Stochastic Process Ch9

2024 Fall

40 Midterm 第8週 — 20241021
40 Final 第16週 — 20241216
20 homework / Participation

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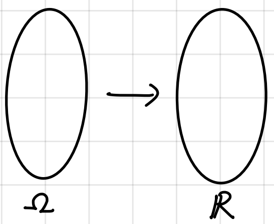
random variable $X : \Omega \to \mathbb{R}$

rv X is a mapping from sample space Ω to real number $\mathbb{R}$



RVs
- discrete : Bernoulli, Binomial, Geometrix, Poisson ex: $P(X=k) = \frac{\lambda^k e^{-\lambda}}{k!}$ $\forall k \in [0,1,2,\cdots k]$
- continuous : uniform $[a,b]$, Gaussian $X \sim N(\mu, \sigma^2)$
- other

$$P(X \le t) = \int_{-\infty}^{t} \boxed{\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}} dx$$

NOT probability ,積分後才是 → $P(X \in [a,b]) = \int_a^b f_X(x)\,dx$

probability density function : $f_X(x)$

Random variable X

$\Downarrow$

Random vector $X = (X_1 X_2 \cdots X_n)$

Ex X is a n-dimensional Gaussian random vector $X \sim N(\mu_X, \Lambda_X)$

- $\mu_X = E[X] \in \mathbb{R}^{n\times 1}$
- $\underset{n\times n}{\Lambda_X} = E[\underset{n\times 1}{(X-\mu_X)}\underset{1\times n}{(X-\mu_X)^T}] \in \mathbb{R}^{n\times n}$ Auto-covariance matrix of X
- $f_X(\vec{x}) = \frac{1}{\sqrt{(2\pi)^n \det(\Lambda_X)}} \cdot e^{-\frac{1}{2}\boxed{(x-\mu_x)^T \Lambda_x^{-1} (x-\mu_x)}}$ ≥ 0

$\vec{x} = (x_1 x_2 \cdots x_n)^T$

$(x-\mu_x)^T (x-\mu_x) = \|x-\mu_x\|^2$

Theorem 1.

$\Lambda_X \triangleq E[(X-\mu_X)(X-\mu_X)^T]$
is a symmetric and semi-definite matrix

- $\Lambda_X^{-1} = \Lambda_X$
- $\vec{u}^T \Lambda_X \vec{u} \geq 0 \quad \forall \vec{u} \in \mathbb{R}^{n\times 1}$

1. Λ_X^{-1} exist $\leftarrow$ $\det(\Lambda_X) > 0$
2. $\det(\Lambda_X) \geq 0$

Chapter 9

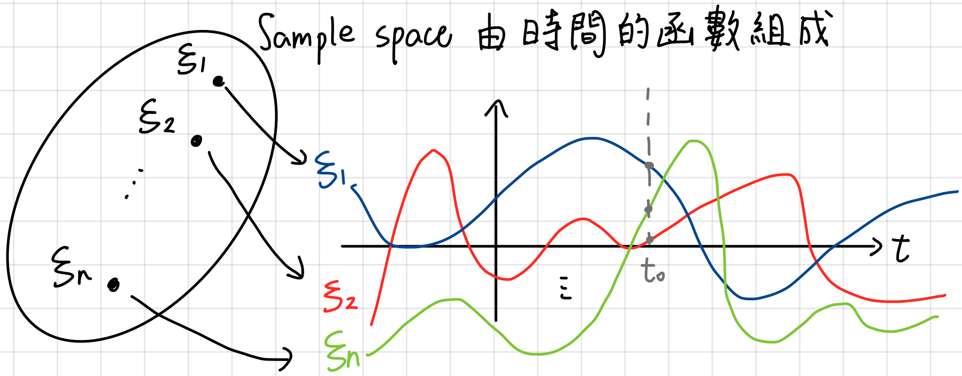
9-1 Definition

Stochastic Process $X(t, \xi)$

is a two-variable function

$X: \mathbb{R} \times \Omega \longrightarrow \Omega$

Sample space 由時間的函數組成



such that

(1) for each fixed $t_0 \in \mathbb{R}$ $X(t_0)$ is a random variable

(2) for each fixed $\xi \in \Omega$, $\{X(t), t \in \mathbb{R}\}$ is a determinstic real-value function

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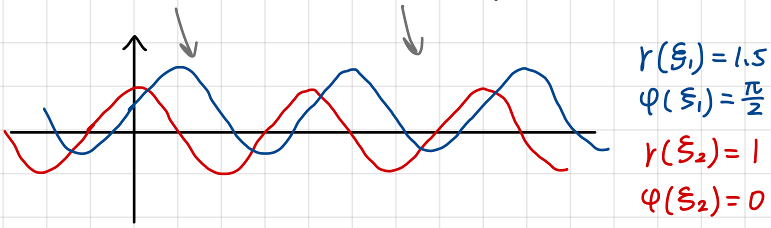
Ex 9-4.

$X(t) = r\cos(\omega t + \varphi), \forall t \in \mathbb{R}$ $\quad$ $X(t, \xi) \triangleq r(\xi)\cos(\omega t + \varphi(\xi))$ $\quad$ $\forall t \in \mathbb{R}$, $\forall \xi \in \Omega$

- r and φ are indep continuous rVs
- $\varphi \sim U(-\pi, \pi)$
- $r \perp \varphi$ (statistically independent)



① $E[X(t)] = E[r\cos(\omega t + \varphi)]$

$= E[r] \cdot \cancel{E[\cos(\omega t + \varphi)]} = 0$

$= 0$ #

常數

- $E[\cos(\omega t + \varphi)] = \int_{-\pi}^{\pi} \cos(\omega t + \theta) \cdot \frac{1}{2\pi} \, d\theta$

$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \cos(\omega t + \theta) \, d\theta$

$= \frac{1}{2\pi} \cdot \sin(\omega t + \theta) \Big|_{-\pi}^{\pi}$

$= \frac{1}{2\pi} \cdot [\sin(\omega t + \pi) - \sin(\omega t - \pi)]$

$= 0$ #

② $R_{XX}(t_1, t_2) = E[r\cos(\omega t_1 + \varphi) \cdot r\cos(\omega t_2 + \varphi)]$

$= E[r^2] \cdot E[\cos(\omega t_1 + \varphi) \cdot \cos(\omega t_2 + \varphi)]$

$= E[r^2] \cdot \frac{1}{2} E[\underbrace{\cancel{\cos(\omega t_1 + \omega t_2 + 2\varphi)}}_{=0} + \cos(\omega t_1 - \omega t_2)]$

$= \frac{1}{2} E[r^2] \cdot \cos(\omega(t_1 - t_2))$ #

$\tau = t_1 - t_2$

- $E[\cos(\omega t_1 + \omega t_2 + 2\varphi)]$

$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \cos(\omega t_1 + \omega t_2 + 2\theta) \, d\theta$

$= 0$

Ex 9-3

$X(t)$ is a random process

$S \triangleq \int_a^b X(t)\,dt$

$$X(t) \longrightarrow \boxed{\int_a^b (\cdot)\,dt} \longrightarrow S$$

$S(\xi) \triangleq \int_a^b X(t,\xi)\,dt,\ \forall t \in \Omega$

(1) $S: \Omega \to R$ is a random variable

$E[X(t)] = \int_{-\infty}^{\infty} u\, f_{X(t)}(u)\,du$

在不同樣本點之間做平均

(2) • $\eta_x(t) \triangleq E[X(t)],\ \forall t \in R$

• $R_{xx}(t_1,t_2) \triangleq E[X(t_1)X(t_2)],\ \forall t_1,t_2 \in R$

• $C_{xx} \triangleq R_{xx}(t_1,t_2) - \eta_x(t_1)\eta_x(t_2)$

Obtain $E[S]$ and $E[S^2]$ in terms of statistical property of $X(t)$

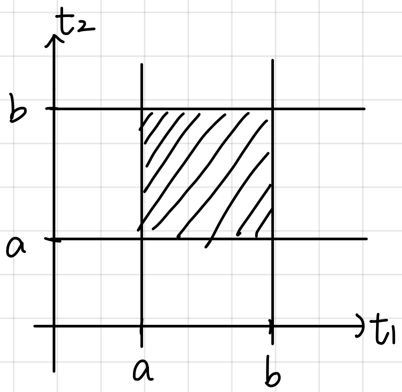
① $E[S] = E\left[\int_a^b X(t)\,dt\right] = \int_a^b \eta_x(t)\,dt$

② $E[S^2] = E\left[\int_a^b X(t_1)\,dt_1 \cdot \int_a^b X(t_2)\,dt_2\right]$

$= E\left[\int_a^b\int_a^b X(t_1)X(t_2)\,dt_1\,dt_2\right]$

$= \int_a^b\int_a^b E[X(t_1)X(t_2)]\,dt_1\,dt_2$

$= \int_a^b\int_a^b R_{xx}(t_1,t_2)\,dt_1\,dt_2$



2024 9/2

Let $X(t)$ and $y(t)$ be two stochastic process (SP)

$Z(t) \triangleq X(t) + jY(t)$ is complex SP

(1) $\eta_z(t) \triangleq E[Z(t)] = E[X(t)] + jE[y(t)], \forall t \in \mathbb{R}$

(2) $R_{zz}(t_1 t_2) \triangleq E[Z(t_1) Z^*(t_2)], \forall t_1 t_2 \in \mathbb{R}$

Lemma 1 : Let $Z(t)$ be complex SP

(1) $R_{zz}(t,t) \geq 0, \forall t \in \mathbb{R}$

posite - definite

(2) $\sum_{i=1}^{n} \sum_{k=1}^{n} a_i a_k^* R_{zz}(t_i, t_k) \geq 0 . \forall n \in N, a \triangleq (a_1 \cdots a_n) \in C^n \quad (t_1 t_2 \cdots t_n) \in \mathbb{R}^n$

proof :

① $R_{zz}(t,t) = E[Z(t) Z^*(t)] = \underbrace{E[|Z(t)|^2]}_{\text{energy / power of } Z(t)} \geq 0$ ※

②

(i) Consider a fixed n, $a = (a_1 a_2 \cdots a_n) \in \mathbb{C}^n$ and $t \triangleq (t_1 t_2 \cdots t_n) \in \mathbb{R}^n$, Define $u = \sum_{i=1}^{n} a_i Z(t_i)$

(ii) Then, u is a complex random variable. Thus $E[|u|^2] \geq 0$

(iii) $E[|u|^2] = E[uu^*]$

$$= E\left[\sum_{i=1}^{n} a_i Z(t_i) \cdot \sum_{k=1}^{n} a_k^* Z^*(t_k)\right]$$

$$= E\left[\sum_{i=1}^{n}\sum_{k=1}^{n} a_i a_k^* Z(t_i) Z^*(t_k)\right]$$

$$= \sum_{i=1}^{n}\sum_{k=1}^{n} a_i a_k^* R_{zz}(t_i, t_k) \geq 0 \quad ※$$

A matrix $A \in \mathbb{C}^{n \times n}$ is positive - definite if $v^H A v \geq 0, \forall v \in \mathbb{C}^n$

Ex 9-7

a is RV, $w \in \mathbb{R}$

$X(t) \triangleq a \cdot e^{jwt}, \forall t \in \mathbb{R}$

depend on t is general.
but become constant iff $E[a] = 0$

(1) $\eta_x(t) = E[a] e^{jwt}$ ※

(2) $R_{xx}(t_1 t_2) = E[a e^{jwt_1} a^* e^{-jwt_2}] = E[|a|^2] e^{jw(t_1 - t_2)}$ ※

Definition:

A complex RP $x(t)$ is said to be WSS, if

① $\eta_x(t) = \eta_x \in \mathbb{C}$, $\forall t \in \mathbb{R}$

② $R_{xx}(t_1+s, t_2+s) = R_{xx}(t_1, t_2) = R_{xx}(t_1 - t_2)$, $\forall t_1 t_2 \in \mathbb{R}$

若 $x(t)$ is WSS,

- $R_{xx}(\tau) \triangleq E[x(t+\tau)x^*(t)]$

$$= E[x(t+\tfrac{\tau}{2})x^*(t-\tfrac{\tau}{2})]$$

- $R_{xx}(0) = E[x(t)x^*(t)]$

$$= \underbrace{E[|x(t)|^2]}_{\text{energy}} \geq 0$$

Ex 9-12

$X(t)$ is WSS

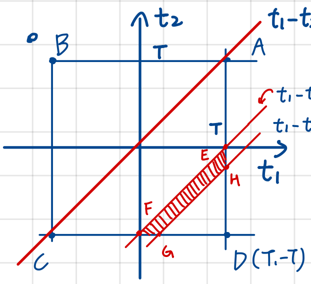
$$S \triangleq \int_{-T}^{T} X(t)\,dt \longleftrightarrow S(\omega) \triangleq \int_{-T}^{T} X(t,\omega)\,dt \quad \forall\, \omega \in \Omega$$

Claim : $$\sigma_S^2 = \int_{-T}^{T}\int_{-T}^{T} C_{xx}(t_1-t_2)\,dt_1 dt_2 = \int_{-2T}^{2T} (2T-|\tau|)\, C_{xx}(\tau)\,d\tau$$

proof

- $$E[S] = \int_{-T}^{T} E[X(t)]\,dt = \int_{-T}^{T} \eta_x(t)\,dt = \int_{-T}^{T} \eta_x\,dt = 2T\cdot \eta_x$$

- $$\sigma_S^2 = E[(S-E[S])^2]$$
$$= E\left[\left(\int_{-T}^{T} X(t)\,dt - \int_{-T}^{T} \eta_x\,dt\right)^2\right]$$
$$= E\left[\int_{-T}^{T} X(t_1)-\eta_x\,dt_1 \cdot \int_{-T}^{T} X(t_2)-\eta_x\,dt_2\right]$$
$$= E\left[\int_{-T}^{T}\int_{-T}^{T} (X(t_1)-\eta_x)(X(t_2)-\eta_x)\,dt_1 dt_2\right]$$
$$= \int_{-T}^{T}\int_{-T}^{T} E[(X(t_1)-\eta_x)(X(t_2)-\eta_x)]\,dt_1 dt_2$$
$$= \int_{-T}^{T}\int_{-T}^{T} C_{xx}(t_1,t_2)\,dt_1 dt_2$$

- 

$E:(T, T-\tau)$

$\overline{ED} = 2T-\tau$

Consider the trapzoid $\triangle$ EFGH

the area $\triangle$ EFGH is

$$= \frac{1}{2}(\overline{EF}+\overline{GH})\cdot \frac{d\tau}{\sqrt{2}}$$
$$\cong \frac{1}{2}\cdot 2\,\overline{EF}\cdot \frac{d\tau}{\sqrt{2}}$$
$$= (2T-\tau)\sqrt{2}\cdot \frac{d\tau}{\sqrt{2}}$$

$$\iint_{\triangle EFGH} C_{xx}(t_1-t_2)\,dt_1 dt_2 \approx (\text{Area} \triangle EFGH)\, C_{xx}(\tau)$$

故 $$\int_{-T}^{T}\int_{-T}^{T} C_{xx}(t_1-t_2)\,dt_1 dt_2 = \int_{-2T}^{2T} (2T-|\tau|)\, C_{xx}(\tau)\,d\tau$$ ※

9-2 System with stochastic input

(1) Memoryless system : $g: \mathbb{R} \longrightarrow \mathbb{R}$

$$X(t) \longrightarrow \boxed{g} \longrightarrow y(t) = g[X(t)]$$

RP SSS   RP SSS

(2) LTI

$$X(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t) = X(t) * h(t)$$

RP   RP

Theorem 1.

Consider a memoryless system characterized by g

- If the input $X(t)$ is SSS, the output process $y(t) = g[X(t)]$ is also SSS
- If in addition, the system g: $y_1 = g(x_1)$, $y_2 = g(x_2) \cdots y_n = g(x_n)$ has a unique solution

then, $\boxed{f_Y(y_1 y_2 \cdots y_n ; t_1 t_2 \cdots t_n) = f_X(x_1 \cdots x_n ; t_1 \cdots t_n) / |g'(x_1) g'(x_2) \cdots g'(x_n)|}$

Theorem 0

Let $X = (X_1 X_2 \cdots X_n)^T$ be a random vector

$g_1 g_2 \cdots g_n : \mathbb{R}^n \to \mathbb{R}^n$ and $Y_k \triangleq g_k(X)$, $\forall\ 1 \le k \le n$

Given $(y_1 y_2 \cdots y_n) \in \mathbb{R}^n$, if there exist a unique $x \in \mathbb{R}^n$ such that

$g_k(X) = Y_k$ has a unique solution

$$\boxed{f_Y(y_1 y_2 \cdots y_n) = \frac{f_X(x_1, x_2 \cdots x_n)}{|J(x_1 x_2 \cdots x_n)|} ; \quad J(x_1 \cdots x_n) = \begin{vmatrix} \frac{\partial g_1}{\partial x_1} & \cdots & \frac{\partial g_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial g_n}{\partial x_1} & \cdots & \frac{\partial g_n}{\partial x_n} \end{vmatrix}}$$

- The n-th order CDF of $X(t)$ at $(t_1, t_2 \cdots t_n)$;

$$F_X(x_1 x_2 \cdots x_n ; t_1 t_2 \cdots t_n) \triangleq P(X(t_1) \le x_1, X(t_2) \le x_2 \cdots X(t_n) \le x_n)$$

$F_{X(t_1) X(t_2) \cdots X(t_n)}(x_1 x_2 \cdots x_n)$

- The n-th order PDF of $X(t)$ at $(t_1 \cdots t_n)$;

$$f_X(x_1 x_2 \cdots x_n ; t_1 t_2 \cdots t_n) \triangleq \frac{\partial^n F_X(x_1 \cdots x_n ; t_1 \cdots t_n)}{\partial x_1 \partial x_2 \cdots \partial x_n}$$

$f_X(x_1 x_2 ; t_1, t_2) \triangleq f_{X(t_1) X(t_2)}(x_1, x_2)$

Definite :

A RP $X(t)$ is said to be "strict-sense stationary" (SSS)

if for each $n \in \mathbb{N}$ $(t_1 t_2 \cdots t_n) \in \mathbb{R}^n$

$$f_X(x_1 x_2 \cdots x_n ; t_1 + s, t_2 + s \cdots t_n + s)$$

$$= f_X(x_1 x_2 \cdots x_n ; t_1, t_2 \cdots t_n)$$

Proof of Theorem 1

① Consider a fixed $n \in N$, $(t_1 t_2 \cdots t_n) \in \mathbb{R}^n$ and $(y_1 y_2 \cdots y_n) \in \mathbb{R}^n$

② If there exist a $k \in \{1.2 \cdots n\}$ such that
$g_k(X) = Y_k$ has no solution, $f_Y(y_1 \cdots y_n) = 0$

③ Consider the non-trivial case in which $g_k(X) = Y_k$
has unique solution for each k

Define $g_k(X) = g(X_k)$, $\forall X = (x_1 x_2 \cdots x_n) \in \mathbb{R}^n$,

$g_k : \mathbb{R}^n \to \mathbb{R}$
$g : \mathbb{R} \to \mathbb{R}$

Then, if $i \neq j$, $\dfrac{\partial g_i}{\partial x_j} = 0$

On the other hand, if $i = j$, $\dfrac{\partial g_i}{\partial x_i} = g'(x_i)$

④ $$J(x_1 x_2 \cdots x_n) = \begin{bmatrix} g'(x_1) & & O \\ & g'(x_2) & \\ & & \ddots \\ O & & g'(x_n) \end{bmatrix} \longrightarrow \det(J(x_1 x_2 \cdots x_n)) = \prod_{i=1}^{n} g'(x_i)$$

⑤ Consider $c \in \mathbb{R}$

$$f_Y(y_1 \cdots y_n; t_1+c \cdots t_n+c)$$
$$= \frac{f_X(x_1 \cdots x_n; t_1+c \cdots t_n+c)}{\prod_{k=1}^{n} g'(x_k)} = \frac{f_X(x_1 \cdots x_n; t_1 \cdots t_n)}{\prod_{k=1}^{n} g'(x_k)}$$
$$= f_Y(y_1 \cdots y_n; t_1 \cdots t_n)$$

⑥ Hence, $y(t)$ is SSS ✱

Ex.

$y(t) = x^2(t)$ $\quad (g(x) \triangleq x^2)$

$$f_Y(y;t) = \left[f_X(\sqrt{y};t) + f_X(-\sqrt{y};t) \right] \frac{1}{2\sqrt{y}}$$
$$= \frac{f_X(\sqrt{y};t)}{|g'(\sqrt{y})|} + \frac{f_X(-\sqrt{y};t)}{|g'(-\sqrt{y})|}$$

$g' = 2X$
$g'(\pm\sqrt{y}) = \pm 2\sqrt{y}$

LTI system

$$\underset{RP}{x(t)} \longrightarrow \boxed{h(t)} \longrightarrow \underset{RP}{y(t)} = x(t) * h(t) = \int x(t-\tau)h(\tau)\,d\tau$$

- $E[y(t)] = \eta_x(t) * h(t)$
- $R_{XY}(t_1,t_2) = E[X(t_1)\cdot Y(t_2)]$, $\forall\, t_1, t_2 \in \mathbb{R}$

> **Theorem 9-2**
>
> $L_2\{R_{XX}(t_1,t_2)\}$
>
> (a) $R_{XY}(t_1,t_2) = \int_{-\infty}^{\infty} R_{XX}(t_1, t_2-\alpha)\,h(\alpha)\,d\alpha$
>
> (b) $R_{YY}(t_1,t_2) = \int_{-\infty}^{\infty} R_{XY}(t_1-\alpha, t_2)\,h(\alpha)\,d\alpha$
>
> $L_1\{R_{XY}(t_1,t_2)\}$

Differentiator

$$x(t) \longrightarrow \boxed{L} \longrightarrow y(t) = x'(t) \longleftrightarrow y(t;\omega) = \frac{\partial X(t,\omega)}{\partial t}$$

- $\eta_{x'}(t) = L\{\eta_x(t)\} = \eta'_x(t)$
- is LTI system : $\frac{d}{dt}[x(t-t_0)] = x'(t-t_0) = y(t-t_0)$
- $H(\omega) = j\omega$

$R_{XX}(t_1-t_2, 0) \triangleq R_{XX}(t_1-t_2)$

(1) $R_{XX'}(t_1,t_2) = L_2[R_{XX}(t_1,t_2)] = \dfrac{\partial R_{XX}(t_1,t_2)}{\partial t_2}$

(2) $R_{X'X'}(t_1,t_2) = L_1[R_{XX'}(t_1,t_2)] = \dfrac{\partial R_{XX}(t_1,t_2)}{\partial t_1\,\partial t_2}$

若 $X(t)$ is WSS , $y(t) \triangleq x'(t)$

- $\eta_Y(t) = 0$
- $R_{XX'}(\tau) = -R'_{XX}(\tau)$
- $R_{X'X'}(\tau) = -R''_{XX}(\tau)$

$R_{XX'}(t_1,t_2) = R_{XX'}(t_1-t_2, 0) = R_{XX'}(t_1-t_2) = \dfrac{\partial R_{XX}(t_1-t_2)}{\partial t_2}$

$= -R'_{XX}(\tau)$

則 $y(t) = x'(t)$ is also WSS

9-3 The Power Spectrum

In 9-3. all RPs are WSS

$$\begin{cases} S_x(\omega) \triangleq \int_{-\infty}^{\infty} R_{xx}(\tau) e^{-j\omega\tau} d\tau = \mathcal{F}\{R_{xx}(\tau)\} \\ R_{xx}(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_x(\omega) e^{j\omega\tau} d\omega = \mathcal{F}^{-1}\{S_x(\omega)\} \end{cases}$$

Table 9-1. 考試會給

$R_{xx}(\tau)$	$S_x(\omega)$
$\delta(\tau)$	1
1	$2\pi\delta(\omega)$
$e^{j\beta\tau}$	$2\pi\delta(\omega-\beta)$
$e^{-\alpha\|\tau\|}$	$\frac{2\alpha}{\alpha^2+\omega^2}$
$e^{-\alpha\tau^2}$	$\sqrt{\frac{\pi}{\alpha}} e^{-\omega^2/4\alpha}$
$\begin{cases} 1-\frac{\|\tau\|}{T}, \|\tau\|<T \\ 0 \end{cases}$	$\frac{4\sin^2(\omega T/2)}{T\omega^2}$

Lamma 1.

If $R_{xx}(\tau) = e^{-\alpha|\tau|}$, where $\alpha > 0$

$$S_x(\omega) = \frac{2\alpha}{\alpha^2+\omega^2}$$

proof.

① $S_x(\omega) = \int_{-\infty}^{\infty} R_{xx}(\tau) e^{-j\omega\tau} d\tau$

$$= \int_{-\infty}^{0} e^{\alpha\tau} e^{-j\omega\tau} d\tau + \int_{0}^{\infty} e^{-\alpha\tau} e^{-j\omega\tau} d\tau$$

$$= \left. \frac{e^{(\alpha-j\omega)\tau}}{\alpha-j\omega} \right|_{-\infty}^{0} + \left. \frac{e^{-(\alpha+j\omega)\tau}}{-(\alpha+j\omega)} \right|_{0}^{\infty}$$

$$= \frac{1}{\alpha-j\omega} + \frac{1}{\alpha+j\omega} = \frac{2\alpha}{\alpha^2+\omega^2} \quad \#$$

Theorem 9-4.

$$R_{xy}(\tau) = R_{xx}(\tau) * h^*(-\tau) \longleftrightarrow S_{xy}(\omega) = S_{xx}(\omega) H^*(\omega)$$

$$R_{yy}(\tau) = R_{xy}(\tau) * h(\tau) \longleftrightarrow S_{yy}(\omega) = S_{xy}(\omega) H(\omega) = S_{xx}(\omega) |H(\omega)|^2$$

proof. $R_{xy}(\tau) = R_{xx}(\tau) * h^*(-\tau) \longleftrightarrow S_{xy}(\omega) = S_{xx}(\omega) H^*(\omega)$

- $R_{xy}(\tau) = E[x(t+\tau) y^*(t)]$

$$= E\left[x(t+\tau) \int_{-\infty}^{\infty} x^*(t-\alpha) h^*(\alpha) d\alpha\right]$$

$$= \int_{-\infty}^{\infty} E[x(t+\tau) x^*(t-\alpha)] h^*(\alpha) d\alpha$$

$$= \int_{-\infty}^{\infty} R_{xx}(\tau+\alpha) h^*(\alpha) d\alpha \text{ —— } ①$$

let $g(\tau) = h^*(-\tau)$

$$R_{xx}(\tau) * h^*(-\tau) = R_{xx}(\tau) * g(\tau)$$

$$= \int_{-\infty}^{\infty} R_{xx}(\tau-\beta) g(\beta) d\beta$$

$$= \int_{-\infty}^{\infty} R_{xx}(\tau+\alpha) g(-\alpha) d\alpha$$

$$= \int_{-\infty}^{\infty} R_{xx}(\tau+\alpha) h^*(\alpha) d\alpha \text{ —— } ②$$

比較 ① & ②. $\underline{R_{xy}(\tau) = R_{xx}(\tau) * h^*(-\tau)}$ ※

- $S_{xy}(\omega) \triangleq \mathcal{F}\{R_{xy}(\tau)\} = \int_{-\infty}^{\infty} R_{xy}(\tau) e^{-j\omega\tau} d\tau$

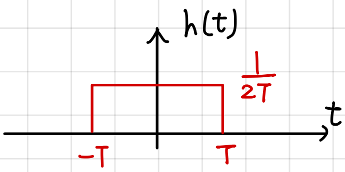
$$\rightarrow \int_{-\infty}^{\infty} h^*(-\tau) e^{-j\omega\tau} d\tau = \left[\int_{-\infty}^{\infty} h(\alpha) e^{-j\omega\alpha} d\tau\right]^* = H^*(\omega)$$

故 $\underline{S_{xy}(\omega) = S_{xx}(\omega) \cdot H^*(\omega)}$ ※

Ex 9-25

- $y(t) = \frac{1}{2T} \int_{t-T}^{t+T} x(\alpha) d\alpha$

- Define $h(t)$



- Then $y(t) = x(t) * h(t)$

Question: $R_{yy}(\tau) = ?$
$S_{yy}(\omega) = ?$

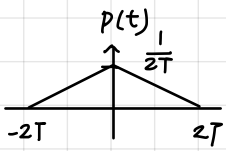
<Sol>

- $H(\omega) = \int_{-T}^{T} \frac{1}{2T} e^{-j\omega t} dt$

$$= \frac{1}{2T}\left[\int_{-T}^{T} \cos\omega t \, dt - \cancel{j\int_{-T}^{T} \sin(\omega t) dt}\right]$$ odd

$$= \frac{\sin\omega T}{\omega T}$$

- $S_{yy}(\omega) = S_{xx}(\omega) |H(\omega)|^2 = S_{xx}(\omega) \frac{\sin^2 \omega T}{\omega^2 T^2}$ ※



- $R_{yy}(\tau) = R_{xx}(\tau) \mathcal{F}^{-1}\left\{\frac{\sin^2 \omega T}{\omega^2 T^2}\right\} = R_{xx}(\tau) * P(t)$

If $y(t)$ is WSS RP

$$0 \le E[|y(t)|^2] = R_{yy}(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{yy}(\omega) d\omega$$

Ex 9-26

- $X(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t) = x'(t)$
- $Y(\omega) = j\omega X(\omega)$

  $H(\omega) = j\omega$

---

(1) $S_{yy}(\omega) = S_{xx}(\omega)\,|H(\omega)|^2$

$= S_{xx}(\omega) \cdot \omega^2$

(2) $R_{yy}(\tau) = \mathcal{F}^{-1}\{S_{yy}(\omega)\}$ $\omega^2 = -(j\omega)^2$

$= \mathcal{F}^{-1}\{\underline{-(j\omega)^2}\, S_{xx}(\omega)\}$

$= -\dfrac{d^2}{d\tau^2} R_{xx}(\tau)$ #

(3) $R_{yy}(0) =$

Ex 9-27

- $y'(t) + c\,y(t) = X(t)$.
- $R_{xx}(\tau) = q\,\delta(\tau)$
- $S_{xx}(\omega) = q$

$R_{yy}(\tau) = ?$ $S_{yy}(\omega) = ?$

$\rightarrow (j\omega + c)\,Y(\omega) = X(\omega)$

$\rightarrow H(\omega) = \dfrac{Y(\omega)}{X(\omega)} = \dfrac{1}{j\omega + c}$

(1) $S_{yy}(\omega) = S_{xx}(\omega)\,|H(\omega)|^2 = \dfrac{q}{\omega^2 + c^2}$ #

(2) $R_{yy}(\tau) = \mathcal{F}^{-1}\left\{\dfrac{q}{2c} \cdot \dfrac{2c}{\omega^2 + c^2}\right\}$

$= \dfrac{q}{2c} \cdot e^{-c|\tau|}$ #

9-4 Discrete Time Process

$x[n]$: discrete-time SP

$x(t)$: continuous-time SP

- $\eta_x[n] \triangleq E\{x[n]\}$
- $R_{xx}[n_1,n_2] \triangleq E\{x[n_1]\cdot x^*[n_2]\} \quad \forall n_1 n_2 \in \mathbb{Z}$
- $C_{xx}[n_1,n_2] \triangleq R_{xx}[n_1,n_2] - \eta_x[n_1]\eta_x^*[n_2]$

$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n] = \sum_{k=-\infty}^{\infty} x[n-k]\,h[k]$$

- $H(z) = \boxed{\mathcal{Z}\{h[n]\} = \sum_{n=-\infty}^{\infty} h[n]\cdot z^{-n}}$

Theorem 1.

$$R_{xy}[n_1,n_2] = \sum R_{xx}[n_1,n_2-k]\,h^*[k]$$

$$R_{yy}[n_1,n_2] = \sum R_{xy}[n_1-r,n_2]\,h[r]$$

$$S_x(z) \triangleq \sum R_x[m]\,z^{-m}$$

- $S_x(\omega) \triangleq S_x(e^{j\omega}) = \sum R_x[m]e^{-j\omega m}$
- $R_x[m] = \frac{1}{2\pi}\int_{-\pi}^{\pi} S_x(e^{j\omega})\,e^{j\omega m}\,d\omega$

$\rightarrow$ $S_x(e^{j\omega})$ is a periodic function with period 2π

Consider $x[n]$, $r_k \triangleq R_x[k]$

$$T_n \triangleq \begin{bmatrix} r_0 & r_1 & r_2 & & r_n \\ r_1^* & r_0 & r_1 & & \\ r_2^* & r_1^* & r_0 & & \\ & & & \ddots & \\ & & & & r_0 \end{bmatrix} \in \mathbb{C}^{(n+1)(n+1)}$$ is autocorrelation matrix

- $T_n = T_n^H$
- T_n is a Toeplitz matrix
- T_n is positive semi-definite, $u^H T_n u \geq 0$

Theorem 2

Let $Y = (Y_1\ Y_2 \cdots Y_n)^T$ be a n-dimensional complex random vector

Define $R_Y \triangleq E[YY^H]$, is a positive semi-definite matrix

proof

1. Consider $u \in \mathbb{C}^{n\times 1}$,

$$u^H R_Y u = E[u^H Y Y^H u]$$
$$= E[|u^H Y|^2] \geq 0$$

Ex 9-31

$R_x[m] = a^{|m|}\ ; \forall m,\ |a| \leq 1$, Find $S_x(z)$ and $S_x(\omega)$

① $S_x(z) = \sum_{-\infty}^{\infty} R_x[m]\, z^{-m}$

$$= \sum_{-\infty}^{-1} a^{-m} z^{-m} + \sum_{0}^{\infty} a^{m} z^{-m}$$

$$= \sum_{n=1}^{\infty} a^{m} z^{m} + \sum_{n=0}^{\infty} a^{m} z^{-m}$$

$$= \underline{\frac{az}{1-az} + \frac{1}{1-az^{-1}}}\ \#$$

ROC: $|az| < 1$ and $\left|\frac{a}{z}\right| < 1$

$$\Rightarrow \underline{|a| < |z| < \left|\frac{1}{a}\right|}$$

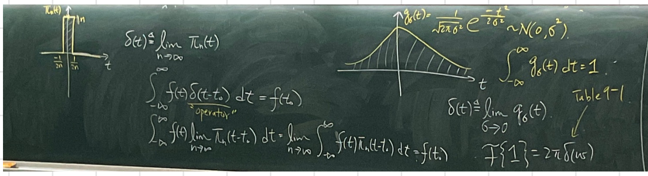
Ex 9-32

C_i is a random variable $\omega_i \in R$

$x[n] = \sum C_i e^{j\omega_i n}$, $\forall n \in Z$

$E[C_i] = 0$, $C_i \perp C_j$

$R_x[m]$, $S_x(\omega) = ?$



① $R_x[n+m, n]$

$$= E\left[\sum_i C_i e^{j\omega_i (n+m)} \cdot \sum_j C_j^* e^{-j\omega_j n}\right]$$

$$= E\left[\sum_i \sum_j C_i C_j^* e^{j[(\omega_i - \omega_j)n + \omega_i m]}\right]$$

$$= \sum_i \sum_j E[C_i C_j^*] e^{j[(\omega_i - \omega_j)n + \omega_i m]}$$

$$= \sum_i E[|C_i|^2] e^{j\omega_i m}$$

$$= \sum_i \sigma_i^2 e^{j\beta_i m} = R_x[m]$$

② $S_x(f)$

$$= \sum_m R_x[m] e^{-j\omega m}$$

$$= \sum_m \sum_i \sigma_i^2 e^{j\beta_i m} e^{-j\omega m}$$

$$= \sum_i \sigma_i^2 \sum_m e^{-j(\omega - \beta_i)m}$$

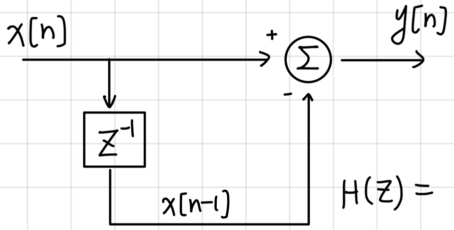
$$= \sum_i \sigma_i^2 \cdot 2\pi \delta(\omega - \beta_i)\ \#$$

Ex 9-33

$x[n]$ is WSS, $E[x[n]]=0$

$y[n]=x[n]-x[n-1]$

$R_{yy}[m], S_{yy}(\omega)=?$



$H(z)=1-z^{-1}$

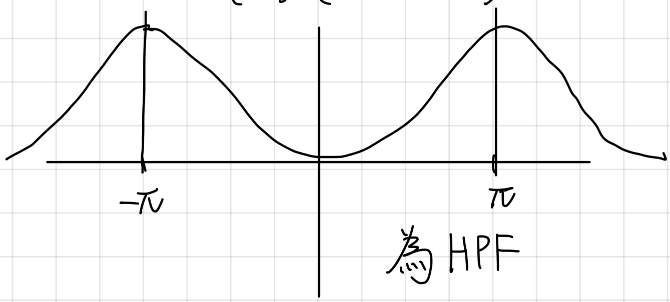
FIR

<Sol>

$$S_{yy}(z)=S_{xx}(z)H(z)H(\tfrac{1}{z})$$
$$=S_{xx}(z)(1-z^{-1})(1-z)$$
$$=S_{xx}(z)(2-z-z^{-1})$$

$$R_{yy}[m]=2R_{xx}[m]-R_{xx}[m+1]-R_{xx}[m-1]$$

$$S_{yy}(\omega)=S_{yy}(e^{j\omega})$$
$$=S_{xx}(e^{j\omega})[2-e^{-j\omega}-e^{j\omega}]$$
$$=S_{xx}(e^{j\omega})(2-2\cos\omega)$$



為 HPF

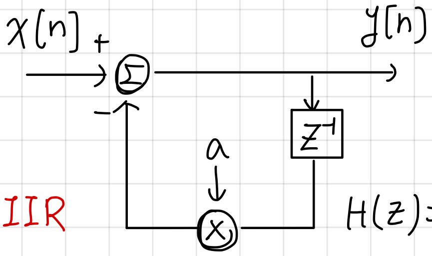
Ex 9-34

$x[n]$ is a WSS RP with zero mean ; $a\in(-1,1)$

$y[n]-a\,y[n-1]=x[n]$

$S_{xx}(z)=q$

IIR



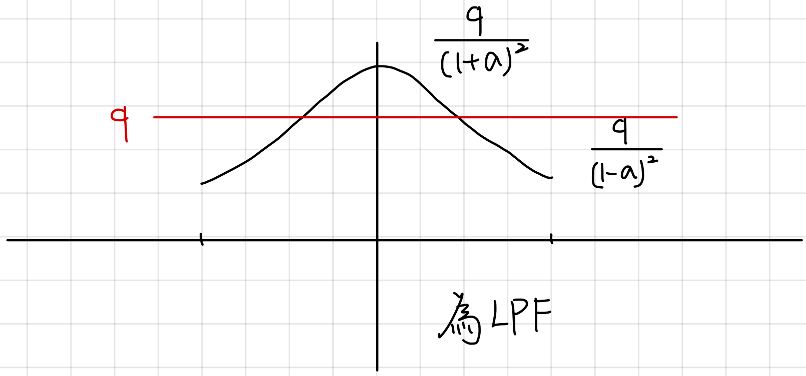
$H(z)=\dfrac{1}{1-az^{-1}}$

<Sol>

$$S_{yy}(z)=S_{xx}(z)\cdot\frac{1}{1-az^{-1}}\cdot\frac{1}{1-az}$$
$$=\frac{q}{1+a^2-a(z+z^{-1})}$$
$$=\frac{qa^{-1}}{(a+a^{-1})-(z+z^{-1})}$$
$$=\frac{a^{-1}-a}{(a+a^{-1})-(z+z^{-1})}\cdot\frac{qa^{-1}}{a^{-1}-a}$$

$$R_{yy}[m]=\frac{qa^{-1}}{a^{-1}-a}\,a^{|m|}$$

$$S_{yy}(\omega)=S_{yy}(e^{j\omega})$$
$$=\frac{q}{1+a^2-2a\cos\omega} \quad \#$$



為 LPF