Digital Signal Processing

Ch8&9 DFS DFT FFT

2024 Fall

modulo N

- 若 $\tilde{x}[n]$ 為週期信號，則滿足 $\tilde{x}[n]=\tilde{x}[n+N]$
- $((n))_N$：將超出範圍的 n 折返到週期內
- EX：假設 $\tilde{x}[n]=(1,2,3,4)$，$N=4$

$$\left.\begin{aligned} x[((0))_4] &= \tilde{x}[0]=1 \\ x[((3))_4] &= \tilde{x}[3]=4 \\ x[\underbrace{((5))_4}_{5\text{超出範圍}}] &= \tilde{x}[1]=2 \end{aligned}\right\} \quad \tilde{x}[n]=x[((n))_N] \text{ 依然為週期信號}$$

DFS

DTFT：
$$\begin{cases} x[n]=\frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}d\omega \\ X(e^{j\omega})=\sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \end{cases}$$

$$\sum e^{-j2\pi nT_0 f}=\frac{1}{T_0}\sum\delta\left(f-\frac{n}{T_0}\right)$$

$$\begin{cases} \tilde{X}[k]=\sum_{n=0}^{N-1}\tilde{x}[n]W_N^{kn} \\ \tilde{x}[n]=\frac{1}{N}\sum_{n=0}^{N-1}\tilde{X}[k]W_N^{-kn} \end{cases} \qquad \boxed{W_N=e^{-j\frac{2\pi}{N}}}$$

$$\Rightarrow \begin{aligned} \tilde{X}[k] &= \tilde{X}[k-N] \\ \tilde{x}[n] &= \tilde{x}[n-N] \end{aligned}$$ 皆為週期信號，故範圍為 $(-\infty,\infty)$

- $e^{j(\frac{2\pi}{N})(k+\ell N)n}=e^{j(\frac{2\pi}{N}k)+\cancel{2\pi k\ell}}$
- $\frac{1}{N}\sum_{n=0}^{N-1}e^{j(\frac{2\pi}{N})\ell n}=\frac{1}{N}\cdot\frac{1-e^{j(\frac{2\pi}{N})\ell n}}{1-e^{j\frac{2\pi}{N}\ell}}=\begin{cases}1 ; & \ell=mN \\ 0 ; & \text{o.w}\end{cases}$
- $$\begin{cases} X(e^{j\omega})=\sum_{n=0}^{N-1}x[n]e^{-j\omega n}=\sum_{n=0}^{N-1}\tilde{x}[n]e^{-j\omega n} \\ \tilde{X}[k]=\sum_{n=0}^{N-1}\tilde{x}[n]W_N^{kn}=\sum_{n=0}^{N-1}\tilde{x}[n]e^{-j\frac{2\pi}{N}kn}=X(e^{j\omega})\Big|_{\omega=\frac{2\pi}{N}k} \end{cases}$$

由於 DFS 的 $\tilde{x}[n]$ 和 $\tilde{X}[k]$ 皆為週期信號 → range 為 $(-\infty,\infty)$
而 DFT 則是選取 DFS 的其中一段，使其為有限區間

$0\le n\le N-1$

DFT

有限個 sequence

- $x[n]=\begin{cases}\tilde{x}[n] ; & 0\le n\le N-1 \\ 0 ; & \text{otherwise}\end{cases}$
- $X[k]=\begin{cases}\tilde{X}[k] ; & 0\le n\le N-1 \\ 0 ; & \text{otherwise}\end{cases}$

or

$$\tilde{x}[n]=x[((n))_N]$$
$$\tilde{X}[k]=X[((k))_N]$$

若 $N=10$，$k=17$ 則
$X[((17))_{10}]=X[7]$

$\tilde{x}[n]$，$\tilde{X}[k]$ 為週期信號，
取長度 N 做為循環。

$$\Rightarrow \begin{cases} X[k]=\sum_{n=0}^{N-1}x[n]W_N^{kn} ; & 0\le n\le N-1 \\ x[n]=\sum_{n=0}^{N-1}X[k]W_N^{-kn} ; & 0\le k\le N-1 \end{cases}$$

DFT property

1. Time-domain circular shift.

$$x_1[n] = \begin{cases} \tilde{x}_1[n] = x[((n-n_0))_N] \ ; & 0 \le n \le N-1 \\ 0 \ ; & \text{otherwise.} \end{cases}$$

EX

$x[n]$, $N=3$ 求 $x[((n-1))_N]$

- $x_1[0] = x[((-1))_3] = x[2]$
- $x_1[1] = x[((0))_3] = x[0]$
- $x_1[2] = x[((1))_3] = x[1]$

① $\tilde{x}[n]$ periodic with N, $\tilde{x}[n] \overset{DFS}{\longleftrightarrow} \tilde{X}[k]$

$\rightarrow \tilde{x}[n-n_0] \overset{DFS}{\longleftrightarrow} \sum_{n=0}^{N-1} \tilde{x}[n-n_0] e^{-j\frac{2\pi}{N}kn}$; 令 $n-n_0 = n'$

$= \tilde{x}_1[n]$

$= \sum_{n'=0}^{N-1} \tilde{x}[n'] e^{-j\frac{2\pi}{N}k(n'+n_0)}$ $\quad \tilde{X}_1[k] = \underline{\tilde{X}[k] \cdot e^{-j\frac{2\pi}{N}kn_0}}$

② $x_1[n] = 0$ for $n \ge N$

$\tilde{x}_1[n] = x[((n))_N] \overset{DFS}{\longleftrightarrow} \tilde{X}_1[k] = X_1[((k))_N]$

若 $0 \le k \le N-1$ $\begin{cases} X_1[k] = X_1[((k))_N] \\ X_1[k] = X[k] \cdot e^{-j\frac{2\pi}{N}kn_0} \end{cases}$ $\Rightarrow \tilde{X}_1[k] = X[\underline{((k))_N}] \cdot e^{-j\frac{2\pi}{N}\underline{((k))_N}n_0}$

③ $\tilde{X}_1[k] = e^{-j\frac{2\pi}{N}\underline{((k))_N}n_0} X[((k))_N]$

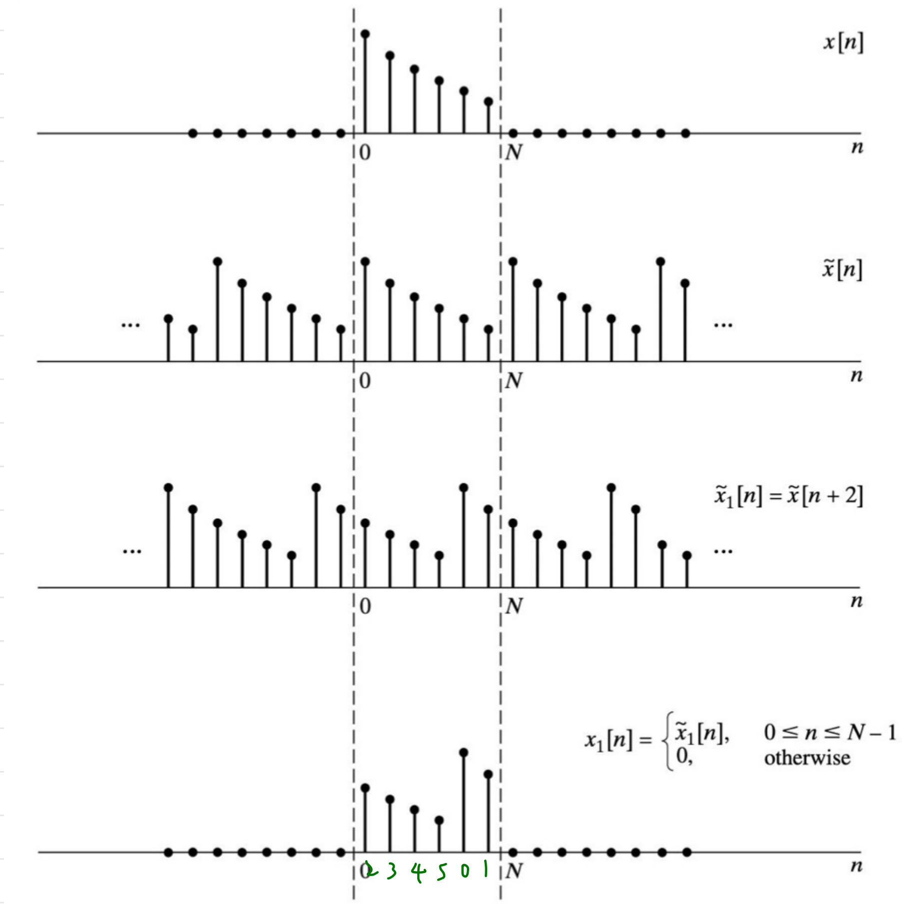
$= e^{-j\frac{2\pi}{N}\underline{k}n_0} \cdot X[((k))_N]$

$\boxed{e^{-j\frac{2\pi}{N}((k))_N n_0} = e^{-j\frac{2\pi}{N}kn_0}}$ 因 $\tilde{X}_1[k]$ 為週期信號，故 $k = k' + qN \Rightarrow ((k))_N = k'$

$\rightarrow \frac{2\pi k}{N} = \frac{2\pi(qN+k')}{N} = \frac{2\pi k'}{N} = \frac{2\pi((k))_N}{N}$

④ $\tilde{x}_1[n] = \tilde{x}[n-n_0] = x[((n-n_0))_N]$

$$\rightarrow x_1[n] = \begin{cases} \tilde{x}_1[n] = x[((n-n_0))_N] \ ; & 0 \le n \le N-1 \\ 0 \ ; & \text{otherwise.} \end{cases}$$



「平移後的信號」對應到「原本信號」

$x_1[0] = x[((0+2))_6] = x[2]$

$x_1[1] = x[((1+2))_6] = x[3]$

⋮

$x_1[4] = x[((4+2))_6] = x[0]$

$x_1[5] = x[((5+2))_6] = x[1]$

Sampling DFT. $\tilde{x}[n] = \sum_{r=-\infty}^{\infty} x[n-rN]$: DFS ; $N >$ length of $x[n]$ 時，we can recover $x(n)$ from $\tilde{x}[n]$

- $\tilde{X}[k] = X(e^{j\omega})\Big|_{\omega = \frac{2\pi}{N}k}$

$$= X(z)\Big|_{z = e^{j\frac{2\pi}{N}k}}$$

- $X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} \xrightarrow{\omega = \frac{2\pi}{N}k} \tilde{X}[k] = \sum_{m=-\infty}^{\infty} x[m] \boxed{e^{-j\frac{2\pi}{N}km}} = \sum_{m=-\infty}^{\infty} x[m] W_N^{km}$ (W_N)

$$\rightarrow \tilde{x}[n] = \frac{1}{N}\sum_{k=0}^{N-1} \tilde{X}[k] W_N^{-kn} = \frac{1}{N}\sum_{k=0}^{N-1}\sum_{m=-\infty}^{\infty} x[m] W_N^{km} \cdot W_N^{-kn}$$

$$= \sum_{m=-\infty}^{\infty} x[m] \boxed{\frac{1}{N}\sum_{k=0}^{N-1} W_N^{-k(n-m)}} = \tilde{p}[n-m]$$

令 $\tilde{p}[n] = \sum_{r=-\infty}^{\infty} \delta[n-rN]$ $\qquad = x[n] * \tilde{p}[n]$ $\qquad \frac{1}{N}\sum_{k=0}^{N-1} W_N^{-kn} = \tilde{p}[n] = \sum_{r=-\infty}^{\infty} \delta[n-rN]$

$\leftrightarrow \tilde{P}(e^{j\omega}) = \frac{2\pi}{N}\sum \delta(\omega - \frac{2\pi}{N}k)$ $\qquad = \sum_{r=-\infty}^{\infty} x[n-rN]$