Notes on **HP Filter:**

$$\overset{\text{in logs}}{y_t} = \overset{\text{cyclical}}{C_t} + \overset{\text{trend}}{T_t} \quad \& \quad T = \#\text{ of observations}$$

$$\min_{T_1, T_2, \dots T_T} \left(\sum_{t=1}^{T} (\overbrace{y_t - T_t}^{\text{cyclical}})^2 + \lambda \sum_{t=2}^{T-1} (\overbrace{(T_{t+1} - T_t) - (T_t - T_{t-1})}^{\text{growth of trend}})^2 \right)$$

or

$$\min_{T_1, T_2, \dots T_T} \left(\sum_{t=1}^{T} (y_t - T_t)^2 + \lambda \sum_{t=2}^{T-1} (T_{t+1} - 2T_t + T_{t-1})^2 \right)$$

Example w/ T=6.

$$\min_{T_1, \dots T_6} \Big((y_1 - T_1)^2 + (y_2 - T_2)^2 + (y_3 - T_3)^2 + (y_4 - T_4)^2 + (y_5 - T_5)^2 + (y_6 - T_6)^2$$
$$+ \lambda (T_3 - 2T_2 + T_1)^2 + \lambda (T_4 - 2T_3 + T_2)^2 + \lambda (T_5 - 2T_4 + T_3)^2 + \lambda (T_6 - 2T_5 + T_4)^2 \Big)$$

First Order Conditions

$\{T_1\}$: $-2(y_1 - T_1) + 2\lambda(T_3 - 2T_2 + T_1) = 0$

$\{T_2\}$: $-2(y_2 - T_2) + -4\lambda(T_3 - 2T_2 + T_1) + 2\lambda(T_4 - 2T_3 + T_2) = 0$

$\{T_3\}$: $-2(y_3 - T_3) + 2\lambda(T_3 - 2T_2 + T_1) - 4\lambda(T_4 - 2T_3 + T_2) + 2\lambda(T_5 - 2T_4 + T_3) = 0$ ☆

$\{T_4\}$: $-2(y_4 - T_4) + 2\lambda(T_4 - 2T_3 + T_2) - 4\lambda(T_5 - 2T_4 + T_3) + 2\lambda(T_6 - 2T_5 + T_4) = 0$ ☆

$\{T_5\}$: $-2(y_5 - T_5) + 2\lambda(T_5 - 2T_4 + T_3) - 4\lambda(T_6 - 2T_5 + T_4) = 0$

$\{T_6\}$: $-2(y_6 - T_6) + 2\lambda(T_6 - 2T_5 + T_4)^2$

☆ For $t > 2$ & $t < (T-1)$ the F.O.C have same pattern.

i.e. $\left(-2(y_t - T_t) + 2\lambda(T_t - 2T_{t-1} + T_{t-2}) - 4\lambda(T_{t+1} - 2T_t + T_{t-1}) + 2\lambda(T_{t+2} - 2T_{t+1} + T_t) = 0\right)$

so we have:

T_1: $(y_1 - T_1) - \lambda(T_3 - 2T_2 + T_1) = 0$

T_2: $(y_2 - T_2) + 2\lambda(T_3 - 2T_2 + T_1) - \lambda(T_4 - 2T_3 + T_2) = 0$

T_3: $(y_3 - T_3) - \lambda(T_3 - 2T_2 + T_1) + 2\lambda(T_4 - 2T_3 + T_2) - \lambda(T_5 - 2T_4 + T_3) = 0$

T_4: $(y_4 - T_4) - \lambda(T_4 - 2T_3 + T_2) + 2\lambda(T_5 - 2T_4 + T_3) - \lambda(T_6 - 2T_5 + T_4) = 0$

T_5: $(y_5 - T_5) - \lambda(T_5 - 2T_4 + T_3) + 2\lambda(T_6 - 2T_5 + T_4) = 0$

T_6: $(y_6 - T_6) - \lambda(T_6 - 2T_5 + T_4) = 0$

& therefore

$y_1 = T_1 + \lambda(T_1 - 2T_2 + T_3)$

$y_2 = T_2 + \lambda(-2T_3 + 4T_2 - 2T_1) + \lambda(T_4 - 2T_3 + T_2)$

$y_3 = T_3 + \lambda(T_3 - 2T_2 + T_1) + \lambda(-2T_4 + 4T_3 - 2T_2) + \lambda(T_5 - 2T_4 + T_3)$

$y_4 = T_4 + \lambda(T_4 - 2T_3 + T_2) + \lambda(-2T_5 + 4T_4 - 2T_3) + \lambda(T_6 - 2T_5 + T_4)$

$y_5 = T_5 + \lambda(T_5 - 2T_4 + T_3) + \lambda(-2T_6 + 4T_5 - 2T_4)$

$y_6 = T_6 + \lambda(T_6 - 2T_5 + T_4)$

& $\therefore$

$$y_1 = T_1 + \lambda(T_1 - 2T_2 + T_3)$$
$$y_2 = T_2 + \lambda(-2T_1 + 5T_2 - 4T_3 + T_4)$$
$$y_3 = T_3 + \lambda(T_1 - 4T_2 + 6T_3 - 4T_4 + T_5)$$
$$y_4 = T_4 + \lambda(T_2 - 4T_3 + 6T_4 - 4T_5 + T_6)$$
$$y_5 = T_5 + \lambda(T_3 - 4T_4 + 5T_5 - 2T_6)$$
$$y_6 = T_6 + \lambda(T_4 - 2T_5 + T_6)$$

Can represent above equations
In matrix form:

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \end{pmatrix} = \left[\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} + \lambda \underbrace{\begin{pmatrix} 1 & -2 & 1 & 0 & 0 & 0 \\ -2 & 5 & -4 & 1 & 0 & 0 \\ 1 & -4 & 6 & -4 & 1 & 0 \\ 0 & 1 & -4 & 6 & -4 & 1 \\ 0 & 0 & 1 & -4 & 5 & -2 \\ 0 & 0 & 0 & 1 & -2 & 1 \end{pmatrix}}_{A_{T\times T}}\right] \begin{pmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \end{pmatrix}$$

or $\quad Y_{T\times 1} = [I_{T\times T} + \lambda A_{T\times T}]\, T_{T\times 1}$

$\therefore \quad T = (I + \lambda A)^{-1} Y$ is the trend component

& $\quad y - T = C$ is the cyclical component.

In general w/ a bigger T. recall ☆, $t > 2$ & $t < (T-1)$ have a pattern of $(1\ -4\ 6\ -4\ 1)$

$$A_{T\times T} = \begin{bmatrix} 1 & -2 & 1 & 0 & 0 & 0 & \cdots & & & 0 \\ -2 & 5 & -4 & 1 & 0 & 0 & \cdots & & & 0 \\ 1 & -4 & 6 & -4 & 1 & 0 & \cdots & & & 0 \\ 0 & 1 & -4 & 6 & -4 & 1 & \cdots & & & 0 \\ 0 & 0 & 1 & -4 & 6 & -4 & 1 & \cdots & & 0 \\ \vdots & & & & & & & & & \vdots \\ 0 & \cdots & 0 & 1 & -4 & 6 & -4 & 1 & 0 & 0 \\ 0 & \cdots & 0 & 0 & 1 & -4 & 6 & -4 & 1 & 0 \\ 0 & \cdots & 0 & 0 & 0 & 1 & -4 & 6 & -4 & 1 \\ 0 & \cdots & 0 & 0 & 0 & 0 & 1 & -4 & 5 & -2 \\ 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 \end{bmatrix}$$

$t > 2$ & $t < T-1$
Same #s keep repeating

$$\overbrace{\underset{(T\times 1)}{T} = (I + \lambda \underset{(T\times T)}{A})^{-1} \underset{(T\times 1)}{Y}}^{\text{TREND}} \quad \& \quad \overbrace{C = Y - T}^{\text{CYCLICAL}}$$