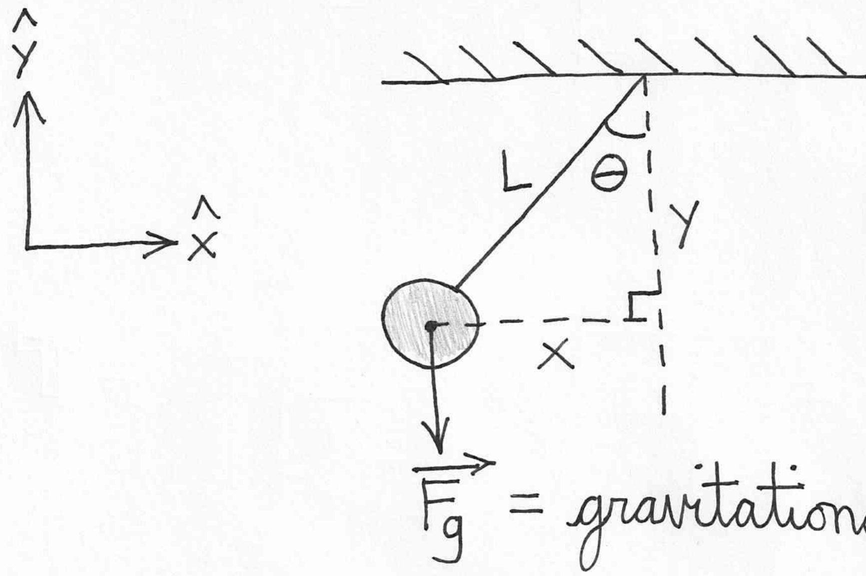
PENDULUM

LAGRANGE METHOD



L = length of the cord

$$x = -L\sin(\theta)$$

$$y = -L\cos(\theta)$$

$$v_x = \frac{dx}{dt} = \frac{d}{dt}\left(-L\sin(\theta)\right) = -L\cos(\theta)\frac{d\theta}{dt}$$

$$v_y = \frac{dy}{dt} = \frac{d}{dt}\left(-L\cos(\theta)\right) = L\sin(\theta)\frac{d\theta}{dt}$$

The kinetic energy is

$$KE = \frac{1}{2}mv^2$$

$$\vec{v} = v_x\hat{x} + v_y\hat{y}$$

$$v^2 = \vec{v}\cdot\vec{v} = (v_x\hat{x} + v_y\hat{y})\cdot(v_x\hat{x} + v_y\hat{y}) = v_x^2 + v_y^2$$

$$KE = \frac{1}{2}mv^2 = \frac{1}{2}m(v_x^2 + v_y^2)$$

$$v_x^2 = L^2\cos^2(\theta)\left(\frac{d\theta}{dt}\right)^2$$

$$v_y^2 = L^2\sin^2(\theta)\left(\frac{d\theta}{dt}\right)^2$$

$$KE = \frac{1}{2}m(v_x^2 + v_y^2)$$

$$= \frac{1}{2}m\left(L^2\cos^2(\theta)\left(\frac{d\theta}{dt}\right)^2 + L^2\sin^2(\theta)\left(\frac{d\theta}{dt}\right)^2\right)$$

$$= \frac{1}{2}mL^2\left(\frac{d\theta}{dt}\right)^2\underbrace{(\sin^2(\theta)+\cos^2(\theta))}_{1}$$

$$= \boxed{\frac{1}{2}mL^2\left(\frac{d\theta}{dt}\right)^2}$$

$$PE = mgy \quad (\text{potential energy})$$

$$= mg(-L\cos(\theta))$$

$$= \boxed{-mgL\cos(\theta)}$$

The Lagrangian is

$$\mathcal{L} = KE - PE$$

$$= \frac{1}{2}mL^2\left(\frac{d\theta}{dt}\right)^2 + mgL\cos(\theta)$$

$$= \boxed{\frac{1}{2}mL^2\dot{\theta}^2 + mgL\cos(\theta)} \qquad \dot{\theta} = \frac{d\theta}{dt}$$

$$\boxed{\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_K}\right) - \frac{\partial \mathcal{L}}{\partial q_K} = 0} \quad (\text{Lagrange Equation})$$

q_K = generalized coordinates

$\dot{q}_K$ = generalized velocities

let $q_K = \theta$ and $\dot{q}_K = \dot{\theta}$

$$\boxed{\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) - \frac{\partial \mathcal{L}}{\partial \theta} = 0}$$

$$\frac{\partial \mathcal{L}}{\partial \theta} = \frac{\partial}{\partial \theta}\left(\frac{1}{2} mL^2\dot{\theta}^2 + mgL\cos(\theta)\right)$$

$$= 0 + mgL(-\sin(\theta))$$

$$= -mgL\sin(\theta)$$

$$\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = \frac{\partial}{\partial \dot{\theta}}\left(\frac{1}{2} mL^2\dot{\theta}^2 + mgL\cos(\theta)\right)$$

$$= \frac{1}{2} mL^2 2\dot{\theta}$$

$$= mL^2\dot{\theta}$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) - \frac{\partial \mathcal{L}}{\partial \theta} = 0$$

$$\frac{d}{dt}\left(mL^2\dot{\theta}\right) - (-mgL\sin(\theta)) = 0$$

$$mL^2\ddot{\theta} + mgL\sin(\theta) = 0$$

$$L\ddot{\theta} + g\sin(\theta) = 0$$

$$\boxed{L\frac{d^2\theta}{dt^2} + g\sin(\theta) = 0}$$ equation of motion of the pendulum

There is no direct method for solving this equation.

If we apply the small angle approximation

$\sin(\theta) \approx \theta$

(valid for $\theta \leq 5^\circ$)

($\theta < 14^\circ$ for less than 1 % error)

we get

$$L\frac{d^2\theta}{dt^2} + g\underbrace{\sin(\theta)}_{\approx\theta} = 0$$

$$L\frac{d^2\theta}{dt^2} + g\theta = 0 \qquad (\theta = \theta(t))$$

$$L\frac{d^2\theta}{dt^2} + 0\frac{d\theta}{dt} + g\theta = 0$$ (second order ordinary differential equation)

The associated auxiliary equation is

$$L\lambda^2 + 0\lambda + g = 0$$

$$\lambda^2 = \frac{-g}{L}$$

$$\lambda = \pm\sqrt{\frac{-g}{L}} = \pm\sqrt{-1}\sqrt{\frac{g}{L}} = \pm i\sqrt{\frac{g}{L}}$$

This is the case of complex roots.

$$\theta(t) = Ae^{\lambda_1 t} + Be^{\lambda_2 t}$$

$$= Ae^{i\sqrt{\frac{g}{L}}t} + Be^{-i\sqrt{\frac{g}{L}}t}$$

$$= A\left(\cos\left(\sqrt{\frac{g}{L}}\,t\right) + i\sin\left(\sqrt{\frac{g}{L}}\,t\right)\right)$$

$$+ B\left(\cos\left(\sqrt{\frac{g}{L}}\,t\right) - i\sin\left(\sqrt{\frac{g}{L}}\,t\right)\right)$$

$$= (A+B)\cos\left(\sqrt{\frac{g}{L}}\,t\right) + i(A-B)\sin\left(\sqrt{\frac{g}{L}}\,t\right)$$

let $A+B \rightarrow A$

let $i(A-B) \rightarrow B$

$$= \boxed{A\cos\left(\sqrt{\frac{g}{L}}\,t\right) + B\sin\left(\sqrt{\frac{g}{L}}\,t\right)}$$

Suppose $\theta = \theta_0$ at $t = 0$.

$$\theta(0) = \theta_0$$

$$\underbrace{\theta(0)}_{\theta_0} = A\underbrace{\cos\left(\sqrt{\frac{g}{L}}\cdot 0\right)}_{1} + B\underbrace{\sin\left(\sqrt{\frac{g}{L}}\cdot 0\right)}_{0}$$

$$\boxed{A = \theta_0}$$

Since the pendulum is not moving at $t=0$, we have

$$\left.\frac{d\theta}{dt}\right|_{t=0} = 0$$

$$\frac{d\theta}{dt} = \frac{d}{dt}\left(A\cos\left(\sqrt{\frac{g}{L}}\,t\right) + B\sin\left(\sqrt{\frac{g}{L}}\,t\right)\right)$$

$$= -A\sin\left(\sqrt{\frac{g}{L}}\,t\right)\sqrt{\frac{g}{L}} + B\cos\left(\sqrt{\frac{g}{L}}\,t\right)\sqrt{\frac{g}{L}}$$

$$\left.\frac{d\theta}{dt}\right|_{t=0} = 0$$

$$0 = -A\underbrace{\sin\left(\sqrt{\frac{g}{L}}\cdot 0\right)}_{0}\sqrt{\frac{g}{L}} + B\underbrace{\cos\left(\sqrt{\frac{g}{L}}\cdot 0\right)}_{1}\sqrt{\frac{g}{L}}$$

$$0 = B\sqrt{\frac{g}{L}}$$

Since $\sqrt{\frac{g}{L}} \neq 0$, it must be the case that

$$\boxed{B = 0}$$

$$\theta(t) = \underbrace{A}_{\theta_0}\cos\left(\sqrt{\frac{g}{L}}\,t\right) + \underbrace{B}_{0}\sin\left(\sqrt{\frac{g}{L}}\,t\right)$$

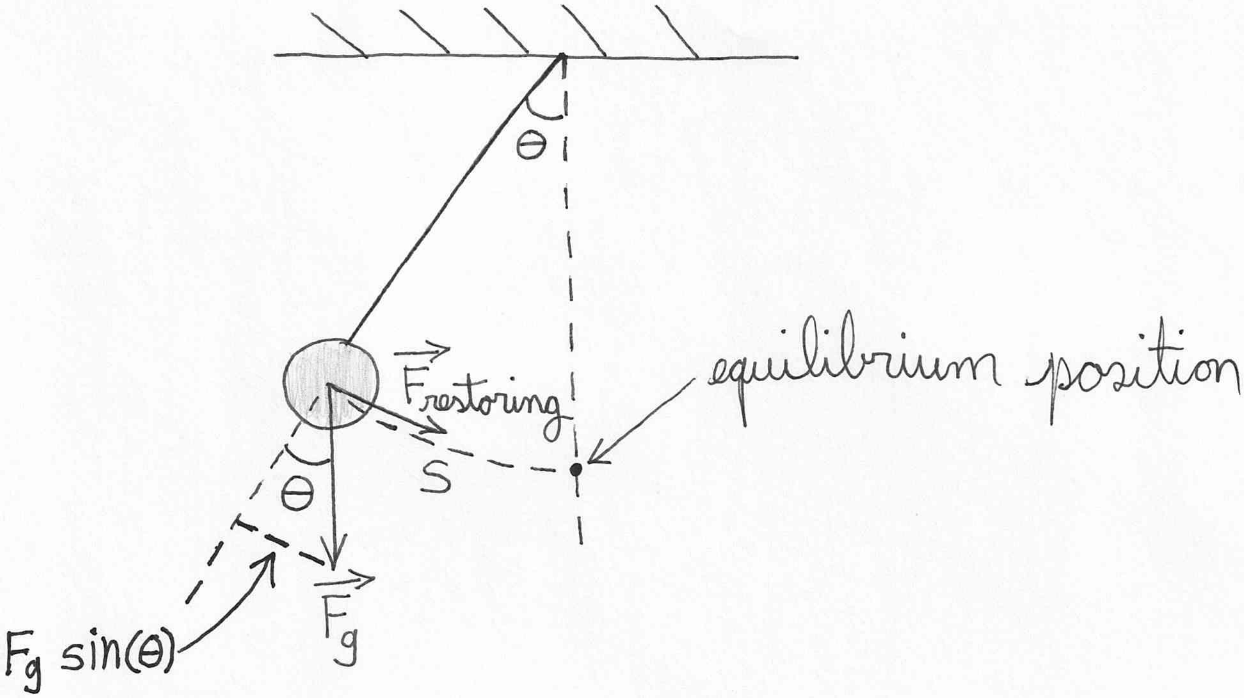
$$= \boxed{\theta_0\cos\left(\sqrt{\frac{g}{L}}\,t\right)}$$

θ_0 = initial angle at $t=0$
= amplitude of the oscillation

g = gravitational constant

L = length of the cord

t = time



The restoring force is the force which brings the pendulum back to its equilibrium position:

$$F_{restoring} = -ks$$

k = proportionality constant

s = arc length

$$s = L\theta$$

$$F_{restoring} = -kL\theta$$

$$F_{restoring} = F_g \sin(\theta)$$

$$-kL\theta = -mg\sin(\theta)$$

$$k = \frac{mg}{L}\frac{\sin(\theta)}{\theta}$$

$$\frac{\sin(\theta)}{\theta} \approx \frac{\theta}{\theta} = 1$$ (small angle approximation)

$$= \frac{mg}{L}$$

For simple harmonic motion, the time period is

$$T = 2\pi\sqrt{\frac{m}{K}} = 2\pi\sqrt{\frac{m}{mg/L}} = 2\pi\sqrt{\frac{L}{g}}$$

The angular frequency is

$$\omega = 2\pi f \qquad f = \text{frequency of oscillation}$$

$$= 2\pi \frac{1}{T}$$

$$= 2\pi \frac{1}{2\pi\sqrt{\frac{L}{g}}}$$

$$= \sqrt{\frac{g}{L}}$$

$$\boxed{\theta(t) = \theta_0 \cos\left(\sqrt{\frac{g}{L}}\, t\right) = \theta_0 \cos(\omega t)}$$

θ_0 = amplitude of oscillation = initial angle at $t = 0$
