Digital Signal Processing

Ch5 LTI analysis

2024 Fall

$$\begin{cases} x[n]=e^{j\omega_0 n} \rightarrow y[n]=\boxed{H(e^{j\omega_0})}\,\boxed{e^{j\omega_0 n}} \\ x[n]=\alpha^n \rightarrow y[n]=\boxed{H(z)}\ \boxed{\alpha^n} \end{cases}$$

eigenvalue. eigenfunction.

Linear Constant Coefficient Difference Equation

- LTI system describe by:
  - $\sum_{k=0}^{N} a_k\, y[n-k] = \sum_{m=0}^{M} b_k\, x[n-m]$
  - finite $\begin{cases} \text{memory unit} \\ \text{multiplication} \\ \text{additions} \end{cases}$
- Applying Z-transform:

$$\begin{cases} \sum_{k=0}^{N} a_k z^{-k} Y(z) = \sum_{m=0}^{M} b_m z^{-k} X(z) \\ H(z) = \dfrac{\sum_{k=0}^{N} a_k z^{-k}}{\sum_{m=0}^{M} b_m z^{-m}} \end{cases}$$

Effect of Phase Response

$$\begin{cases} \text{Linear phase : "time delay"} \\ \text{Nonlinear phase : be understood via "group delay".} \end{cases}$$

- zero phase system.
  1. zero phase : $\measuredangle H(e^{j\omega})=0$
  2. $H(e^{j\omega}) = |H(e^{j\omega})|\,e^{j0} = |H(e^{j\omega})|$

  $\rightarrow Y(e^{j\omega}) = |H(e^{j\omega})|\,X(e^{j\omega})$

  EX: ideal lowpass
- Linear phase system
  - Linear phase : $\measuredangle H(e^{j\omega}) = \alpha\omega$
  - This is time delay system

    $y[n] = x[n-\alpha]$

Group Delay

- $\tau(\omega) = -\frac{d}{d\omega}\measuredangle H(e^{j\omega})$
- A measure of linearity of $\measuredangle H(e^{j\omega})$

Linear Phase System

$H(e^{j\omega}) = H_R(e^{j\omega})\cdot e^{-j(\alpha\omega+\beta)}$

$$\begin{cases} \measuredangle H(e^{j\omega}) = \alpha\omega+\beta \leftarrow \text{線性} \\ |H(e^{j\omega})| = H_R(e^{j\omega}) \end{cases}$$

Zero and Pole for Filter

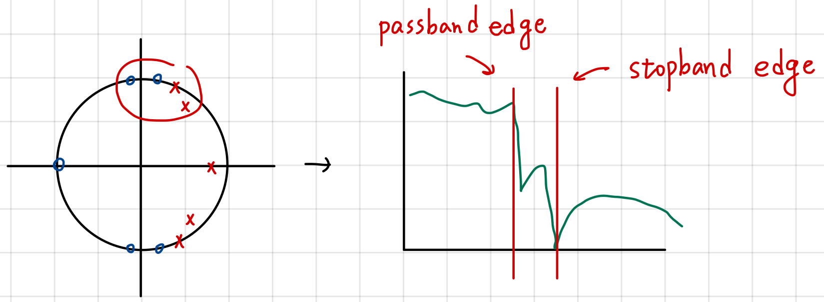
$\begin{cases} \text{Pole in the } \underline{\text{passband}} \\ \text{zero in the } \underline{\text{stopband}} \end{cases}$

← pole 讓 passband 的值較大

← zero 讓 stopband 的值較小

If transition band is narrow

$\begin{cases} \text{pole cluster around passband edge} \\ \text{zero cluster around stopband edge} \end{cases}$



Linear Phase System

$\tilde{X}(z) = X^*(1/z^*) = X^*(e^{j\omega})$

- $H(e^{j\omega}) = H_R(e^{j\omega}) \cdot e^{-j\underline{(\alpha\omega+\beta)}}$

$\begin{cases} \measuredangle H(e^{j\omega}) = \alpha\omega + \beta \\ |H(e^{j\omega})| = H_R(e^{j\omega}) \end{cases}$ ← 線性

- What would h[n] be like if it is linear phase.

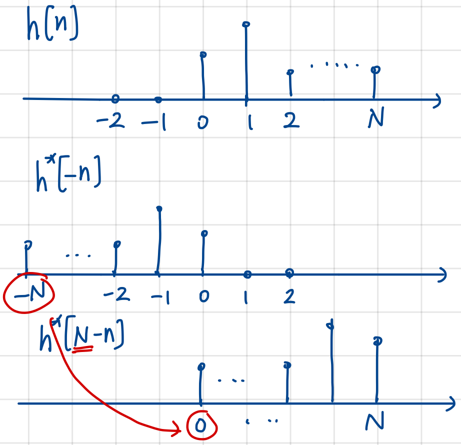
①

Tilde Notation

$\begin{cases} x[n] \longleftrightarrow X(e^{j\omega}) \\ x^*[-n] \longleftrightarrow X^*(e^{j\omega}) = \text{"}X^*(1/z^*)\text{"} \end{cases}$

- $\tilde{X}(z) \triangleq X^*(1/z^*) = \sum x^*[n]\left(\frac{1}{z}\right)^{-n}$
- 若 $H(z) = \sum_{n=0}^{N} h(n) z^{-n}$ is a causal FIR

則 $\underline{h^*[N-n]}$ 亦是 ⟶ 說明



整体往右平移N
故依然為 causal.

②

x(n) real.

$x^*[-n] = x[-n] \longleftrightarrow X^*(1/z^*) = \tilde{X}(z)$

- $\tilde{X}(z) = X^*(1/z^*) = X(1/z) = \sum_{n=-\infty}^{\infty} x[n](1/z)^{-n}$

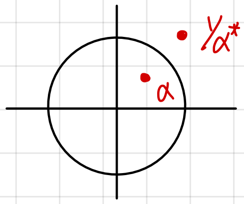
- If $H(z) = \sum_{n=0}^{N} h[n] z^{-n}$ is a causal FIR

則 $\underline{h(N-n)}$ 亦是.

Zero and Pole for Tilde Notation $\tilde{X}(z) \triangleq X^*(1/z^*)$

$x^*[-n]$ 的 z-transform

- $\alpha = re^{j\theta}$ and $1/\alpha^* = \frac{1}{r}e^{j\theta}$
- $X(\alpha) = 0$ iff $\tilde{X}(1/\alpha^*) = 0$
- zero of $X(z)$ and $\tilde{X}(z)$ are conjugate reciprocals.
- pole of $X(z)$ and $\tilde{X}(z)$ are conjugate reciprocals.



$\longrightarrow$ if α is on the unit circle, then $\alpha = 1/\alpha^*$

Properties of Tilde Notation.

- if $H(z) = F(z) + G(z)$, then $\tilde{H}(z) = \tilde{F}(z) + \tilde{G}(z)$
- if $H(z) = F(z)G(z)$, then $\tilde{H}(z) = \tilde{F}(z)\tilde{G}(z)$
- if $G(z) = H(z)\tilde{H}(z)$, then $g[n] = h[n] * h^*[-n]$ $G(e^{j\omega}) = H(e^{j\omega})H^*(e^{j\omega}) = |H(e^{j\omega})|$

為 zero-phase filter.

Rational Linear Phase System :

$H(e^{j\omega})$ has linear phase iff

$$h[n]=e^{j\theta}h^*[N-n] \longleftrightarrow H(z)=e^{j\theta}z^{-N}\tilde{H}(z)$$

<pf>

- $H(e^{j\omega})$ has linear phase $\longleftrightarrow$ $H(e^{j\omega}) = H_R(\omega)e^{-j\alpha\omega}\cdot e^{-j\beta}$
- This is equivalent to $\longrightarrow$ $H(e^{j\omega}) = H^*(e^{j\omega})e^{-j2\alpha\omega}\cdot e^{-j2\beta}$

$$\begin{cases} H(e^{j\omega}) = H_R(e^{j\omega})e^{j\phi(\omega)} \leftarrow \text{Any } H(e^{j\omega}) \text{ can be expressed as that} \\ H(e^{j\omega}) = H^*(e^{j\omega})\cdot e^{-j2\alpha\omega}\cdot e^{-j2\beta}\text{, then} \end{cases}$$

$$\begin{aligned} &H_R(e^{j\omega})e^{j\phi(\omega)} \\ &=\left(H_R(e^{j\omega})e^{j\phi(\omega)}\right)^* e^{-j2\alpha\omega}\cdot e^{-j2\beta} \\ &= H_R(e^{j\omega})\cdot e^{-j(2\alpha\omega+2\beta+\phi(\omega))} \end{aligned}$$

$\rightarrow \phi(\omega) = -2\alpha\omega-2\beta-\phi(\omega)$

$\rightarrow \phi(\omega) = -\alpha\omega-\beta$

$\Rightarrow H(e^{j\omega}) = H_R(e^{j\omega})\cdot e^{-j\alpha\omega}\cdot e^{-j\beta}$ 為 linear phase.

- Then $H(e^{j\omega}) = H^*(e^{j\omega})e^{-j2\alpha\omega}\cdot e^{-j2\beta}$

$$\begin{cases} h[n] = h^*[2\alpha-n]e^{-j2\beta} \\ H(z) = e^{-j2\beta}z^{-2\alpha}\cdot\tilde{H}(z) \end{cases}$$ ※

<觀念彙總>

① 若 $H(e^{j\omega})$ 為 linear phase $\longleftrightarrow$ $H(e^{j\omega}) = H_R(e^{j\omega})e^{-j\alpha\omega}\cdot e^{-j\beta}$

② $\begin{cases} H(e^{j\omega}) = H^*(e^{j\omega})e^{-j2\alpha\omega}\cdot e^{-j2\beta} \\ h[n] = h^*[\underbrace{2\alpha}_{N}-n]\cdot e^{-j2\beta} \end{cases}$ $\leftarrow$ linear phase system 的 impulse response

③ $H(z) = \tilde{H}(z)z^{-2\alpha}e^{-j2\beta}$

④ 結論. $H(e^{j\omega})$ has linear phase iff

$$h[n]=e^{j\theta}h^*[N-n] \longleftrightarrow H(z)=e^{j\theta}z^{-N}\tilde{H}(z)$$

FIR Linear phase Filter

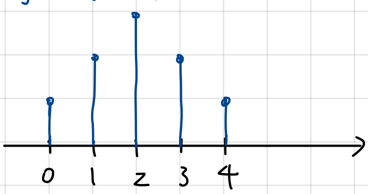
- $H(z) = \sum_{n=0}^{N} h[n] z^{-n}$
- $H(e^{j\omega})$ has linear phase, iff $h[n] = e^{j\theta} h^*[N-n] \longleftrightarrow H(z) = e^{j\theta} z^{-N} \tilde{H}(z)$

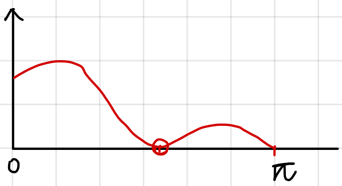
Real Coefficient FIR Linear Phase Filter

- $h[n] = e^{j\theta} h^*[N-n]$ 可簡化成 $\boxed{h[n] = \pm h[N-n]}$
- real-coefficient FIR filter $h[n]$ has linear phase
  $\longleftrightarrow h[n] = \pm h[N-n]$

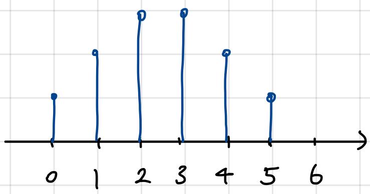
	$h[n] = h[N-n]$	$h[n] = -h[N-n]$
Even N	Type I	Type III
Odd N	Type II	Type IV

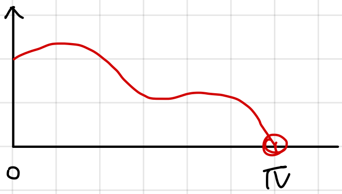
<Type I> $h[n] = h[N-n]$ even N No predetermined zero



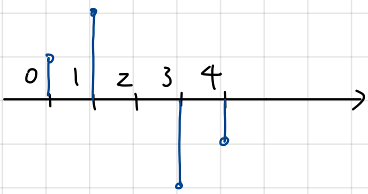


<Type II> $h[n] = h[N-n]$ odd N $H(e^{j\pi}) = 0 \leftarrow$ NOT for highpass

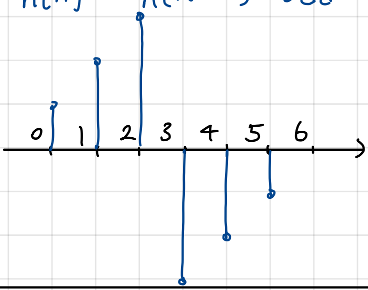


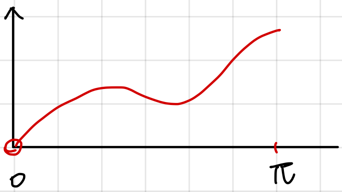


<Type III> $h[n] = -h[N-n]$ even N $H(e^{j0}) = H(e^{j\pi}) = 0 \leftarrow$ NOT for highpass and lowpass



<Type IV> $h[n] = -h[N-n]$ odd N $H(e^{j0}) = 0 \leftarrow$ NOT for lowpass





Zero of FIR Linear Phase Filter

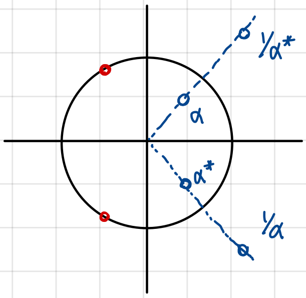
α和$1/\alpha^*$

- The zero of $H(z)$ are "conjugate reciprocal pair"

  pf: $H(z) = e^{j\theta} z^{-N} \tilde{H}(z) \rightarrow H(\alpha) = e^{j\theta} \alpha^{-N} H^*(1/\alpha^*) = 0$

- The pole of $H(z)$ can only be 0 or ∞

Zero of Real Coefficient FIR Linear Phase Filter



Interconnection of Linear Phase System.

- Cascade: If $H_1(z)$ and $H_2(z)$ are two linear phase system, then $H(z) = H_1(z) \cdot H_2(z)$ also
- Parallel: If $H_1(z)$ and $H_2(z)$ are two linear phase system, then $H(z) = H_1(z) + H_2(z)$ in general are NOT.

Rational IIR Linear phase Filter.

- $H(z)$ can not be "causal stable"
- Since $H(z) = e^{j\theta} z^{-N} \tilde{H}(z)$, $H(\alpha) = e^{j\theta} \alpha^{-N} H^*(1/\alpha^*)$

  $\rightarrow |H(\alpha)| = \infty \longleftrightarrow |H^*(1/\alpha^*)| = \infty$

  $\rightarrow H(z)$ must have a finite pole outside the unit circle.
- In practice, we are interested only in "causal stable" system

  $\rightarrow$ linear phase property is consider only for FIR filter.

Allpass Filter

Definition

- $H(z)$ is said to be allpass if $|H(e^{j\omega})| = c$, $\forall \omega$
- $H(e^{j\omega}) = c \cdot e^{j\phi(\omega)}$

EX1 : $H(z) = 1$

EX2 : $H(z) = e^{j\phi}$

EX3 : $H(z) = z^{-k}$

EX4 : $H(z) = \dfrac{-\alpha^* + z^{-1}}{1 - \alpha z^{-1}}$; zero $= 1/\alpha^*$, pole $= \alpha$

$H(z) = \beta \prod_{k=1}^{N} \dfrac{-\alpha_k^* + z^{-1}}{1 - \alpha_k z^{-1}}$ 為 allpass 的 general form.

$$H(e^{j\omega}) = \frac{-\alpha^* + e^{-j\omega}}{1 - \alpha e^{-j\omega}} = e^{-j\omega} \frac{1 - e^{j\omega}\alpha^*}{1 - e^{-j\omega}\alpha} = e^{-j\omega} \frac{(1 - \alpha e^{-j\omega})^*}{1 - \alpha e^{-j\omega}}$$

$\rightarrow |H(e^{j\omega})| = 1$

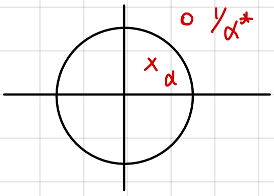
Zero and Pole of Allpass filter

- A rational $H(z)$ is allpass $\longleftrightarrow$ $H(z)\tilde{H}(z) = c^2$
- The zero and pole are conjugate reciprocal pair

$\rightarrow$ if $H(z)$ causal stable , all pole inside the unit circle and all zero outside

$\rightarrow H(z)\tilde{H}(z) = c^2$

$$H(\alpha) = 0 \Rightarrow H^*(1/z^*) = \tilde{H}(\alpha) = \frac{c^2}{H(\alpha)} = \infty$$



General form of Allpass filter

- $H(z) = \beta \prod_{k=1}^{N} \dfrac{-\alpha_k^* + z^{-1}}{1 - \alpha_k z^{-1}}$

$$\begin{cases} \text{zero}: \dfrac{1}{\alpha_1^*}, \dfrac{1}{\alpha_2^*} \cdots \dfrac{1}{\alpha_N^*} \\ \text{pole}: \alpha_1, \alpha_2 \cdots \alpha_N \end{cases}$$

互為 conjugate reciprocal pair

- Energy Blance property

→ $E_y = c^2 E_x$

→ The energy amplification c^2 is independent to input $x[n]$

- Monotone Phase property

外側 含unit circle zero/pole 為 conjugate reciprocal pair

- For a causal stable allpass filter $H(z)$, all pole are inside the unit circle.
- The phase $\phi(\omega)$ are monotone decreasing. (單調遞減)

- Replace a zero by its Conjugate Reciprocal

- Let $H(z)$ with zero at b : $H(z) = (1 - bz^{-1})\, G(z)$
- If replace the zero b by $1/b^*$: $H_1(z) = (b^* - z^{-1})\, G(z)$

→ They have same magnitude response $H_1(z) = H(z) \underbrace{\dfrac{(b^* - z^{-1})}{1 - bz^{-1}}}_{\text{unit gain allpass.}}$

- Minimum Phase property 作業4.

- All zero and pole are inside the unit circle

→ $H(z)$ and $1/H(z)$ are both "causal stable"

- Any rational stable causal system $H(z)$ can be decompose as

$$\boxed{H(z) = H_{ap}(z) \cdot H_{min}(z)}$$

- $H_{ap}(z)$: allpass system.
- $H_{min}(z)$: minimum phase system.

- A minimum phase system $H_{min}(z)$ has

1. Minimum group delay property : $\boxed{grd[H(e^{j\omega})] \geq grd[H_{min}(e^{j\omega})]}$
2. Minimum Phase-lag property : $\boxed{-arg[H(e^{j\omega})] \geq -arg[H_{min}(e^{j\omega})]}$
3. Minimum Energy Delay property : $\boxed{|h[0]| \leq |h_{min}[0]|}$