Royden Solutions

Real Analysis, 3rd edition

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NOT A COMPLETE SOLUTION MANUAL. ADDITIONALLY, SOME PROOFS COULD BE INACCURATE OR INCOMPLETE

Contents

1 Set Theory

1.1 Introduction

1.) Show:

$$\{x : x \neq x\} = \emptyset$$

Proof: Let $A = \{x : x \neq x\}$. Suppose $A \neq \emptyset$. This implies $\exists y, y \neq y$, which is not true for any y. This poses a contradiction. Therefore, $A = \{x : x \neq x\} = \emptyset$.

2.) Show:

$$x \in \emptyset \to x \text{ is a green-eyed lion}$$

Proof: For any x, $x \in \emptyset$ is false. Therefore, $x \in \emptyset \to x$ is a green-eyed lion is always true. (Refer to the truth table of an implication).

3.) Show that in general $(X \times Y) \times Z \neq X \times (Y \times Z)$ and explain the correspondence between them and the set $X \times Y \times Z$.

Let's give an example where $(X \times Y) \times Z \neq X \times (Y \times Z)$. Let $X = \{1\}, Y = \{2\}, Z = \{3\}$.

$$X \times (Y \times Z) = X \times (\{(2,3)\}) = \{(1,(2,3))\}$$

$$(X \times Y) \times Z = (\{(1,2)\}) \times Z = \{((1,2),3)\}$$

$$X \times Y \times Z = \{(1,2,3)\}$$

4.) Show that the well-ordering principle implies the principle of mathematical induction.

Proof: Assume the well-ordering principle is true. That is, assume any non-empty subset of $\mathbb{N}$ has a smallest element.

Suppose $P(1)$ is true and $P(n) \rightarrow P(n+1)$ for all $n \in \mathbb{N}$. Consider $\{n \in \mathbb{N} : \neg P(n)\}$ and suppose that $\{n \in \mathbb{N} : \neg P(n)\} \neq \emptyset$. By the well-ordering property, it has a smallest element, which we denote as q. Since $q \in \{n \in \mathbb{N} : \neg P(n)\}$, $P(q)$ must be false. Notice that $q \neq 1$ since $P(1)$ is true. Let $r = q - 1$. Notice $r \in \mathbb{N} \ \wedge \ P(r)$. However, it follows from our hypothesis that $P(r) \rightarrow P(r+1) = P(q)$, which means q cannot be the smallest element. This contradicts our initial assumption that q is the smallest element, so $\{n \in \mathbb{N} : \neg P(n)\}$ has no smallest element. By the well-ordering property, this set must then be empty. This implies that $P(n)$ is true for all $n \in \mathbb{N}$. Q.E.D

1.2 Functions

1.) Let $f : X \to Y$ be a mapping of a nonempty space X into Y. Show that f is one-to-one iff there is a mapping $g : Y \to X$ such that $g \circ f$ is the identity map on X, that is, such that $g(f(x)) = x, \ \forall x \in X$.

Proof: We must prove both directions of this statement.

($\to$) Suppose $f : X \to Y$ is one-to-one. For each $y \in f(X)$, there exists a unique element $x_a \in X$ such that $f(x_a) = y$. Define a function $g : Y \to X$ where $g(y) = x_a$ if $y \in f(X)$ and $g(y) = x_0$ otherwise. This is a well-defined function for all possible y.

($\leftarrow$) Suppose there is a mapping $g : Y \to X$ such that $g(f(x)) = x, \ \forall x \in x$. Suppose that $f(x_1) = f(x_2)$. It follows that $g(f(x_1)) = g(f(x_2)) \to x_1 = x_2$ by the definition of g. Therefore, f is one-to-one.

1.3 Unions, Intersections, and Complements

3.) Show:

$$A \subset B \leftrightarrow \tilde{B} \subset \tilde{A}$$

Proof: We must show that both directions of the statement are true.

($\rightarrow$) Suppose $A \subset B$. By the definition of a subset, if $x \in A$, then $x \in B$. Taking the contra-positive of this statement, we get $x \notin B \rightarrow x \notin A$. Applying the definition of set complement, $x \in \tilde{B} \rightarrow x \in \tilde{A}$. Therefore, $\tilde{B} \subset \tilde{A}$.

($\leftarrow$) Simply reverse the steps done above (try it out yourself).

NOTE: Most other proofs in the section rely on similar tedious set arguments, so I won't prove them here.

1.4 Algebras of Sets

1.) Given any collection $\mathscr{C}$ of subsets of X, there is a smallest σ-algebra that contains $\mathscr{C}$; that is, there is a σ-algebra $\mathscr{A}$ containing $\mathscr{C}$ such that if $\mathscr{B}$ is any σ-algebra containing $\mathscr{C}$, then $\mathscr{A} \subset \mathscr{B}$

Proof: We follow in a similar manner to the proof of Proposition 1. **COME BACK LATER**

1.5 Axiom of Choice and Infinite Direct Products

1.) Let $f : X \to Y$ be a mapping onto Y. Then there is a mapping $g : Y \to X$ such that $f \circ g$ is the identity map on Y.

Proof: Let $f : X \to Y$ be a mapping onto Y.

Consider the collection $C = \{A : (\exists y \in Y) \text{ with } A = f^{-1}[\{y\}]\}$. First note that each set $A \in C$ is the inverse image of some $y \in Y$ and is nonempty since f is onto. By the axiom of choice, there is some function F which assigns to each A some $a \in A$. The choice of such function is arbitrary here, but we now assume we've picked some fixed F. Define $g : Y \to X$ such that $g(y) = F(f^{-1}[\{y\}])$. Notice that $f(g(y)) = y$, so we've found our desired identity map. Q.E.D

1.6 Countable Sets

1.) Show that every subset of a finite set is finite.

Proof: Let A be an arbitrary finite set and $B \subset A$. If $A = \emptyset$ or $B = \emptyset$, then B must be a finite set by definition. Suppose $A \neq \emptyset$ and $B \neq \emptyset$. Let $A = \{x_n\}$. Choose $x \in B$. Define a new finite sequence $\langle y_n \rangle$ by setting $y_n = x_n$ if $x_n \in B$ and $y_n = x$ if $x_n \notin B$. Then, B is the range of the finite sequence $\langle y_n \rangle$ and is therefore finite. Q.E.D

2.) Proof that the set of all rational numbers is countable (using propositions 4 and 5 from the text).

Proof: Our goal is to describe a one-to-one correspondence between an countable set and the rational numbers. Let's find our domain (countable set) first.

Let $A = \{\langle 1, 1, 3 \rangle\} \cup \{\langle p, q, r \rangle : gcd(p, q) = 1 \wedge r \in \{1, 2\}\}$. Recall from Proposition 5 that the set of all finite sequences from a countable set is also countable. Therefore, $\mathbb{N}^3$ must be countable since it's the set of all finite sequences of length 3 from $\mathbb{N}$. Notice that $A \subset \mathbb{N}^3$, so by Proposition 4, A must be countable. Define the following function $f : A \to \mathbb{Q}$ as the following:

$\langle p, q, 1 \rangle = \frac{p}{q}$, $\langle p, q, 2 \rangle = -\frac{p}{q}$, $\langle 1, 1, 3 \rangle = 0$

From our careful choice of our domain A, we see that this is indeed a one-to-one correspondence from a countable set to $\mathbb{Q}$. Therefore, the rationals are countable. Q.E.D

3.) Show that the set E of infinite sequences from $\{0, 1\}$ is not countable.

Readers reference: Think of a_{vn} as an element in an infinite 2-d grid where v denotes the row and n denotes the column.

Proof: We'll use proof by contradiction. Suppose that E is countable. Then, there exists a one-to-one correspondence $f : \mathbb{N} \to E$ such that $f(v) = \langle a_{vn} \rangle_{n=1}^{\infty}$ is a unique sequence in E (each a_{vn} is either 0 or 1). Define a new sequence $\langle b_n \rangle_{n=1}^{\infty}$ such that $b_n = 1 - a_{nn}$ for each n. $\langle b_n \rangle_{n=1}^{\infty} \in E$, but $\langle b_n \rangle_{n=1}^{\infty} \neq \langle a_{vn} \rangle_{n=1}^{\infty}$ for any v (their v^{th} entry differs since $b_v \neq a_{vv}$ by the definition of b). Therefore, f is not a one-to-one correspondence, which is a contradiction. It follows that E is uncountable. Q.E.D

4.) Let f be a function from a set X to the collection $P(X)$ of subsets of X. Then, there is a set $E \subset X$ that is not in the range of f.

Proof: Let $E = \{x : x \notin f(x)\}$. Assume that E is in the range of $f(x)$. Then, there is some x_1 such that $f(x_1) = E$. Say $x_1 \in E$. Then, $x_1 \notin f(x_1)$. However, $f(x_1) = E$. This is a contradiction. Say $x_1 \notin E$. Then, $x_1 \in f(x_1)$. However, $f(x_1) = E$. Another contradiction. Since $x_1 \in E$ and $x_1 \notin E$ cannot both be false, it follows that $f(x_1) \neq E$. Therefore, E is not in the range of f.

1.7 Relations

2.) Let X be an Abelian group under $+$. Then $\equiv$ is compatible with $+$ iff $x = x'$ implies $x + y \equiv x' + y$. The induced operation then makes the quotient space into a group.

Proof: We must show that $+$ on the quotient space satisfies the requirements to be a group. Let Q be the quotient space of X (elements of Q are equivalence classes on X).

Property 1: Closure.

Let E, F be elements of Q. $E + F$ is defined as $E_{(x+y)}$, where $x \in E$ and $y \in F$. Since X is an Abelian group, $x + y \in X$. Therefore, $E_{(x+y)} \in Q$.

Property 2: Identity.

Let $E \in Q$. **Finish after learning more about groups**

2 Real Numbers

2.1 Axioms for the Real Numbers

1.) Show that $1 \in P$.

Proof: Recall 0 is unique in the real number system. So, $1 \neq 0$. We now show that $-1 \notin P$. Let $x \in P$. We use axioms (A2), (A3), (A4), (A9) freely.

$$(-1)x = (-1)x + (x + (-x)) = ((-1)x + x) + (-x) = (x((-1)+1)) + (-x) = x * 0 + (-x) = -x$$

By (B3), $x \in P \to -x \notin P$. The contra-positive of (B2) is $xy \notin P \to x \notin P \ \vee \ y \notin P$. Since $(-1)x \notin P$, either $x \notin P \ \vee \ -1 \notin P$. We defined $x \in P$, so it follows that $-1 \notin P$.

We now apply axiom (B4). For any real number x, either $x = 0$ or $x \in P$ or $-x \in P$. We know $1 \neq 0$ and $-1 \notin P$. Therefore, it directly follows that $1 \in P$. Q.E.D

2.) Use Axiom C to show that every nonempty set of real numbers with a lower bound has a greatest lower bound.

NOTE: We use without proof that $x \leq y \rightarrow -x \geq -y$.

Lemma 2.1: c is a lower bound on $S \rightarrow -c$ is an upper bound on $S' = \{x : -x \in S\}$.

Proof: We prove both directions of the statement.

($\rightarrow$): Since c is a lower bound on S, we have $c \leq x, \forall x \in S$. We now take a look at $S' = \{x : -x \in S\}$. If $-x \in S$, then for all $x \in S'$, $c \leq -x \rightarrow -c \geq x$. Therefore, $-c$ is an upper bound on S'.

($\leftarrow$): Since $-c$ is an upper bound on S', we have $-c \geq x, \forall x \in S'$. It follows that $c \leq -x, \forall x \in S'$. By the definition of S', $-x \in S$, and c is then a lower bound on S.

Corollary: c is an upper bound on $S' = \{x : -x \in S\} \leftrightarrow -c$ is a lower bound on S.

Proof: Let S be a set of numbers with a lower bound. Let s be a lower bound on S. By the lemma, $-s$ is an upper bound on $S' = \{x : -x \in S\}$. By axiom C, S' has a least upper bound, which we will call s', that satisfies $s' \leq c$ where c is any possible upper bound on S'. By the corollary, $-s'$ is a lower bound on S. Suppose there were a larger upper bound.

INCOMPLETE

3.) Prove the following proposition: Let L, U be nonempty subset of $\mathbb{R}$ with $R = L \cup U$ such that for each $l \in L$ and $u \in U$ we have $l < u$. Then either L has a greatest element or U has a least element.

Proof: Notice that any $u \in U$ serves as a upper bound for L. The completeness axiom then implies that L has a least upper bound, which we will call l'. Since $l' \in \mathbb{R}$, either $l' \in L$ or $l' \in U$. If $l' \in L$, then l' is the greatest element in L by definition. Suppose l' in U. Since l' is the least upper bound of L, $l' \leq c$ for any upper bound c of L. Since any $u \in U$ is an upper bound on L, $l' \leq u$, $\forall u \in U$. Therefore, l' is the least element of U.

Therefore, l' is either the greatest element of L or the least element in U. Q.E.D

2.2 The Natural and Rational Numbers as Subsets of $\mathbb{R}$

There are no problems in this section.

2.3 The Extended Real Numbers

1.) Show that $\inf E \leq \sup E$ iff $E \neq \emptyset$

Proof: We prove both directions of the statement

($\rightarrow$): $\inf E \leq \sup E \rightarrow E \neq \emptyset$

We attempt to prove the contrapositive, that is $E = \emptyset \rightarrow \inf E > \sup E$. Suppose $E = \emptyset$. By definition of the extended real numbers, $\sup E = -\infty$ and $\inf E = \infty$ and $\inf E = \infty > -\infty = \sup E$.

($\leftarrow$): $E \neq \emptyset \rightarrow \inf E \leq \sup E$

Suppose $E \neq \emptyset$. By definition of supremum and infimum, $\sup E \geq x$ and $\inf E \leq x$, $\forall x \in E$. Fix $x \in E$. Then, $\sup E \geq x \geq \inf E \rightarrow \sup E \geq \inf E$. Q.E.D

2.4 Sequences of Real Numbers

7.) Show that a sequence can have at most one limit.

Proof: Let $\langle x_n \rangle$ be a sequence. Suppose l_1, $l_2 \in \mathbb{R}$ are limits of this sequence.

$$|x_n - l_1| < \epsilon/2, \forall n \geq N_1$$

$$|x_n - l_2| < \epsilon/2, \forall n \geq N_2.$$

Pick $N_0 = \max(N_1, N_2)$. Then:

$$|x_n - l_1| < \epsilon/2 \text{ and } |x_n - l_2| < \epsilon/2, \forall n \geq N_0.$$

Applying the triangle inequality:

$$|l_1 - l_2| \leq |l_1 - x_n| + |x_n - l_2| = |x_n - l_1| + |x_n - l_2| < \epsilon/2 + \epsilon/2 = \epsilon$$

This holds for any $\epsilon > 0$, so $|l_1 - l_2| = 0 \to l_1 = l_2$. This shows that if l_1 and l_2 are limits of $\langle x_n \rangle$, then $l_1 = l_2$. It follows directly that the sequence can have at most one (unique) limit. Q.E.D

8.) Show that l is a cluster point of $\langle x_n \rangle$ if and only if there is a sub-sequence that converges to l.

Proof: We must prove both directions of this statement.

($\rightarrow$): Suppose that l is a cluster point of $\langle x_n \rangle$. That is, given any $\epsilon > 0$ and any $N \in \mathbb{N}$, $\exists n \geq N$ such that $|x_n - l| < \epsilon$. We now define n_k recursively as follows:

$$n_1 = \inf(\{n : |x_n - l| < 1/1 = 1\})$$

$$n_k = \inf(\{n : n \geq n_{k-1} + 1 \wedge |x_n - l| < 1/k\})$$

Every n_k is a non-empty set of natural numbers, so there exists a smallest element (equal to the infimum). We now show that this subsequence converges to l. Notice that $\forall \epsilon > 0$, $\exists N \in \mathbb{N}$ such that $\frac{1}{N} < \epsilon$ and that for such N, we have:

$$|x_{nk} - l| < 1/N < \epsilon, \forall k \geq N$$

Therefore, l is the limit of our sequence.

($\leftarrow$):

11.)

a.) Show that a sequence $\langle x_n \rangle$ which converges to a real number l is a Cauchy sequence.

Proof: Let $\langle x_n \rangle$ be a sequence with limit l and let $\epsilon > 0$.

$$|x_n - l| < \epsilon/2, \, \forall n \geq N$$

$$|x_m - l| < \epsilon/2, \, \forall m \geq N$$

Applying the triangle inequality:

$$|x_n - x_m| \leq |x_n - l| + |l - x_m| = |x_n - l| + |x_m - l| < \epsilon/2 + \epsilon/2 = \epsilon, \, \forall n, m \geq N$$

This satisfies the requirements of a Cauchy sequence. Q.E.D

b.) Show that each Cauchy sequence is bounded

Lemma 2.11.b: $|x_n| = |x_n - x_m + x_m| \leq |x_n - x_m| + |x_m|, \, \forall n, m \in \mathbb{N}$

Proof (11b): Let $\langle x_n \rangle$ be a Cauchy sequence. We must find some number h such that for all $n \in \mathbb{N}$, $h > |x_n|$. Fix $\epsilon > 0$ for $\langle x_n \rangle$. Then,

$$|x_n - x_m| < 1, \, \forall n, m \geq N$$

We now fix $m = N$. Therefore, by Lemma 2.11.b:

$$|x_n| \leq |x_n - x_m| + |x_m| < 1 + |x_N|, \forall n \geq N$$

It follows that $h_1 = 1 + |x_N|$ bounds $|x_n|$ when $n \geq N$. Now consider all $n < N$. This is a finite set, so we take our bound to be $h_2 = \max(x_1, x_2, ..., x_n)$. A bound for the entire sequence is therefore $h = \max(c_1, c_2)$. We've found a suitable bound h for any Cauchy sequence, so every Cauchy sequence is bounded. Q.E.D

c.) Show that if a Cauchy sequence has a sub-sequence that converges to l, then then the original sequence converges to l.

NOTE: $\langle x_{nk} \rangle$ is shorthand for $\langle x_{n_k} \rangle$ where $n_k = n(k)$ is a monotone increasing function $f : \mathbb{N} \to \mathbb{N}$. For example, $k = 2$ could produce $n(k) = n(2) = 3$ and $x_{nk} = x_3$.

Proof: Suppose $\langle x_n \rangle$ is a Cauchy sequence. Let $\langle x_{nk} \rangle$ be a sub-sequence that converges to l. Recall that $k \leq n_k, \forall k \in \mathbb{N}$ since n_k is monotone increasing on $\mathbb{N}$. Then, for any $\epsilon > 0$:

$$|x_{nk} - l| < \epsilon/2, \forall k \text{ s.t. } k \geq N_1 \to n_k \geq N_2$$

$$|x_k - x_{nk}| < \epsilon/2, \forall k \geq N_3$$

We now choose $N = \max(N_1, N_3)$ and the apply the triangle inequality:

$$|x_k - l| \leq |x_k - x_{nk}| + |x_{nk} - l| < \epsilon/2 + \epsilon/2 = \epsilon, \, \forall k \geq N$$

$$|x_k - l| < \epsilon, \, k \geq N$$

Therefore, $\langle x_n \rangle$ converges to the limit l. Q.E.D

d.) Establish the Cauchy Criterion: There is a real number l to which the sequence $\langle x_n \rangle$ converges if and only if $\langle x_n \rangle$ is a Cauchy sequence.

Proof: We proved the forward direction (i.e. $\rightarrow$) in part (a), so we only need to prove the reverse direction. That is, if $\langle x_n \rangle$ is a Cauchy sequence, then there exists some number l such that $\langle x_n \rangle$ converges to l. Fortunately, we've done most of the heavy lifting already and we can chain together previously proved results.

Let $\langle x_n \rangle$ be a Cauchy sequence. By (11b), this sequence is bounded. Applying the Bolzano-Weierstrass theorem (refer to the Appendix), we know that it has a sub-sequence that converges to some limit l. By (11c), it then follows that the Cauchy sequence $\langle x_n \rangle$ converges to l. Q.E.D

18.) Show that if each $x_v \geq 0$, there is always an extended real number s where $s = \sum_{v=1}^{\infty} x_v$

Lemma 18.1: Every bounded, monotone non-decreasing sequence converges to its least upper bound $\sup(\langle x_n \rangle)$.

The following lemma which is used in the proof of (18) is a part of the Monotone Convergence theorem. A full description can be found in the Appendix.

Proof (Lemma 18.1): Let $\langle x_n \rangle$ be a bounded, monotone non-decreasing sequence. Let c be an upper bound on $\langle x_n \rangle$. By the Completeness Axiom, there exists a least upper bound $a = \sup(\langle x_n \rangle)$. It follows that no number $a - \epsilon$ where $\epsilon > 0$ is an upper bound. In other words, $\forall \epsilon > 0$, $\exists N$, $x_N > a - \epsilon$. Recall that x_n is monotone non-decreasing, which means $x_{n+1} \geq x_n$. Therefore, $x_{N+1} \geq x_N > a - \epsilon$ and we can prove by induction that $\forall n \geq N$, $x_n > a - \epsilon$. Also, since a is an upper bound, $a + \epsilon$ is also an upper bound. We can then naturally extend the previous statement to be $\forall \epsilon > 0$, $\exists N$, $\forall n \geq N$, $a - \epsilon < x_n < a + \epsilon$. This matches the definition of a limit, which means $\lim x_n \to a = \sup(\langle x_n \rangle)$. Q.E.D

Proof (18): We split the sum $\langle s_n \rangle$ into two cases.

Case 1 - $\langle s_n \rangle$ has no upper bound

In mathematical terms, we have $\forall c$, $\exists N$, $s_N > c$. We now prove (somewhat informally) that $\forall n \geq N$, $s_n > c$. When $n = N$, the statement is trivially true. Suppose the statement is true $\forall n$, $N \leq n \leq k$. Then, $s_k > c$. It follows that $s_{k+1} = s_k + x_{k+1} > s_k > c$. By induction the statement is true $\forall n$.

We now replace c by $\triangle$ and get:

$$\forall \triangle, \exists N, \forall n \geq N, s_n > \triangle$$

This is precisely the definition for $\lim s_n = \infty$. Therefore, s_n converges to an extended real number ∞.

Case 2 - s_n has an upper bound

$\langle s_n \rangle$ is monotone non-decreasing since $s_{n+1} = s_n + x_{n+1} \geq s_n$. We apply Lemma 18.1 and see that s_n converges to a real number $\sup(\langle x_n \rangle)$.

Summary: In both cases, we found that s_n converges to an extended real number. Q.E.D

3 Appendix

Here exists a collection of important theorems and results that one should remember as they progress through the book. This is not concerned with definitions (e.g. definition of a limit) but rather theorems proven from these definitions.

3.1 Set Theory

Well-Ordering Principle (for $\mathbb{N}$): Every nonempty subset of $\mathbb{N}$ has a smallest element.

Well-Ordering Principle (General): Every set X can be well-ordered; that is, there is a relation $<$ that well orders X.

3.2 The Real Number System

Bolzano-Weierstrass Theorem: Every bounded sequence in R has a convergent subsequence.

Monotone Convergence Theorem: There are two parts to the theorem:

(i) Every bounded, monotone non-decreasing sequence converges to a real number $\sup(\langle x_n \rangle)$.
(ii) Every bounded, monotone non-increasing sequence converges to a real number $\inf(\langle x_n \rangle)$