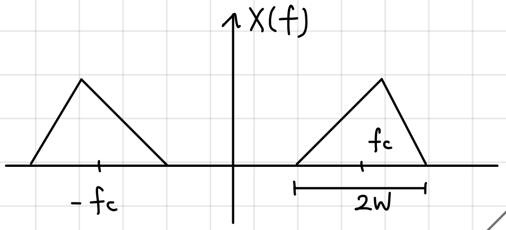
Digital Communication Ch2

2024 Fall

Bandpass signal $x(t)$

Def
Bandpass signal $x(t)$ is real-value signal



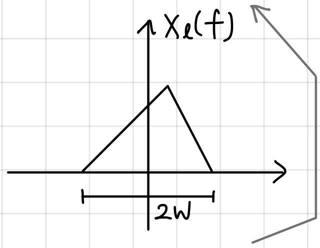
- $x(t)$ real $\longleftrightarrow X(f) = X^*(-f)$ Hermitian symmetric
- $x(t)$ complex $\longleftrightarrow X(f) \neq X^*(-f)$

Baseband signal $x_l(t) = x_i(t) + j x_q(t)$

Def
Baseband or lowpass (equivalent) signal

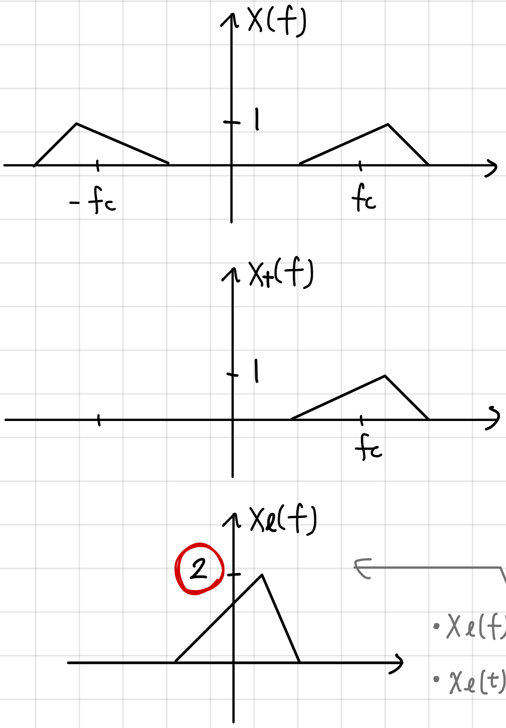
「等效」基頻不一定存在，只是為了方便分析

$x_l(t)$ is complex-value signal



- $X_l(f)$ 未必 Hermitian symmetric
- $x_l(t) = x_i(t) + j x_q(t)$

$x_l(t) \longleftrightarrow x_+(t) \longleftrightarrow x(t)$



$$x_+(t) = \frac{1}{2}\left[x(t) + j\hat{x}(t)\right]$$

$$\begin{cases} X_l(f) = 2X_+(f+f_c) \\ x_l(t) = 2x_+(t)e^{-j2\pi f_c t} \end{cases}$$

$$= \left[x(t) + j\hat{x}(t)\right]e^{-j2\pi f_c t}$$

$$= x_i(t) + j x_q(t)$$

$$\boxed{\begin{cases} x(t) = Re\{x_l(t)e^{j2\pi f_c t}\} \\ X(f) = \frac{1}{2}\left[X_l(f-f_0) + X_l^*(-f-f_0)\right] \end{cases}}$$

$$\begin{cases} x(t) = x_I(t)\cos(2\pi f_c t) - j x_Q(t)\sin(2\pi f_c t) \\ x_I(t) = x(t)\cos(2\pi f_c t) + j\hat{x}(t)\sin(2\pi f_c t) \end{cases}$$

- $X_l(f)$ 不對稱 $\longleftrightarrow x_l(t)$ is complex
- $x_l(t) = x_I(t) + j x_q(t)$

- $x_l(t)$ 為 $x(t)$ 的等效基頻
- $x(t)$ 為 $x_I(t)$ 的等效基頻

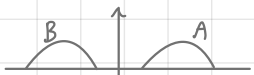
Energy of signal

$$\boxed{E_{x_l} = 2E_x = 4E_{x_+}}$$

<pf> $E_s = \int_{-\infty}^{\infty}|s(t)|^2 dt = \int_{-\infty}^{\infty}|S(f)|^2 df$

- $E_x = \int_{-\infty}^{\infty}|X(f)|^2 df = 2E_{x_+}$ —— ①
- $E_{x_l} = 4E_{x_+}$ —— ②

$\Rightarrow E_{x_l} = 2E_x = 4E_{x_+}$



$$|X(f)|^2 = X(f)X^*(f) = (A+B)(A^*+B^*)$$

$$= AA^* + \cancel{AB^*} + \cancel{BA^*} + BB^*$$

$$= AA^* + BB^* = 2E_{x_+}$$

Inner product

Def

$$\begin{cases} \langle x(t), y(t)\rangle = \int_{-\infty}^{\infty} x(t)\, y^*(t)\, dt \\ \langle x(t), y(t)\rangle = \langle X(f), Y(f)\rangle \end{cases}$$

必滿足下列性質

1. 左線性 : $\langle \alpha\vec{x}, \vec{y}\rangle = \alpha\langle \vec{x}, \vec{y}\rangle$
2. 右共軛線性 : $\langle \vec{x}, \alpha\vec{y}\rangle = \alpha^*\langle \vec{x}, \vec{y}\rangle$
3. 共軛交換性 : $\langle \vec{x}, \vec{y}\rangle = \langle \vec{x}, \vec{y}\rangle^*$
4. 恆正線 : $\forall \vec{x} \neq 0,\ \langle \vec{x}, \vec{x}\rangle > 0$

$$\langle x(t), y(t)\rangle = \frac{1}{2} Re\{\langle x_\ell(t), y_\ell(t)\rangle\}$$

• 射頻正交 $\overset{\times}{\rightleftarrows}$ 基頻正交

<pf>

$\langle x(t), y(t)\rangle$

$= \langle X(f), Y(f)\rangle$

$= \langle \frac{1}{2}X_\ell(f-f_c) + \frac{1}{2}X^*_\ell(-f-f_c), \frac{1}{2}Y(f-f_c) + \frac{1}{2}Y^*(-f-f_c)\rangle$

$= \frac{1}{4}X_\ell(f-f_c)Y_\ell(f-f_c) + \frac{1}{4}X^*_\ell(-f-f_c)Y^*_\ell(-f-f_c)$

$= \frac{1}{4}\langle x_\ell(t), y_\ell(t)\rangle + \frac{1}{4}\langle x_\ell(t), y_\ell(t)\rangle^* = \frac{1}{2}Re\{\langle x_\ell(t), y_\ell(t)\rangle\}$

Bandpass system

Def

Bandpass system is LTI system with Real impulse response h(t)

$$y(t) = x(t) * h(t) \longleftrightarrow y_\ell(t) = \frac{1}{2} x_\ell(t) * h_\ell(t)$$

<pf>

- $X_\ell(f) = 2X_+(f+f_0) = 2X(f+f_0)\,u(f+f_0)$
- $H_\ell(f) = 2H_+(f+f_0) = 2H(f+f_0)\,u(f+f_0)$
- $Y_\ell(f) = 2Y_+(f+f_0) = 2Y(f+f_0)\,u(f+f_0)$

其中 $Y(f+f_0) = X(f+f_0)\cdot H(f+f_0)$

$\rightarrow Y_\ell(f) = \frac{1}{2}[2X(f+f_0)u(f+f_0)]\cdot[2H(f+f_0)u(f+f_0)]$

$= \frac{1}{2}X_\ell(f)\cdot H_\ell(f)$ #

Cross-correlation

Def

$$\rho_{x,y} = \frac{\langle x(t), y(t)\rangle}{\sqrt{E_x E_y}}$$

$$\rho_{XY} = 0 \overset{\times}{\rightleftarrows} \rho_{X_\ell Y_\ell} = 0$$

Random process

- $m_X(t) = E[X(t)]$
- $R_X(t_1,t_2) = E[X(t_1)X^*(t_2)]$
- $K_X(t_1,t_2) = E[(X(t_1)-m_X(t_1))(X(t_2)-m_X(t_2))^*]$

$$\begin{cases} R_{XY}(t_1,t_2) = E[X(t_1)\cdot Y^*(t_2)] \\ R_{YX}(t_2,t_1) = E[Y(t_2)\cdot X^*(t_1)] \end{cases}$$

$R_{XY}(t_1,t_2) = R^*_{YX}(t_2,t_1)$

- $K_{XY}(t_1,t_2) = E[(X(t_1)-m_X(t_1))(Y(t_2)-m_Y(t_2))^*]$

Stationary Random Process

$F_X(x_1 \cdots x_k; t_1, t_2 \cdots t_k) = F_X(x_1 \cdots x_k; t_1-\tau, t_2-\tau \cdots)$

任意平移後不改變 Joint distribute function

Wide Sence Stationary (WSS)

- $E[X(t)]$ is constant
- $R_X(t_1,t_2) = R_X(\tau)$: 時間差τ的函數

Joint correlation

Def

$R_{XY}(t_1,t_2)$

- rv X, Y 皆 WSS
- 為 $\tau = t_1 - t_2$ 的函數 $R_{XY}(\tau)$

Joint correlation

定理

若 rv X, Y 為 Joint WSS

$$\boxed{R_{XY}(\tau) = R^*_{YX}(-\tau)}$$

- $R_{XY}(\tau) \triangleq E[X(t+\tau)\cdot Y^*(t)]$
  $= (E[Y^*(t)X(t+\tau)])^*$
  $= R^*_{YX}(-\tau)$
- $R_X(\tau) \triangleq E[X(t+\tau)X^*(t)]$
  $= (E[X(t)X^*(t+\tau)])^*$
  $= R^*_X(-\tau)$

Power Spectrum Density

Def

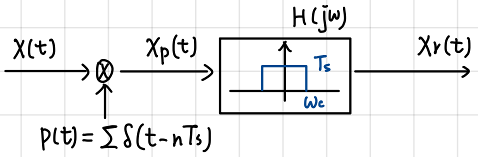
若 rv X 為 WSS，則 $\boxed{S_X(f) = \mathcal{F}\{R_X(\tau)\}}$

Theorem

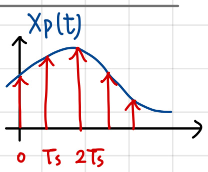
- $S_X(f)$ is real / non-negative
- $R_X(0) = E[|X(t)|^2] = \int_{-\infty}^{\infty} S_X(f)\,df$ ：總功率
- if rv X real，$S_X(f)$ even
- $S_X(f) = \lim_{T\to\infty} \frac{1}{2T} E[|X_T(f)|^2]$

- $P = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} |X_T(t)|^2 dt = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} |X_T(f)|^2 df$

Ideal Sampling



理想取樣

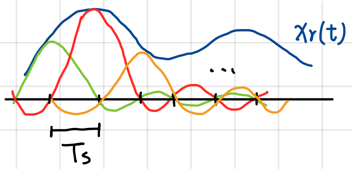


- $p(t) = \sum_{n=-\infty}^{\infty} \delta(t-nT_s)$
- $x_p(t) = \sum x(nT_s)\cdot\delta(t-nT_s)$
- $X_p(j\omega) = \frac{1}{2\pi} X(j\omega) * P(j\omega) = \boxed{\frac{1}{T_s}\sum X(j(\omega-k\omega_s))}$

訊號重建

- $x_r(t) = x_p(t) * h(t) = \sum x(nT_s)\cdot h(t-nT_s)$
- $H(j\omega) = T_s\cdot\Pi\left(\frac{\omega}{2\omega_c}\right)$
- $h(t) = \mathrm{sinc}\left(\frac{t}{T_s}\right)$; if $\omega_s = 2\omega_c$

$\rightarrow \boxed{x_r(t) = \sum x(nT_s)\,\mathrm{sinc}\left(\frac{t-nT_s}{T_s}\right)}$



LTI system with WSS Random Process

$$\underset{\text{WSS}}{\text{rv } X(t):} \longrightarrow \boxed{h(t)}_{\text{LTI}} \longrightarrow \underset{\text{WSS}}{\text{rv } Y(t):}$$

m_X	$m_Y = m_X\cdot\lvert H(0)\rvert$
$R_X(\tau)$	$R_Y(\tau) = R_X(\tau) * h(\tau) * h^*(-\tau)$ $R_{XY}(\tau) = R_Y(\tau) * h^*(-\tau)$
$S_X(f)$	$S_Y(f) = S_X(f)\lvert H(f)\rvert^2$ $S_{XY}(f) = S_X(f)\cdot H^*(f)$

$$R_{X,Y}(\tau) = E\left[X(t+\tau)\left(\int_{-\infty}^{\infty} h(u)X(t-u)\,du\right)^*\right]$$
$$= \int_{-\infty}^{\infty} h^*(u) R_X(\tau+u)\,du = \int_{-\infty}^{\infty} h^*(-v) R_X(\tau-v)\,dv$$
$$= R_X(\tau) * h^*(-\tau)$$

$$R_Y(\tau) = E\left[\int_{-\infty}^{\infty} h(u)X(t+\tau-u)\,du \cdot \left(\int_{-\infty}^{\infty} h(v)X(t-v)\,dv\right)^*\right]$$
$$= \int_{-\infty}^{\infty} h(u)\left(\int_{-\infty}^{\infty} h^*(v) R_X((\tau-u)+v)\,dv\right) du$$
$$= \int_{-\infty}^{\infty} h(u) R_{XY}(\tau-u)\,du = R_{XY}(\tau) * h(\tau)$$
$$= R_X(\tau) * h(\tau) * h^*(-\tau) \quad \#$$

Bandpass Signal Fundamental Assumption (2-9)

Fundamental assumption

$N(t)$ is bandpass WSS r.p. Real-value

則 $\begin{cases} N_\ell(t) \text{ is WSS} \\ N_i(t)\ N_q(t) \text{ are WSS} \end{cases}$

$\longrightarrow$

- $m_x = 0 \longleftrightarrow m_{xi} = m_{xq} = 0$
- $\begin{cases} R_{N_i}(\tau) = R_{N_q}(\tau) \\ R_{N_i N_q}(\tau) = -R_{N_q N_i}(\tau) \end{cases}$
- $R_x(\tau) = \frac{1}{2}Re\{R_{x\ell}(\tau)e^{j2\pi f_c t}\}$
- $S_x(f) = \frac{1}{4}(S_{x\ell}(f-f_c) + S_{x\ell}^*(-f-f_c))$
- if $S_{x\ell}(f) = S_{x\ell}(-f) \longleftrightarrow R_{x_i x_q}(\tau) = 0$

〈說明〉

$$\boxed{m_N = 0 \leftrightarrow m_{Ni} = m_{Nq} = 0}$$

$$m_N = m_{Ni}\cos(2\pi f_c t) - m_{Nq}\sin(2\pi f_c t)$$

$$\boxed{\begin{aligned} &R_{Ni}(\tau) = R_{Nq}(\tau) \\ &R_{NiNq}(\tau) = -R_{NqNi}(\tau) \end{aligned}}$$

- $\cos(A)\cos(B) = \frac{\cos(A-B)+\cos(A+B)}{2}$
- $\sin(A)\cos(B) = \frac{\sin(A-B)+\sin(A+B)}{2}$
- $\sin(A)\sin(B) = \frac{\cos(A-B)-\cos(A+B)}{2}$

$$\begin{aligned} &R_N(\tau) \\ &\triangleq E[N(t+\tau)N(t)] \\ &= E[Re\{N_\ell(t+\tau)e^{j2\pi f_c(t+\tau)}\}\cdot Re\{N_\ell(t)e^{j2\pi f_c t}\}] \\ &= E[(N_i(t+\tau)\cos(2\pi f_c(t+\tau)) - N_q(t+\tau)\sin(2\pi f_c(t+\tau))) \\ &\qquad (N_i(t)\cos(2\pi f_c t) - N_q(t)\sin(2\pi f_c t))] \end{aligned}$$

$$\begin{aligned} &= \frac{R_{xi}(\tau)+R_{xq}(\tau)}{2}\cos(2\pi f_c\tau) \\ &+ \frac{R_{xixq}(\tau)-R_{xqxi}(\tau)}{2}\sin(2\pi f_c\tau) \end{aligned} \Rightarrow \boxed{\begin{aligned} R_x(\tau) = &R_{xi}(\tau)\cos(2\pi f_c\tau) \\ &- R_{xqxi}(\tau)\sin(2\pi f_c\tau) \end{aligned}}$$

$$+ \frac{R_{xi}(\tau)-R_{xq}(\tau)}{2}\cancel{\cos(2\pi f_c(2t+\tau))}^{\,0} \Rightarrow R_{xi}(\tau) = R_{xq}(\tau)$$

$$- \frac{R_{xixq}(\tau)+R_{xqxi}(\tau)}{2}\cancel{\sin(2\pi f_c(2t+\tau))}^{\,0} \Rightarrow R_{xqxi}(\tau) = -R_{xixq}(\tau)$$

$$\boxed{R_N(\tau) = \frac{1}{2}Re\{R_{N\ell}(\tau)e^{j2\pi f_c\tau}\}}$$

- $R_{Ni}(\tau) = R_{Nq}(\tau)$
- $R_{NiNq}(\tau) = -R_{NqNi}(\tau)$

$$\begin{aligned} R_{N\ell}(\tau) &= E[N_\ell(t+\tau)\cdot N_\ell^*(t)] \\ &= E[(N_i(t+\tau)+jN_q(t+\tau))\cdot(N_i(t)-jN_q(t))] \\ &= E[N_i(t+\tau)N_i(t)] - jE[N_i(t+\tau)N_q(t)] \\ &\quad + E[N_q(t+\tau)N_q(t)] + jE[N_q(t+\tau)N_i(t)] \\ &= R_{Ni}(\tau) + R_{Nq}(\tau) + jR_{NqNi}(\tau) - jR_{NiNq}(\tau) \\ &= 2R_{Ni}(\tau) + 2jR_{NqNi}(\tau) \end{aligned}$$

$$\begin{aligned} R_N(\tau) &= R_{Ni}(\tau)\cos(2\pi f_c\tau) - R_{xqxi}(\tau)\sin(2\pi f_c\tau) \\ &= Re\{\tfrac{1}{2}R_{N\ell}(\tau)e^{j2\pi f_c\tau}\} \text{ ※} \end{aligned}$$

$$\boxed{S_N(f) = \frac{1}{4}[S_N(f-f_c) + S_N^*(-f-f_c)]}$$

$$R_N(\tau) = \frac{1}{2}Re\{R_{N\ell}(\tau)e^{j2\pi f_c\tau}\}$$

FT

$$= \frac{1}{4}[R_{N\ell}(\tau)e^{j2\pi f_c\tau} + R_{N\ell}^*(\tau)e^{-j2\pi f_c\tau}]$$

$$S_N(f) = \frac{1}{4}[S_{N\ell}(f-f_c) + S_{N\ell}^*(-f-f_c)] \text{ ※}$$

NOTE: