Rational Linear Phase System.

- $h[n] = e^{j2\beta} h^*[2\alpha - n] \longleftrightarrow H(z) = e^{j2\beta} z^{-2\alpha} \tilde{H}(z)$

$\rightarrow \tilde{H}(z) \triangleq H^*(1/z^*) = H^*(e^{j\omega})$

〈證明〉

$H(e^{j\omega}) = H_R(e^{j\omega}) \cdot e^{-j\alpha\omega} \cdot e^{-j\beta}$

$\rightarrow H(e^{j\omega}) = H^*(e^{j\omega}) \cdot e^{-j2\alpha\omega} \cdot e^{-j2\beta}$

$\rightarrow h[n] = e^{-j2\beta} \cdot h^*[2\alpha - n]$ ※

FIR linear phase filter

- $H(z) = \sum_{n=0}^{N} h(n) z^{-n}$
- $H(e^{j\omega})$ has linear phase iff

$h[n] = e^{j\theta} h^*[N-n] \leftrightarrow H(z) = e^{j\theta} z^{-N} \tilde{H}(z)$

- zero : "conjugate reciprocal pair"
- pole : only 0 or ∞

Rational IIR Linear Phase Filter

- Pole outside the unit circle

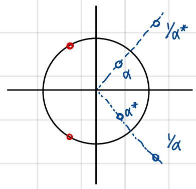
$\rightarrow$ IIR linear phase filter $H(z)$ cannot be causal / stable 矛盾

$\rightarrow$ linear phase property only consider FIR

Real Coefficient FIR linear phase filter

- $h[n] = \pm h[N-n]$: symmetric or anti-symmetric

	$h[n] = h[N-n]$	$h[n] = -h[N-n]$
Even N	1	3
Odd N	2	4

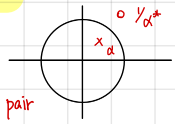


- zero : set of 4 conjugate reciprocal zero $\{\alpha, \alpha^*, 1/\alpha, 1/\alpha^*\}$

Allpass filter

- $H(z) = \beta \prod_{k=1}^{N} \frac{-\alpha_k^* + z^{-1}}{1 - \alpha_k z^{-1}}$

zero : $1/\alpha_1^*, 1/\alpha_2^*, \cdots 1/\alpha_N^*$
pole : $\alpha_1, \alpha_2, \cdots \alpha_N$
conjugate reciprocal pair



- $|H(e^{j\omega})| = C \rightarrow H(z)\tilde{H}(z) = C^2$

Replace Zero by its Conjugate Reciprocal.

- Let $H(z) = (1 - bz^{-1}) G(z)$, zero at $z = b$
- If replace b by $1/b^*$, $H_1(z) = (b^* - z^{-1}) G(z)$

$\Rightarrow$ They still have same magnitude response.

$H_1(z) = H(z) \frac{(b^* - z^{-1})}{(1 - bz^{-1})}$ unit gain allpass

Pole-zero plot with causal LTI system

IIR	pole at place other than 0 and ∞
FIR	pole can only be 0 or ∞
stable	ROC include unit circle
minimum phase	All zero and pole inside the unit circle
generalized linear phase	• FIR : zero are conjugate reciprocal pair • ~~IIR : pole outside the unit circle~~ 矛盾 ★ in practice, we use causal & stable system
allpass	pole and zero are conjugate reciprocal pair
shortest response	• IIR : 不需考慮 • FIR : $\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$ 長度最短者

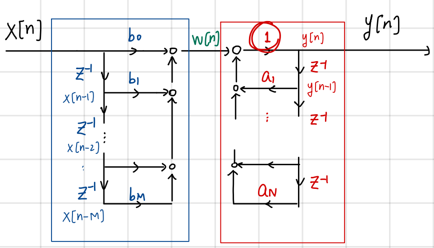
$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 - \boxed{\sum_{k=1}^{N} a_k z^{-k}}}$$

Structure of Rational System.

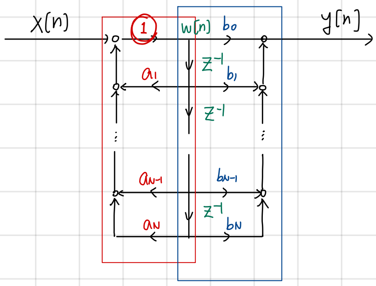
- Direct form I / II
- Cascade form
- Paralle form : FIR 沒有 Paralle form

- Feedback :
  - close path
  - begin and end at same node
- Transpose

Direct form I

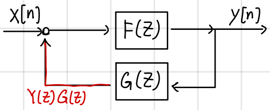


Direct form II



Discrete-time network.

- Cascade
- Parallel
- Feedback



$\rightarrow [X(z) + Y(z)G(z)] F(z) = Y(z)$

IIR Filter design.

- Butterworth Filter
- Chebyshev Filter
- Elliptic Filter.

- Bilinear Transformation. $\Omega = \frac{2}{T_d} \tan(\frac{\omega}{2})$
- Impulse invariance Transformation $h_d[n] = T_d h_c(nT_d)$

FIR Filter design

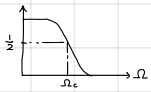
- Window design : Kaiser window.

Bilinear transformation

- $\Omega = \frac{2}{T_d} \tan(\frac{\omega}{2})$
- $S = \frac{2}{T_d}\left[\frac{1 - z^{-1}}{1 + z^{-1}}\right]$ ★

Butterworth Filter

- $|H_c(j\Omega)|^2 = \frac{1}{1 + (\Omega/\Omega_c)^{2N}}$



- $H(s) \cdot H(-s) = \frac{1}{1 + (S/j\Omega_c)^{2N}}$

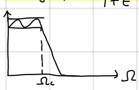
pole : $(S/j\Omega_c)^{2N} = e^{j(\pi + 2\pi k)}$

$\rightarrow S/j\Omega_c = e^{j\frac{\pi}{2N} + \frac{\pi}{N}k}$

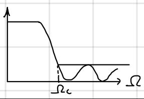
$\rightarrow S = \Omega_c e^{j\frac{\pi}{2N} + \frac{\pi}{N}k + \frac{\pi}{2}}$

Chebyshev Filter

- $|H_c(j\Omega)|^2 = \frac{1}{1 + \epsilon^2 T_N(\Omega/\Omega_c)}$

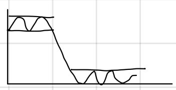


- $|H_c(j\Omega)|^2 = \frac{1}{1 + 1/\epsilon^2 T_N(\Omega/\Omega_s)}$



Elliptic Filter

- $|H_c(j\Omega)|^2 = \frac{1}{1 + \epsilon^2 R_N^2(\Omega/\Omega_c)}$



Kaiser Window.

step 0. Given $\{\delta_p, \delta_s, \omega_p, \omega_s\}$

step 1. $\omega_c = (\omega_p + \omega_s)/2$

step 2. Determine β

$$\beta = \begin{cases} 0.1102(A_s - 8.7), & A_s > 50 \\ 0.5842(A_s - 21)^{0.4} + 0.07886(A_s - 21), & 21 < A_s < 50 \\ 0, & A_s < 21 \end{cases}$$

- $\delta = \min(\delta_p, \delta_s)$
- $A_s = -20\log_{10}\delta$

step 3. $M = \frac{A_s - 8}{2.285\Delta\omega}$

$$W[n] = \frac{I_0\left(\beta\sqrt{1 - \left(\frac{n - M/2}{M/2}\right)^2}\right)}{I_0(\beta)}$$

DFS

$W_N = e^{-j\frac{2\pi}{N}}$

$$\begin{cases}\tilde{X}[k] = \sum_{n=0}^{N-1} \tilde{x}[n] W_N^{kn} & ; -\infty < n < \infty \\ \tilde{x}[n] = \frac{1}{N}\sum_{n=0}^{N-1} \tilde{X}[k] W_N^{kn} & ; -\infty < k < \infty\end{cases}$$

- 皆以 N 為週期,故序列為無限長.

DFT

僅範圍不同

$$\begin{cases}X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn} & ; 0 < n < N-1 \\ x[n] = \frac{1}{N}\sum_{n=0}^{N-1} X[k] W_N^{-kn} & ; 0 < k < N-1\end{cases}$$

- 從 DFS 選取其中一段來做訊號處理

Circular Convolution $x[n] \circledast h[n] \longleftrightarrow X[k]H[k]$

Sampling the DTFT

- Given $x[n]$; $0 \le n \le N-1$

$$\begin{cases}X(e^{j\omega}) = \sum_{n=0}^{N-1} x[n] e^{-j\omega n} = \sum_{n=0}^{N-1} \tilde{x}[n] e^{-j\omega n} \\ \tilde{X}[k] = \sum_{n=0}^{N-1} \tilde{x}[n] W_N^{kn} = \sum_{n=0}^{N-1} \tilde{x}[n] e^{-j\frac{2\pi}{N}kn}\end{cases}$$

Then,

- $\tilde{X}[k] = X(e^{j\omega})\big|_{\omega = \frac{2\pi}{N}k}$
- $\tilde{x}[n] = \sum_{r=-\infty}^{\infty} x[n-rN]$

- If $N >$ length of $x[n]$ we can recover $x[n]$
- Otherwise, aliasing occur

Modulo N

- If $\tilde{x}[n]$ periodic sequence with $N \rightarrow \tilde{x}[n] = \tilde{x}[n+N]$
- $((n))_N$: 將超出週期的 n 折返回週期內

EX. $\tilde{x}[n] = (1, 3, 7, 8)$, $N = 4$

$$\left.\begin{aligned} x[((0))_4] &= 1 \\ x[((3))_4] &= 8 \\ x[((5))_4] &= x[1] = 3 \end{aligned}\right\}, \ \tilde{x}[n] = x[((n))_N]$$

FFT Fast computation of DFT

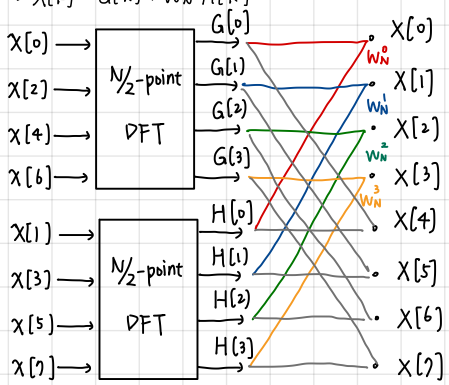
- Decimation-in-Time (DIT)
- Decimation-in-Freq. (DIF)

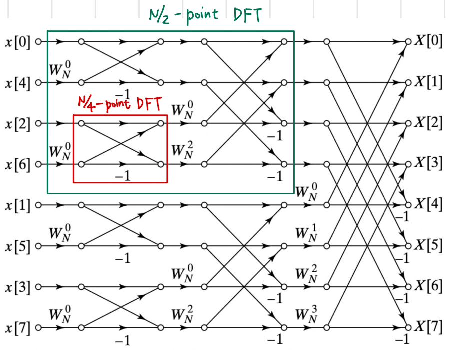
DIT.

Consider the subsequence of $x[n]$

$$\begin{cases} g[n] = x[2n] \\ h[n] = x[2n+1] \end{cases} \Rightarrow \begin{aligned} G[k] &= \sum_{n=0}^{N/2-1} x[2n] W_{N/2}^{nk} \\ H[k] &= \sum_{n=0}^{N/2-1} x[2n+1] W_{N/2}^{nk} \end{aligned}$$

$\rightarrow X[k] = G[k] + W_N^k H[k]$





- $\log_2 N$ stage
- $N/2$ butterflies/stage $\Rightarrow$ Total $\begin{cases}\frac{N}{2}\log_2 N \text{ MUL} \\ N\log_2 N \text{ ADD}\end{cases}$
- 1 MUL, 2 ADD each butterfly

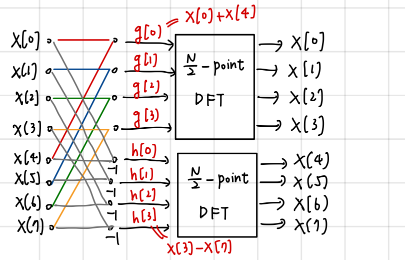
DIF.

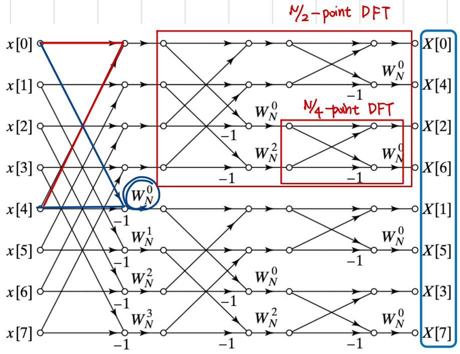
Consider the subsequence of $X[k]$

$$\begin{cases} X[2r] & ; 0 < r < N/2 - 1 \\ X[2r+1] & ; 0 < r < N/2 - 1 \end{cases}$$

$\rightarrow X[2r] = \sum_{n=0}^{N/2-1} (x[n] + x[n+N/2]) W_{N/2}^{rn}$

$\rightarrow X[2r+1] = \sum_{n=0}^{N/2-1} (x[n] - x[n+N/2]) W_{N/2}^{rn} \cdot \boxed{W_N^n}$





- $\log_2 N$ stage
- $N/2$ butterflies/stage $\Rightarrow$ Total $\begin{cases}\frac{N}{2}\log_2 N \text{ MUL} \\ N\log_2 N \text{ ADD}\end{cases}$
- 1 MUL, 2 ADD each butterfly

Type I: Symmetric, M Even, Odd Length

$$\begin{aligned} H(e^{j\omega}) &= \sum_{n=0}^{M} h[n]e^{-j\omega n} \\ &= \sum_{n=0}^{(M-2)/2} h[n]e^{-j\omega n} + \sum_{n=(M+2)/2}^{M} h[n]e^{-j\omega n} + h[M/2]e^{-j\omega(M/2)} \\ &= \sum_{n=0}^{(M-2)/2} h[n]e^{-j\omega n} + \sum_{m=0}^{(M-2)/2} h[M-m]e^{-j\omega(M-m)} + h[M/2]e^{-j\omega(M/2)} \\ &= e^{-j\omega(M/2)}\left(\sum_{m=0}^{(M-2)/2} h[m]e^{j\omega((M/2)-m)} + \sum_{m=0}^{(M-2)/2} h[m]e^{-j\omega((M/2)-m)} + h[M/2]\right) \\ &= e^{-j\omega(M/2)}\left(\sum_{m=0}^{(M-2)/2} 2h[m]\cos\omega((M/2)-m) + h[M/2]\right) \\ &= e^{-j\omega(M/2)}\left(\sum_{n=1}^{M/2} 2h[(M/2)-n]\cos\omega n + h[M/2]\right) \end{aligned}$$

Let

$$a[n] = \begin{cases} h[M/2], & n = 0 \\ 2h[(M/2)-n], & n = 1, \ldots, M/2 \end{cases}$$

Then

$$H(e^{j\omega}) = e^{-j\omega(M/2)} \sum_{n=0}^{M/2} a[n]\cos\omega n$$

and we have

$$A(\omega) = \sum_{n=0}^{M/2} a[n]\cos(\omega n), \quad \alpha = \frac{M}{2}, \quad \beta = 0$$

Type II: Symmetric, M Odd, Even Length

$$\begin{aligned} H(e^{j\omega}) &= \sum_{n=0}^{M} h[n]e^{-j\omega n} \\ &= \sum_{n=0}^{(M-1)/2} h[n]e^{-j\omega n} + \sum_{n=(M+1)/2}^{M} h[n]e^{-j\omega n} \\ &= \sum_{n=0}^{(M-1)/2} h[n]e^{-j\omega n} + \sum_{m=0}^{(M-1)/2} h[M-m]e^{-j\omega(M-m)} \\ &= e^{-j\omega(M/2)}\left(\sum_{m=0}^{(M-1)/2} h[m]e^{j\omega((M/2)-m)} + \sum_{m=0}^{(M-1)/2} h[m]e^{-j\omega((M/2)-m)}\right) \\ &= e^{-j\omega(M/2)}\left(\sum_{m=0}^{(M-1)/2} 2h[m]\cos\omega((M/2)-m)\right) \\ &= e^{-j\omega(M/2)}\left(\sum_{n=1}^{(M+1)/2} 2h[(M+1)/2-n]\cos\omega(n-(1/2))\right) \end{aligned}$$

Let

$$b[n] = 2h[(M+1)/2-n], \quad n = 1, \ldots, (M+1)/2$$

Then

$$H(e^{j\omega}) = e^{-j\omega(M/2)} \sum_{n=1}^{(M+1)/2} b[n]\cos\omega(n-(1/2))$$

and we have

$$A(\omega) = \sum_{n=1}^{(M+1)/2} b[n]\cos\omega(n-(1/2)), \quad \alpha = \frac{M}{2}, \quad \beta = 0$$

Type III: Antisymmetric, M Even, Odd Length

$$\begin{aligned} H(e^{j\omega}) &= \sum_{n=0}^{M} h[n]e^{-j\omega n} \\ &= \sum_{n=0}^{(M-2)/2} h[n]e^{-j\omega n} + 0 + \sum_{(M+2)/2}^{M} h[n]e^{-j\omega n} \\ &= \sum_{n=0}^{(M-2)/2} h[n]e^{-j\omega n} + \sum_{m=0}^{(M-2)/2} h[M-m]e^{-j\omega(M-m)} \\ &= e^{-j\omega(M/2)}\left(\sum_{m=0}^{(M-2)/2} h[m]e^{j\omega((M/2)-m)} - \sum_{m=0}^{(M-2)/2} h[m]e^{-j\omega((M/2)-m)}\right) \\ &= e^{-j\omega(M/2)}\left(j\sum_{m=0}^{(M-2)/2} 2h[m]\sin\omega((M/2)-m)\right) \\ &= e^{-j\omega(M/2)}e^{j(\pi/2)}\left(\sum_{n=1}^{M/2} 2h[(M/2)-n]\sin\omega n\right) \end{aligned}$$

Let

$$c[n] = h[(M/2)-n], \quad n = 1, \ldots, M/2$$

Then

$$H(e^{j\omega}) = e^{-j\omega(M/2)}e^{j(\pi/2)} \sum_{n=1}^{M/2} c[n]\sin\omega n$$

and we have

$$A(\omega) = \sum_{n=1}^{M/2} c[n]\sin(\omega n), \quad \alpha = \frac{M}{2}, \quad \beta = \frac{\pi}{2}$$

Type IV: Antisymmetric, M Odd, Even Length

$$\begin{aligned} H(e^{j\omega}) &= \sum_{n=0}^{M} h[n]e^{-j\omega n} \\ &= \sum_{n=0}^{(M-1)/2} h[n]e^{-j\omega n} + \sum_{n=(M+1)/2}^{M} h[n]e^{-j\omega n} \\ &= \sum_{n=0}^{(M-1)/2} h[n]e^{-j\omega n} + \sum_{m=0}^{(M-1)/2} h[M-m]e^{-j\omega(M-m)} \\ &= e^{-j\omega(M/2)}\left(\sum_{m=0}^{(M-1)/2} h[m]e^{j\omega((M/2)-m)} - \sum_{m=0}^{(M-1)/2} h[m]e^{-j\omega((M/2)-m)}\right) \\ &= e^{-j\omega(M/2)}\left(j\sum_{m=0}^{(M-1)/2} 2h[m]\sin\omega((M/2)-m)\right) \\ &= e^{-j\omega(M/2)}e^{j(\pi/2)} \sum_{n=1}^{(M+1)/2} 2h[(M+1)/2-n]\sin\omega(n-(1/2)) \end{aligned}$$