Stochastic Process Ch12

2024 Fall

Ch12 Spectrum Estimation ($X(t)$ is WSS RP)

Given a realization of $\{X(t) : a \le t \le b\}$

we would like to estimate statistical properties of $X(t)$

such as $E\{X(t)\}$, $R_{XX}(\tau)$, $S_{XX}(\omega)$

12-1. Ergodicity.

$X(t)$ is a WSS RP with unknow η_x, $T > 0$

To estimate η_x, we use η_T as follow:

$$\boxed{\eta_T \triangleq \frac{1}{2T}\int_{-T}^{T} X(t)\,dt} \quad (12\text{-}1)$$

① $E[\eta_T] = \eta_x$, $\forall T > 0$

on average, η_T is "unbiased estimated"

② $\sigma_T^2 \triangleq Var[\eta_T] \ge 0$, $\lim\limits_{T\to\infty} \sigma_T^2 = 0$

$\lim\limits_{T\to\infty} P_r(\eta_T = \eta_x) = 1$.

$\rightarrow \eta_T = \eta_x$ almost surely.

> Definition:
>
> A RP is said to be mean-ergodic
>
> if $\lim\limits_{T\to\infty} \sigma_T^2 = 0$

One can use $\{X(t, \xi), -\infty < t < \infty\}$ is perfectly

estimate $E[X(t)]$

Theorem 12-1

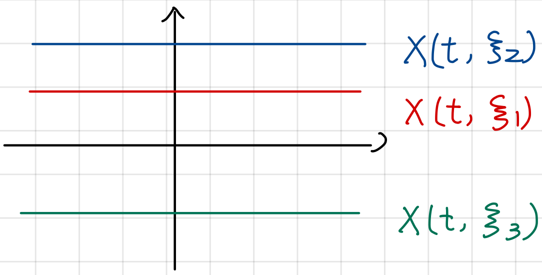
A RP $X(t)$ is mean-ergodic if and only if.

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \underbrace{C_{xx}(\tau)}_{R_{xx}(\tau)-\eta_x^2}\, d\tau = 0 \qquad (12\text{-}7)$$

EX12-1

Suppose C is random variable $X(t)=C \quad \forall t \in R$

$\longleftrightarrow X(t,\xi) = C(\xi), \ \forall t\in R, \ \xi\in\Omega$



Q: is $X(t)=C$ a mean-ergodic RP?

A: ① Consider $T>0$

$$\eta_T \triangleq \frac{1}{2T}\int_{-T}^{T} X(t)\,dt = \frac{1}{2T}\int_{-T}^{T} c\,dt = c$$

Namely, $\eta_T(\xi) = c(\xi)$.

② $E[\eta_T] = E[c]$

$var(\eta_T) = var(c) > 0 \ \forall T>0$

$$\lim_{T\to\infty} \sigma_T^2 = \lim_{T\to\infty} var(c) = var(c) > 0$$

$\Rightarrow X(t)$ here is not mean-ergodic

Consider EX12-1, $X(t)=C$

- $\eta_x = E[C]$

  $R_{xx}(\tau) = E[X(t+\tau)X(t)]$

  $= E[C^2]$

- $C_{xx}(\tau) = R_{xx}(\tau) - \eta_x^2$

  $= E[C^2] - E[C]^2$

  $= var(C)$

- $\frac{1}{T}\int_0^T C_{xx}(\tau)\,d\tau = \frac{1}{T}\int_0^T var(c)\,d\tau = var(c) > 0$

$$\Rightarrow \lim_{T\to\infty}\frac{1}{T}\int_0^T C_{xx}(\tau)\,d\tau = \lim_{T\to\infty} var(c) = var(c) \neq 0 \quad \#$$

$\mathcal{F}$

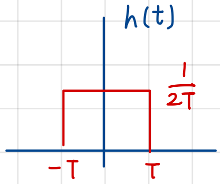
Theorem 1.

$$\sigma_T^2 = \frac{1}{2T}\int_{-2T}^{2T} C_{xx}(\alpha)\left(1-\frac{|\alpha|}{2T}\right)d\alpha = \frac{1}{T}\int_{0}^{2T} C_{xx}(\alpha)\left(1-\frac{|\alpha|}{2T}\right)d\alpha$$

<proof>

① Define $W(t) = \frac{1}{2T}\int_{t-T}^{t+T} X(\alpha)\,d\alpha$, $\forall t \in R$ (12-2)

Then, $W(0) = \frac{1}{2T}\int_{-T}^{T} X(\alpha)\,d\alpha = \eta_T$



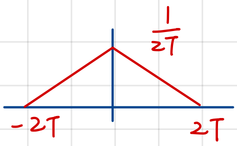
Since $X(t)$ is WSS, and $W(t) = X(t) * h(t)$ where

$\rightarrow W(t)$ is also a WSS RP (Ch9)

② $C_{WW}(\tau) = C_{XX}(\tau) * h(\tau) * h(-\tau)$

$\rho(\tau) \triangleq h(\tau) * h(\tau)$

$= h(\tau) * h(\tau) =$



③ Thus

$C_{WW}(\tau)$

$= C_{XX}(\tau) * \rho(\tau)$

$= \int_{-\infty}^{\infty} C_{XX}(\tau-\alpha)\rho(\alpha)\,d\alpha$

$= \frac{1}{2T}\int_{-2T}^{2T} C_{XX}(\tau-\alpha)\left(1-\frac{|\alpha|}{2T}\right)d\alpha$

④ $\eta_T = W(0)$

Assume $\eta_X = 0$,

for simplicity

$\sigma_T^2 = E[\eta_T^2] = E[W(0)W(0)] = C_{WW}(0) = \frac{1}{2T}\int_{-2T}^{2T} C_{XX}(\alpha)\left(1-\frac{|\alpha|}{2T}\right)d\alpha.$

$X(t)$ is a WSS RP

- $\eta_T \triangleq \frac{1}{2T}\int_{-T}^{T} X(t)\,dt$, $\forall T > 0 \rightarrow E[\eta_T] = \eta_x$
- $\sigma_T^2 \triangleq \mathrm{var}(\eta_T)$

$$= \frac{1}{T}\int_0^{2T} C_{xx}(\alpha)\left(1-\frac{\alpha}{2T}\right)d\alpha \quad (12-4)$$

- $X(t)$ is mean-ergodic iff

$$\lim_{T\to\infty} \sigma_T^2 = 0, \text{ iff}$$

$$\lim_{T\to\infty} \frac{1}{T}\int_0^{2T} C_{xx}(\alpha)\left(1-\frac{\alpha}{2T}\right)d\alpha = 0$$

Theorem 12-1 (Slutsky Theorem)

A RP $X(t)$ is mean-ergodic iff $\lim \frac{1}{T}\int_0^{T} C_{xx}(\alpha)\,d\alpha = 0$ (13-7)

---

proof.

$Cov(X,Y) \triangleq E\{(X-\mu_x)(Y-\mu_y)\}$

① $Cov[\eta_T, X(0)]$

$$= E\left\{\left[\frac{1}{2T}\int_{-T}^{T} X(t)\,dt - \eta_x\right][X(0)-\eta_x]\right\}$$

$$= E\left\{\frac{1}{2T}\int_{-T}^{T} (X(t)-\eta_x)(X(0)-\eta_x)\,dt\right\}$$

$$= \frac{1}{2T}\int_{-T}^{T} E\{(X(t)-\eta_x)(X(0)-\eta_x)\}\,dt$$

$$= \frac{1}{2T}\int_{-T}^{T} C_{xx}(t)\,dt = \frac{1}{T}\int_0^{T} C_{xx}(t)\,dt$$

② Based on (6-168)

$$Cov^2[\eta_x, X(0)] \le var(\eta_T)\,var(X(0))$$

$$= \sigma_T^2 \cdot C_{xx}(0)$$

③ Based on ① and ②

$$\left|\frac{1}{T}\int_0^{T} C_{xx}(t)\,dt\right|^2 \le \sigma_T^2 C_{xx}(0), \text{ Then}$$

- $0 \le \left|\frac{1}{T}\int_0^{T} C_{xx}(t)\,dt\right|^2 \le \sigma_T^2 C_{xx}(0)$

$$\lim_{T\to\infty} \sigma_T^2 C_{xx}(0) = 0 \quad 因為 \lim_{T\to\infty} \sigma_T^2 = 0$$

- $\lim \frac{1}{T}\int_0^{T} C_{xx}(t)\,dt = 0$

Discrete-time process

Theorem 12-2

$X[n]$ is mean-ergodic iff $\lim\limits_{M\to\infty} \frac{1}{M}\sum\limits_{m=0}^{M} C_{xx}[m] = 0$ (12-13)

Variance-ergodic RPs.

- $X(t)$ is a SSS process $\longleftrightarrow (X(t_1+c), X(t_2+c) \cdots) = (X(t_1), X(t_2) \cdots)$
- It is assumed that $\eta_x = 0$

$$V \triangleq var[X(t)] = E[X^2(t)] - \eta_x^2$$

$$= E[X^2(t)] \quad \because \eta_x = 0$$

- $V_T \triangleq \frac{1}{2T}\int_{-T}^{T} X^2(t)\,dt \rightarrow E[V_T] = V$ (12-20)

Definition:

A RP $X(t)$ is said to be variance-ergodic if

$\lim\limits_{T\to\infty} Var(V_T) = 0$ (iff $X^2(t)$ is mean-ergodic)

- Based on Theorem 12-1

A SSS RP $X(t)$ is variance-ergodic iff

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T C_{x^2x^2}(t)\,dt = 0 \longleftrightarrow \lim_{T\to\infty} \frac{1}{T}\int_0^T E[X^2(t+\tau)X^2(t)]\,d\tau = [C_{xx}(0)]^2 \quad (12\text{-}22)$$

- $C_{x^2x^2}(\tau) = C_{x^2x^2}(t+\tau, t)$

$$= R_{x^2x^2}(t+\tau, t) - E[X^2(t+\tau)]E[X^2(t)]$$

$$= E[X^2(t+\tau)X^2(t)] - E^2[X^2(t)] \quad \because E[X^2(t+\tau)] = E[X^2(t)]$$

since $X(t)$ is SSS

- $E^2[X^2(t)] = [E[X(t)X(t)]]^2$

$$= [R_{xx}(0)]^2$$

$$= [C_{xx}(0)]^2$$

since $\eta_x = 0$, $C_{xx}(0) = R_{xx}(0) - \eta_x^2$

Special case: $X(t)$ is a normal SSS RP.

- Then, based on (9-77) $C_{x^2x^2}(\tau) = 2C_{xx}^2(\tau)$ (12-23)
- In addition, (12-23) becomes $\lim_{T\to\infty} \frac{1}{T}\int_0^T C_{xx}^2(\tau)\,d\tau = 0$ (12-24)

Based on Prob 12-10, we have

$$\left| \frac{1}{T}\int_0^T C_{xx}(\tau)\,d\tau \right|^2 \leq \frac{1}{T}\int_0^T C_{xx}^2(\tau)\,d\tau$$

to check mean ergodic of $X(t)$

To check Variance-ergodic of $X(t)$

$X(t)$ is normal and SSS

variance-ergodic $\longrightarrow$ mean-ergodic.

- $C_T(\lambda) \triangleq \frac{1}{2T}\int_{-T}^{T} Z(t)\,dt$ (as an estimator for $C_{xx}(\lambda)$ when $\eta_x = 0$)
- $Z(t) = X(t+\lambda)X(t)$ (12-35)

12-2 Spectrum Estimation. (不考)

- Suppose one is given a sample path of $\{X(t), -T \le t \le T\}$

- $R^T(\tau) \triangleq \frac{1}{2T-|\tau|} \int_{-T+\frac{|\tau|}{2}}^{T+\frac{|\tau|}{2}} X(t+\frac{\tau}{2}) X(t-\frac{\tau}{2}) dt \quad (12\text{-}37)$

- One could use $R^T(\tau)$ to estimate $R_x(\tau) \triangleq E\{X(t+\frac{\tau}{2}) X(t-\frac{\tau}{2})\}$

$$\tau > 0 \begin{cases} -T \le t+\frac{\tau}{2} \le T \longrightarrow -T-\frac{\tau}{2} \le t \le T-\frac{\tau}{2} \\ -T \le t-\frac{\tau}{2} \le T \longrightarrow -T+\frac{\tau}{2} \le t \le T+\frac{\tau}{2} \end{cases}$$

- One proposed the second approach for estimating $R_{xx}(\tau)$ as follow

$$R_T(\tau) \triangleq \left(1-\frac{|\tau|}{2T}\right) R^T(\tau) \quad (12\text{-}38)$$

- The periodogram is used to estimate $S_{xx}(\omega)$

$$S_T(\omega) = \frac{1}{2T}\left|\int_{-T}^{T} X(t) e^{-j\omega t} dt\right|^2 \quad (12\text{-}39)$$

Theorem 12-3

$$S_T(\omega) = \int_{-2T}^{2T} R_T(\tau) e^{-j\omega\tau} d\tau \quad (12\text{-}40)$$

12-3 System Identification (SI)

- AR: $x[n] + \sum_{k=1}^{N} a_n x[n-k] = b_0 i[n]$
- MA
- ARMA

∘ SI problem for an AR process $x(t)$

$$\begin{cases} \text{input} : x[n] \text{ is an AR process of order } N \\ \text{output} : a = (a_1, a_2 \cdots a_n) \text{ and } b_0 > 0 \end{cases}$$

$$P_N \triangleq b_0^2$$

∘ We use the Yule Walker equation as follow

$$\left.\begin{aligned} R[0] + a_1 R[1] + \cdots + a_N R[N] &= P_N \\ R[1] + a_1 R[0] + \cdots + a_N R[N-1] &= 0 \\ &\vdots \\ R[N] + a_1 R[N-1] + \cdots + a_N R[0] &= 0 \end{aligned}\right\} \quad (12\text{-}82)$$

∘ $P_N = \dfrac{\Delta_{N+1}}{\Delta_N}$ (12-83)

$$1 = a_0 = \begin{vmatrix} P_N & R[1] & \cdots & R[N] \\ 0 & R[0] & & \vdots \\ \vdots & \vdots & & \\ 0 & R[N-1] & & R[0] \end{vmatrix} \Bigg/ \begin{vmatrix} R[0] & R[1] & \cdots & R[N] \\ R[1] & R[0] & & R[N-1] \\ \vdots & \vdots & & \vdots \\ R[N] & & & R[0] \end{vmatrix} = \frac{P_N \cdot \Delta_N}{\Delta_{N+1}}$$

Δ_N

AR

MA $\quad x[n] = \sum_{k=0}^{M} b_k\, i[n-k]$

ARMA

- SI problem for an MA process $x(t)$

$$\begin{cases} \text{input}: & x[n] \text{ is a MA process of order } M, \\ \text{output}: & b \triangleq (b_0\ b_1 \cdots b_M) \end{cases}$$

- Algorithm :

① $S(z) = \sum_{m=-M}^{M} R_{xx}[m]\, z^{-m}$

② Factorize $S(z)$ into $L(z)\cdot L(\frac{1}{z})$ such that $L(z)$ has no zero/pole outside unit circle.

③ $L(z) = \alpha\,(z^{-1}-z_1)\cdot(z^{-1}-z_2)\cdots(z^{-1}-z_M) = \sum_{k=0}^{M} b_k z^{-k}$

AR

MA

ARMA $\quad x[n] + \sum_{k=1}^{N} a_k\, x[n-k] = \sum_{k=0}^{M} b_k\, i[n-k] \quad (12\text{-}87)$

- System identification for an ARMA process $x[n]$

$$L(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1+\sum_{k=1}^{N} a_k z^{-k}} = \frac{N(z)}{D(z)} \qquad (12\text{-}86)$$

- Lemma 1 : $R[m] + \sum_{k=1}^{N} a_k R[m-k] = 0,\ \forall m > M \ (12\text{-}88)$

$$\underbrace{i[n] \longrightarrow \boxed{\frac{N(z)}{D(z)}} \xrightarrow{x[n]} \boxed{D(z)}}_{N(z)} \longrightarrow y[n]$$

- Algorithm :

① Find a_k's based on (12-88)

- $d[m] \triangleq \sum_{k=0}^{N} a_k\, \delta[m-k]$
- $D(z) \triangleq \sum_{k=0}^{N} a_k z^{-k}$

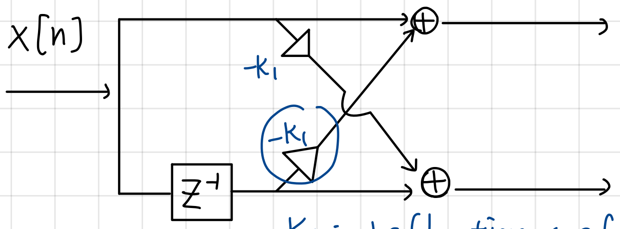
② $y[n] \triangleq x[n] * d[m]$, $\rho[m] \triangleq \sum_{k=m}^{N} a_{k+m}\cdot a_k$

$R_{yy}[m] = \sum_{i=-N}^{N} R[m-i]\, \rho[i]$

③ $S_{yy}(z) = \sum_{m=-M}^{M} R_{yy}[m]\, z^{-m} = N(z)\cdot N(\frac{1}{z})$

Lattice Filter

- First-order lattice filter

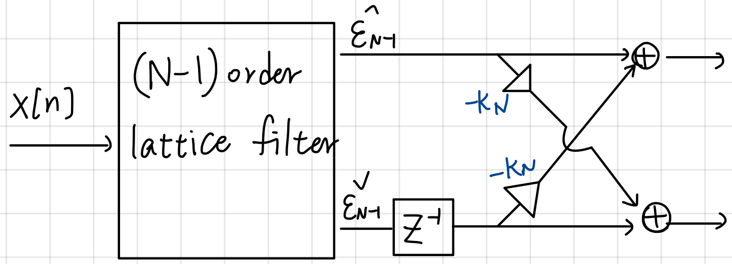


$$\hat{\varepsilon}_1[n] = x[n] - K_1 x[n-1]$$

$$\check{\varepsilon}_1[n] = -K_1 x[n] + x[n-1]$$

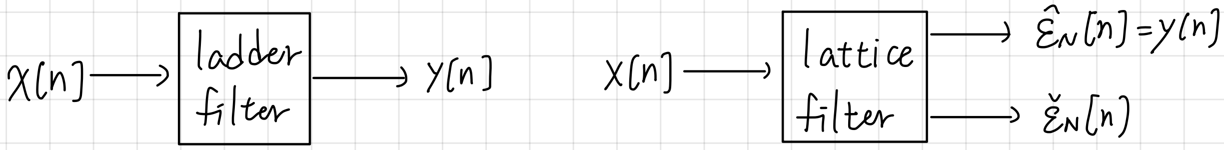
K_1 : reflection coefficient

- N-order lattice filter



$$\hat{\varepsilon}_N[n] = \hat{\varepsilon}_{N-1}[n] - K_N \check{\varepsilon}_{N-1}[n-1] \quad \text{— (12-92a)}$$

$$\check{\varepsilon}_N[n] = \check{\varepsilon}_{N-1}[n-1] - K_N \hat{\varepsilon}_{N-1}[n] \quad \text{— (12-92b)}$$



$$\begin{cases} \hat{E}_N(z) \triangleq Z\{\hat{\varepsilon}_N[n]\} \\ \check{E}_N(z) \triangleq Z\{\check{\varepsilon}_N[n]\} \end{cases}$$

- Since $\hat{\varepsilon}_N[n] = \hat{\varepsilon}_{N-1}[n] - K_N \check{\varepsilon}_{N-1}[n-1]$

$$\longleftrightarrow \hat{E}_N(z) = \hat{E}_{N-1}(z) - K_N \check{E}_{N-1}(z) z^{-1} \longrightarrow (12\text{-}93a)$$

- Similarly, based on (12-92b)

$$\longleftrightarrow \check{E}_N(z) = z^{-1} \check{E}_{N-1}(z) - K_N \hat{E}_{N-1}(z) \longrightarrow (12\text{-}93b)$$

$$\begin{cases} \hat{E}_1(z) = 1 - K_1 z^{-1} \\ \check{E}_1(z) = -K_1 + z^{-1} \end{cases}$$ → first order lattice filter is FIR filter.

$$\rightarrow \check{E}_N(z) = z^{-N} \hat{E}_N\left(\tfrac{1}{z}\right) \quad (12\text{-}94)$$

(The proof is based on mathematical induction) 期末考

Theorem 1、

(1) $\hat{E}_N(z)$ is a polynomial of z^{-1} with degree N, $\forall N$

(2) $\check{E}_N(z)$ is a polynomial of z^{-1} with degree N

We will consider inverse lattice filter that could implement IIR filter

Task:

(1) Given N, $K_1, K_2 \cdots K_N$, obtain $\hat{E}_N(z)$

(2) Given $H(z)$, a polynomial of z^{-1} with degree N, find a lattice filter such that $\hat{E}_N(z) = H(z)$

- Lavinson Algorithm

$$\hat{E}_N(z) = 1 - \sum_{k=1}^{N} a_k^N z^{-k} \quad (12\text{-}95)$$

$$\begin{cases} a_k^N = a_k^{N-1} - K_N a_{N-k}^{N-1} \quad \forall k \in \{1, 2, \cdots N-1\} \\ a_N^N = K_N \end{cases} \quad (12\text{-}97)$$

① $\hat{E}_1(z) = 1 - \underline{K_1} z^{-1} = 1 - \underline{a_1^1} z^{-1}$

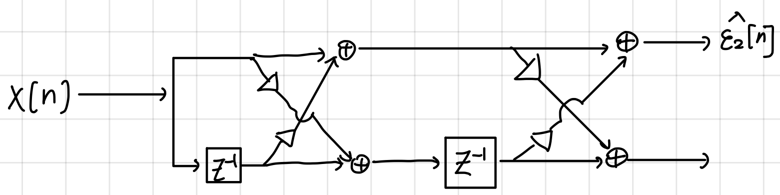
② When $k=2$ $\quad K_1 = a_1^1$

$a_2^2 = K_2$

$a_1^2 = a_1^1 - K_2 a_1^1 = (1-K_2)K_1$

③ $\hat{E}_2(z) = 1 - a_1^2 z^{-1} - a_2^2 z^{-2}$

$= 1 - (1-K_2)K_1 z^{-1} - K_2 z^{-2}$



∘ Lavinson Algorithm for task 2

Input : N, $\{a_k^N\}_{k=1}^N$, $\left(H(z) = 1 - \sum_{k=1}^{N} a_k^N z^{-k}\right)$

Output : $(K_1\ K_2 \cdots K_N)$; $a_N^N \to K_N$

for $i = N$, down to 2

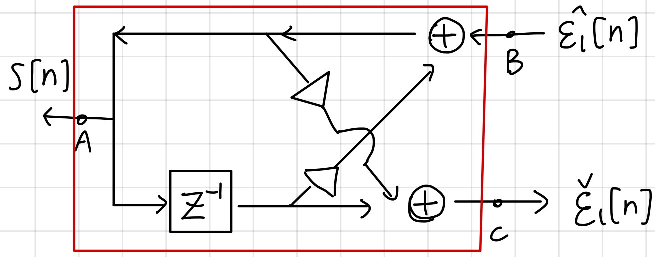
$-a_N^N \cdot z^{-N}$

// $(1-K_i^2)\hat{E}_{i-1}(z) = \hat{E}_i(z) + K_i z^{-1} \hat{E}_i\left(\frac{1}{z}\right)$ (12-98)

use (12-98) to obtain a_k^{i-1}. $K_{i-1} \leftarrow a_{i-1}^{i-1}$

end

∘ Inverse Lattice filter (For realizing an IIR filter)



The system function from B to A is

$$\frac{1}{\hat{E}(z)} = \frac{1}{1-a_1^1 z^{-1}} = \frac{1}{1-K_1 z^{-1}}$$ ← IIR filter

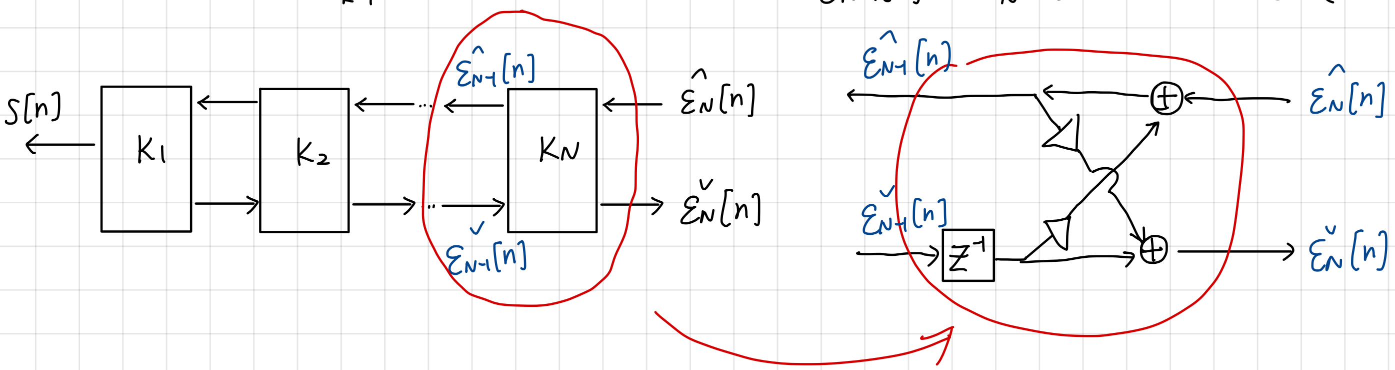
---

For an inverse lattice filter of order N,
The system function from B to A is

$$\frac{1}{\hat{E}_N(z)} = \frac{1}{1-\sum_{k=1}^{N} a_k^N z^{-k}}$$

For the N-th inverse lattice filter

$\hat{\varepsilon}_{N-1}[n] = \hat{\varepsilon}_N[n] + K_N \hat{\varepsilon}_{N-1}[n-1]$ (12-104a)



$\check{\varepsilon}_{N-1}[n) = \check{\varepsilon}_{N-1}[n-1] - K_N \hat{\varepsilon}_{N-1}[n]$ (12-104b)

- Equation (12-104) is identical to equation 12-92
  If we set $\mathcal{E}_N[n] = i[n]$, then $S[n]$ is an AR process
  with system function $L(Z) = \dfrac{1}{1-\sum_{k=1}^{N} a_k^N Z^{-k}}$

- A variant of the Levinson algorithm can be used to solve the Yule-walker equation with N variable in $O(N^2)$ time.

$$\underset{R^{N\times N}}{A}\,\underset{R^{N}}{x} = b$$

Gauss elimination method required $O(N^3)$ time.

- Levinson algorithm for solving the Yule-Walker equation.

  Input : N, $R[m]$

  Output : $(a_1 a_2 \cdots a_N)$

  Step 1. $a_1^1 = K_1 = \dfrac{R[1]}{R[0]}$, $P_1 = (1-K_1^2)P_0$; $P_0 = R[0]$

  Step 2. for $i=2$ to N

$$P_{N-1}K_N = R[N] - \overset{O(N)}{\boxed{\sum_{k=1}^{N-1} a_k^{N-1} R[N-k]}} \quad (12-107)$$

$$P_N = (1-K_N^2)P_{N-1} \quad (12-108)$$

  end

  step3. $a_k \overset{O(N)}{\longleftarrow} a_k^N$

  $\forall K \in \{1,2,\cdots N\}$

$X[n]$ is an AR process with $L_x(Z) = \dfrac{\sqrt{P_N}}{D(Z)} = \dfrac{\sqrt{P_N}}{1-\sum_{k=1}^{N} a_k^N Z^{-k}}$

$$\left.\begin{aligned} \hat{\mathcal{E}}_N[n] &= X[n] - \sum_{k=1}^{N} a_k^N X[n-k] \\ \check{\mathcal{E}}_N[n] &= X[n-N] - \sum_{k=1}^{N} a_k^N X[n+k-N] \end{aligned}\right\} \text{12-105}$$

12/2

Multiple-input Multiple-output (MIMO) Filter

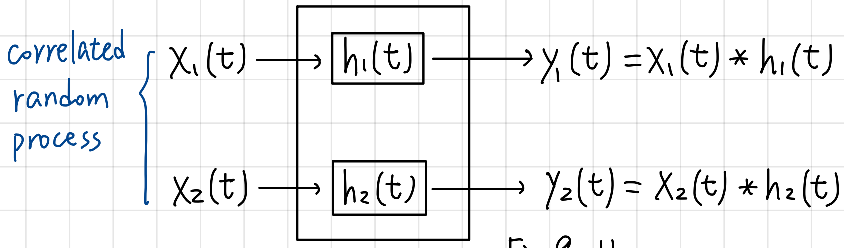


Fig 9-11

- $H(t) \triangleq \begin{bmatrix} h_1(t) & 0 \\ 0 & h_2(t) \end{bmatrix}$: impulse response matrix

- $$\begin{cases} R_{x_1y_2}(t_1,t_2) = \int_{-\infty}^{\infty} R_{x_1x_2}(t_1, t_2-\alpha) \cdot h_2^*(\alpha)\, d\alpha & (9\text{-}130) \\ R_{y_1y_2}(t_1,t_2) = \int_{-\infty}^{\infty} R_{x_1y_2}(t_1-\alpha, t_2) \cdot h_1(\alpha)\, d\alpha & (9\text{-}131) \end{cases}$$

- When $R_{x_1y_2}(t_1,t_2)$ depend on (t_1, t_2) only through $t_1 - t_2$,

  based on (9-130), we have

  $R_{x_1y_2}(t_1-t_2) = \int_{-\infty}^{\infty} R_{x_1x_2}(t_1-t_2+\alpha)\, h_2^*(\alpha) d\alpha \rightarrow R_{x_1y_2}(\tau) = \int_{-\infty}^{\infty} R_{x_1y_2}(\tau+\alpha)\, h_2^*(\alpha)\, d\alpha$

- Note that

  $R_{x_1y_2}(\tau) \neq R_{x_1x_2}(\tau) * h_2^*(\tau)$, However, defining $\tilde{h_2}(\tau) = h_2(-\tau)\ \forall \tau \in R$

  we have $R_{x_1y_2}(\tau) = R_{x_1x_2}(\tau) * \tilde{h_2}(\tau)$ $\leftrightarrow \tilde{H}(s) = H(-s)$

$$\mathcal{L}\{f(t)\} \triangleq \int_{-\infty}^{\infty} f(t) e^{-st}\, dt$$

$$\tilde{h_2}(t) = h_2(-t)$$

$$\tilde{H_2}(s) = \int_{-\infty}^{\infty} \tilde{h_2}(t)\, e^{-st}\, dt$$

$$= \int_{-\infty}^{\infty} h_2(-t)\, e^{-st}\, dt$$

$$= \int_{\infty}^{-\infty} h_2(u)\, e^{su} \cdot -du$$

$$= H_2(-s) \quad ※$$

① Choose $X_1(t) = S(t)$ and $X_2(t) = X(t)$, $h_2(t) = r(t)$

$$S(t) \longrightarrow \boxed{h_1(t)} \longrightarrow Y_1(t)$$

$$X(t) \longrightarrow \boxed{r(t)} \longrightarrow i_x(t) = X(t) * r(t)$$

$$\hat{S}(t+\lambda) \triangleq E\{ S(t+\lambda) \mid X(t-\tau),\ \tau \geq \}$$

Lemma 1. $S_{six}(s) = S_{sx}(s) \cdot \Gamma_x(-s)$ $\quad$ (13-94)

proof :

② based on (9-130) and $\hat{r}_x(\tau) \triangleq r_x(-\tau)$, $\forall \tau \in R$

$$R_{six}(\tau) = R_{sx}(\tau) * \hat{r}_x(\tau)$$

$$\xrightarrow{\mathcal{L}} S_{six}(s) = S_{sx}(s) \cdot \Gamma_x(-s)$$

$$X(t) \longrightarrow \boxed{\Gamma_x(s)} \xrightarrow{i_x(t)} \boxed{H_{ix}(s)} \longrightarrow \hat{S}(t+\lambda)$$

- Recall

$$h_{ix}(\tau) = R_{six}(\tau+\lambda)\, u(\tau) \qquad \text{shift property} \qquad (13\text{-}93)$$

$$S_x(s) = \mathcal{L}\{ R_{si}(\tau+\lambda) \}(s) = S_{six}(s) e^{\lambda s} \qquad (13\text{-}96)$$

- Given $S_{xx}(s)$, $S_{ss}(s)$, $S_{sx}(s) \rightarrow S_{six}(s) = S_{sx}(s)\, \Gamma_x(-s)$

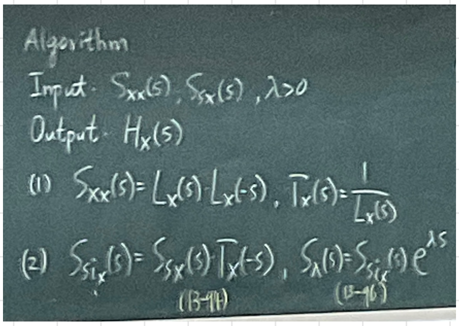
$$\begin{array}{ccccc}
\circ\ S_{xx}(s),\ S_{sx}(s) & \xrightarrow{(13\text{-}94)} & S_{six}(s) & \xrightarrow{(13\text{-}96)} & S_x(s) \\
 & & & & \downarrow \\
 & & R_{six}(\tau+\lambda)\, u(\tau) = h_{ix}(\tau) & \xleftarrow{\mathcal{L}^{-1}} & S_x^+(s)
\end{array}$$

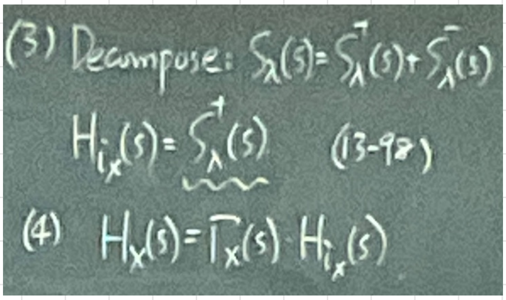
$$R_{six}(\tau+\lambda) = R_{six}(\tau+\lambda) \cdot [u(\tau) + u(-\tau)]$$

$$= \underbrace{R_{six}(\tau+\lambda)\, u(\tau)}_{\text{causal}} + \underbrace{R_{six}(\tau+\lambda)\, u(-\tau)}_{\text{anti-causal}}$$

$$\begin{cases} S_x^+(s) \triangleq \mathcal{L}\{ R_{six}(\tau+\lambda)\, u(\tau) \} \leftarrow \text{analytic in } Re\{s\} > 0 \rightarrow \text{has no pole in } Re(s) > 0 \\ S_x^-(s) \triangleq \mathcal{L}\{ R_{six}(\tau+\lambda)\, u(-\tau) \} \leftarrow \text{analytic in } Re\{s\} < 0 \rightarrow \text{has no pole in } Re(s) < 0 \end{cases}$$

$$\rightarrow S_x(s) = S_x^+(s) + S_x^-(s)$$





EX 13-7

$$x(t)=s(t)+v(t)$$
$$S_{ss}(\omega)=\frac{N_0}{\alpha^2+\omega^2}\ \forall\omega\in\mathbb{R} \longleftrightarrow R_{ss}(\tau)=N_0e^{-\alpha|\tau|}$$
$$S_{vv}(\omega)=N>0,\ \forall\omega\in\mathbb{R} \longleftrightarrow R_{vv}(\tau)=N\delta(\tau)$$
$$S_{sv}(\omega)=0\quad,\ \forall\omega\in\mathbb{R}\longleftrightarrow R_{sv}(\tau)=0$$

(1) Find $S_{xx}(\omega)$, $S_{sx}(\omega)$
(2) Find $H_x(s)$ for $\hat{s}(\tau+\lambda)$

<Sol>

① $R_{xx}(t_1,t_2)$

$$\begin{aligned}
&= E\{x(t_1)x(t_2)\}\\
&= E\{[s(t_1)+v(t_1)]\cdot[s(t_2)+v(t_2)]\}\\
&= E[s(t_1)\cdot s(t_2)]+E[s(t_1)v(t_2)]\\
&\quad +E[v(t_1)s(t_2)]+E[v(t_1)\cdot v(t_2)]\\
&= R_{ss}(t_1,t_2)+\cancel{R_{sv}(t_1,t_2)}+\cancel{R_{sv}(t_2,t_1)}+R_{vv}(t_1,t_2)\\
&= R_{ss}(t_1-t_2)+N\delta(t_1-t_2)\\
&= N_0e^{-\alpha|t_1-t_2|}+N\delta(t_1-t_2)
\end{aligned}$$

② $S_{xx}(\omega)$

$$\begin{aligned}
&= \mathcal{F}\{R_{ss}(\tau)+R_{vv}(\tau)\}\\
&= S_{ss}(\omega)+S_{vv}(\omega)\\
&= \frac{N_0}{\alpha^2+\omega^2}+N \ \#
\end{aligned}$$

$S_{xx}(s)$

$$\begin{aligned}
&= \frac{N_0}{\alpha^2-s^2}+N = N\left(\frac{N_0/N}{\alpha^2+s^2}+1\right)\\
&= N\left(\frac{\overbrace{N_0/N+\alpha^2}^{\beta^2}-s^2}{\alpha^2-s^2}\right)=N\left(\frac{\beta^2-s^2}{\alpha^2-s^2}\right)
\end{aligned}$$

③ $R_{sx}(\tau)\xrightarrow{\mathcal{F}}S_{sx}(\omega)\xrightarrow{s=j\omega}S_{sx}(s)$

- $R_{sx}(t_1,t_2)$

$$\begin{aligned}
&\triangleq E[s(t_1)\cdot x(t_2)]\\
&= E\{s(t_1)\cdot[s(t_2)+v(t_2)]\}\\
&= R_{ss}(t_1,t_2)+\cancel{R_{sv}(t_1,t_2)}\\
&= R_{ss}(t_1-t_2)
\end{aligned}$$

④ $S_{xx}(s)$

$$=\sqrt{N}\left(\frac{\beta+s}{\alpha+s}\right)\sqrt{N}\left(\frac{\beta-s}{\alpha-s}\right)$$

$-\beta \quad -\alpha \quad 0 \quad \alpha \quad \beta$

$$\rightarrow L_x(s)=\sqrt{N}\left(\frac{\beta+s}{\alpha+s}\right)\rightarrow \Gamma_x(-s)=\frac{\alpha-s}{\sqrt{N}(\beta-s)}$$

⑤ • $S_{si_x}(s)$

$$\begin{aligned}
&= S_{sx}(s)\cdot\Gamma_x(-s)\\
&= S_{ss}(s)\cdot\Gamma_x(-s)\\
&= \frac{N_0}{\alpha^2-s^2}\cdot\frac{\alpha-s}{\sqrt{N}(\beta-s)}\\
&= \frac{N_0}{\sqrt{N}}\cdot\frac{1}{(\alpha+s)(\beta-s)}\\
&= \frac{A_1}{s+\alpha}-\frac{A_2}{s+\beta}\ ;\ A_1=A_2=\frac{N_0}{\sqrt{N}}\cdot\frac{1}{\alpha+\beta}
\end{aligned}$$

⑥ $S_\lambda(s)=S_{si_x}(s)e^{\lambda s}$

$$=\left(\frac{A}{s+\alpha}-\frac{A}{s-\beta}\right)e^{\lambda s}$$

$$\Rightarrow S_\lambda(s)=S_\lambda^+(s)+S_\lambda^-(s)$$

Theorem 1.

If $S_{six}(s) = \sum_i \frac{a_i}{s-s_i} + \sum_k \frac{b_k}{s-z_k}$ (13-99),

where $Re\{s_i\} < 0$ and $Re\{z_k\} > 0$, $\forall\, i, k$, then

(2) $S_\lambda^+(s) = \sum_i \frac{a_i e^{s_i\lambda}}{s-s_i}$

(1) $R_{six}(\tau+\lambda)u(\tau) = \sum_i a_i e^{s_i(\tau+\lambda)} u(\tau) = \sum_i a_i e^{s_i\lambda} \cdot e^{s_i\tau} u(\tau)$

⑦ Applying Theorem 1, with $a_1 = A$, $s_1 = -\alpha < 0$, $b_1 = -A$, $z_1 = \beta > 0$,

we have $S_\lambda^+(s) = \frac{Ae^{-\alpha\lambda}}{s+\alpha}$

⑧ $H_x(s) = \Gamma_x(s) \cdot S_\lambda^+(s) = \left(\frac{1}{\sqrt{N}} \frac{\cancel{\alpha+s}}{\beta+s}\right) \cdot \frac{Ae^{-\alpha\lambda}}{\cancel{s+\alpha}}$, then,

$H_x(s) = \frac{A}{\sqrt{N}} \frac{e^{-\alpha\lambda}}{s+\beta} = \frac{\beta-\alpha}{s+\beta} \cdot e^{-\alpha\lambda}$ #

$\begin{cases} \beta^2 = \alpha^2 + \frac{N_0}{N} \\ \beta^2 - \alpha^2 = \frac{N_0}{N} \end{cases} \rightarrow \frac{A}{\sqrt{N}} = \frac{N_0}{N} \frac{1}{\alpha+\beta} = \beta - \alpha$

$(\alpha+\beta)(\beta-\alpha)$, $\alpha+\beta = N_0(\beta-\alpha)/N$

證明 期末會考

Hint:

(1) $R_{six}(\tau) = \mathcal{L}^{-1}\{S_{six}(s)\}$

$= \underbrace{\sum_i a_i e^{s_i\tau} u(\tau)}_{\text{causal}} + \underbrace{\sum_k b_k e^{z_k\lambda} u(-\tau)}_{\text{anti-causal}}$

(2) $u(\tau+\lambda)u(\tau) = u(\tau)$

$u(-\tau-\lambda)u(\tau) = 0$

Discrete-time Process

$$\hat{S_r}[n+r] = \hat{E}\{S[n+r] \mid x[n-k],\ k \geq 0\} = \sum_{k=0}^{\infty} h_x^r[k]\, x[n-k] \quad (13\text{-}105)$$

- $S_r[n+r] - \hat{S_r}[n+r] \perp x[n-m]$, $\forall m \geq 0$

$$\rightarrow R_{sx}[m+r] = \sum_{k=0}^{\infty} h_x^r[k]\, R_{xx}[m-k], \ \forall m \geq 0 \quad (13\text{-}106)$$

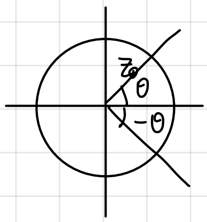
$$\hat{S_r}[n+r] = \sum_{k=0}^{\infty} h_{ix}^r[k]\, i_x[n-k] \quad (13\text{-}107)$$

$$X[n] \longrightarrow \boxed{\Gamma_x(z)} \xrightarrow{i_x[n]} \boxed{H_{ix}^r(z)} \xrightarrow{\hat{S_r}[n-r]}$$

- $h_{ix}^r[m] = R_{six}[m+r]\, u[m]$, $\forall m \in \mathbb{Z} = \{0, \pm1, \pm2 \cdots\}$ $\quad (13\text{-}108)$

$h_{ix}(\tau) = R_{six}(\tau+\lambda)\, u(\tau)$ ← continuous case

$$S_{six}(z) = Z\{R_{six}[m]\} = S_{sx}(z) \cdot \Gamma_x(z^{-1}) \quad (13\text{-}109)$$



$z = re^{j\theta}$

$z^{-1} = \frac{1}{r} e^{-j\theta}$

$S_r(z) = S_{six}(z) \cdot z^r \quad (13\text{-}110)$

$S_\lambda(s) = S_{six}(s) \cdot e^{\lambda s} \quad (13\text{-}96)$

$S_r(z) = S_r^+(z) + S_r^-(z)$

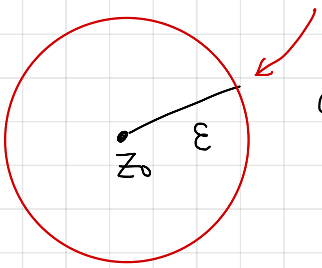
$S_r^+(z)$ is analytic for $|z| > 1$

$S_r^-(z)$ is analytic for $|z| < 1$

Definition: A function $f: C \rightarrow C$ is said to be analytic at z_0 if

(1) $f(z)$ is differentiable at z_0

(2) There exist $\varepsilon > 0$ such that f is differentiable at z_1 whenever $|z_1 - z_0| \leq \varepsilon$



$q(z) \triangleq \frac{1}{z - z_0}$ is not differentiable at z_0