TUTORIAL 2 Trigonometry A (geometry)

Background and motivation: Trigonometry relates angular and linear measurements, thus is ubiquitous and unavoidable in biomechanics. This is a crucial mathematical building block for mathematical models which we'll build upon in future tutorials. Trig functions (sin, cos, tan) give lengths of line segments if the angle between them is known. Likewise, inverse trig functions (arcsin, arccos, arctan) give angles between line segments if the segment lengths are known. You can always check your calculations by 1) make a drawing to scale and measuring with a ruler/protractor OR 2) drawing a diagram on a computer (e.g. ImageJ) and digitally measuring angles/segment lengths.

<u>Why do we care?</u>

a). Vectors: Inevitably, all biomechanists are faced with vector problems (e.g. velocity, force). For example, trig analysis on the time-varying orientation of muscle fibers (and their resulting force vectors) revealed how muscles can automatically adapt their mechanics depending on the task [1].

b). We can use trig to simplify complex motion into linear (translational) and angular (rotational) components. This vastly simplifies statistical analysis and makes the system easier to model. E.g., the leg joint movements of a swimming frog are a headache to analyze. However, simple trig analysis reveals that individual joint actions don't really matter – what is important is the net effect of all joints on foot movement [2].

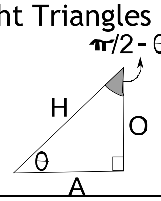
c). Coordinate systems (later tutorial): Biomechanists often rely on video measurements of animals who rarely move in a straight line parallel to the camera view. Our measurements would be off if we didn't correct for animal re-orientation. Thus in the post-processing stage, we often have to transform our measurements to re-align the animal's body coordinate system (e.g. it's A-P axis) in a way that is useful for analysis. For example, coordinate transformations of aerodynamic forces revealed how birds re-orient forces during turning [3].

Aims of this tutorial:
*** Review basic trigonometry**
*** Develop facility deriving equations relating range-of-motion to geometry/anatomy**
*** Calculate the angle between vector components using experimental force plate data**
*** Build a simple mathematical limb model (morphological model) using above equations**

Skills required prior to this tutorial: Geometry of vectors; How to open, manipulate and plot data in Mathematica (*MM*). See bottom of this document for links if you need refreshing.

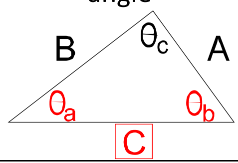
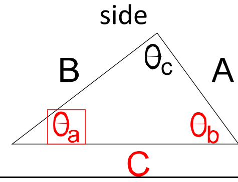
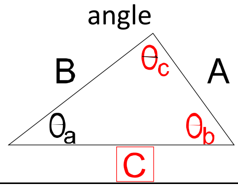
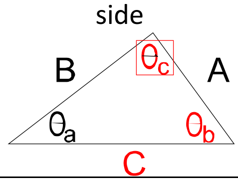
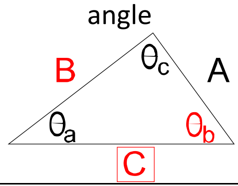
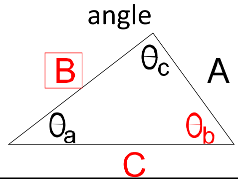
Skills gained from this tutorial:
*** Trig, Inverse Trig, Law of Sines & Cosines**
*** Mathematica software:**
-How to do trig operations on real experimental data to obtain kinematics information

If you have forgotten the following, you <u>must</u> memorize it now – know it like you know the alphabet:

Right Triangles Only	Trig:	Inverse Trig:
	Sin(θ) = O/H Cos(θ) = A/H Tan(θ) = O/A	ArcSin(O/H) = θ ArcCos(A/H) = θ ArcTan(O/A) = θ

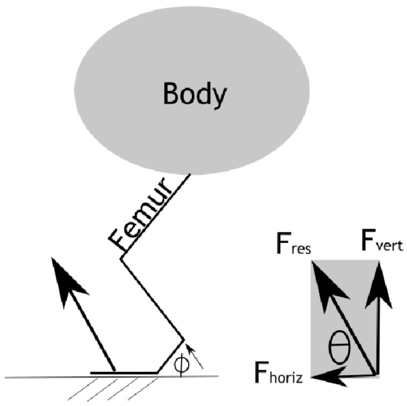
My favorite memory trick ("<u>O</u>h <u>h</u>ell, <u>a</u>nother <u>h</u>our <u>o</u>f <u>a</u>lgebra")

Rule-of-thumb chart for geometry problems – <u>for reference, not for memorization</u>! Note for diagrams: Black = known; Red = unknown Also note that some have alternative methods Look up Law of Cosines (LOC) and Sines (LOS) on the internet (e.g. hyperphysics website)

SHAPE	KNOWN	SOLVE FOR	METHOD
Right triangle	2 angles	unknown angle	sum of angles = π
	2 sides	unknown side	Pythagorean theorem
	1 angle, 1 side (not hypotenuse)	unknown angle	Inverse Trig
		unknown side	Trig
Generic triangle	2 sides, 1 angle	unknown side opposite known angle 	LOC
		unknown angle opposite known side 	Step 1. LOC → Solve for unknown side, C Step 2. LOS → A/Sin(θa) = C/Sin(θc) Step 3. Inverse Trig
		unknown side opposite unknown angle 	Step 1. LOS → A/Sin(θa) = B/Sin(θb) Step 2. Inverse Trig → θb Step 3. θa + θb + θc = π Step 4. LOC
		unknown angle opposite unknown side 	Step 1. LOS → A/Sin(θa) = B/Sin(θb) Step 2. Inverse Trig → θb Step 3. θa + θb + θc = π
	2 angles, 1 side	unknown angle	sum of angles = π
		unknown side opposite known angle 	Step 1. LOS → A/Sin(θa) = C/Sin(θc)
		unknown side opposite unknown angle 	Step 1. θa + θb + θc = π Step 2. LOS → A/Sin(θa) = B/Sin(θb)

Examples

[a, b, c] At each instant in time, a force plate can measure force as a foot pushes against the ground. Using trig, the total (resultant) force can be broken down into horizontal and vertical forces for forward propulsion and weight support, respectively. For this, we've assumed there is only negligible medio-lateral force. **Note:** If you are rusty on how to add vectors, check the link at the bottom of this tutorial for an online interactive tool.



[a] If the magnitude of the resultant (Fres) is 80N what is the value of the horizontal component (Fhoriz) if the angle θ = 1.1 radians?

We know that this concerns the hypotenuse ('H') and its adjacent side ('A' = Fhoriz in this case) which means we can use the Cos function

Cos θ = A/H = Fhoriz/Fres = Fhoriz/80N

Solving for Fhoriz = 80N x Cos θ

If you're skeptical of your calculation, convince yourself by drawing the vectors with a ruler (or digitally with ImageJ or Inkscape) with correct proportions and measure the unknown vector length or angle.

[b] Alternatively, what if Fvert = 55N and Fhoriz = 23N, what would the resultant angle θ be with respect to the ground?

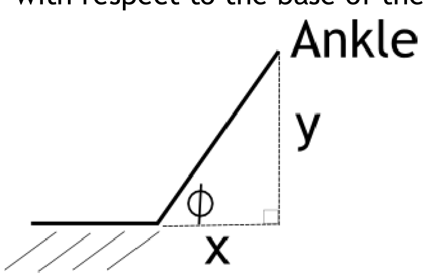
We know this concerns the relationship between the adjacent ('A') and opposite ('O') sides meaning we can use the Tan function

Tan θ = O/A = Fvert/Fhoriz = 55/23

Now, solving for we need to use the inverse Tan:

$$\theta = \text{ArcTan}(55/23) = 1.17 \text{ radians}$$

[c] What if the ankle joint were raised such that the metatarsals are at an angle ϕ = 0.95 radians with respect to the ground. What would be the {x, y} coordinates of the ankle joint with respect to the base of the foot, given a metatarsal length lmeta = 164 mm?

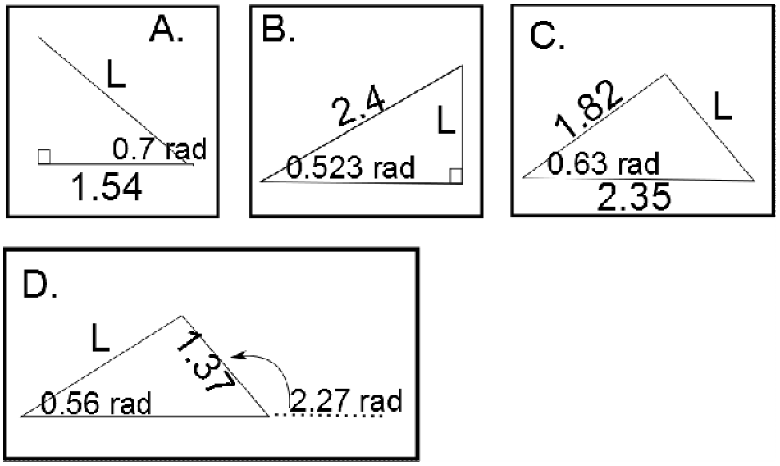


In this case, we make an imaginary right triangle with a height equal to the y coordinate of the ankle joint and a base length of x.

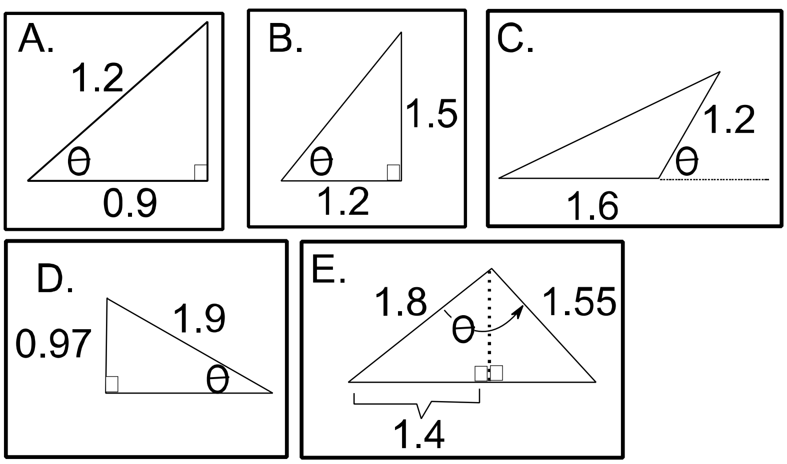
As above, knowing metatarsal length (hypotenuse), we can find the x component using Cos ϕ = x/lmeta whereas we can find the y component using Sin ϕ = y/lmeta. Assuming the base of the foot is the origin {0, 0}, then Ankle {x, y} = {lmeta x Cos ϕ, lmeta x Sin ϕ} = {164 mm x Cos 0.95, 164 mm x Sin 0.95 } = {95.4 mm, 133.4 mm}

Exercises (Easy):

1. Solve for L in the following triangles (angles are in radians and drawings are to scale):



2. Solve for θ in the following triangles (angles are in radians and drawings are to scale):



2. Exercise (in Mathematica, '*MM*'):

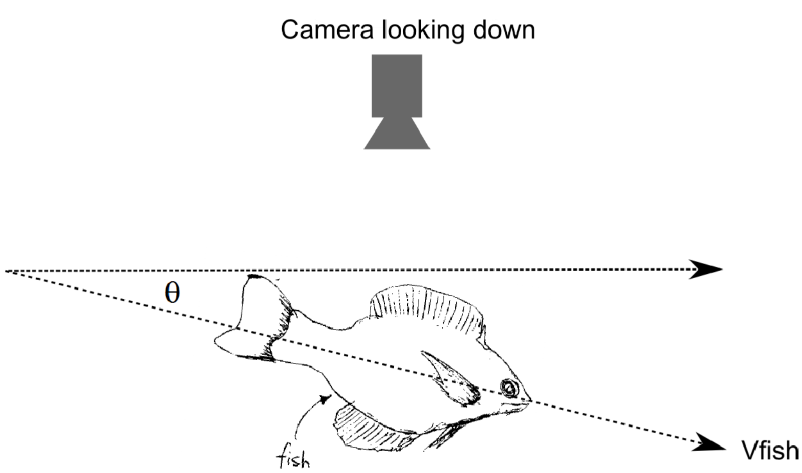
Task: Using the sample data of ground reaction force components (FrogForceData), calculate the angle of the ground reaction force vector with respect to the ground (i.e. horizontal = 0 degrees). This is actual data measuring the vector sum of horizontal + vertical force (similar to the example above) produced by a jumping frog. The resultant vector is the sum of both horizontal and vertical components where the horizontal component is forward thrust and vertical component is the upward thrust. Combined together, they push the frog body upwards and forwards. Ideally, a jumping animal should jump at 45 degrees to maximize upward and forward distances traveled. We're going to use *MM* to find out whether or not real frogs angle their resultant force at 45 degrees.

Step-by-step

1. Import your data into *MM* using a variable called "DATA". If you don't know how to do this, refer to the following youtube videos associated with Tutorial 1 (see links below). Make sure to transpose the data such that time values are rows, NOT columns. Check the dimensions (Dimensions[DATA]). Should be {4 rows, 116 columns}

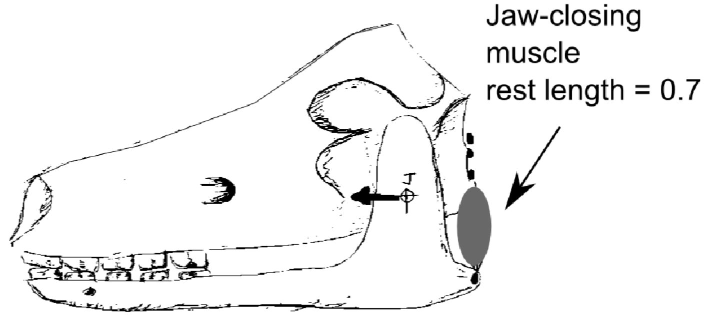
2. Put the x values (time values) for the resultant force data (i.e. Col 1) into a variable “xdata” (or whatever you wish to name it).
   xdata = DATA[[1]];
3. Do the same thing for the y values (resultant force values; i.e. Col 2), but change the variable name so we know it’s the resultant values.
   ydataRes
4. Make x-y pairs of data.
5. XYdataRes = Thread[{xdata, ydataRes}];
6. Plot the data. Since it’s discrete data points, we’ll use ‘ListPlot[]’
   ListPlot[XYdataRes]
7. Do the same (copy/paste your code from steps 1-6), but modify it slightly to load in the horizontal and vertical components.
8. On paper, sketch a resultant vector (the force vector that the frog produced) and its corresponding horizontal and vertical components. Make the angle of the resultant any angle you like between 0 to 90 deg drawn with respect to the ground such that it makes a diagonal arrow pointing up and to the right. Remember, we want to calculate this angle. We know all the sides of the triangle, but don’t know the angle. There are different ways to calculate it, but we can say (on paper): tan (forceangle) = horizontal force/vertical force. Then we use inverse trig to solve for the angle.
9. *MM*, fortunately, will perform Trig over an entire list of data, so treat your ydataRes, ydataHoriz and ydataVert as if they were single numbers and plug the appropriate ones into the appropriate trig expression to calculate data values for force angle. The output should be a list of values (in radians) with the same length (same # time points) as the original data.
10. You’ll see that the angle of the resultant force vector changes through time – it is not consistently 45 degrees as we predicted. Maybe the frog is doing something wrong, but likely not. We’ll further discuss the significance of this changing force angle, but for that we’ll need calculus which will be in a later tutorial.

<u>**Exercises (Harder):**</u> Hypothetical biomechanics problems:
Hint: Look for right triangles first. If not present, find generic triangles and use LOC & LOS.



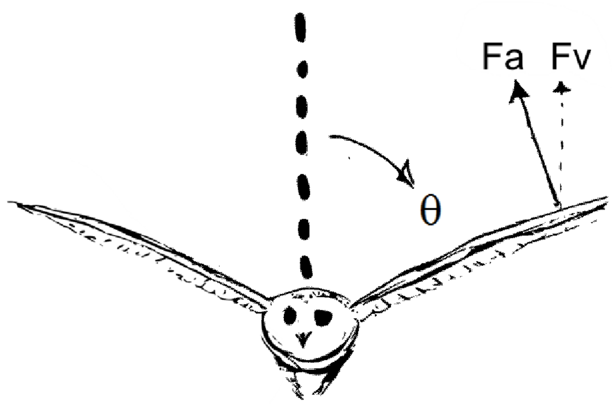
1. You want to measure the forward swimming speed of a fish, so you place a single camera above the water and film. You assume that the fish wants to swim horizontally (parallel to the camera), but of course the fish does whatever it wants. Given an actual speed (V_{fish}) of 1.5 m/s, what would be the measured speed if the fish were swimming at a slight downward angle (θ) of 0.23 radians?

2. In the diagram of the skull, the mandible joint (J) can translate forward and backwards, causing the jaw-opening muscle to rotate. If the jaw were to slide forward, what would be the resulting muscle angle (with respect to vertical) if it were to be stretched by 10% of its original length? Note that the muscle's original resting length is 0.7 and is oriented at 0 rad with respect to vertical (dotted line).

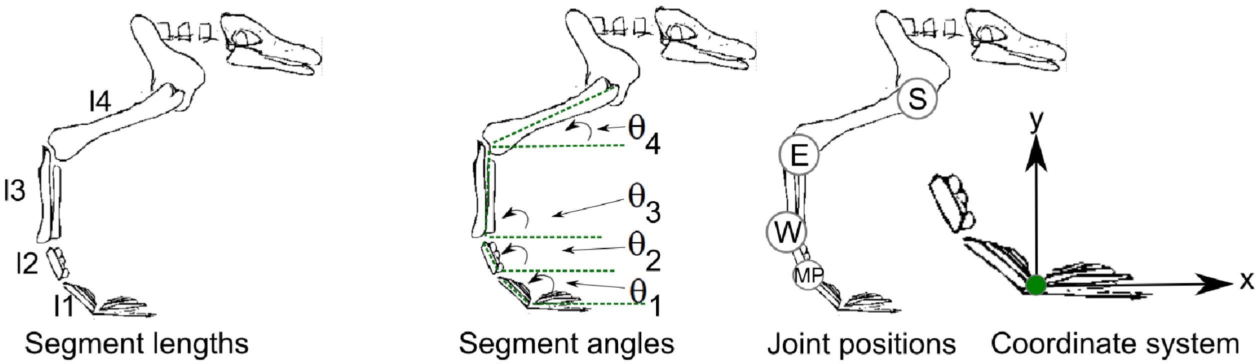


3. A predator skull has a maximum gape angle of 0.92 rad and a jaw length of 0.6. What is the maximum prey diameter that could fit between the jaws? In other words, what is the linear distance between the distal tips of the jaws at maximum gape?

4. As a certain bird is taking off, its wing produces its maximum lift when the wings are at an angle (θ) of 1.8 radians from vertical (vertical = 0 radians). If one wing is producing 2.6 N of aerodynamic force (Fn) normal to the wing, how much vertical force (Fv) is the wing producing to lift the bird (i.e. what is the magnitude of the vertical component)?



5A. Regarding the vertebrate skeleton below and its bone lengths, we're going to set up a morphological model (soon to become a kinematic model and eventually a tool to try some simple dynamics). This is a busy exercise, but since it will serve as the basis for future tutorials, please don't skip it. This exercise is on paper, but in the next tutorial we'll draw the limb using *MM* software to interactively 'play' with the anatomy and limb posture.



You can assume the following (see diagram):
- The bones are all in a single plane (sagittal plane).
- The x,y coordinates of the foot base is the origin of the coordinate system {0,0}.
- Each segment angle is the angle between the horizontal and the limb segment – e.g. θ_1 is the angle between the horizontal and l1. θ_2 is the angle between horizontal and l2, etc.
- The foot always stays at the ground (for now).
- The lengths of the segments are l1, l2, l3, l4

A. First, let's find equations giving the {x, y} coordinates of the MP joint as a function of the angle of segment #1 (θ_1,) and segment length l1. You'll have two equations. The x coordinate of the MP joint = l1*Cos (θ_1), so MP_X = l1*Cos (θ_1). What is MP_y ? Leave the equation in variable form (we'll put values in later).

B. Regarding the skeleton above write similar equations for the wrist joint {W_x, W_y} in terms of θ_1, θ_2 and the relevant limb segments. Hint: you can substitute equations for Part A (i.e. W_x = MP_x + ….).

C. Similarly, regarding the same skeleton above write equations for the elbow joint in terms of θ_1, θ_2, θ_3 and the relevant limb segments.

D. Finally, do the same as above for the shoulder joint position in terms of θ_1, θ_2, θ_3, θ_4 and the relevant limb segments. Congratulations, you've just constructed a morphological model (perhaps your first mathematical model of a biological system). It's not super useful yet, so we'll build things onto it in future tutorials.

Further reading:
Biewener, Andrew A. Animal locomotion. Oxford University Press, 2003.Chapter 3.

References:

1. Azizi, Emanuel, Elizabeth L. Brainerd, and Thomas J. Roberts. "Variable gearing in pennate muscles." *Proceedings of the National Academy of Sciences* 105.5 (2008): 1745-1750.

2. Richards, Christopher T. "The kinematic determinants of anuran swimming performance: an inverse and forward dynamics approach." *Journal of Experimental Biology* 211.19 (2008): 3181-3194.

3. Ros, Ivo G., et al. "Pigeons steer like helicopters and generate down-and upstroke lift during low speed turns." *Proceedings of the National Academy of Sciences* 108.50 (2011): 19990-19995.

Useful links:
hyperphysics trigonometry section (http://hyperphysics.phy-astr.gsu.edu/hbase/trig.html)

vector geometry (incredibly cool interactive tool)
(http://phet.colorado.edu/en/simulation/vector-addition)

Mathematica Video Tutorials:

Welcome to Mathematica (6:38):
http://youtu.be/UTM_FDBPEjc

How to Load Spreadsheet (text) Data to Mathematica (3:30)
http://youtu.be/CEolAaIE8vw

How to do a linear regression (best fit line) of XY data in Mathematica (5:09)
http://youtu.be/BypTzePw2lc