Introduction to Graph Theory

Chapter 1

Paths, Cycles and Trails

Lemma 1.2.5: Every u-v walk has a u-v path

Proof.

If the walk has no repeated vertex then it is a path.

Let the length of the walk be 'l'.

Basis Step: $l = 1$. A u-v walk of length 1 is the same as a u-v path of length 1.

Therefore the lemma is true for walks of length 1.

Induction Step: For some $l > 1$, let the lemma be true $\forall\ m \leq l$.

Consider a u-v walk of length $l+1$.

If the walk has no repeated vertices then it is already a path and the lemma becomes true for walks of length $l+1$.

Let w be a repeated vertex. There exists a closed walk with w as its starting and ending vertex. Removing all edges in this closed walk leaving only a single appearance of the vertex w results in a shorter u-v walk. By induction hypothesis, this walk contains a u-v path. This path is also present in the original walk of length $l+1$.

$\therefore$ By induction, every u-v walk contains a u-v path $\square$

Proposition 1.2.11: Every graph with n vertices and k edges has atleast $n-k$ connected components

Proof:

If a graph has m connected components, creating an edge between any 2 non-adjacent vertices present in the graph, results in the reduction in the number of connected components by atmost 1 $\longrightarrow$ (1)

A graph with n vertices and 0 edge has n connected components. (all are trivially connected)

Creating k edges can result in the reduction in the number of connected components by atmost k. (Applying (1) k times)

$\therefore$ There are atleast $n-k$ connected components left. $\square$

1.2.12. Definition. A **cut-edge** or **cut-vertex** of a graph is an edge or vertex whose deletion increases the number of components. We write $G - e$ or $G - M$ for the subgraph of G obtained by deleting an edge e or set of edges M. We write $G - v$ or $G - S$ for the subgraph obtained by deleting a vertex v or set of vertices S. An **induced subgraph** is a subgraph obtained by deleting a set of vertices. We write $G[T]$ for $G - \overline{T}$, where $\overline{T} = V(G) - T$; this is the subgraph of G **induced by** T.

1.2.14. Theorem. An edge is a cut-edge if and only if it belongs to no cycle.

Proof:

If an edge is a cut-edge then removing must increase the number of connected components by exactly 1.

Consider a connected component of the graph.
If an edge uv is a cut-edge and a part of a cycle through u, v, then removing it still leaves u, v connected by a path obtained by deleting an edge uv from the cycle. Therefore uv cannot be a cut-edge.

$\therefore$ An edge is a cut-edge $\Rightarrow$ It does not belong to any cycle.

Now let uv be an edge that does not belong to any cycle. If u and v are connected after removing uv then there exists a uv walk not passing through uv. Therefore by lemma 1.2.5, there exist a uv path in the graph without edge uv. This path also exists in the original graph. The path along with the edge uv results in a cycle through u, v with uv belonging to the cycle. This is a contradiction. Therefore u, v are no longer connected after the removal of edge uv.

∴ An edge does not belong to a cycle

$\Rightarrow$ It is a cut edge $\square$

1.2.15. Lemma. Every closed odd walk contains an odd cycle.

Proof:

Let the length of a walk be denoted by 'l'.

Basis Step:

For a closed walk of length 1, there is a single vertex and this can be considered an odd cycle of length 1.

Induction Step:

Let the statement be true for all closed odd walks of length m where $m \leq l$ for some odd number l.

Now consider a closed odd walk of length $l+2$.

If there are no repeated vertices then the walk is already a cycle.

Let w be a repeated vertex in the walk. Now there are 2 closed walks passing through w. Since the walk under consideration is of odd length, the two walks obtained now must be such that exactly one of them is an odd walk and another is an even walk.

By induction hypothesis, the smaller closed odd walk has an odd cycle and therefore the closed odd walk of length $l+2$ also has an odd cycle.

By induction,

Every closed odd walk contains an odd cycle. □

1.2.18. Theorem. (König [1936]) **A graph is bipartite if and only if it has no odd cycle.**

Proof:

Necessity: If a graph is bipartite, then it cannot have an odd cycle because an odd cycle cannot be split into two independent sets.

Say an odd cycle of length $2m+1$ exists whose vertices are $v_1, v_2, \dots, v_{2m+1}$ in order.

Let the 2 partite sets be P_1, P_2

WLOG $v_1 \in P_1$

$\Rightarrow v_2 \in P_2$

$\Rightarrow v_3 \in P_1$

$\vdots$

$\Rightarrow v_{2i+1} \in P_1$

$\Rightarrow v_{2i+2} \in P_2 \quad \forall\, i \in \{0, 1, 2, \dots, m-1\}$

$\vdots$

$v_{2m-1} \in P_1$ and $v_{2m} \in P_2$

$\Rightarrow v_{2m+1} \in P_1$

But v_1 and v_{2m+1} are adjacent vertices.

$\therefore$ An odd cycle cannot be partition into 2 partite sets $\square$

Sufficiency:

Let G be a graph with connected components and no odd cycles in any of the connected components.

Consider one of these connected components H.

Choose any vertex u from $V(H)$.

$\forall\, v \in V(H)$ $\exists$ a path uv as H is connected.

Let $f(v)$ denote the length of the shortest path from u to v $\forall\, v \in V(H)$ with $f(u)=0$

Consider 2 sets X and Y with

$$X = \{ v \mid v \in V(H) \text{ and } f(v) \text{ is even} \}$$

$$Y = \{ v \mid v \in V(H) \text{ and } f(v) \text{ is odd} \}$$

Let $v, v' \in X$ with $\{v, v'\} \in E(H)$.

The closed walk consisting of the shortest path from u to v, the edge vv' and the shortest path from v' to u is of length $f(v) + f(v') + 1$ which is an odd number.

$\therefore$ There is an odd walk in X if vv' is an edge

$\Rightarrow$ $\exists$ an odd cycle in X

A contradiction

$\therefore$ vv' is not an edge

$\Rightarrow$ No two vertices in X are adjacent.

Similarly for Y, no 2 vertices Y cannot have an edge between them.

$\therefore$ X and Y are 2 bipartite sets. $\square$

1.2.20. Definition. The **union** of graphs $G_1, \dots, G_k$, written $G_1 \cup \dots \cup G_k$, is the graph with vertex set $\bigcup_{i=1}^{k} V(G_i)$ and edge set $\bigcup_{i=1}^{k} E(G_i)$.

1.2.23. Theorem. The complete graph K_n can be expressed as the union of k bipartite graphs if and only if $n \leq 2^k$.

Proof:

We prove this by inducting on k.

For $k=1$, $2^k = 1$

If $n=3$, K_3 is an odd cycle C_3 and is therefore not bipartite

$\therefore$ $n \leq 2$

And if $n \leq 2$, the resulting complete graph K_n can be partitioned into a union of 1 bipartite graph

$\therefore$ The theorem is true for $k=1$.

Let the theorem be true $\forall m < k$ for some $k \in \mathbb{N}$.

Suppose K_n can be expressed as a union of k bipartite graphs $G_1, G_2, \ldots, G_k$.

Partition the vertices into 2 sets X and Y with the property that no edge in G_k is present in X and Y.

Consider the subgraphs induced X and Y, $G[X]$ and $G[Y]$.

$G[X] = \bigcup_{i=1}^{k-1} G_i$ and $G[Y] = \bigcup_{i=1}^{k-1} G_i$ because it cannot contain any edge present in G_k.

Such a partition of $V(G)$ clearly exists which $X_k \subseteq X$, $Y_k \subseteq Y$ where are the partite sets of G_k.

By induction hypothesis, $|V(G[X])| \leq 2^{k-1}$ and $|V(G[Y])| \leq 2^{k-1}$

$$\Rightarrow |X| \leq 2^{k-1} \;\&\; |Y| \leq 2^{k-1}$$

$$\Rightarrow |X| + |Y| \leq 2^k$$

$$\Rightarrow n \leq 2^k$$

We have proved that if a graph can be expressed as a union of k bipartite graphs then $|V(G)| = n \leq 2^k$.

Now for the converse,

we are given that $n \leq 2^k$

Consider a partition of $V(G)$ into X and Y with

$|X| \leq 2^{k-1}$ and $|Y| \leq 2^{k-1}$.

Consider the induced sub-graphs $G[X]$ and $G[Y]$.

By induction hypothesis, $G[X] = \bigcup_{i=1}^{k-1} G_{x_i}$ and $G[Y] = \bigcup_{i=1}^{k-1} G_{Y_i}$ for some Bipartite graphs G_{x_i}, G_{Y_j}.

Take pairwise union of G_{x_i} and G_{Y_i} $\forall i$, $1 \leq i \leq k-1$.

Now consider $\bigcup_{i=1}^{k-1} G_i$ where $G_i = G_{x_i} \cup G_{Y_i}$ is Bipartite

Edges of the form xy with $x \in X$ and $y \in Y$ have not been covered yet.

Consider a Biclique with the partite sides X, Y. Call this G_k

$\left(\bigcup_{i=1}^{k-1} G_i\right) \cup G_k$ covers all vertices and edges of K_n.

$$\Rightarrow \boxed{K_n = \bigcup_{i=1}^{k} G_k}$$

We have proved that if $n \leq 2^k$ then K_n can be expressed as a union of k Bipartite graphs

By induction,

$\forall n$, $K_n = \bigcup_{i=1}^{k} G_i$, G_i is a Bipartite graph $\Longleftrightarrow n \leq 2^k$ $\square$

1.2.24. Definition. A graph is **Eulerian** if it has a closed trail containing all edges. We call a closed trail a **circuit** when we do not specify the first vertex but keep the list in cyclic order. An **Eulerian circuit** or **Eulerian trail** in a graph is a circuit or trail containing all the edges.

An **even graph** is a graph with vertex degrees all even. A vertex is **odd** [**even**] when its degree is odd [even].

1.2.25. Lemma. If every vertex of a graph G has degree at least 2, then G contains a cycle.

1.2.26. Theorem. A graph G is Eulerian if and only if it has at most one nontrivial component and its vertices all have even degree.

Proof:

There are 2 parts to the proof: Necessity and Sufficiency.

Necessity: A graph G is Eulerian $\Rightarrow$ It has atmost one nontrivial component and its vertices all have even degree.

Proof:

Since there is an Eulerian trail in the graph, every edge must be present in the trail

$\Rightarrow$ Every vertex in the trail must be connected to every other vertex in the trail.

$\Rightarrow$ There is atmost one non-trivial connected component.

There must be an even number of edges from every vertex because every time a vertex occurs in the trail, it must be leading to another vertex through an edge different from the ones already visited □

Sufficiency: A graph has atmost one connected component and all it's vertices have even degrees $\Rightarrow$ The graph is Eulerian.

Proof:

We prove this by induction on the number of vertices of the graph 'n'.

Base Case: $n=1$. There is one trivial connected component and there is an Eulerian trail consisting of this single vertex.

Inductive Hypothesis: If a graph satisfies all the required conditions and has m vertices where $m < n$ for some n, then it is Eulerian.

Now consider a graph with n vertices.

Case 1: There are no non-trivial connected components.

In this case we are done as there are no edges.

Case 2: There is one non-trivial connected component.

Now since every vertex has even degree we can find a cycle in it.

Let this cycle be 'C'.

Now delete all edges that are present in C. We get a graph G' which still has the degree of every vertex as an even number. Now the resulting connected components of G' all have lesser number of vertices than G and by induction hypothesis have an Eulerian cycle.

Construct an Eulerian cycle for G by starting at any vertex of C and whenever the vertex of a nontrivial connected component is reached, go along the Eulerian cycle of that component and then continue along C.

$\therefore$ There is an Eulerian cycle for G

By Induction, there are also sufficient conditions $\square$