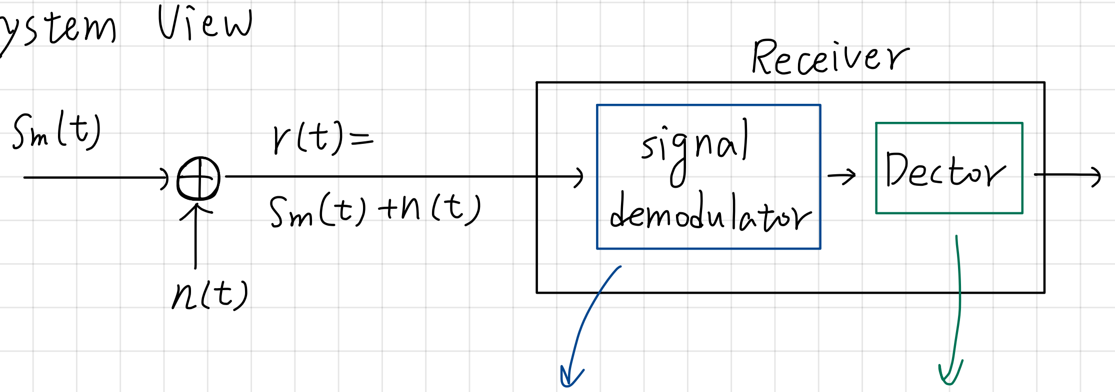
Digital Communication

Ch4: Optimum Receivers for AWGN Channel

2024 Fall

4-1 Waveform and vector channel model.

System View



- Signal demodulator:

  Vectorization receive signal $r(t)$

  $r(t) \longrightarrow [r_1\ r_2 \cdots r_N]^T$

- Dector:

  minimize the probability of error

- AWGN

$$\begin{cases} S_N(f) = \frac{N_0}{2} \\ R_N(\tau) = \frac{N_0}{2}\delta(\tau) \end{cases}$$

- $n_i = \langle n(t), e_i(t) \rangle$
  - $E[n_i] = 0$
  - $E[n_i n_j] = \frac{N_0}{2}\delta(i-j)$

- So, we have

$$n(t) = \sum_{i=1}^{N} n_i e_i(t) + \tilde{n}(t)$$

→ 因為雜訊可能無法用原本 N 個基底表示

→ $\tilde{n}(t)$ is independent of $\sum^{N} n_i e_i(t)$

→ NOT affect the detection of $s_m(t)$

- Rewrite the waveform channel to a discrete channel

$$r(t) = s_m(t) + n(t) \longrightarrow r = s_m + n$$

- $n \sim N(0, \frac{N_0}{2} I_N)$
- $f_N(n) = \left(\frac{1}{\sqrt{\pi N_0}}\right)^N \exp\left(-\frac{\|n\|^2}{N_0}\right)$

Optimal Decision

- MAP

事後機率 "a posterori probability"

$$g_{opt}(r) = \arg\max_{m} Pr\{ S_m \text{ sent} \mid r \text{ receive} \}$$

已知接收到 r 的情況(事後機率)
看哪個 m 能使機率最大 i.e. 最有可能的發射信號

$$Pr(S_m|r) = \frac{f(r|S_m)\cdot P(S_m)}{f(r)} \Rightarrow f(r|S_m)\,P(S_m)$$

$$g_{MAP}(r) = \arg\max_{m} Pr\{S_m\} f(r|S_m)$$ MAP

- Bit level decision (BER)

只關心 b_i 是否等於 ℓ

$$g_{MAP,i}(r) = \arg\max_{\ell} \left(\sum_{b_i=\ell} Pr(S_m|r) \right)$$ MAP, i

	b_1	b_2	b_3
S_1	0	0	0
S_2	0	0	1
S_3	0	1	1
S_4	0	1	0
S_5	1	0	0
S_6	1	0	1
S_7	1	1	0
S_8	1	1	1

已知傳送 $S_1 = 000$. 但現在只在乎 $b_2 = 0$

故 $g_{MAP,i}$

$$= \arg\max_{\ell} \sum_{b_i=\ell} Pr(S_m)\, f(r|S_m)$$

$$= Pr(S_1)\, f(r|S_1) + Pr(S_2) f(r|S_2) + Pr(S_5)\, f(r|S_5) + Pr(S_6)\, f(r|S_6)$$

Theorem:

$$P_b \le P_e \le k \cdot P_b$$

average BER　　symbol error

Sufficient Statistic:

Given S_m transmitted, received $r=(r_1,r_2)$ with $S_m \to r_1 \to r_2$

$$f(r|S_m) = \underbrace{f(r_1,r_2|S_m)}_{P(A,B)=P(A)\cdot P(A|B)} = f(r_1|S_m)\cdot f(r_2|\cancel{S_m}, r_1) = f(r_1|S_m)f(r_2|r_1)$$

r_2和S_m無關

$g_{opt}(r)$

$$= \arg\max_m P_r(S_m|r) = \arg\max_m P_r(S_m)\cdot f(r|S_m)$$

$$= \arg\max_m P_r(S_m) f(r_1|S_m)\, \cancel{f(r_2|r_1)}$$

和m無關

$$= \arg\max_m P_r(S_m) f(r_1|S_m) \quad \#$$

opt和r_2無關

→ r_1 is called "sufficient statistics"

→ r_2 is called "irrelevant data",
since optimal decision can made without r_2

- ML: if S_m equally likely.

$$\boxed{g_{ML}(r) = \arg\max f(r|S_m)}$$

4-2 Optimal Detection for vector AWGN Channel.

$$g_{MAP}(r)$$

$$= \arg\max_m P_m \cdot f(r|s_m)$$

$$= \arg\max_m \left\{ P_m \cdot \left(\frac{1}{\sqrt{\pi N_0}}\right)^N \exp\left(-\frac{\|r-s_m\|^2}{N_0}\right) \right\}$$

$$= \arg\max_m \left\{ \ln P_m - \frac{\|r-s_m\|^2}{N_0} \right\}$$

$$= \arg\max_m \left\{ \frac{N_0}{2} \ln P_m - \frac{1}{2}\|r-s_m\|^2 \right\}$$

$$= \arg\max_m \left\{ \frac{N_0}{2} \ln P_m - \frac{1}{2} E_m + r^T s_m \right\}$$

MD decision rule $P_m = \frac{1}{M}$

$$\hat{m} = \arg\max_m \left\{ \cancel{\frac{N_0}{2} \ln P_m} - \frac{1}{2}\|r-s_m\|^2 \right\}$$

$$= \arg\min_m \|r-s_m\|^2$$

Correlation rule $P_m = \frac{1}{M}$ 且 $E_m = E$

$$\hat{m} = \arg\max_m \left\{ \cancel{\frac{N_0}{2} \ln P_m} - \cancel{\frac{1}{2} E_m} + r^T s_m \right\}$$

$$= \arg\max_m r^T s_m$$

There are two factor affect the error probability

- Euclidean distance：信號之間距離越大，錯誤率越小
- Position of signal vector：周圍越多信號，錯誤率高

Binary Equal-Probable Signal in General.

- Consider $Pr\{x_1(t)\} = Pr\{x_2(t)\} = \frac{1}{2}$
- $x_1(t)$ and $x_2(t)$ are linearly indenpendent $\rightarrow$ 2 orthogonal basis

— $\boxed{P_e = Q\left(\sqrt{\frac{d_{12}^2}{2N_0}}\right)}$ 用於所有的高斯環境

BPSK vs BFSK

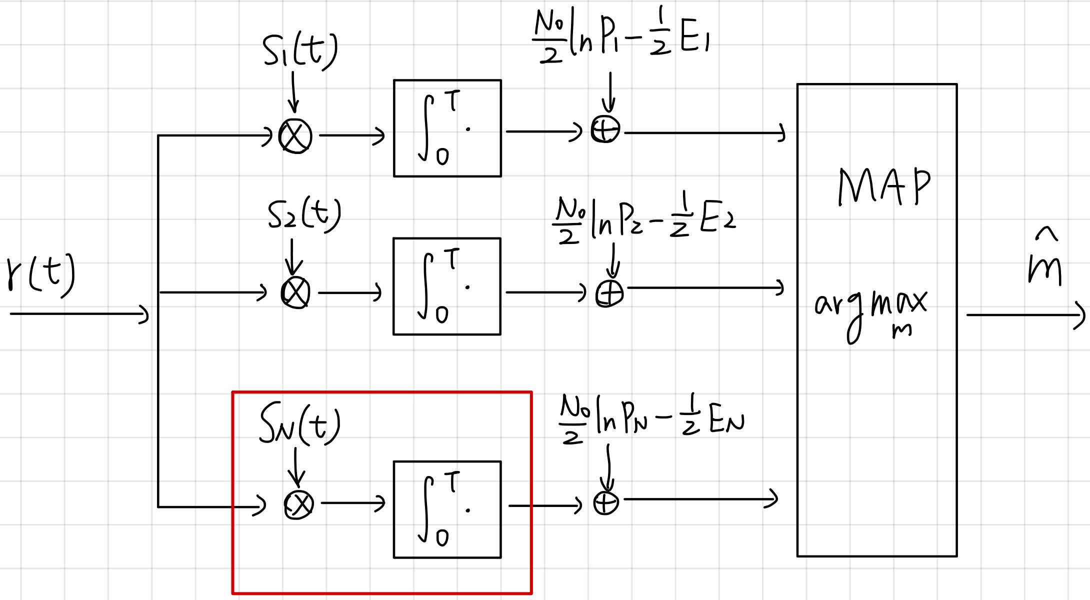
$Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$ $\quad$ $Q\left(\sqrt{\frac{E_b}{N_0}}\right)$

BPSK is 3dB better than BFSK

Correlation receiver

∘ Optimal decision rule :

$$g_{opt}(t) = \arg\max_m \left\{ \frac{N_0}{2}\ln P_m - \frac{1}{2}E_m + r^T s_m \right\} \qquad \int_0^T r(t)\, s_m(t)\, dt$$

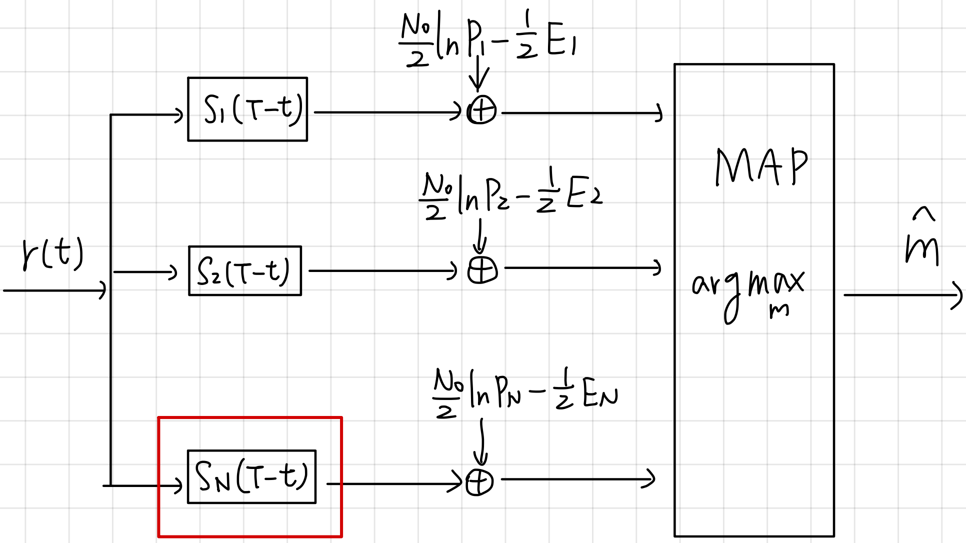


Matched filter Receiver

- 用信號 $s_m(t)$ 本身的波型來 matched 接收訊號 r.
- 目的是讓 Filter 的 SNR_{out} 最大

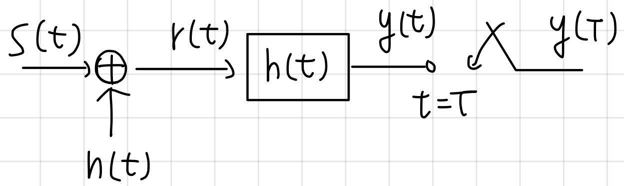
∘ matched filter : $h_m(t) = s_m(T-t)$

$$\longrightarrow r(t) * h_m(t)\Big|_{t=T} = \int_{-\infty}^{\infty} r(\tau)\, h_m(T-\tau)\, d\tau = \int_0^T r(t)\, s_m(t)\, dt$$



Matched filter 證明

訊號模型



- $r(t) = S(t) + n(t)$
- $y(t) = r(t) * h(t) = \underline{S(t) * h(t)} + \underline{n(t) * h(t)}$

- 訊號：

$$S(t) * h(t)\Big|_{t=T} = \mathcal{F}^{-1}\{H(f)\,S(f)\}\Big|_{t=T} = \int_{-\infty}^{\infty} H(f)\,S(f)\,e^{j2\pi f t}\,df\Big|_{t=T}$$

- 雜訊：

$$E[n^2(t)] = \int_{-\infty}^{\infty} S_N(f)\,|H(f)|^2\,df = \frac{N_0}{2}\int_{-\infty}^{\infty} |H(f)|^2\,df$$

- $$SNR_{out} = \frac{\left|\int_{-\infty}^{\infty} H(f)\,S(f)\,e^{j2\pi f T}\,df\right|^2}{\frac{N_0}{2}\int_{-\infty}^{\infty}|H(f)|^2\,df}$$

$$\leq \frac{\cancel{\int_{-\infty}^{\infty}|H(f)|^2 df} \cdot \int_{-\infty}^{\infty}|S(f)\,e^{j2\pi f T}|^2\,df}{\frac{N_0}{2}\cancel{\int_{-\infty}^{\infty}|H(f)|\,df}}$$

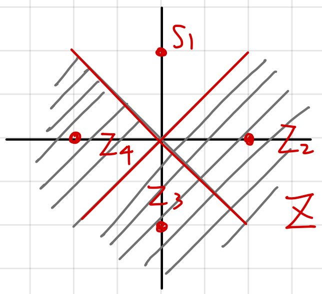
$$= \underline{\frac{2}{N_0}\int_{-\infty}^{\infty}|S(f)|^2\,df} \leftarrow \text{最佳 SNR}$$

- Cauchy-schwartz：

when $\underline{H(f) = \alpha S^*(f)\,e^{-j2\pi f T}}$ inequality hold with equal.

$\rightarrow \boxed{h(t) = \alpha S^*(T-t)}$

Union bound.



$$P_e(m_1) = P(m' \mid m_1)$$

$$= P(r \in Z \mid m_1)$$

$$= P(r \in Z_2 \cup Z_3 \cup Z_4 \mid m_1)$$

union bound

$$\leq \underline{P(r \in Z_2 \mid m_1) + P(r \in Z_3 \mid m_1) + P(r \in Z_4 \mid m_1)}$$

- $P(r \in Z_k \mid m_i) = Q\left(\sqrt{\dfrac{d_{ik}^2}{2N_0}}\right)$

$$\Rightarrow P_e(m_i) \leq \sum_{\substack{k=1 \\ k \neq i}}^{N} P(r \in Z_k \mid m_i) = \sum_{k=1\, k\neq i}^{N} Q\left(\sqrt{\frac{d_{ik}^2}{2N_0}}\right)$$

$$\Rightarrow P_e = \sum_{i=1}^{N} P(m_i) \cdot P_e(m_i) \leq \boxed{\sum_{i=1}^{N} P(m_i) \sum_{k=1\, k\neq i}^{N} Q\left(\sqrt{\frac{d_{ik}^2}{2N_0}}\right)}$$

general form

4種不同的 union bound

〈討論 1〉 $P(m_i) = \frac{1}{M}$

$$\boxed{P_e \leq \frac{1}{M} \sum_{i=1}^{N} \sum_{\substack{k=1 \\ k\neq i}}^{N} Q\left(\sqrt{\frac{d_{ik}^2}{2N_0}}\right)}$$

〈討論2〉 $Q(x) \leq \frac{1}{2} e^{-\frac{x^2}{2}}$

$$\boxed{P_e \leq \frac{1}{2M} \sum_{i=1}^{N} \sum_{k\neq i}^{N} \exp\left(-\frac{d_{ik}^2}{4N_0}\right)}$$

〈討論 3〉 $d_{min} = \min \| S_i - S_{i'} \|^2$

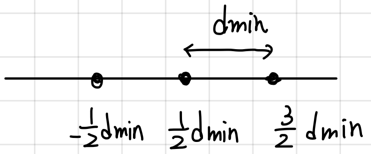
$$\begin{cases} Q\left(\sqrt{\dfrac{d_{ik}^2}{2N_0}}\right) \leq Q\left(\sqrt{\dfrac{d_{min}^2}{2N_0}}\right) \\ \exp\left(-\dfrac{d_{ik}^2}{4N_0}\right) \leq \exp\left(-\dfrac{d_{min}^2}{4N_0}\right) \end{cases}$$

$$\Rightarrow P_e \leq \boxed{(M-1)\, Q\left(\sqrt{\frac{d_{min}^2}{2N_0}}\right)} \leq \boxed{\frac{M-1}{2} \exp\left(-\frac{d_{min}^2}{4N_0}\right)}$$

4-3 Optimal Detection and Error Probability for bandlimited Signaling

$\boxed{\text{Consider M-PAM}}$

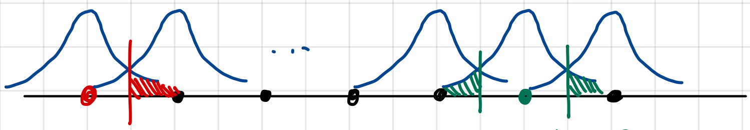
- $S = \{ \pm\frac{1}{2}d_{min} \pm \frac{3}{2}d_{min} \cdots \}$



- average bit signal energy

$$\boxed{E_b = \frac{M^2-1}{12\log_2 M} d_{min}^2}$$

- Error event
  - Inner point : $P_{e,i} = P_r(|n| > \frac{d_{min}}{2}) = 2Q(\frac{d_{min}}{\sqrt{2N_0}})$
  - Outer point : $P_{e,o} = P_r(n > \frac{d_{min}}{2}) = Q(\frac{d_{min}}{\sqrt{2N_0}})$



$Q(\sqrt{\frac{d_{min}^2}{2N_0}})$ $\quad$ $2Q(\sqrt{\frac{d_{min}^2}{2N_0}})$

- The symbol error probability : Pe

$$P_e = \frac{1}{M}\sum_{m=1}^{M} P_r(\text{error} \mid m \text{ sent})$$

$$= \frac{1}{M}\left[(M-2)\cdot 2Q\left(\sqrt{\frac{d_{min}^2}{2N_0}}\right) + 2\cdot Q\left(\sqrt{\frac{d_{min}^2}{2N_0}}\right)\right]$$

$$= \frac{2(M-1)}{M} Q\left(\sqrt{\frac{d_{min}^2}{2N_0}}\right)$$ ← 在M-QAM會用到

$$= \frac{2(M-1)}{M} Q\left(\sqrt{\frac{6\log_2 M}{(M^2-1)}\cdot\frac{E_b}{N_0}}\right)$$

- Efficiency

For large M
- To increase rate by 1 bit
- To keep same Pe.

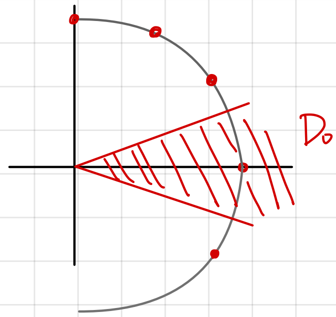
⇒ Increase Eb by 6dB (4倍)

M	2	4	8	16
$\frac{6\log_2 M}{(M^2-1)}$	2	$\frac{4}{5}$	$\frac{2}{7}$	$\frac{8}{85}$

2.5 $\quad$ 2.8 $\quad$ 3 $\cdots$ 4

較小的M提升bit時,
所需的能量放大倍數較少

Consider M-PSK



- $S = \{ S_m = \sqrt{E}\left(\cos\left(\frac{2\pi m}{M}\right), \sin\left(\frac{2\pi m}{M}\right)\right) \}$
  - assume $S_0 = (\sqrt{E}, 0)$ was transmited
  - received $r = (r_1, r_2) = (\sqrt{E} + n_1, n_2)$

- $f(n_1 n_2) = \frac{1}{\pi N_0} \exp\left(-\frac{n_1^2 + n_2^2}{N_0}\right)$
- $f(r = (r_1, r_2) | S_0) = \frac{1}{\pi N_0} \exp\left(-\frac{(r_1 - \sqrt{E})^2 + r_2^2}{N_0}\right)$

$\downarrow$ 令 $V = \sqrt{r_1^2 + r_2^2}$, $\theta = \tan^{-1}\left(\frac{r_2}{r_1}\right)$

- $f(V, \theta | S_0) = \frac{V}{\pi N_0} \exp\left(-\frac{V^2 + E - 2\sqrt{E} V \cos\theta}{N_0}\right)$

Efficient:

- $P_e = 1 - \int_{-\frac{\pi}{M}}^{\frac{\pi}{M}} \int_0^{\infty} f(V, \theta | S_0)\, dV\, d\theta \Rightarrow$ Have NO close form
- for large M, we can approximate P_e as

$$P_e \approx 2Q\left(\sqrt{\frac{2\pi^2 \log_2 M}{M^2} \cdot \frac{E_b}{N_0}}\right)$$

- To increase rate by 1 bit
- To keep same P_e

$\Longrightarrow$ Increase E_b by 6dB

有 close form 的特例：

- When $M=2$, BPSK is antipodal $P_e = Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$

$1 - (1 - P_{e.i})(1 - P_{e.q})$

- When $M=4$, QPSK is $P_e = 1 - \left[1 - Q\left(\sqrt{\frac{2E_b}{N_0}}\right)\right]^2$

Consider M-ary QAM

$E[|n_{\ell}|^2] = 2N_0$

$E_{\ell m} = 2E_m$

- 假設為方形 QAM : $M = N^2$
- $S_{PAM} = \{\pm\frac{1}{2}d_{min}, \pm\frac{3}{2}d_{min}\cdots\}$
- Average bit signal Energy for M-QAM

$$\boxed{E_b = \frac{M-1}{6\log_2 M} d_{min}^2}$$

$$S_{M\text{-}QAM} = S_{N\text{-}PAM} \times S_{N\text{-}PAM}$$

- $P_{e,M\text{-}QAM} = 1-[1-P_{e,PAM}]^2 = 2P_{e,PAM} - \underbrace{\cancel{P_{e,PAM}^2}}_{}$ 可忽略 $\approx 2P_{e,PAM}$
- $P_{e,PAM} = 2(1-\cancel{\frac{1}{M}})Q\left(\frac{d_{min}}{\sqrt{2N_0}}\right) \approx 2Q\left(\frac{d_{min}}{\sqrt{2N_0}}\right)$ ← P10 (large M)

$$\Rightarrow P_{e,M\text{-}QAM} \lesssim 4Q\left(\frac{d_{min}}{\sqrt{2N_0}}\right) = \boxed{4Q\left(\sqrt{\frac{3\log_2 M}{M-1}\cdot\frac{E_b}{N_0}}\right)}$$

Efficient :

$$P_{e,M\text{-}QAM} \lesssim \boxed{4Q\left(\sqrt{\frac{3\log_2 M}{M-1}\cdot\frac{E_b}{N_0}}\right)}$$

M	4	16	64
$\frac{\log_2 M}{M-1}$	$\frac{2}{3}$	$\frac{3}{15}$	$\frac{2}{21}$

$\Rightarrow$

$4\to16$	$16\to64$	$\cdots$	$M\to4M$
2.5	2.8	$\cdots$	4

- To increase 2 bit, $M \to 4M$
- To keep same P_e

$\Rightarrow$ Increase k by 1 bit, increase E_b by 3dB

$\Rightarrow$ QAM is more power efficient than PAM and PSK

4.3-4 Demodulation and dection.

- For bandpass signal,
- We need two orthogonal basis $\begin{cases} \phi_1(t) = \sqrt{\frac{2}{T}}\, g(t)\cos(2\pi f_c t) \\ \phi_2(t) = \sqrt{\frac{2}{T}}\, g(t)\sin(2\pi f_c t) \end{cases}$

NOTE:

- $\vec{S} = \pm\sqrt{E} + \frac{N_0}{2}$ ← 射頻
- $\vec{S} = \pm\sqrt{2E} + 2N_0$ ← 基頻

因為 $s(t) = \mathrm{Re}\{ S_\ell(t)\, e^{j2\pi f_c t} \}$

PAM signal

Transmission of PAM signal (bandpass)

- $S_{PAM} = \left\{ \pm\frac{1}{2}d_{min} \quad \pm\frac{3}{2}d_{min} \;\cdots\; \frac{M-1}{2}\, d_{min} \right\}$ ← 向量化

  where $d_{min} = \sqrt{\frac{12\log_2 M}{M^2-1} E_{avg}}$

- $S_{PAM}(t) = \left\{ \pm\frac{1}{2}d_{min}\,\phi_1(t),\ \pm\frac{3}{2}d_{min}\,\phi_1(t) \cdots \right\}$

Define the set of baseband

- $S_{PAM.\ell}(t) = S_{m.\ell}\cdot\phi_{1.\ell}(t)$ where $\begin{cases} \phi_{1.\ell}(t) = \sqrt{\frac{1}{E_g}}\, g(t) \\ S_{m.\ell} = \sqrt{2}\, S_m \end{cases}$
- $S_{PAM}(t) = \mathrm{Re}\{ S_{PAM.\ell}(t)\, e^{j2\pi f_c t} \}$

baseband 和 bandpass
的定義不同

Demodulation & detection

- $r(t) = S_m(t) + n(t) \rightarrow r = \langle r(t), \phi_1(t) \rangle$
- The bandpass MAP rule:

$$\hat{m} = \underset{S_m \in S_{PAM}}{\arg\max} \left[r\cdot S_m + \frac{N_0}{2}\ln P_m - \frac{1}{2}|S_m|^2 \right]$$

PSK signal

Transmission of PSK signal. (bandpass)

能量為$\sqrt{E}$ → 射頻 domain 的投影

- $S_{PSK} = \left\{ S_m = \sqrt{E} \left[\cos\left(\frac{2\pi m}{M}\right), \sin\left(\frac{2\pi m}{M}\right) \right]^T \right.$
- $S_{PSK}(t) = \left\{ S_m(t) = \sqrt{E} \left[\cos\left(\frac{2\pi m}{M}\right) \phi_1(t) + \sin\left(\frac{2\pi m}{M}\right) \phi_2(t) \right] \right.$

Define the set of baseband

能量為$\sqrt{2E}$ → 基頻 domain 的投影

- $S_{PSK,\ell} = \left\{ S_{m,\ell} = \sqrt{2E}\, e^{j\frac{2\pi m}{M}} \right\}$, where $\phi_\ell(t) = \sqrt{\frac{1}{E_g}}\, g(t)$
- $S_{PSK,\ell}(t) = \left\{ S_{m,\ell}(t) = \sqrt{2E}\, e^{j\frac{2\pi m}{M}} \cdot \phi_\ell(t) \right\}$

$\longrightarrow S_{PSK}(t) = \left\{ Re\left[S_{m,\ell}(t)\, e^{j2\pi f_c t} \right] ; S_{m,\ell}(t) \in S_{PSK,\ell}(t) \right\}$

Demodulation & Detection

- $S_m(t) \in S_{PSK}(t)$ was transmitted (bandpase)

  The bandpass received signal is

  - $r(t) = S_m(t) + n(t)$
- let $r_\ell(t)$ be the lowpass equivalent received signal.
  - $r_\ell(t) = S_{m,\ell}(t) + n_\ell(t)$
- The baseband MAP rule :

每點的能量相同

$$\hat{m} = \arg\max_m \left\{ Re\{ r_\ell \cdot S_{m,\ell}^* \} + N_0 \ln P_m - \cancel{\frac{1}{2} \cdot 2E} \right\}$$ → 可忽略

QAM signal

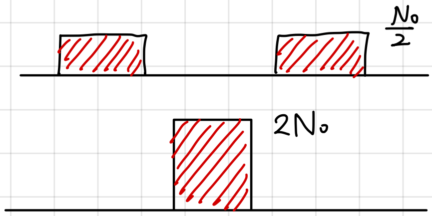
Transmission of QAM signal (bandpass)

- $S_{PAM,i} = \{ \pm\frac{1}{2} d_{min}, \pm\frac{3}{2} d_{min} \cdots \pm\frac{M_i - 1}{2} d_{min} \}$
- $S_{QAM} = \{ (x, y) ; x \in S_{PAM,1} ; y \in S_{PAM,2} \}$

where $d_{min} = \sqrt{\frac{6 \log_2 M}{M-1} E_b}$

$\rightarrow S_{QAM}(t) = \{ x\phi_1(t) + y\phi_2(t) \}$

Summary : Theorem for lowpass MAP detection.



- $\begin{cases} S_\ell = \{ S_{1.\ell} \cdots S_{M.\ell} \} \\ \phi_{n.\ell} = \{ \phi_{1.\ell} \cdots \phi_{N.\ell} \} \end{cases}$

$\rightarrow S_{m.\ell} = \sum_{n=1}^{N} S_{m.n.\ell} \cdot \phi_{n.\ell}(t)$

- $S_m(t) = Re\{ S_{m.\ell}(t) e^{j2\pi f_c t} \}$
- $E_m = \| S_m(t) \|^2 = \frac{1}{2} \| S_{m.\ell}(t) \|^2 = \frac{1}{2} E_{m.\ell}$

- $S_\ell(f) = \frac{1}{4} \{ S(f - f_c) + S^*(-f - f_c) \}$
- $E_m = \frac{1}{2} E_{m.\ell}$

- $r_\ell(t) = S_{m.\ell}(t) + n_\ell(t) = \pm\sqrt{2E_b} + n_\ell(t)$

$\longrightarrow E[|n_\ell(t)|^2] = 2N_0$

證明 baseband 和 bandpass 錯誤率相同.

- $r_\ell = \langle r_\ell(t), \phi_\ell(t) \rangle$

$= \langle S_\ell(t), \phi(t) \rangle + \langle n_\ell(t), \phi_\ell(t) \rangle$

$= S_{m.\ell} + n_\ell = \pm\sqrt{2E_b} + n_\ell$

$\underset{2N_0}{n_\ell} = \underset{N_0}{n_{x.\ell}} + \underset{N_0}{i n_{y.\ell}}$

- $Re\{ \frac{1}{\sqrt{2}} r_\ell \} = \pm\sqrt{E_b} + \frac{1}{\sqrt{2}} n_{x.\ell}$

$\rightarrow E[| \frac{n_{x.\ell}}{\sqrt{2}} |^2] = \frac{1}{2} N_0$

$\rightarrow$ 訊號依然為 $\sqrt{E_b}$ $\Rightarrow$ 和 bandpass 相同

4.4 Optimal Detection and error probability for power limited signaling.

- $S_{FSK} = \{ S_1 = [\sqrt{E}, 0 \cdots 0]^T, \cdots, S_m = [0, \cdots \sqrt{E}]^T \}$
- Given S_1 is transmitted, the received signal vector is $r = S_1 + n$

$$\begin{cases} r_1 = [\sqrt{E} + n_1, n_2, \cdots, n_M] \\ r_2 = [n_1, \sqrt{E} + n_2, \cdots n_M] \\ \vdots \\ r_M = [n_1, n_2, \cdots, \sqrt{E} + n_M] \end{cases}$$

- Assume S_m are equiproble. MAP decision rule

$$\hat{m} = \arg\max_m \left(\cancel{\frac{N_0}{2} \ln P_m} - \cancel{\frac{1}{2} E_m} + r^T S_m \right)$$

各點到原點距離皆相同

→ Correlated decision : $\underline{r^T S_1 = E + \sqrt{E} n_1} > \underline{r^T S_m = \sqrt{E} n_m}$; $m \neq 1$

$$[\sqrt{E} + n_1, n_2 \cdots n_M] \begin{bmatrix} \sqrt{E} \\ 0 \\ \vdots \\ 0 \end{bmatrix} = E + \sqrt{E} n_1$$

→ $Pr(\text{correct} | S_1 \text{ sent}) = Pr\{\sqrt{E} + n_1 > n_2, \cdots \sqrt{E} + n_1 > n_M\}$

→ $Pr(\text{correct}) = \cancel{\frac{1}{M}} [Pr(\text{correct} | S_1 \text{ sent}) + \cancel{Pr(\text{correct} | S_2 \text{ sent})} \vdots \cancel{Pr(\text{correct} | S_M \text{ sent})}]$

→ $Pr(\text{correct}) = Pr\{\sqrt{E} + n_1 > n_2, \cdots \sqrt{E} + n_1 > n_M\}$ 貝式定理

$$= \int_{-\infty}^{\infty} Pr\{\sqrt{E} + n_1 > n_2, \cdots \sqrt{E} + n_1 > n_M | n_1\} f(n_1) \, dn_1$$

$$= \int_{-\infty}^{\infty} \left[1 - Q\left(\frac{n_1 + \sqrt{E}}{\sqrt{\frac{N_0}{2}}} \right) \right]^{M-1} f(n_1) \, dn_1 \quad \#$$

→ $P_e = 1 - Pr(\text{correct})$

$$P_b = \frac{P_e}{M-1} \approx \frac{1}{2} P_e \; *$$

4.6 Comparison of Digital Signaling Method.

- In general, time-limited and band-limited do not exist.
- We could relax as:

$$\to \frac{\int_{-W}^{W} |X_e(f)|^2 df}{\int_{-\infty}^{\infty} |X_e(f)|^2 df} \geq 1-\eta$$ 能量大多集中在[-W, W]

→ $\boxed{\text{Theorem 5. Prolate spheroidal function.}}$

Theorem 5. Prolate spheroidal function.

- for a signal $X(t)$ in time $[-\frac{T}{2}, \frac{T}{2}]$ and η band-limited to W and exist N-orthogonal signal $\{\phi_j(t) ; 1 \leq j \leq N\}$, s.t.

$$\frac{\int_{-\infty}^{\infty} \Big| \underbrace{x(t)}_{\text{原始信號}} - \underbrace{\sum_{j=1}^{N} \langle X(t), \phi_j(t)\rangle \cdot \phi_j(t)}_{\text{合成信號}} \Big|^2 dt}{\int_{-\infty}^{\infty} |X_e(f)|^2 df} \leq \epsilon$$ ← 合成訊號造成的損失，需小於 ϵ

- $N = 2WT + 1$

- Why $N = 2WT+1$?
  - for bandwidth W, the Nyquest rate $2W$ can perfect reconstruction.
    - → We get $2W$ sample/sec
    - → for time duration T, we get $2WT$ sample

故只能有 2WT 個 sample
⇒ $N = 2WT+1$

- Bandwidth efficiency

$$\begin{cases} N = 2WT \to \frac{1}{W} = \frac{2T}{N} \\ \text{since } R = \frac{1}{T}\log_2 M \quad (\text{bit/sec}) \end{cases}$$

$$\to \boxed{\frac{R}{W} = \frac{2}{N}\log_2 M \ (\text{bit/dimension}) \quad \text{bandwidth efficiency}}$$

- N : the dimensionality of constellation

	PAM	PSK and QAM	FSK
N	1	2	M
$\frac{R}{W} = \frac{2}{N}\log_2 M$	$2\log_2 M$	$\log_2 M$	$\frac{2\log_2 M}{M}$
Bandwidth efficiency.	大於 1	大於 1	小於 1

用頻寬效益換其它效能. e.g. P_b

Shanno Channel Coding Theorem.

- Given an average power constraint P over $[-W, W]$ the maximum number bit/channel use is

$$\boxed{C=\frac{1}{2}\log_2\left(1+\frac{\frac{P}{2W}}{N_0/2}\right)} \text{ bit/channel use}$$

〈討論〉

$$C=\frac{1}{2}\log_2\left(1+\frac{P}{N_0W}\right) \text{ bit/channel use}$$

$\times\, 2WT$ channel use/transmission

$$=WT\log_2\left(1+\frac{P}{N_0W}\right) \text{ bit/transmission}$$

$\times\, \frac{1}{T}$ transmission/sec

$$=W\log_2\left(1+\frac{P}{N_0W}\right) \text{ bit/sec}$$

$$\rightarrow R\leq W\log_2\left(1+\frac{P}{N_0W}\right)$$

$$\rightarrow \frac{R}{W}\leq \log_2\left(1+\frac{P}{N_0W}\right)$$

$$E_b=\frac{E}{\log_2 M}=\frac{PT}{\log_2 M}=\frac{P}{R}$$

$$\rightarrow \frac{R}{W}\leq \log_2\left(1+\frac{E_b}{N_0}\cdot\frac{R}{W}\right)$$

$$故\quad \frac{E_b}{N_0}\geq\frac{2^{\frac{R}{W}}-1}{R/W}$$

$$\rightarrow \frac{E_b}{N_0}\geq \lim_{R/W\to 0}\frac{2^{R/W}-1}{R/W}=\ln 2=-1.6\,dB$$

當 $R/W\to 0$，代表 $W\to\infty$ 時

$\frac{E_b}{N_0}$ 須超過 $-1.6\,dB$，則 $P_b\to 0$

Shannon limit for AWGN Channel.

if $\frac{E_b}{N_0}\geq \ln 2=-1.6\,dB$,

then $k\to\infty$ can implement $P_b\to 0$

BPSK 在 $P_e=10^{-5}$ 時需要 $\frac{E_b}{N_0}=9.6\,dB$.

4.5 Optimal noncoherent dection.

- We assumed that all communication is well synchronized
  $\rightarrow r(t) = s_m(t) + n(t)$

- In practice, the signal $s_m(t)$ could be delay
  $\rightarrow r(t) = s_m(t-t_d) + n(t)$

- We use θ to capture such impairment

  $\rightarrow r(t) = s_m(t;\theta_m) + n(t)$

MAP rule

$$\hat{m} = \arg\max_m \Pr(s_m \mid r)$$

$$= \arg\max_m P_m \cdot f(r \mid s_m, \theta)$$ ↓ Bayes rule

$$= \arg\max_m P_m \cdot \int f_n(r - s_m) \cdot \underline{f_\theta(\theta)}\, d\theta$$

$$\Rightarrow \boxed{P_e = E\left[Q\left(\underline{A}\sqrt{\frac{2E_b}{N_0}}\right)\right]}$$ ← 對 A 做期望值，但錯誤率很差.

4-5-1 Noncoherent Detection of Carrier Modulated Signals.

- $S_m(t) = Re\{S_{\ell,m}(t) \cdot e^{j2\pi f_c t}\}$

- Assume received signal delay by t_d

$$r(t) = S_m(t-t_d) + n(t)$$
$$= Re\{S_{\ell,m}(t-t_d) e^{j2\pi f_c(t-t_d)}\} + n(t)$$
$$= Re\{[S_{\ell,m}(t-t_d) e^{-j2\pi f_c t_d} + n_\ell(t)] e^{j2\pi f_c t}\}$$

並且 $S_{\ell,m}(t-t_d) \approx S_{\ell,m}(t)$

$$\Rightarrow r_\ell(t) = S_{\ell,m}(t) e^{j\phi} + n_\ell(t)$$

旋轉 ϕ，故沒有損失能量

- Assume $\phi \sim U(-\pi,\pi)$, the MAP rule as :

$$\hat{m} = \arg\max_m P_m \cdot \int_{-\pi}^{\pi} f_{n_\ell}(r_\ell - S_{\ell,m} e^{j\phi}) \cdot \frac{1}{2\pi} d\phi$$
$$= \arg\max_m \frac{P_m}{\cancel{2\pi}} \cdot \int_{-\pi}^{\pi} \frac{1}{\cancel{(2\pi N_0)^N}} e^{-\frac{\| r_\ell - S_{\ell,m} e^{j\phi} \|^2}{2N_0}} d\phi$$

$$\cancel{\|r_\ell\|^2} + \|S_{m,\ell} \cancel{e^{j\phi}}\|^2 + \| 2 r_\ell^\dagger S_{m,\ell} e^{j\phi} \|$$

和m無關 $= E_{m,\ell} = 2E_m$ $= 2Re\{r_\ell^\dagger S_{m,\ell} e^{j\phi}\}$

$= 2Re\{\underbrace{|r_\ell^\dagger S_{m,\ell}|}_{振幅} \underbrace{e^{j\theta_m}}_{相位} \cdot e^{j\phi}\}$

$$= \arg\max_m P_m e^{-\frac{E_m}{N_0}} \int_{-\pi}^{\pi} e^{\frac{|r_\ell^\dagger S_{\ell,m}|}{N_0} \cos(\theta_m + \phi)} d\phi$$
$$= \arg\max_m P_m e^{-\frac{E_m}{N_0}} \cdot \underbrace{I_0\left(\frac{|r_\ell^\dagger S_{\ell,m}|}{N_0}\right)}_{\text{Bessel function.}} \text{ ※}$$

- $$\boxed{g_{MAP}(r_\ell) = \arg\max_m P_m e^{-\frac{E_m}{N_0}} \cdot I_0\left(\frac{|r_\ell^\dagger S_{\ell,m}|}{N_0}\right)}$$

  - if equal-energy and equiprobable

$$\hat{m} = \arg\max_m |r_\ell^\dagger S_{\ell,m}|$$
$$= \arg\max_m \left| \int_0^T r_\ell^*(t) S_{\ell,m}(t)\, dt \right|$$

Compair with coherent detection.

- $r_\ell(t) = S_{m,\ell}(t) + n(t)$, $r_\ell = S_{m,\ell} + n_\ell$

沒有delay造成的 $e^{j\phi}$

- MAP rule :

$$\hat{m} = \arg\max_m P_m \cdot f_{n,\ell}(r_\ell - S_{m,\ell})$$

$$\cancel{\|r_\ell\|^2} + \cancel{\|S_{m,\ell}\|^2} + 2\| r_\ell^\dagger S_{m,\ell} \|$$

$E_{m\ell} = \cancel{2E_m}$ $2Re(r_\ell^\dagger S_{m,\ell})$

$$= \arg\max_m \frac{P_m}{(4\pi N_0)^N} \cdot \exp\left(-\frac{\| r_\ell - S_{m,\ell} \|^2}{4N_0}\right)$$

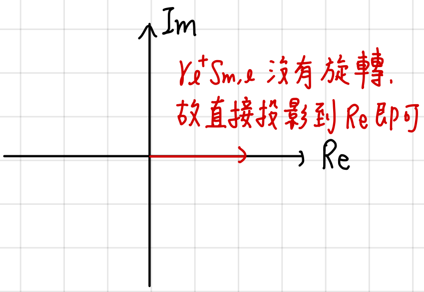
, if equal-energy and equiprobable.

$$= \arg\max_m Re(r_\ell^\dagger \cdot S_{m,\ell}) \text{ ※}$$

〈結論〉

- Coherent ML detection : $\hat{m} = \arg\max_{m} \mathrm{Re}\{ r_\ell^{\dagger} s_{m,\ell} \}$
- Non-coherent ML detection : $\hat{m} = \arg\max_{m} | r_\ell^{\dagger} \cdot s_{m,\ell} |$

Coherent detection



Non-coherent detection