- impulse response $h[n]$ has infinite duration
- Cost of IIR may smaller than FIR
- Assume $h[n] = u[n]$

$$\begin{cases} y[n] = \sum_{n=-\infty}^{\infty} x[n]\,u[n-k] = \sum_{k=-\infty}^{n} x[k] \\ y[n-1] = \sum_{k=-\infty}^{n-1} x[k] \end{cases}$$



$$\Rightarrow x[n] = y[n] - y[n-1]$$

DTFT

- $X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$
- $x[n] = \frac{1}{2\pi}\int_0^{2\pi} X(e^{j\omega})e^{j\omega n}d\omega$

DTFT Property

- Time shift: $\boxed{x[n-n_0] \longleftrightarrow X(e^{j\omega})e^{-j\omega n_0}}$

$$\sum x[n-n_0]e^{-j\omega n} = \sum x[n']e^{-j\omega(n'+n_0)} = X(e^{j\omega})e^{-j\omega n_0}$$

- Frequency shift: $\boxed{x[n]e^{j\omega_0 n} \longleftrightarrow X(e^{j(\omega-\omega_0)})}$

$$\sum x[n]e^{j\omega_0 n}e^{-j\omega n} = \sum x[n]e^{-j(\omega-\omega_0)n} = X(e^{j(\omega-\omega_0)})$$

- Time reversal: $\boxed{x[-n] \longleftrightarrow X(e^{-j\omega})}$

$$\sum x[-n]e^{-j\omega n} = \sum x[n']e^{j\omega n'} = X(e^{-j\omega})$$

$$\sum x^*[-n]e^{-j\omega n} = \left(\sum x[n']e^{-j\omega n'}\right)^* = X^*(e^{j\omega})$$

$$\Rightarrow x^*[-n] \leftrightarrow X^*(e^{j\omega})$$

- Parserval theorem: $\boxed{\sum |x[n]|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(e^{j\omega})| d\omega}$

let $y[n] = x[n] * x^*[-n] = \sum x[n-k]\,x^*[-k]$

$$\begin{cases} y[0] = \sum x[k]\,x^*[-k] = \sum |x[k]|^2 \\ y[0] = \frac{1}{2\pi}\int_0^{2\pi} Y(e^{j\omega})d\omega = \frac{1}{2\pi}\int_0^{2\pi} |X(e^{j\omega})|^2 d\omega \end{cases}$$

- Differentiation: $\boxed{n\,x[n] \longleftrightarrow j\frac{dX(e^{j\omega})}{d\omega}}$

$$j\frac{d}{d\omega}X(e^{j\omega}) = j\frac{d}{d\omega}\sum x[n]e^{-j\omega n} = j\sum -jn\,x[n]e^{-j\omega n}$$

Theorem 1.

if $x[n]$ is real, then

$\boxed{X(e^{j\omega}) = X^*(e^{-j\omega})}$ ← Hermitian symetric

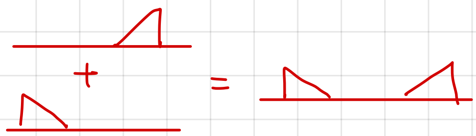
- $|X(e^{j\omega})| = |X^*(e^{-j\omega})|$: even

  $\measuredangle X(e^{j\omega}) = -\measuredangle X^*(e^{-j\omega})$. odd

- $X_R(e^{j\omega}) = X_R^*(e^{-j\omega})$: even

  $X_I(e^{j\omega}) = -X_I^*(e^{-j\omega})$: odd

Theorem 2 ☆☆☆☆☆



$$\begin{cases} x^*[-n] \longleftrightarrow X^*(e^{j\omega}) \\ x^*[n] \longleftrightarrow X^*(e^{-j\omega}) \end{cases}$$

$$\Rightarrow \mathrm{Re}\{x(n)\} = \frac{x[n]+x^*[n]}{2} \longleftrightarrow \frac{X(e^{j\omega})+X^*(e^{-j\omega})}{2} = X_e(e^{j\omega})$$

$$\Rightarrow \mathrm{Odd}\{x(n)\} = \frac{x[n]-x^*[n]}{2} \longleftrightarrow \frac{X(e^{j\omega})-X^*(e^{-j\omega})}{2} = X_o(e^{j\omega})$$