MAT220 manditory 4

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1
- Show that $I \cap J = \{a | \, a \in I \wedge a \in J\}$ is an ideal.
  Given $n, m \in I \cap J = (n, m \in I \wedge n, m \in J)$.
  $I \cap J$ is an ideal if;
  - **$I \cap J$ is closed under subtraction**
    Presume $n - m \in I \cap J$, then $n - m \in I \wedge n - m \in J$ which is true by theorem 6.1.
    $\therefore I \cap J$ is closed under subtraction.
  - **And $I \cap J$ absorbs products.** Given $r \in R$, then
    * $rm \in I \cap J = rm \in I \wedge rm \in J$ which is true by theorem 6.1.
      $\therefore rm \in I \cap J$
    * $mr \in I \cap J = mr \in I \wedge mr \in J$ which is true by theorem 6.1.
      $\therefore mr \in I \cap J$

    $\therefore I \cap J$ absorbs products.

  $\therefore I \cap J$ is an Ideal, by theorem 6.1
- Show that $I + J = \{i + j | \, i \in I, j \in J\}$ is an ideal.
  Given $n, m \in I + J$, where $m = a + b$, $n = x + y$, where $a, x \in I$ and $b, y \in J$ and I, J are Ideals.
  Then $I + J$ is an ideal if;
  - **$I + J$ is closed under subtraction** Meaning $n - m \in I + J$.
    $n - m$
    $= (a + b) - (x + y)$
    $= a + b - x - y$
    $= (a - x) + (b - y)$
    By theorem 6.1, $(a - x) \in I$ and $(b - y) \in J$
    $\therefore n - m \in I + J$
  - **And $I + J$ absorbs products.** Given $r \in R$, then
    * $rm = r(a + b) = ra + rb$
      Where $ra \in I$ and $rb \in J$ by theorem 6.1.
      $\therefore ra + rb \in I + J$. and
    * $mr = (a + b)r = ar + br$
      Where $ar \in I$ and $br \in J$ by theorem 6.1.
      $\therefore ar + br \in I + J$.

    $\therefore I + J$ absorbs products.

  $\therefore I + J$ is an Ideal, by theorem 6.1

2 Let $P_m = \{p_1, p_2, \ldots p_n\}$ such that $m = p_1 \cdot p_2 \cdots p_n$ where $p_i \in \{\text{prime}\}$.

- Given $S = \{\frac{x}{y} | \text{when } \frac{x}{y} \text{ is reduced } = \frac{a}{b} \implies b = \text{odd}\}$. $b = \text{odd} \iff p_i \neq 2$, in P_b
  Given $\frac{a}{b}, \frac{c}{d} \in S$. S is a subring of $\mathbb{Q}$ if;
  - **S is closed under subtraction.**
    $\frac{a}{b} \cdot \frac{c}{d} = \frac{ad - bc}{bd}$
    Since $b, d = \text{odd} \implies bd = \text{odd}$ and therefore any subset of P_{bd} is also odd
    $\implies$ when $\frac{ac}{bd}$ is reduced, say $\frac{ad - bc}{bd} = \frac{r}{s}$, then $s = \text{odd}$, because P_s is a subset of P_{bd}.
    $\therefore S$ is closed under subtraction
  - **S is closed under multiplication.** $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$
    The same argument as for the closure under subtraction applies here.
    $\therefore S$ is closed under multiplication

  $\therefore S$ is a subring of $\mathbb{Q}$
- Given $I \subset J, I = \{\frac{x}{y} | \text{when } \frac{x}{y} \text{ is reduced } = \frac{a}{b} \implies b = \text{odd} \wedge a = \text{even}\}$.
  $b = \text{odd} \iff p_i \neq 2$, in P_b
  $a = \text{even} \iff 2 * q_1 \cdot q_2 \cdots q_k$ where $q_i \in \{\text{prime}\}$
  Given $\frac{a}{b}, \frac{c}{d} \in S$. Then I is an ideal if;

- **I is closed under subtraction**
  $\frac{a}{b} - \frac{c}{d} = \frac{ad-cb}{bd}$
  Form our earlier argument, bd is odd. and
  $ad - cd$
  $= (2 \cdot q_1 \cdot q_2 \cdots q_k)d - (2 \cdot t_1 \cdot t_2 \cdots t_l)b$
  $= 2((q_1 \cdot q_2 \cdots q_k)d - (t_1 \cdot t_2 \cdots t_l)b)$
  $\implies ad - cb$ is even, also when reduced, because the 2 does not get removed.
  $\therefore \frac{a}{b} - \frac{c}{d} \in I$
- **And I absorbs products.** Given $\frac{r}{s} \in S$, then
  * $\frac{r}{s} \cdot \frac{a}{b} = \frac{ra}{sd}$
    Form our earlier argument, sb is odd. $ra = r \cdot 2(q_1 \cdot q_2 \cdots q_n)$ is even also when reduced, because the 2 does not get removed.
    $\therefore \frac{r}{s} \cdot \frac{a}{b} = \frac{ra}{sd} \in I$. and
  * $\frac{a}{b} \cdot \frac{r}{s} = \frac{ar}{bs}$
    Form our earlier argument, bs is odd. $ar = 2(q_1 \cdot q_2 \cdots q_n) \cdot r$ is even also when reduced, because the 2 does not get removed.
    $\therefore \frac{a}{b} \cdot \frac{r}{s} \in I$.

  $\therefore I + J$ absorbs products.

$\therefore I$ is an Ideal, by theorem 6.1. Interesting to note is that I is just a principle Ideal generated by $\frac{2}{1} \in S$

- Show that $S/I \cong \mathbb{Z}_2$
  Given any element $\frac{a}{b} \in S$, consider the coset $\frac{a}{b} + I$. Then $\frac{a}{b}$ in reduced form is either a multiple of $\frac{2}{1}$, in which case $\frac{a}{b} \in I$, so that $\frac{a}{b} \equiv 0(mod\, I)$
  or $\frac{a}{b} - 1 \in I$, because $\frac{a}{b} - \frac{1}{1} = \frac{a+b}{b} \implies a + b$ is even, $\implies \frac{a}{b} \equiv 1(mod\, I)$
  Therefore S/I consists of two disjoint sets, (1+I) and (0+I).
  As we can se $(1 + I) + (1 + I) = ((1 + 1) + I) = 2 + I$, but $2 \in I \implies 2 \equiv 0(mod\, I)$
  $\therefore$ by choosing $[1] = 1 + I$ and $[0] = 0 + I$ we get that $S/I \cong \mathbb{Z}_2$ Calculated by hand.

3 Show that $\mathbb{Z}_6/([3]) \cong \mathbb{Z}_3$ Let $f : \mathbb{Z}_6 \to \mathbb{Z}_3$ and $f :=$
$[3], [0] \to [0]$
$[4], [1] \to [1]$
$[5], [2] \to [2]$
$\implies$ f is surjectiv homomorphism
$\implies$ ([3]) is the kernel of f.
In addition, I am assuming that [0] is implicitly within the kernel.
$\therefore \mathbb{Z}_6/([3]) \cong \mathbb{Z}_3$ by the first isomorphism theorem.

4 TODO

- Show that every ideal of R is principal (even if R is not necessarily a domain)
- List all the prime and maximal ideal of $\mathbb{Z}_{12}$

5 Let$(G, *)$be a group. Consider the following operation on $G : a\sharp b = b * a$. Given $a, b, c \in (G, *)$ $(G, \sharp)$ is a group if these axioms are satisfied ;

- **Closure under the operation**
  $a\sharp b \in G \implies b * a \in G$ which is true. ✓
- **Associative under the operation**
  $(a\sharp b)\sharp c = c * (b * a) = (c * b) * a = a\sharp(b\sharp c)$ ✓
- **Has Identity element**
  Given an identity $e \in G$
  $a\sharp e = e * a = a = a * e = e\sharp a$ ✓

- **Has an inverse**
  Given an inverse of $a, d \in G$
  $a \sharp d = d * a = e = a * d = d \sharp a$ $\checkmark$

$\therefore (G, \sharp)$ is a group. $\square$

6 Both $\mathbb{Z}_4$ and $\mathbb{Z}_2$ are groups under addition. Therefore the identity of $\mathbb{Z}_4 \times \mathbb{Z}_2$ is $(0, 0)$.
$(0, 0)$ has order 1
$(1, 0)$ has order 4
$(2, 0)$ has order 2
$(3, 0)$ has order 4
$(0, 1)$ has order 2
$(1, 1)$ has order 4
$(2, 1)$ has order 2
$(3, 1)$ has order 4

7 Important to note that $4 \in [4]$ and that $\mathbb{Z}_{15}$ is a group by addition.
f is group isomorphsim if;

- $f(a * b) = f(a) * f(b)$
  Given any $[n], [m] \in \mathbb{Z}_{15}$
  $f([n] \oplus [m]) = f([n + m]) = 4[n + m] = [4][n + m] = [4(n + m)] = [4n + 4m] = [4n] + [4m] = [4][n] \oplus [4][m] = f([n]) \oplus f([m])$ $\checkmark$
- f injective
  Given any $[n], [m] \in \mathbb{Z}_{15}$
  $f([n]) = f([m]) \iff 4[n] = 4[m] \iff [n] = [m]$ $\checkmark$
- f is surjective
  Given any $[n] \in \mathbb{Z}_{15}$,and the inverse of $[4] = d$ which is $[4]$
  There exists $[m] = [n]d$ such that $f([m]) = f(d[n]) = 4d[n] = [n]$ $\checkmark$

$\therefore f$ is a group isomorphism.

8 Show that if f is a group homomorphism then $f \equiv 0$ TODO

9 $(14)(27)(523)(34)(1472) = (1573)(24) \in S_8$

Keep in mind. I am behind in this subject,because I also had mandatory assignments in 3 other subjects, therefore I am behind in my reading for this assignment. This does not mean that I am struggling to understand this subject, I just had limited time to do this properly.