Wireless Communication

Ch5

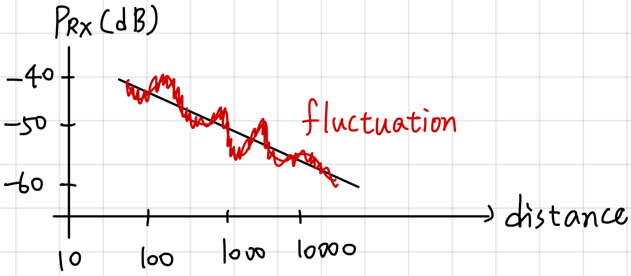
2024 Fall

Lec 5

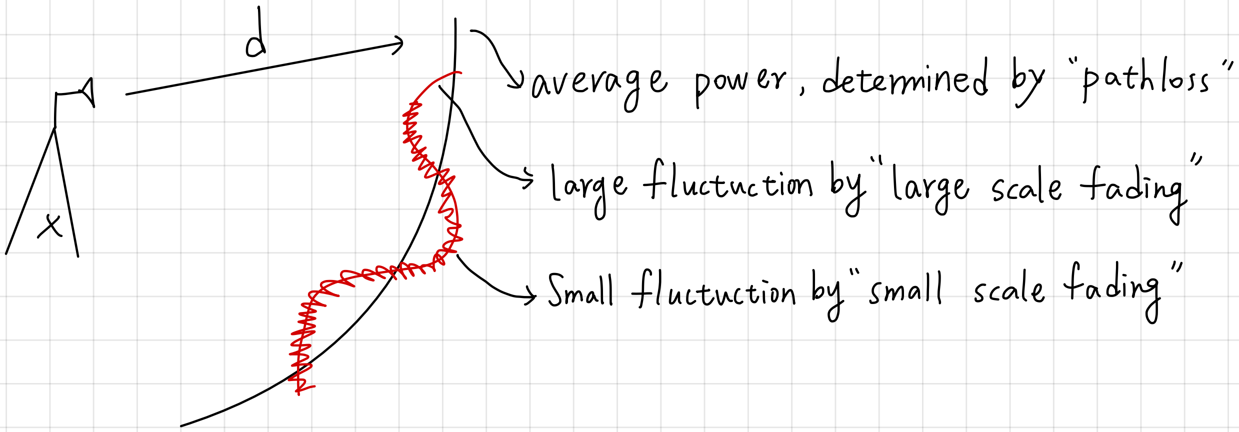
- In many case, it is too complicated to compute all the reflection, diffraction, scattering, etc
  → the matheology of exact computation is called "Ray-tracing"

- It is perfectable that we describle the probability that a channel coefficient would be
  — statistical description of channel

- Typically, when we do measurement, we can see



When we move a small distance, we have small fluctuation : small-scale fading
When we move a large distance, we have large fluctuation : large-scale fading
on average, the power reduction comes from : pathloss



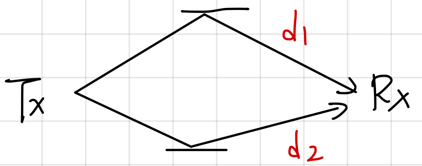
Small-scale fading.

- Illustrate a two path model (distance effect)

$$S(t,d) = A\exp\left(j2\pi f_c t + j\frac{2\pi}{\lambda}d\right)$$

△ d is the distance between Tx and Rx

- Suppose we have two paths each give you distance d_1 and d_2



$$S_1(t) = A\exp\left(j2\pi f_c t - j\frac{2\pi}{\lambda}d_1\right) = Ae^{-j\frac{2\pi}{\lambda}d_1}\cdot e^{j2\pi f_c t}$$

$$S_2(t) = A\exp\left(j2\pi f_c t - j\frac{2\pi}{\lambda}d_2\right) = Ae^{-j\frac{2\pi}{\lambda}d_2}\cdot e^{j2\pi f_c t}$$

$$\rightarrow r(t) = S_1(t) + S_2(t) = A\underbrace{\left[e^{-j\frac{2\pi}{\lambda}d_1} + e^{-j\frac{2\pi}{\lambda}d_2}\right]}_{h}e^{j2\pi f_c t}$$

△ In the best case, we have $|h| = 2$ — constructive effect

△ In the worst case, we have $|h| = 0$ — destructive effect

⟶ These lead to fading effect

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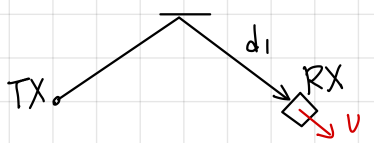
one-path model with Dopper effect

- Suppose the RX is moving



$$S(t) = Ae^{j2\pi f_c t}\cdot e^{-j\frac{2\pi}{\lambda}(d_1 + vt)}$$

$$= Ae^{-j\frac{2\pi}{\lambda}d_1}\cdot e^{j2\pi\left(f_c - \frac{v}{\lambda_c}\right)t}$$

Dopper shift

- The carrier freq is changed because RX is moving

Small-scale fading without dominant path

實驗背景

- Rayleigh fading model
- A large number of path
- No dominant path $\longrightarrow$ power from different path are average similar.
- Due to "central limit theorem", we have a good approximation by Gaussian RV

$\Rightarrow$ by CLT, we have

$$\boxed{h = \sum_{i=1}^{N} a_i e^{j\phi} = h_R + j h_I = R e^{j\phi}}$$

△ Path are statistically independent
△ Number of path $N \longrightarrow \infty$

- h_R and $h_I \sim \text{Gauss}(0, \frac{\sigma_s^2}{2})$ $\longrightarrow$ σ_s^2 is the received signal power
  $\sigma_s^2 = I$ after normalization $\longrightarrow P_{Rx} \cdot |h|^2$
- $R = \sqrt{h_R^2 + h_I^2} \sim$ Rayleigh distribution 振幅
- ϕ is uniform distribution $\sim U[0, 2\pi]$
- R and ϕ are independent
  - $f_R(r) = \frac{r}{\sigma^2} \exp\left(-\frac{r^2}{2\sigma^2}\right)$; $r \geq 0$
    $h_R \sim N(0, \sigma^2)$
    $h_I \sim N(0, \sigma^2)$
  - $f_\phi(\phi) = \frac{1}{2\pi}$

$rv\,R = \sqrt{h_I^2 + h_R^2}$ 且 $rv\,h_I \sim \text{Gauss}(0, \frac{\sigma_s^2}{2})$
$rv\,h_R \sim \text{Gauss}(0, \frac{\sigma_s^2}{2})$

故 $rv\,R \sim$ Rayleigh

• Property of Rayleigh distribution $f_R(r) = \frac{r}{\sigma^2} \exp\left(-\frac{r^2}{2\sigma^2}\right)$

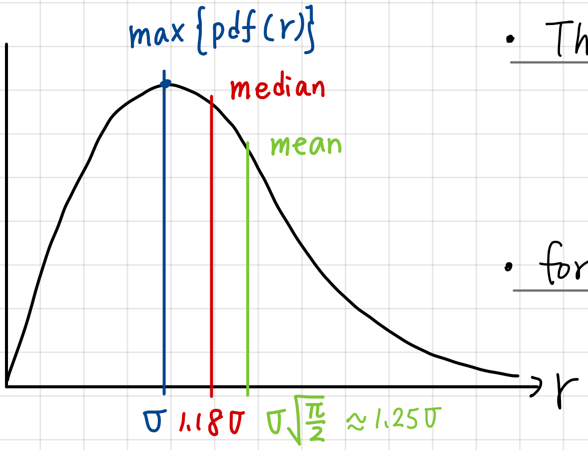
• Mean : $\bar{R} = E[R] = \sigma\sqrt{\frac{\pi}{2}}$

• Mean square : $\overline{R^2} = E[R^2] = 2\sigma^2$

• Variance : $\overline{R^2} - (\bar{R})^2 = E[(R-\bar{R})^2] = \left(2 - \frac{\pi}{2}\right)\sigma^2 = 0.429\sigma^2$

• Median : $R_{50} = \sigma\sqrt{2\ln 2} = 1.18\sigma$

• Max{pdf(r)} : $\arg\max_r f_R(r) = \sigma$



• The cdf of Rayleigh distribution :

$$cdf_R(r) = \int_0^r pdf_R(u)\,du = \boxed{1 - e^{-\frac{r^2}{2\sigma^2}}}$$

• for small value of r, this can be approximated as

$$cdf(r) = 1 - \exp\left(-\frac{r^2}{2\sigma^2}\right) \approx \frac{r^2}{2\sigma^2}$$

Taylor series

Remark 1 :

(i) R is the amplitude of the channel, (received signal) $\Rightarrow R \geq 0$

(ii) The power of $|h|^2 = R^2 = P$, follow by exponential distribution , in which

$\boxed{pdf_P(p) = \frac{1}{\Omega}\exp\left(-\frac{P}{\Omega}\right)}$, where $\Omega = 2\sigma^2$ is the mean power $2\sigma^2$為平均功率

Remark 2: Rayleigh fading model is widely used in wireless communication

(i) Effective approximation in many pratical case — NLOS case

(ii) Worst case scenario in the sence, that there is no obvious dominant path

(iii) It has a very simple description — its pdf is only related to σ

(iv) Mathematically convenience → easy to compute

$pdf_R(r) = \frac{r}{\sigma^2}\exp\left(-\frac{r^2}{2\sigma^2}\right)$, $2\sigma^2$ is mean received power

$pdf_P(p) = \frac{1}{\Omega}\exp\left(-\frac{P}{\Omega}\right)$, Ω is mean received power

Fading Margin Computation for Rayleigh fading

- We see from link-budget, that we need fading margin to accommadate the fading effect

- Because we are having statistical, the concept become to find:

  What is the minimum power (or fading margin) we need for ensuring the transmission is failed only in X % of the situation

- We want to find r_{min}, such that $P_{out} = X$

$$\boxed{X = P(r \le r_{min}) = cdf(r_{min}) \approx \frac{r_{min}^2}{2\sigma^2}} \quad \frac{P_{min}}{P}$$

我們需要求出平均功率，以確保在X%的情況下接收器的接收功率足夠高

由於環境因素，接收信號強度會隨機變化
因此需要確保發射器的平均功率夠高

如果接收信號小於這個值，則無法通訊

Interference-limited case with Rayleigh fading

- Both signal and interference have Rayleigh fading

Denote S as a Rayleigh distributed received signal

- $pdf_R(s) = \frac{s}{\sigma_s^2} \exp\left(-\frac{s^2}{2\sigma_s^2}\right)$

Denote I also as a Rayleigh distributed received interference

- $pdf_I(I) = \frac{I}{\sigma_I^2} \exp\left(-\frac{I^2}{2\sigma_I^2}\right)$

- Then, we denote $r = \frac{S}{I}$, we have

$$\begin{cases} pdf(r) = \dfrac{2\tilde{\sigma}^2 r}{(\tilde{\sigma}^2 + r^2)^2}, \quad \tilde{\sigma}^2 = \dfrac{\sigma_s^2}{\sigma_I^2} \text{ average SIR} \\ cdf(r) = 1 - \dfrac{\tilde{\sigma}^2}{\tilde{\sigma}^2 + r^2} \end{cases}$$

Ex.

— Required SNR = 5dB , Rayleigh fading

— $P_{out} = 1\%$

Q. What is the required average SNR ?

- $P_{out} = Pr(r < r_{min}) = cdf(r_{min})$

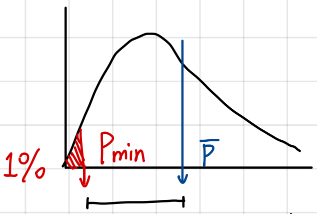
$$= 1 - \exp\left(-\frac{r_{min}^2}{2\sigma^2}\right) \approx \frac{r_{min}^2}{2\sigma^2} = 0.01$$

$$\frac{r_{min}^2}{2\sigma^2} = \frac{P_{min}}{\bar{P}}$$

- Because $2\sigma^2 = \bar{P}$,

we need $\frac{P_{min}}{\bar{P}} = 0.01 \Rightarrow$ fading margin $= 10\log_{10}100 = 20dB$

意思是 $\bar{P}$ 需比 P_{min} 高出 100倍，
也就是 20dB



$\bar{P}$ 為 P_{min} 的 100倍 , i.e. 20dB

$\Rightarrow$ required average SNR at RX = 20+5 = 25dB※

EX 5.1

Suppose we have signal with Rayleigh distributed amplitude ,

Q. What is the prob , that the received power is at least 20dB , 6dB , 3dB below the mean

A. Note that the mean power $= 2\sigma^2 = \overline{r^2}$

- If 20dB = 100倍

$$1-\exp\left(-\frac{r_{min}^2}{2\sigma^2}\right) \approx \frac{r_{min}^2}{2\sigma^2}$$ 只在 r 很小才能近似

$$\frac{r_{min}^2}{2\sigma^2} = \frac{1}{100} \Rightarrow cdf(r_{min}) = 1-\exp\left(-\frac{1}{100}\right) = 9.95\times10^{-3} \approx 10^{-2}$$

- If 6dB = 4倍

$$\frac{r_{min}^2}{2\sigma^2} = \frac{1}{4} \Rightarrow cdf(r_{min}) = 0.221$$

- If 3dB = 2倍

$$\frac{r_{min}^2}{2\sigma^2} = \frac{1}{2} \Rightarrow cdf(r_{min}) = 0.393$$

Ex.

$f_c = 5\text{GHz}$. $B = 20\text{MHz}$, $N_0 = -174\,\text{dBm/Hz}$

$G_{RX} = G_{TX} = 2\text{dB}$, $\underline{P_{TX} = 0.2\text{W}}$ (23dB), $L = 5\text{dB}$

$F = 5\text{dB}$, $d_{break} = 10\text{m}$, $d = 200\text{m}$, $n = 3.8$. Consider Rayleigh

a) Find the average received SNR

b) Suppose we increase $G_{TX} = 22\text{dB}$. we want $\underline{SNR_{min} = 12\text{dB}}$, what is our outage prob?

c) If we want $P_{out} = 2\%$, what is the required fading margin?

(a)

- $P_n = -174 + 10\log_{10} 20\times 10^6 + 5 = -96\,\text{dBm}$
- $PL = \underbrace{20\log_{10}\left(\frac{4\pi}{\lambda_c}\right)}_{32+2.7} + \underbrace{20\log_{10}(d_{break})}_{20} + \underbrace{n\cdot 10\log_{10}\left(\frac{d}{d_{break}}\right)}_{3.8\cdot(3+10)} = 115.4\,\text{dB}$
- $P_{RX} = 23+2+2-5-115.4 = -93.4\,\text{dBm}$
- Received mean SNR: $-93.4-(-96) = 2.6\,\text{dB}$ ※

(b) G_{TX} increase 20dB, so Average SNR: $2.6\,\text{dB} \rightarrow 22.6\,\text{dB}$

$P_{out} = cdf(r_{min}) = 1 - \exp\left(\frac{-r_{min}^2}{2\sigma^2}\right) = 1 - \exp\left(\frac{-P_{min}}{\bar{P}}\right)$

- $\frac{\bar{P}}{P_{min}} = (22.6-12)\,\text{dB} = 10.6\,\text{dB} = 11.5$
- $P_{out} = 1-\exp\left(\frac{-1}{11.5}\right) = 0.083 = 8.3\%$

(c) $P_{out} = 2\%$

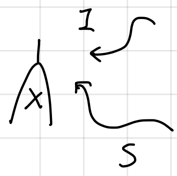
$\Rightarrow P_{out} = 1-\exp\left(\frac{-r_{min}^2}{2\sigma^2}\right) = 0.02$

$\Rightarrow \frac{r_{min}^2}{2\sigma^2} \approx 0.02 \Rightarrow$ fading margin $= \frac{1}{0.02} = 50 = 17\,\text{dB}$ ※

Ex.

Consider a 2-user system, both transmit signal to the BS

We consider Rayleigh fading for both user



a) Suppose we want $SIR_{min} \geq 20dB$, What is outage prob. if the distance between BS and 2 users are the same.

$$\tilde{\sigma} = \frac{\sigma_S^2}{\sigma_I^2} = 1 \Rightarrow P_{out} = cdf(\gamma_{min}) = 1 - \frac{\tilde{\sigma}^2}{\tilde{\sigma}^2 + \gamma_{min}^2}$$

$$\gamma_{min}^2 = 20dB = 100 \Rightarrow P_{out} = 1 - \frac{1}{1+100} = \frac{100}{101} \approx 0.99 \#$$

b) If we increase P_1 by 10dB, What is P_{out}?

$$\tilde{\sigma} = \frac{\sigma_S^2}{\sigma_I^2} = 10 \Rightarrow P_{out} = 1 - \frac{10}{10+100} = \frac{10}{11} = 0.91 \#$$

c) If we want $P_{out} = 0.05$, how much should P_1 be increased by?

$$0.05 = 1 - \frac{\tilde{\sigma}^2}{\tilde{\sigma}^2 + 100} \Rightarrow \frac{\tilde{\sigma}^2}{\tilde{\sigma}^2 + 100} = 0.95$$

$$\tilde{\sigma}^2 = 1900 = 32.8dB \#$$

d) If user I is at d_1 and user 2 is at d_2, then how much we need to increase

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Ex 5.2

Compute the fading margin for a Rice distributed signal with $K_r = 0.3, 3, 20dB$, such that the outage prob is less then 5%

$$P_{out} = cdf(\gamma_{min})$$

For Rician. we know

$$\frac{\bar{r}^2}{r_{min}^2} = \frac{A^2 + 2\sigma^2}{\gamma_{min}^2} = \frac{2\sigma^2(1+K_r)}{\gamma_{min}^2} = \text{fading margin}$$

Therefore, we need

$$P_{out} = 0.05 = cdf(\gamma_{min}) = \int_0^{\gamma_{min}} \frac{r}{\sigma^2} \exp\left(-\frac{r^2 + A^2}{2\sigma^2}\right) I_0\left(\frac{rA}{\sigma}\right)$$

Small-Scale fading with dominant path

- Rician fading
- Basically happens in LoS case

$$h = \underbrace{A}_{\text{dominant path}} + \underbrace{Re^{j\phi}}_{\text{Rayleigh fading}} = re^{j\psi} \qquad r = |h| = \sqrt{(A+R_{Re})^2 + (R_{Im})^2}$$

- Without loss of generally, we assume the dominant component have zero phase
- The joint pdf of r and ψ is:

$$pdf_{r,\psi}(r,\psi) = \frac{r}{2\pi\sigma^2} \exp\left(-\frac{r^2 + A^2 - 2rA\cos\psi}{2\sigma^2}\right)$$

- The pdf of amplitude r is rice distribution:

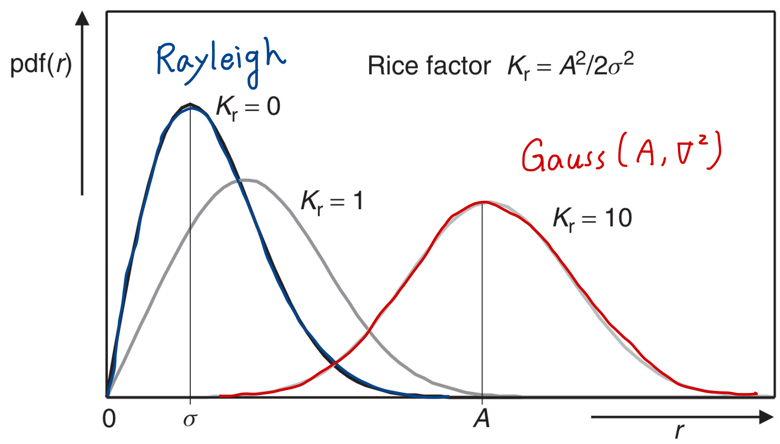
$$pdf_r(r) = \frac{r}{\sigma^2} \exp\left(-\frac{r^2+A^2}{2\sigma^2}\right) I_0\left(\frac{rA}{\sigma^2}\right) \; ; \; 0 < r < \infty$$

  - $I_0(x)$ is the modified Bessel function of the first kind, zero order
  - $\overline{r^2} = E[r^2] = 2\sigma^2 + A^2$

- Define $K_r = \frac{A^2}{2\sigma^2}$, called Rician factor

→ indicate the power ratio between dominant component and other path

  - If $K_r \to 0$, Become back as Rayleigh fading
  - If K_r large, Become as $r \sim Gauss(A, \sigma^2)$



Dopper spectrum and temporal variation

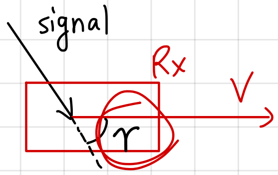
- We have seen the physical interpretation for freq shift by movement.

  This lead to Dopper effect :

$$S(t) = A e^{j2\pi f_c t} e^{-j\frac{2\pi}{\lambda}(d_1+vt)} = A e^{-j\frac{2\pi}{\lambda}d_1} \cdot e^{j2\pi(f_c - \boxed{\frac{v}{\lambda_c}})t}$$

← Dopper effect

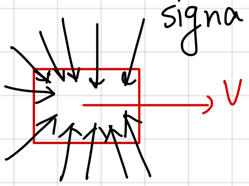
- Equation of Dopper shift



$\cdot\ f = f_c(1 - \frac{V\cos\gamma}{c}) = f_c - \nu$; $\boxed{\nu = f_c \frac{V}{c}\cos\gamma}$; $\gamma \in [-\pi, \pi]$

$\cdot\ |\nu| \leq f_c \frac{V}{c} = \nu_{max}$

- If we have multiple path, we have to sum up the effects of different incoming signal



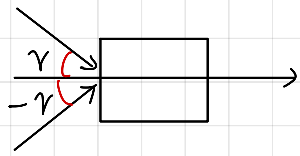
- Then, suppose the incoming signal are described by using statistic, i.e. distribution of power of incoming signal from different angle.

- Define the distribution of incoming signal in terms of power as : $\boxed{pdf_\gamma(\gamma)}$

- Suppose we have antenna pattern $G(\gamma)$

- let $\bar{\Omega}$ be the mean power of total arrival signal.

- The received power spectrum is expressed as :



$$S_D(\nu)\, d\nu = \bar{\Omega}\left[\int_{\gamma \in T(\nu)} pdf(\gamma)\, G(\gamma)\, d\gamma\right]$$

  - $T(\nu)$ is the γ such that it Dopper effect results in the Dopper shift ν
  - By variable transformation, we have $d\gamma = \left|\frac{d\gamma}{d\nu}\right| d\nu$

Also,

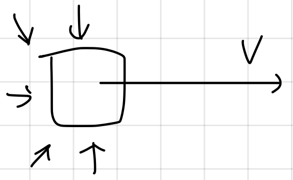
$$\left|\frac{d\gamma}{d\nu}\right| = \left|\frac{1}{\frac{d\nu}{d\gamma}}\right| = \left|\frac{1}{\frac{v f_c}{c}\sin\gamma}\right| = \frac{1}{\sqrt{(\frac{v f_c}{c})^2 - (\frac{v f_c}{c})^2\cos^2\gamma}} = \frac{1}{\sqrt{\nu_{max}^2 - \nu^2}}$$

$$S_D(\bar{\Omega}) = \begin{cases} \bar{\Omega}\left[\mathrm{pdf}(\gamma)G(\gamma) + \mathrm{pdf}(-\gamma)G(-\gamma)\right]\dfrac{1}{\sqrt{v_{max}^2 - v^2}} \\ 0 \end{cases}$$

Example:

If we have $\mathrm{pdf}(\gamma) = \frac{1}{2\pi}$, correspond to Rayleigh fading.

- Jake's model.



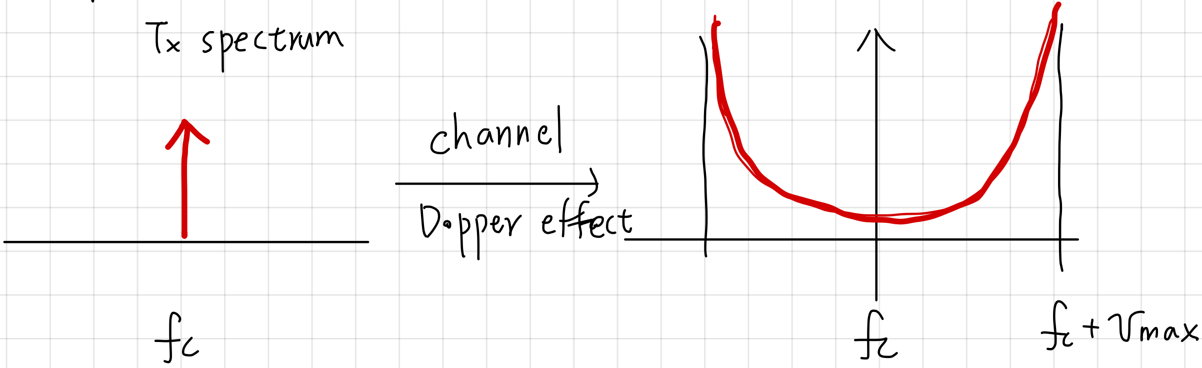
Suppose $G(\gamma) = 1.5, \ \forall \gamma$, then

$$S_D(v) = \bar{\Omega}\left[\frac{1}{2\pi} + \frac{1}{2\pi}\right] 1.5 \frac{1}{\sqrt{v_{max}^2 - v}} = \frac{1.5\,\bar{\Omega}}{\pi\sqrt{v_{max}^2 - v}}$$

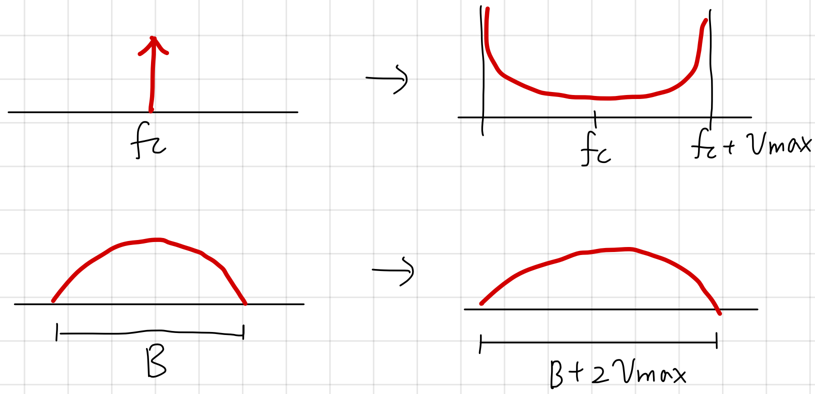
→ uniform distribution on angle γ, give ◯

to a non-uniform distribution on Dopper spectrum.

→ Dopper spectrum describe the frequency dispersion.



- Such frequency dispersion (Dopper shift) can cause error when communication, especially for narrowband signal. eg: OFDM



Dopper shift 在 narrowband 影響很多
⇒ 不同頻寬有不同表現

- Dopper Spectrum is a measure of temporal variation of the channel.

→ Notice that the temporal dependency of fading descrease fast w.r.t time difference

→ We use correlation function to describe the channel variation.

$$\frac{E[I(t)\cdot I(t+\Delta t)]}{E[I(t)^2]} \overset{\text{Jalce's model}}{=} J_0(2\pi \nu_{max} \Delta t)$$

$J_0(\cdot)$ is propotional to the inverse Fourier transform of $S_D(\nu)$

$$\nu_{max}\Delta t = \frac{f_c V}{c}\cdot \Delta t = \underbrace{\frac{f_c}{c}}_{1/\lambda_c} \underbrace{V\Delta t}_{\Delta d} = \frac{\Delta d}{\lambda_c}$$

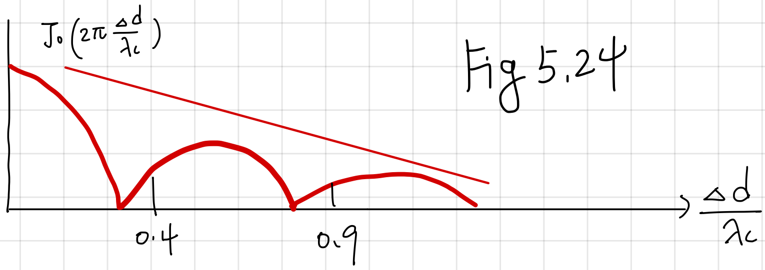


Fig 5.24

- Correlation between different location reduce to a a small number, when $\Delta d \geq 0.5\lambda_c$