Stochastic Process Ch11

2024 Fall

Chapter 11.

11-1 All RPs are WSS unless I say no

Question:

How to create a WSS RP $X(t)$ so that $S_{XX}(\omega) = G(\omega)$, where $G(\omega)$ is given and $G(\omega) \geq 0 \ \forall \omega \in \mathbb{R}$

$X(t)$ is WSS if

(1) $E[X(t)] = c \ \forall t \in R$

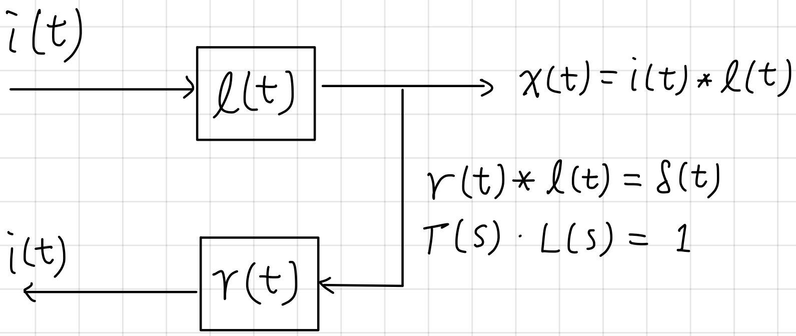
(2) $R_{XX}(t_1, t_2) = R_{XX}(t_1 + \tau, t_2 + \tau) = R_{XX}(t_1 - t_2, 0)$

$$\xrightarrow{i(t)} \boxed{h(t)} \xrightarrow{X(t) = i(t) * h(t)}$$

- $E[i(t)] = 0$
- $R_{ii}(\tau) = \delta(\tau)$

Given a function $G(\omega) \geq 0$, $\forall \omega \in R$

There exists a LTI system with impulse response $h(t)$, such that $X(t) = i(t) * h(t)$ and $S_{XX}(\omega) = G(\omega)$



$r(t) * l(t) = \delta(t)$

$T(S) \cdot L(S) = 1$

- Definition. Let $x(t) = i(t) * l(t)$

  The RP $x(t)$ is said to be a regular process, if

  (1) $l(t)$ is causal and finite energy.

  (2) The inverse of $l(t) \to r(t)$ is causal and finite energy

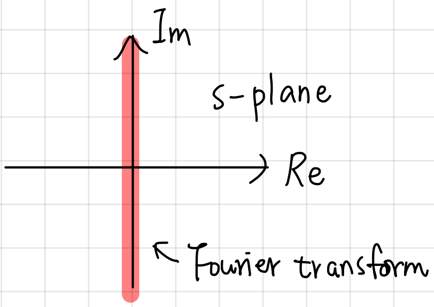
- $S_{xx}(\omega) = S_{ii}(\omega)\,|L(j\omega)|^2 = |L(j\omega)|^2$

  $S_{xx}(s) = L(s) \cdot L(-s)$ where

  $S_{xx}(s)$ is the analytic extension of $S_{xx}(\omega)$ to complex plane

$$\mathcal{L}\{R_x(\tau)\} = S_{xx}(s) = \int R_x(\tau) e^{-s\tau} d\tau$$

$$S_{xx}(\omega) = \int R_x(\tau) e^{-j\omega\tau} d\tau \leftarrow s = j\omega$$



Given a function f defined on $S \triangleq \operatorname{Im} \mathbb{C} \in \{j\omega \mid \omega \in \mathbb{R}\}$

an analytic extension F

1. defined on $\mathbb{C}$
2. differentiable function over $\mathbb{C}$
3. $F(z) = f(z), \forall z \in \operatorname{Im} \mathbb{C}$

Ex 11-1

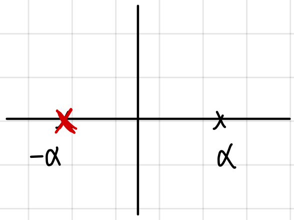
$S(\omega) = \dfrac{N}{\alpha^2 + \omega^2}$, $\alpha > 0$,

Find $L(s)$ such that $L(s) \cdot L(-s) = S(s)$

and $L(s)$ can NOT have pole on the right half plane

① Replace $j\omega$ by s

② Then, $S(s) = \dfrac{N}{\alpha^2 - s^2} = \dfrac{\sqrt{N}}{\alpha + s} \cdot \dfrac{\sqrt{N}}{\alpha - s}$

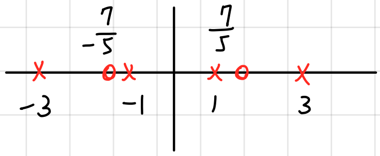


③ $L(s) = \dfrac{\sqrt{N}}{\alpha + s} \longleftrightarrow \ell(t) = \mathcal{L}^{-1}\left\{\dfrac{\sqrt{N}}{\alpha + s}\right\} = \underline{\sqrt{N} e^{-\alpha t} \; ; \; t \geq 0}$ #

Ex 11-2

$S(\omega) = \dfrac{25\omega^2 + 49}{\omega^4 + 10\omega^2 + 9}$. Find $L(s)$ and $\ell(t)$

- $S(s) = \dfrac{49 - 25s^2}{s^4 - 10s^2 + 9} = \dfrac{(7+5s)(7-5s)}{(s+1)(s+3)(s-1)(s-3)}$



- Since $L(s)$ can NOT have pole on the right half plane

$$\begin{cases} L(s) = \dfrac{7+5s}{(s+3)(s+1)} = \dfrac{4}{s+3} + \dfrac{1}{s+1} \quad \# \\ \ell(t) = \left(4e^{-3t} + e^{-t}\right) u(t) \quad \# \end{cases}$$

Ex 11-3

$S(\omega) = \dfrac{25}{\omega^4 + 1}$. Find $L(s)$

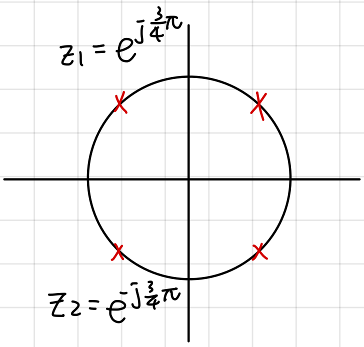
$S(s) = \dfrac{25}{s^4 + 1}$

$s^4 = -1$

$s^4 = e^{j(\pi + 2\pi k)}$

$s = e^{j\frac{\pi}{4} + \frac{2\pi}{4}k}$; $k = 1, 2, 3, 4$

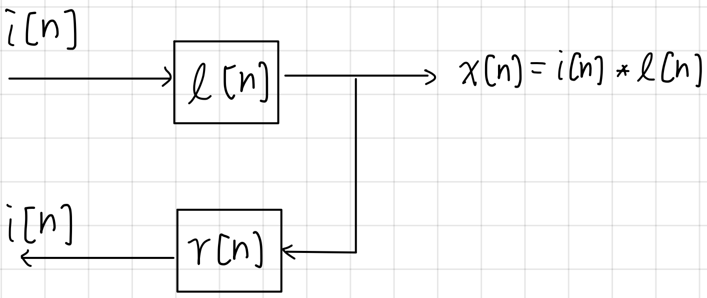
$s = e^{j\frac{3}{4}\pi}, e^{j\frac{5}{4}\pi}, e^{j\frac{7}{4}\pi}, e^{j\frac{\pi}{4}}$



Then $L(s) = \dfrac{5}{(s - z_1)(s - z_2)} = \dfrac{5}{s^2 - (z_1 + z_2)s + z_1 z_2} = \dfrac{5}{s^2 - 2\cos(\frac{3}{4}\pi)s + 1}$

$= \dfrac{5}{s^2 - \sqrt{2} - 1}$ #

Discrete-time regular process



Definition:

$x[n] \triangleq i[n] * \ell[n]$ is said to be a regular process, if

(1) $\ell[n]$ is causal and $\sum_{n=-\infty}^{\infty} \ell[n] \leq \infty$

(2) There exist $r[n]$ such that $\ell[n] * r[n] = \delta[n]$

$r[n]$ is causal and $\sum_{n=-\infty}^{\infty} \ell[n] \leq \infty$

Ex 11-4.

$S(\omega) = \dfrac{5 - 4\cos\omega}{10 - 6\cos\omega}$, $\forall\, \omega \in [-\pi, \pi]$, Find $L(z)$ such that

(1) $L(z) \cdot L(\frac{1}{z}) = S(z)$

(2) $L(z)$ can not have pole or zero outside the unit circle.

<Sol>.

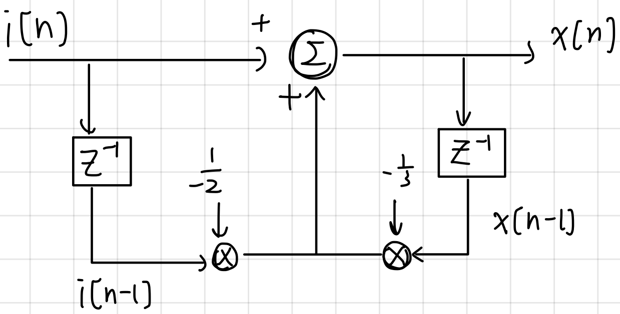
$z = e^{j\omega}$

$$S(z) = \frac{5 - 2(z + z^{-1})}{10 - 3(z + z^{-1})} = \frac{(1-2z)(1-2z^{-1})}{(1-3z)(1-3z^{-1})}$$; zero: $\frac{1}{2}, 2$ pole: $\frac{1}{3}, 3$

Then

$$L(z) = \frac{z - \frac{1}{2}}{z - \frac{1}{3}} = \frac{1 - \frac{1}{2}z^{-1}}{1 - \frac{1}{3}z^{-1}} \#$$

$\Rightarrow X(z)(1 - \frac{1}{3}z^{-1}) = I(z)(1 - \frac{1}{2}z^{-1}) \Rightarrow x[n] - \frac{1}{3}x[n-1] = i[n] - \frac{1}{2}i[n-1]$



11-2 Discrete time finite-order process

$$\boxed{L(z) = \frac{N(z)}{D(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{n=1}^{N} a_k z^{-k}} = \frac{X(z)}{I(z)}}$$

→ The corresponding $x[n]$ is a finite-order process

$$\longrightarrow \boxed{x[n] + \sum_{k=1}^{N} a_k \, x[n-k] = \sum_{k=0}^{M} b_k \, i[n-k]}$$

$$\longrightarrow R[m] = \ell[m] * \ell[-m] = \sum_{k=-\infty}^{\infty} \ell[k] \cdot \ell[|m|+k]$$

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Autoregressive process $x[n]$

$$x[n] + \sum_{k=1}^{N} a_k x[n-k] = b_0 i[n]$$

- $x[n]$ is an autoregressive process of order N
- $R_{ii}[m] = \delta[m]$ $\qquad \underset{\text{WSS}}{\xrightarrow{i[n]}} \boxed{L(z)} \underset{\text{WSS}}{\xrightarrow{x[n]}}$

Theorem 1 Yule-Walker Equation

$$\left.\begin{aligned} R[0] + a_1 R[1] + \cdots + a_N R[N] &= b_0^2 \\ R[1] + a_1 R[0] + \cdots + a_N R[N-1] &= 0 \\ &\vdots \\ R[N] + a_1 R[N-1] + \cdots + a_N R[0] &= 0 \end{aligned}\right\} \text{(11-41a)}$$

$$R[m] + a_1 R[m-1] + \cdots + a_N R[m-N] = 0, \ \forall m > N \quad \text{(11-41b)}$$

We obtain $R[m]$'s as follows:

(1) Solve (11-41a) to obatian $R[0], R[1] \cdots R[N]$

(2) We use (11-41b) to sequentially obtain $R[N+1], R[N+2], \cdots$

(3) $R[-m] = R[m] \ \forall m \in N$

$$\underset{A}{\begin{bmatrix} R[0] & R[1] & \cdots & R[N] \\ R[1] & R[0] & \cdots & R[N-1] \\ \vdots & & & \\ R[N] & R[N-1] & \cdots & R[0] \end{bmatrix}} \times \underset{X}{\begin{bmatrix} 1 \\ a_1 \\ a_2 \\ \vdots \\ a_N \end{bmatrix}} = \underset{b}{\begin{bmatrix} b_0^2 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}}$$

Proof of Theorem 1 :

$$x[n]+\sum_{k=1}^{N}a_k x[n-k]=b_0 i[n]$$

$x[n]$

$$x[n]\cdot x[n]+\sum_{k=1}^{N}a_k x[n]\,x[n-k]=b_0 i[n]\,x[n]$$

$E[\,]$

$$R[0]+\sum_{k}^{N}a_k R[k]=b_0 \underline{E[x(n)\,i[n]]}$$

$$\rightarrow E[x(n)\,i[n]]$$

$$=E\left[\left(\underline{b_0 i[n]+\sum_{k=1}^{\infty}C_k i(n-k)}\right)i(n)\right]$$

$$=b_0 R_{ii}(0)+\underbrace{\sum_{k=1}^{\infty}C_k R_{ii}[k]}_{0}$$

$$=b_0\cdot 1$$

$$\rightarrow R[0]+\sum_{k=1}^{N}a_k R[k]=b_0^2 \ \#$$

$$x[n]=b_0 i[n]-\sum_{k=1}^{N}a_k x[n-k]$$

$$x(n-1)=b_0 i[n-1]-\sum_{k=1}^{N}a_k x[n-1-k]$$

$\vdots$

$$x(n)=b_0 i[n]-a_1 x[n-1]-a_2 x[n-2]\cdots$$

$$x[n-1]=b_0 i[n-1]-a_1 x[n-2]-a_2 x[n-3]$$

$$x(n-2)=b_0 i[n-2]-a_1 x[n-3]-a_2$$

故 $x[n]=b_0 i[n]$

$$-a_1 b_0 i[n-1]+a_1^2 x(n-2)+a_1 a_2 x[n-3]$$

$$-a_2 b_0 i[n-2]+a_1 a_2 x[n-3]\ \cdots$$

Consider $m>N$

$$x(n)+\sum_{k=1}^{N}a_k x[n-k]=b_0 i(n)$$

$$x[n]\cdot x[n-m]+\sum_{k=1}^{N}a_k x[n-k]\,x(n-m]=b_0 i(n)\,x[n-m]$$

$$R(m)+\sum_{k=1}^{N}a_k R(m-k]=b_0 E[i(n)\,x(n-m)]=0$$

$$x[n-m]=\sum_{k=0}^{\infty}C_k\, i[n-m-k]$$

Moving-average process of order $M \geq 1$

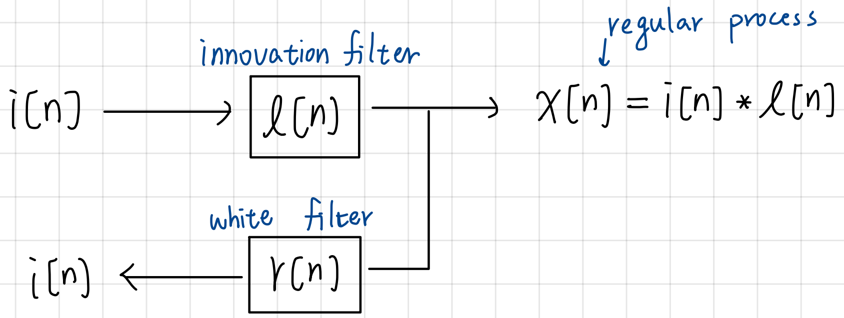
$$x[n] = \sum_{k=0}^{M} b_k i[n-k] \qquad (11\text{-}45)$$

Lemma 1. Consider a MA process as in (11-45)

(1) $R[m] = \sum_{k=0}^{M-m} \ell[m+k]\,\ell[k] = \sum_{k=0}^{M-m} b_{k+m} \cdot b_k \quad \forall\, 0 \leq m \leq M$

(2) $R[m] = 0 \quad \forall\, m > M$

$$L(z) = \underset{\substack{\uparrow \\ \ell[0]}}{\underline{b_0}} + \underset{\substack{\uparrow \\ \ell[1]}}{\underline{b_1}} z^{-1} + \underset{\substack{\uparrow \\ \ell[2]}}{\underline{b_2}} z^{-2} \cdots + \underset{\substack{\uparrow \\ \ell[M]}}{\underline{b_M}} z^{-k}$$



$x[n]$ Autoregress moving average (ARMA) process of order (M, N)

$$x[n] + \sum a_k x[n-k] = \sum_{k=0}^{M} b_k i[n-k] = b_0 i[n] + \sum_{k=1}^{M} b_k i[n-k] \qquad (11\text{-}48)$$

Before (11-49)

$$E\big[x[n-m]\; i[n-r]\big] = 0 \quad \forall\, m > r$$

$$R[m] + \sum_{k=1}^{N} a_k R[m-k] = 0\,, \ \forall\, m > M \qquad (11\text{-}49)$$

Ex 11-7 $a \in (-1, 1)$, $R_W[m] = b\delta[m]$

$X[m] - aX[n-1] = \nu[n] = \sqrt{b}\, i[n]$

Obtain $R_{xx}[m], \forall m$ $\quad L(z) = \dfrac{\sqrt{b}}{1-az^{-1}}$

① $X[n] - aX[n-1] = \sqrt{b}\, i[n]$

$x[n], E\{\}$

$X[n] = \sqrt{b}\, i[n] + \sum_{k=1}^{\infty} C_k i[n-k]$

$R[0] - aR[1] = \sqrt{b}\, E[\underline{x[n]}\, i[n]] = b$

② $X[n] - aX[n-1] = \sqrt{b}\, i[n]$

$x[n-1], E\{\}$

$R[1] - aR[0] = 0$

③ $R[1] = aR[0]$

$R[0] - a^2 R[0] = b \quad \Longrightarrow$

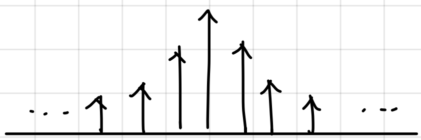
- $R[0] = \dfrac{b}{1-a^2}$
- $R[1] = a\dfrac{b}{1-a^2}$
- $R[2] = a^2\dfrac{b}{1-a^2}$

$\vdots$

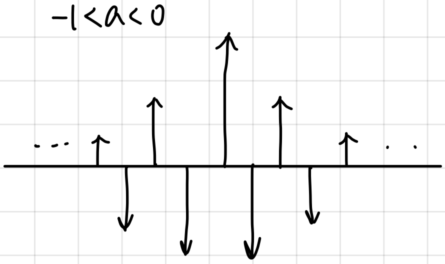
④ 同理. $R[2] - aR[1] = 0$

故 $R[m] = a^{|m|}\dfrac{b}{1-a^2}$

$0<a<1$



$-1<a<0$



11-3.

Definition :

mean square

A RP $x(t)$ is MS periodic with period $T>0$, if

$E\{[x(t+T)-x(t)]^2\}=0 \;, \forall t\in R$

Theorem 2.

A RP $x(t)$ is MS periodic with $T>0$ iff

$\boxed{R_{xx}(\tau) \text{ is periodic with period } T}$

Consider a RP $x(t)$ that is MS periodic with T and WSS

$$\left.\begin{aligned} &\bullet\; R_{xx}(\tau)=\sum_{n=-\infty}^{\infty} r_n \cdot e^{jnw_0\tau} \\ &\bullet\; r_n=\frac{1}{T}\int_0^T R_{xx}(\tau)e^{-jnw_0\tau}d\tau \end{aligned}\right\} (11\text{-}50)$$

$\beta \triangleq \{e^{jnw_0t} \mid n\in\mathbb{Z}\}$

is an orthogonal basis for periodic function with period T

random variable

$$\left.\begin{aligned} &\bullet\; C_n \triangleq \frac{1}{T}\int_0^T x(t)e^{-jnw_0t}dt \\ &\bullet\; \hat{x}(t) \triangleq \sum_{n=-\infty}^{\infty} C_n e^{jnw_0t} \end{aligned}\right\} (11\text{-}51)$$

estimate of $x(t)$

Theorem 3,

(1) $E\{|\hat{x}(t)-x(t)|^2\}=0 \;, \forall t\in R$

(2) $E[C_n]=\begin{cases} \eta_x \;; & n=0 \\ 0 \;; & n\neq 0 \end{cases}$

(3) $E[C_nC_m^*]=\begin{cases} r_n \;; & n=m \\ 0 \;; & n\neq m \end{cases}$

Key result from linear algebra

1. if $A^H = A$ and $Av = \lambda v$ then $\lambda \in R$
2. if A is positive semi-define and $Av = \lambda v$, then $\lambda \geq 0$
3. if $A^H = A$, there exist $\beta = \{v_1\, v_2 \cdots v_n\}$ such that

   $A v_k = \lambda_k v_k$ and $\langle v_i, v_j \rangle = 0 \ \forall i \neq j$ and $\|v_k\| = 1$

Consider a RP $\{x(t),\ 0 \leq t \leq T\}$ with $R_{xx}(t_1\, t_2)$, find $\hat{x}(t)$ such that

positive semi-define function

$R_{xx}(t_2, t_1) = R_{xx}(t_1, t_2)^*$

- $E\left\{|\hat{x}(t) - x(t)|^2\right\} = 0 \ , \forall t \in [0, T]$
- $\hat{x}(t) = \sum_{n=1}^{\infty} C_n \varphi_n(t)$

$$R(t,t) = \sum_{n=1}^{\infty} \lambda_n |\varphi_n(t)|^2$$

$$\int_0^T R(t_1\, t_2)\, \varphi_n(t_2)\, dt_2 = \lambda_n \varphi_n(t_1) \ , \forall\, 0 \leq t \leq T$$

$$C_n \triangleq \int_0^T x(t)\, \varphi_n^*(t)\, dt$$ ← random variable

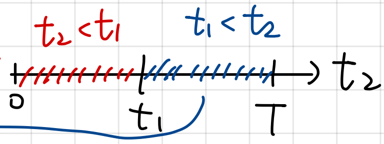
$\hat{x}(t) = \sum_{n=1}^{\infty} C_n \varphi_n(t)$ is called

the Karhunen-Loève expansion of $x(t)$

Ex 11-10

$W(t)$ is a Brownian motion · with parameter α

$R_{WW}(t_1 t_2) = \alpha \min(t_1 t_2)$, Find $\hat{W}(t)$



- $\int_0^T R_{WW}(t_1, t_2)\varphi(t_2)\,dt_2 = \lambda \varphi(t_1)$

$\rightarrow \int_0^{t_1} \alpha t_2 \varphi(t_2)\,dt_2 + \int_{t_1}^{T} \alpha t_1 \varphi(t_2)\,dt_2 = \lambda \varphi(t_1)$, Then

$\alpha \int_0^{t_1} t_2\varphi(t_2)\,dt_2 + \alpha t_1 \int_{t_1}^{T} \varphi(t_2)\,dt_2 = \lambda \varphi(t_1)$

- $\frac{d}{dt_1} \Rightarrow \cancel{\alpha \cdot t_1 \varphi(t_1)} + \alpha \left[\int_{t_1}^{T} \varphi(t_2)\,dt_2 + \cancel{t_1 \cdot -\varphi(t_1)} \right] = \lambda \varphi'(t_1)$

Then, $\alpha \int_{t_1}^{T} \varphi(t_2)\,dt_2 = \lambda \varphi'(t_1)$

- $\frac{d}{dt_1} \Rightarrow \alpha \cdot -\varphi(t_1) = \lambda \varphi''(t_1)$

Then, $\lambda \varphi''(t) + \alpha \varphi(t_1) = 0$ #

① Setting $t_1 = 0$, then $\lambda\varphi(0) = 0 \Rightarrow \varphi(0) = 0$

② Setting $t_1 = T$, then $\lambda\varphi'(T) = 0 \Rightarrow \varphi'(T) = 0$

$\Rightarrow$ Guess $\varphi(t) = \cancel{a\cos(wt)} + b\sin(wt) \rightarrow \varphi(0) = a = 0$

$\varphi'(t) = \cancel{-aw\sin(wt)} + bw\cos(wt) \rightarrow \varphi'(T) = bw\cos(wT) = 0$

$\varphi''(t) = \cancel{-aw^2\cos(wt)} - bw^2\sin(wt)$

$wT = \frac{\pi}{2} + n\pi$

$w = \frac{(2n+1)\pi}{2T}$ #

且 $\|\varphi(t)\|^2 = 1 = \int_0^T \varphi^2(t)\,dt$

$\Rightarrow b^2 \int_0^T \sin^2(wt)\,dt = \frac{T}{2} b^2 = 1.$

$\Rightarrow b = \sqrt{\frac{2}{T}}$ #

$\varphi_n(t) = \sqrt{\frac{2}{T}} \sin(wt)$

$W_n \triangleq \sqrt{\frac{\alpha}{\lambda_n}} = \frac{(2n+1)\pi}{2T}$

$\rightarrow \hat{W}(t) = \sum_{n=-\infty}^{\infty} C_n \sqrt{\frac{2}{T}} \sin\left(\frac{(2n+1)\pi}{2T} t\right)$ #