DISCRETE STRUCTURES

DISCRETE STRUCTURES

CS207

Quiz 2 Solutions

1- An $n \times n$ board with n^2 unit squares is supposed to be coloured by 2 colours red or blue. The group of symmetries are:

Rotating about centre by $90°, 180°, 270°, 360°$

Reflecting about each of the 2 diagonals

Reflecting about a line passing through midpoints of pair of opposite sides. There are 2 two such lines

$\therefore |G| = 8$

There are 2 cases: n is even; n is odd.

Now do case work.

2. Given a poset of size $km+1$ prove that there exists a chain of length $k+1$ or antichain of length $m+1$.

Proof:

For every poset, the minimum number of antichains that the poset can be partitioned into equal to the length of the longest maximal chain in the poset.

Let the length of the longest chain $\leq k$.

There exist a minimum of k antichains into which the poset can be partitioned.

In every such partition one of the antichains must contain an antichain of length $\geq m+1$ $\square$

We have proved,

$\sim$(A chain of length $k+1$ exists)

$\Rightarrow$ (A chain of length $m+1$ exists)

which is equivalent to proving what was asked

Q1. Define a relation $\Leftarrow\!\Rightarrow$, with $\wedge, \vee$ operations that are commutative, associative, and are idempotent and absorptive.

Finite boolean lattice: Set of all subsets of a finite set ordered by a relation.

Distributive properties are not satisfied by $\vee, \wedge$ in general.

There are lattices that are not Boolean lattices but satisfy these properties. Additional properties are required to define a Boolean lattice.

Identity: $\exists$ elements 0 and 1 $\;\; a \vee 0 = a,$
$a \wedge 1 = a$

$\forall a$

0 – Empty set, 1 – whole set

Complement: $\forall a \; \exists \; a^c \ni a \vee a^c = 1$
$a \wedge a^c = 0$

A partial order with $\wedge, \vee$ with the above properties (on a finite set) is a Boolean lattice with $\wedge$ corresponding to intersection and $\vee$ corresponding to union.

23/10/2023

RECURRENCE RELATIONS

A way of defining functions on natural numbers (one or more variables) such that the value for n is defined in terms of values for numbers < n and the value for 0 is defined explicitly.

* Counting Problems:

For each n, we have a finite subset S_n defined of some objects.

Ex: $|S_n| = 2|S_{n-1}|$ is a possible recurrence relation for some problem of counting.

* Running time of recursively defined algorithms are calculated using recurrence relations.

* Solving a recurrence relation is finding an explicit solution of f(n) in terms of n itself.

* Most counting problems are reduced to coming up with a recurrence relation. Solving a recurrence is fairly easy if it is solvable.

Q) S_n: Set of all subsets of $\{1, 2, \ldots, n\}$ that do not contain two consecutive numbers. Find $|S_n|$.

Every subset can either include 'n' or not.

Say a subset A contains 'n'. The remaining elements are from $\{1, 2, \ldots, n-2\}$ and there are $|S_{n-2}|$ sets.

Say it does not include n. Then, the remaining elements are from $\{1, 2, \ldots, n-1\}$ and there are $|S_{n-1}|$ sets.

$$\therefore |S_n| = |S_{n-1}| + |S_{n-2}| \quad \forall n > 2$$

$|S_1| = 2$, $|S_2| = 3$

$\downarrow$ $\qquad\qquad \downarrow$

$\phi, \{1\}$ $\qquad \phi, \{1\}, \{2\}$

This is a part of Fibonacci numbers starting from 2, 3.

Catalan Numbers:

C_n = Number of binary trees with n nodes.

$C_0 = 1$

C_n be calulated by counting the number of trees with i nodes in the left subtree and $n-1-i$ nodes in the right subtree with $i \in \{0, 1, \ldots, n-1\}$

Number of trees with i nodes in the left and $n-1-i$ nodes in the right are $C_i \cdot C_{n-1-i}$

$$\therefore \boxed{C_n = \sum_{i=0}^{n-1} C_i C_{n-1-i} \quad \forall n \geq 1, C_0 = 1}$$

There are other problems that have their solution as the Catalan numbers.

Consider the problem of balanced paranthesis with n pairs of paranthesis.

Either a bijection can be drawn from binary trees to paranthesis or it can be shown that the recurrence relation for both are the same.

For every valid paranthesis, the first bracket must be a left bracket. Consider it's corresponding closing bracket.

$$\checkmark(\underbrace{\ldots\ldots}_{i \text{ pairs inside}})\checkmark \; {}^*(\underbrace{\ldots\ldots)^*}_{n-1-i \text{ including } (\,)^*}$$

For every valid paranthesis both of the substrings must also be valid.

$$\therefore \quad \boxed{P_n = \sum_{i=0}^{n-1} P_i P_{n-1-i}, \forall n \geq 1, P_0 = 1}$$

which is the same as the Catalan numbers.

Consider a $2 \times n$ matrix with entries $1, 2, 3, \cdots, 2n$. such that each row and each column is increasing with increase in index.

This problem can be converted into the problem of lattice paths:

Consider the xy plane. Find the number of paths from $(0,0)$ to (n,n) through points with integer coordinates (x,y) such that $x \geq y$ at every point in the path and only right and up directions are allowed.

Essentially the path can never lie above $y=x$.

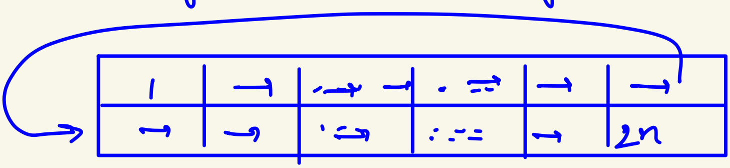
Let the path be partitioned into $(0,0) \to (i,i)$, $(i,i) \to (n,n)$ where (i,i) is the first time the path has intersected with the line $y=x$.

Number of such paths $= P_i P_{n-1-i}$

This can be done for $\forall i \in \{0,1,2,\dots,n-1\}$

$$\therefore \boxed{P_n = \sum_{i=0}^{n-1} P_i P_{n-1-i} \quad \forall n \geq 1, \quad P_0 = 1}$$

At each coordinate, the sum $x+y$ is increasing and every entry can be filled with the sum of the coordinates excluding $(0,0)$ in this way:



Running time complexity of Quicksort:

Average time complexity $= T(n)$

$$T(n) = \frac{1}{n} \sum_{i=1}^{n} \left(T(i-1) + T(n-i)\right)$$

where each term in the sum represents the case when the i^{th} largest element is the pivot and $\frac{1}{n}$

is present because the probability of any element being the pivot is $1/n$.

28/10/2023

Number of subsets of $\{1, 2, \ldots, n\}$ with no two consecutive elements:

Consider a bit string of length 'n' where i^{th} bit being 1 implies the given subset has the element 'i'.

Number of strings with no occurrences of '11' is equal to the number of subsets with no two consecutive numbers of the set $\{1, 2, \ldots, n\}$

Let the bit string have last bit 0.
Now, the remaining $n-1$ bits is any valid bit string with no occurrence of '11'.
$\therefore$ There are a_{n-1} such bitstrings

Let the last bit be 1.
Now the second last bit must be 0 and the remaining $n-2$ bits must be any valid bitstring with no occurrences of '11'.
$\therefore$ There are a_{n-2} such bit strings

$$\Rightarrow \boxed{a_n = a_{n-1} + a_{n-2}, \ a_1 = 2, a_2 = 3}$$

In matrix form,

$$\vec{T}(n) = \begin{bmatrix} T_0(n) \\ T_1(n) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^{n-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$T_0(n)$ = Number of bit strings that end with 0

$T_1(n)$ = Number of bit strings that end with 1

$$\begin{bmatrix} T_0(n) \\ T_1(n) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} T_0(n-1) \\ T_1(n-1) \end{bmatrix}$$

This is obtained by $T_0(n) = T_1(n-1) + T_0(n-1)$

$T_1(n) = T_0(n-1)$

By induction, it can be proved that:

$$\begin{bmatrix} T_0(n) \\ T_1(n) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^{n-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The characteristic polynomial of $\underset{(A)}{\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}}$ is:

$$(1-\lambda)(-\lambda) - 1 \times 1 = 0$$

$$\Rightarrow \lambda^2 - \lambda - 1 = 0$$

By Cayley Hamilton Theorem,

$$A^2 = A + I$$

$\Rightarrow A^{n+2} = A^{n+1} + A^{n} \ \forall n \in \mathbb{N}$

The matrix can be simplified to get a closed form solution for the Fibonacci numbers.

Q) Count the number of strings of length n in which no substring of length 3 contains all the letters of an alphabet of size 3.

Ex: If the alphabet is a, b, c, any occurrence of abc, bca, cab, cba, bac, acb are forbidden.

Consider the strings which end with a.
The substring of $n-1$ must not end with "bc" or "cb"
The conditions are symmetric in a, b, c

Strings of length $n-1$ can be extended to length n in some cases only.

If the string ends with 2 distinct letters then it can be extended in two ways by adding one of the last 2 letters.
If the string ends with 2 letters that are identical then it can be extended by adding any of the 3 letters.

So there are 2 possible classes of strings

$T_{xx}(n)$ = Number of such strings in which last two letters are same where $x \in \{a, b, c\}$

$T_{xy}(n)$ = Number of strings with last two letters being distinct where $x \neq y$, $x, y \in \{a, b, c\}$

$$T_{xy}(n) = T_{xx}(n-1) + T_{xy}(n-1)$$ → n-1 length string can only end with xx

$$T_{xx}(n) = T_{xx}(n-1) + 2T_{xy}(n-1)$$ → n-1 length string can end with yx or zx.

$$\therefore \begin{bmatrix} T_{xx}(n) \\ T_{xy}(n) \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} T_{xx}(n-1) \\ T_{xy}(n-1) \end{bmatrix}$$

and by induction,

$$\begin{bmatrix} T_{xx}(n) \\ T_{xy}(n) \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}^{n-2} \begin{bmatrix} T_{xx}(2) \\ T_{xy}(2) \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$$

$$(1-\lambda)(1-\lambda) - 2 = 0$$

$$\lambda^2 - 2\lambda - 1 = 0$$

$$A^2 = 2A + I$$

$$\boxed{A^{n+2} = 2A^{n+1} + A^n \quad \forall n \in \mathbb{N}}$$

Clearly $T_{xx}(2) = 1$ and $T_{xy}(2) = 1$

$$\therefore \boxed{T(n) = 3T_{xx}(n) + 6T_{xy}(n)}$$

and $$\begin{bmatrix} T_{xx}(n) \\ T_{xy}(n) \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}^{n-2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \forall n > 2$$

$$\boxed{a_n = 2a_{n-1} + a_{n-2} \quad \forall n > 2,\ a_1 = 3,\ a_2 = 6}$$

Obtained from the characteristic equation of the matrix.

Find a direct explanation for the recurrence relation for this problem.

30/10/2023

Solving Linear Recurrence Relations with Constant Coefficients:

Consider the previous problem with recurrence relation being

$$a_n = 2a_{n-1} + a_{n-2} \;\forall n > 2$$

Let $f_0, f_1, f_2, \ldots$, be an infinite sequence of numbers.

Denote by $f(x) = \sum_{i=0}^{\infty} f_i \cdot x^i$ which is the generating function of the sequence.

If $f(x)$ and $g(x)$ are the generating functions of $f_0, f_1, \ldots$ and $g_0, g_1, \ldots$, we can define $f(x) + g(x)$ is the generating function of $f_0 + g_0, f_1 + g_1, \ldots$,

$h(x) = f(x) \cdot g(x)$ is the generating function of the sequence $h_0, h_1, h_2, \ldots$ where $h_n = \sum_{i=0}^{n} f_i \cdot g_{n-i}$

Use recurrence relations to find the generating function of the sequence. Then use a known generating function to get the closed form solution.

Consider the recurrence,

$$T(n) = 2T(n-1) + T(n-2) \ \forall \ n \geq 2 \text{ with}$$

the initial conditions $T(0)=1, T(1)=3$

Since this a homogeneous linear equation, every term $T(n)$ is a linear combination of $T(0)$ and $T(1)$

So it is sufficient to find a solution for the conditions $T(0)=1, T(1)=0$, $T(0)=0, T(1)=1$ and then combine the two solutions by multiplying by 1, 3 the respective solutions.

$T(x)$ would be the generating function for the sequence.

$$\overset{n \geq 2}{T(n)} = 2T(n-1) + T(n-2)$$

$$\Rightarrow T(n)x^n = 2T(n-1)x^n + T(n-2)x^n$$

$$\Rightarrow \sum_{n=2}^{\infty} T(n)x^n = 2\sum_{n=2}^{\infty} T(n-1)x^n + \sum_{n=2}^{\infty} T(n-2)x^n$$

$$\Rightarrow T(x) - T(0) - T(1)x = 2x\left(T(x) - T(0)\right) + x^2 T(x)$$

$$\Rightarrow T(x)\left[x^2 + 2x - 1\right] = x\left(2T(0) - T(1)\right) - T(0)$$

$$\Rightarrow T(x) = \frac{x(2T(0) - T(1)) - T(0)}{x^2 + 2x - 1}$$

Some generating functions: $\frac{1}{1-cx} \equiv c^n$

$\frac{1}{(1-cx)^2} \equiv 1, 2c, 3c^2, \ldots$

Let $T(0) = 1$ and $T(1) = 0$

$$\frac{1-2x}{1-2x-x^2} = \frac{c_1}{1-r_1 x} + \frac{c_2}{1-r_2 x}$$

$\left.\begin{array}{l} r_1 + r_2 = 2 \\ r_1 r_2 = -1 \end{array}\right\}$ roots of $x^2 - 2x - 1 = 0$ (obtained by replacing x with $\frac{1}{x}$ in $1-2x-x^2$)

$r_1 = 1+\sqrt{2}$, $r_2 = 1-\sqrt{2}$

c_1 and c_2 depend on the initial conditions

$\left.\begin{array}{l} c_1 + c_2 = 1 \\ -c_1 r_2 - c_2 r_1 = -2 \end{array}\right\}$ As $T(0) = 1$ and $T(1) = 0$ was taken

After solving, $T(n) = c_1 (1+\sqrt{2})^n + c_2 (1-\sqrt{2})^n$

Similarly for Fibonacci sequence,

$T(n) = T(n-1) + T(n-2) \; \forall n \geq 2$, $T(0) = 1$, $T(1) = 1$

We have,

$$T(x) - T(0) - T(1)x = x\left(T(x) - T(0)\right) + x^2 T(x)$$

$$\Rightarrow T(x)(1 - x - x^2) = (T(1) - T(0))x + T(0)$$

$$\Rightarrow T(x) = \frac{(T(1) - T(0))x + T(0)}{1 - x - x^2}$$

With $T(n) = c_1\left(\frac{\sqrt{5}-1}{2}\right)^n + c_2\left(-\frac{\sqrt{5}-1}{2}\right)^n + n$

$$\left.\begin{array}{l} c_1 + c_2 = T(0) \\ (c_1 + c_2)(-1) = T(1) \end{array}\right\} \Rightarrow c_1 + c_2 = 1,$$

$$\frac{c_1}{4}(6 - 2\sqrt{5}) + \frac{c_2}{4}(6 + 2\sqrt{5}) = T(2)$$

$$\Rightarrow (c_1 + c_2)(6) + 2\sqrt{5}(c_2 - c_1) = 4 \times 2 = 8$$

$$\Rightarrow 6 + 2\sqrt{5}(c_2 - c_1) = 8$$

$$\Rightarrow c_2 - c_1 = \frac{1}{\sqrt{5}}$$

$$\Rightarrow c_2 = \frac{\sqrt{5}+1}{2\sqrt{5}}, \quad c_1 = \frac{\sqrt{5}-1}{2\sqrt{5}}$$

$$\therefore T(n) = \frac{(\sqrt{5}-1)^{n+1} + (-1)^n(\sqrt{5}+1)^{n+1}}{4\sqrt{5} \cdot 2^n} + n$$

This method can also be used for non linear recurrence relations.

Consider Catalan numbers.

$$C_0 = 1,$$

$$C_n = \sum_{i=0}^{n-1} C_i C_{n-1-i}$$

Let $C(x)$ be the generating function of the sequence $C_0, C_1, \ldots$

$$C(x) = \sum_{i=0}^{\infty} C_i x^i$$

$$\sum_{n=1}^{\infty} C_n x^n = \sum_{n=1}^{\infty} \left(\sum_{i=0}^{n-1} C_i C_{n-1-i} \right) x^n$$

$$\Rightarrow C(x) - C_0 = \sum_{n=1}^{\infty} \left(\sum_{i=0}^{n-1} C_i x^i \cdot C_{n-1-i} x^{n-1-i} \cdot x \right)$$

$$\Rightarrow C(x) - C_0 = \sum_{n=1}^{\infty} x^n \left(\text{Coefficient of } x^{n-1} \text{ in } C^2(x) \right)$$

$$\Rightarrow C(x) - C_0 = \sum_{n=1}^{\infty} x^n \times C'_{n-1}$$

$$\Rightarrow C(x) - C_0 = x \sum_{n=1}^{\infty} C'_{n-1} \cdot x^{n-1}$$

$$\Rightarrow C(x) - C_0 = x \sum_{n=0}^{\infty} C'_n x^n$$

where C'_n is the coefficient of x^n in $C^2(x)$

$C^2(x)$ is the generating function for the series $A_0, A_1, A_2 \ldots$ where

$$A_n = \sum_{i=0}^{n-1} C_i \cdot C_{n-1-i} \quad \text{but } C_n' = \sum_{i=0}^{n-1} C_i C_{n-1-i}$$

$$\therefore \sum_{n=0}^{\infty} C_n' x^n = C^2(x) \quad \text{as } C_n' = A_n$$

$$\Rightarrow C(x) - C_0 = x\,C^2(x)$$

$$\Rightarrow x\,C^2(x) - C(x) + C_0 = 0$$

$$\Rightarrow x\,C^2(x) - C(x) + 1 = 0$$

$$\Rightarrow C(x) = \frac{1 \pm \sqrt{1-4x}}{2x}$$

There are 2 possible solutions for $C(x)$. However generating functions are unique.

$\lim_{x \to 0} C(x)$ must equal C_0 which is 1

$$\lim_{x \to 0} \frac{1+\sqrt{1-4x}}{2x} \text{ is not defined.}$$

$$\lim_{x \to 0} \frac{1-\sqrt{1-4x}}{2x} = 1$$

$$\therefore C(x) = \frac{1-\sqrt{1-4x}}{2x}$$

$$(1-x)^{\frac{1}{2}} = 1 + \frac{\left(\frac{1}{2}\right)(-x)}{1} + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)(-x)^2}{1 \cdot 2}$$

$$+ \ldots$$

Coefficient of $x^n = \frac{-1}{n!} \prod_{i=1}^{n-1} \frac{(2i-1)}{2}$

$$= \frac{-1}{2^n n!} \prod_{i=1}^{n-1} \frac{(2i-1)\cdot i}{i}$$

$$= \frac{-1}{2^n \cdot n! \cdot n!} \times \frac{1}{2^n} \times \frac{1}{2n-1} \prod_{i=1}^{n} (2i-1)\cdot(2i)$$

$$= \frac{-1}{4^n} \cdot \frac{{}^{2n}C_n}{2n-1}$$

$$\therefore (1-x)^{\frac{1}{2}} = \sum_{n=1}^{\infty} \frac{(-1)\, x^n\, {}^{2n}C_n}{4^n (2n-1)} + 1$$

$$(1-4x)^{\frac{1}{2}} = \sum_{n=0}^{\infty} \frac{(-1)\cdot x^n\, {}^{2n}C_n \cancel{4^n}}{(2n-1)\cancel{4^n}} + 1$$

$$\Rightarrow (1-4x)^{\frac{1}{2}} = \sum_{n=0}^{\infty} \frac{(-1)\, x^n\, {}^{2n}C_n}{2n-1} + 1$$

$$\frac{1-(1-4x)^{\frac{1}{2}}}{2x} = \frac{1}{2x}\left(1 - \left(\sum_{n=1}^{\infty} \frac{(-1)\, x^n\, {}^{2n}C_n}{2n-1} + 1\right)\right)$$

$$= \frac{1}{2x} \sum_{n=1}^{\infty} \frac{x^n\, {}^{2n}C_n}{2n-1}$$

$$\Rightarrow C(x) = \frac{1}{2} \sum_{n=0}^{\infty} \frac{{}^{2n+2}C_{n+1}\, x^n}{2n+1}$$

$$\therefore C_n = \frac{{}^{2n+2}C_{n+1}}{2(2n+1)}$$

$$= \frac{(\cancel{2n+2})(\cancel{2n+1})(2n)!}{\cancel{2}(\cancel{2n+1})\cdot(\cancel{n+1})\cdot(n+1)(n!)^2}$$

$$= \frac{(2n)!}{(n+1)n!\,n!}$$

$$\Rightarrow \boxed{C_n = \frac{{}^{2n}C_n}{n+1} \quad \forall n}$$

The method of generating functions can be used to solve recurrence relations of quadratic types also.

Another example:

$F_0 = 1, F_1 = 2, F_n = F_{n-1} + F_{n-2} \ \forall n \geq 2$. Find the closed form for the Fibonacci sequence.

Let $f(x) = \sum_{n=0}^{\infty} F_n x^n$ be the generating function.

$$F_n = F_{n-1} + F_{n-2}$$

$$\Rightarrow F_n x^n = F_{n-1} x^n + F_{n-2} x^n$$

$$\Rightarrow \sum_{n=2}^{\infty} F_n x^n = \sum_{n=2}^{\infty} F_{n-1} x^n + \sum_{n=2}^{\infty} F_{n-2} x^n$$

$$\Rightarrow f(x) - F_0 - F_1 x = x(f(x) - F_0) + x^2 f(x)$$

$$\Rightarrow f(x)(x^2 + x - 1) = xF_0 - xF_1 - F_0$$

$$\Rightarrow \quad f(x)(x^2+x-1) = x(F_0-F_1)-F_0$$

$$\Rightarrow \quad f(x) = \frac{(-1)(x)-1}{x^2+x-1}$$

$$\Rightarrow f(x) = \frac{-(x+1)}{x^2+x-1} = \frac{1+x}{1-x-x^2}$$

$$\Rightarrow f(x) = \frac{C_1}{1-r_1x} + \frac{C_2}{1-r_2x}$$

$$\frac{1}{r_1}+\frac{1}{r_2} = -1, \quad \frac{1}{r_1r_2} = -1$$

$$\Rightarrow r_1+r_2 = 1, \quad r_1r_2 = -1$$

They are the roots of $x^2-x-1=0$

$$x = \frac{1\pm\sqrt{5}}{2}, \; r_1 = \frac{1-\sqrt{5}}{2}, \; r_2 = \frac{1+\sqrt{5}}{2}$$

$$C_1+C_2 = 1, \quad -C_1r_2+(-C_2)r_1 = 1$$

$$\Rightarrow C_1r_2+C_2r_1 = -1$$

$$\Rightarrow \frac{C_1}{2}+\frac{C_2}{2}+\frac{\sqrt{5}}{2}(C_1-C_2) = -1$$

$$\Rightarrow \frac{1}{2}+\frac{\sqrt{5}}{2}(C_1-C_2) = -1$$

$$\Rightarrow (C_1-C_2)\sqrt{5} = -3$$

$$\Rightarrow C_1-C_2 = \frac{-3}{\sqrt{5}}$$

$$\therefore C_1 = \frac{\sqrt{5}-3}{2\sqrt{5}}, \quad C_2 = \frac{\sqrt{5}+3}{2\sqrt{5}}$$

The sequence given by $c_1 r_1^n + c_2 r_2^n$ has the generating function $\dfrac{c_1}{1-r_1 x} + \dfrac{c_2}{1-r_2 x}$.

$$\therefore \quad F_n = c_1 r_1^n + c_2 r_2^n$$

$$\Rightarrow F_n = \frac{1}{2\sqrt{5}}\left((\sqrt{5}-3)\left(\frac{1-\sqrt{5}}{2}\right)^n + (\sqrt{5}+3)\left(\frac{1+\sqrt{5}}{2}\right)^n \right)$$

$$= \frac{1}{2\sqrt{5}} \times \frac{1}{2^n} \times \frac{1}{2}\Big((2\sqrt{5}-6)(1-\sqrt{5})^n + (2\sqrt{5}+6)(1+\sqrt{5})^n \Big)$$

$$\Rightarrow \frac{1}{2^{n+2}\sqrt{5}}\left(-(1-\sqrt{5})^{n+2} + (1+\sqrt{5})^{n+2} \right)$$

$$\boxed{F_n = \frac{(1+\sqrt{5})^{n+2} - (1-\sqrt{5})^{n+2}}{2^{n+2}\sqrt{5}} \quad \forall\, n}$$

31/10/2023

GRAPHS

* A graph G consists of a set V of vertices and a set E of edges which are unordered pairs of vertices.
* Edges define a symmetric and irreflexive relation on the set of vertices. This relation is called an adjacency relation: i.e. If $\{u, v\}$ is an edge, vertex u is said to be adjacent to v and vice versa.

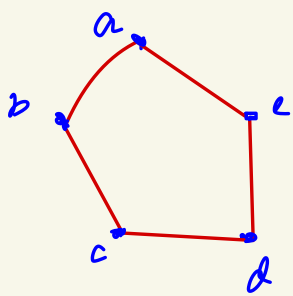
  By definition of the relation (irreflexive), a vertex cannot be adjacent to itself.
* These graphs are also called undirected graphs.
* In directed graphs, the relation is not symmetric in general.
* The edge $\{u, v\}$ is denoted by uv and is said to be incident with vertices u and v.
* A graph G is said to be isomorphic to a graph H if there exists a bijection from $V(G)$ to $V(H)$. This is like renaming the vertices of the graph G to obtain graph H.

$$uv \in E(G) \iff f(u)\,f(v) \in E(H) \quad \text{for some bijection } f$$

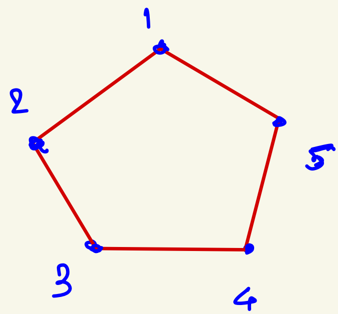
* Properties satisfied by a graph G is also satisfied by graphs that are isomorphic to G.

* Isomorphic graphs can be considered to belong to an equivalence class of graphs with respect to some equivalence relation.

* Ex: Number of edges, number of vertices etc. are properties that do not depend on the label.



and



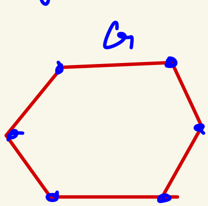
are isomorphic

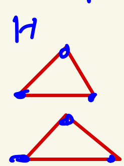
* **Degree** of a vertex is the number of vertices adjacent to it.

* The sequence of degrees (or) degree sequence of a graph is the degrees of vertices arranged in non-increasing order.

Isomorphic graphs have the same degree sequences. However this just this much is not enough to conclude if a pair of graphs are isomorphic to each other or not.

Ex:





G and H are not isomorphic

* There is no simple/efficient algorithm known to determine if two graph are Isomorphic to each other or not.
* A brute force algorithm would be to test every Bijection. This is $O(n!)$ in time where n is the number of vertices
* The best known algorithm is of $O(n^{\log n})$ complexity.
* For graphs with small degrees, there is a polynomial time algorithm is known.

* A graph H is a subgraph of graph G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. A subgraph is obtained by deleting edges from G and some vertices and all edges incident at deleted vertices.

* A spanning sub-graph is obtained by only deleting edges from a graph.
* An induced sub-graph is obtained by only deleting vertices. The adjacency relations in the sub-graphs are present in the original graph. $E(I) \subseteq E(G)$ where I is any induced sub-graph.

Some Specific Graphs:

* P_n: Path with n vertices.

$$V = \{v_1, v_2, \ldots, v_n\}$$

$$E = \{\{v_i, v_{i+1}\} \mid i \in \{1, 2, \ldots, n-1\}\}$$

P_n is also called a path of order "n". (number of vertices)

There is only one path with n vertices. All others are isomorphic. Number of edges is usually called the size.

* C_n : Cycle with n vertices with $n \geq 3$.

$V = \{v_1, v_2, \ldots, v_n\}$

$E = \{\{v_i, v_{i+1}\} \mid i \in \{1, 2, \ldots, n-1\}\} \cup \{v_n, v_1\}$

* We deal with finite graphs in this course.

Q Every graph with n vertices and atleast n edges contains a cycle C_m for some $m \geq 3$.

Proof:

The statement means, there exists a sub-graph that is isomorphic to C_m for some $m \geq 3$.

Another statement: There exists graphs with n vertices and n-1 edges that do not contain any cycle. Such graphs are called trees.

Base Case: $n = 1$

This is trivially true since there is no graph with one vertex and atleast one edge.

Let G be any graph with n vertices and atleast n edges with $n > 1$.

We use induction on 'n' the number of vertices.

There are 2 cases:

(1) Suppose there exists a vertex of degree ≤ 1 in G.

Let G' be the graph obtained by deleting this vertex.

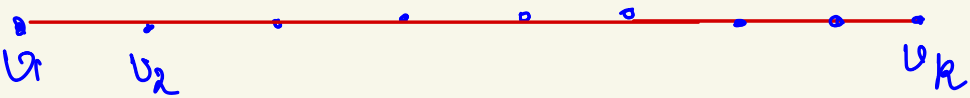
G' has $n-1$ vertices and removal of the vertex resulted in the removal of atmost one edge.

So G' has atleast $n-1$ edges.

By Induction, G' has a cycle which is also contained in G.

(2) Every vertex in G has degree atleast 2.

Let P be the longest path contained in the graph.



Consider the endpoints v_1, v_k.

Since v_1 has degree 2, it must be connected to some other vertex apart from v_2.

This vertex v must be from the path itself.

If it were not in the path, then the path P can be extended. Therefore there is a cycle.

2/11/2023

TREES

For all $n \geq 1$, there exists a graph with n vertices and $n-1$ edges that does not contain a cycle.

Proof:

Basis Step: For $n=1$, there exists a graph with a single vertex and 0 edge that satisfies the property.

Inductive Step: For some $n \geq 1$, let there exist a graph satisfying the given property. Let this graph be G_n.
Now add a vertex v to the graph and join it to any vertex in G_n.

The resulting graph G_{n+1} has $n+1$ vertices and n edges. It does not have a cycle because G_n does not have a cycle.

And no new cycle was created by the introduction of v because degree of v is 1.

By Induction, such a graph exists $\forall n \geq 1$. $\square$

* These graphs are called 'Trees'.

* This proof confirms the existence of trees.

Property: Every tree with $n \geq 2$ vertices has a vertex of degree 1.

Proof:

Claim: There are no vertices with degree 0.

Proof:

If a vertex has degree 0 then removing it results in a graph with $n-1$ vertices and $n-1$ edges. This graph must contain a cycle.

$\Rightarrow$ The tree contains a cycle

$\therefore$ There is no vertex with degree 0. $\square$

If every vertex has degree ≥ 2 then there are atleast n edges which is again a contradiction.

$\therefore$ There exist atleast one vertex with degree 1. $\square$

* Vertices with degree 1 are called leaf vertices.

Construction of Trees :

Every tree with $n \geq 2$ vertices can be obtained from a tree with $n-1$ vertices by adding a vertex and joining it to some vertex in the tree with $n-1$ vertices.

Proof :

Let T_n denote a tree with n vertices.

T_n has a vertex with degree 1.

Removing this vertex results in a graph with $n-1$ vertices and $n-2$ edges. This a tree with $n-1$ edges.

$\therefore$ For every tree T_n $\exists$ a tree T_{n-1}. $\square$

* A weighted graph is a graph in which every vertex and every edge is assigned an integral weight.

* The weight of a subgraph is the sum of weights of vertices and edges contained in the sub-graph.

Conjecture: For all $k \geq 1$, in any weighted graph with $n \geq k$ vertices and edges $> kn - {}^{k+1}C_2$, there exists a cycle with weight divisible by k.

This has been proven for small values of k but there is no proof known for any value of k in general.

For $k=1$, number of edges $> n-1$. Clearly such a graph has a cycle and every possible weight is divisible by k as $k=1$.

The conjecture is true for $k=1$.

For $k=2$, $n \geq 2$ and number of edges $> 2n-3$.

Let $n \geq 3$ as there are no cycles for $n=2$.

Consider an arbitrary weighted graph with n vertices and atleast $2n-2$ edges. If there are more than $2n-2$ edges, then they can be removed until the number of edges $= 2n-2$.

Case 1: There is a vertex of degree atmost 2.

Deleting this vertex results in a graph with $n-1$ vertices and $2(n-1)-2$ edges.

By induction hypothesis this subgraph must have a cycle of even weight and we are done.

Case 2: Every vertex has degree atleast 3.

Claim: There exists atleast one vertex of degree 3.

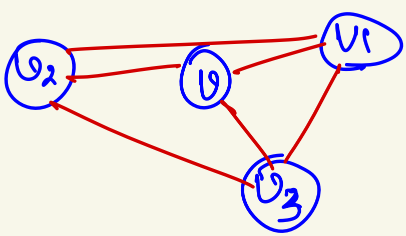
If no vertex had degree 3 then every vertex will have degree atleast 4.

$\Rightarrow$ Number of edges $\geq \frac{1}{2} \times 4n = 2n > 2n-2$

$\therefore \exists$ a vertex with degree 3.

Now just removing this vertex will result in a graph that does not satisfy the induction hypothesis.

Case 2.1: Every vertex adjacent to the vertex with degree 3 is adjacent to the other 2 vertices.



This a complete graph K_4

Claim: K_4 always has a cycle of even weight.

Proof:

In any combination of 3 vertices, atleast 2 of them must have the same parity. For example: In $\{v_1, v_2, v_3\}$, v_1, v_2 might both be even or odd

WLOG let v_1, v_2 both have the same parity.

Consider 2 possible 3-cycles passing through v_1, v_2:

$\{v, v_1, v_2\}$, $\{v_3, v_1, v_2\}$.

If either $w(v_1 v) + w(v) + w(v v_2) + w(v_1 v_2)$

or $w(v_1 v_3) + w(v_3) + w(v_3 v_2) + w(v_1 v_2)$ is even

then we get a 3-cycle of even weight and we are done.

If $w(v_1 v) + w(v) + w(v v_2) + w(v_1 v_2)$ and

$w(v_1 v_3) + w(v_3) + w(v_3 v_2) + w(v_1 v_2)$ are both odd

then $w(v_1 v) + w(v) + w(v v_2)$ and $w(v_1 v_3) + w(v_3)$

$+ w(v_3 v_2)$ must have the same parity.

$\Rightarrow w(v_1 v_3) + w(v_3) + w(v_3 v_2) + w(v_2 v) + w(v)$

$+ w(v v_1)$ is even

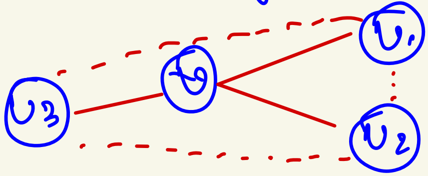
$\Rightarrow w(v_1 v_3) + w(v_3) + w(v_3 v_2) + w(v_2) + w(v_2 v)$

$+ w(v) + w(v v_1) + w(v_1)$ is even

$\Rightarrow$ A 4-cycle of even weight exists and we are done.

* In fact K_4 is the smallest such graph that has a cycle of even weight and the presence of K_4 in any graph ensures the presence of a cycle of even weight.

Case 2.2: There exists atleast a pair of vertices from the set of 3 vertices v_1, v_2, v_3 that are adjacent to v but are not adjacent to each other.



Atmost 2 of the dotted lines are edges.

Let v_1 and v_2 be the pair without an edge between them.

Let the weights of v_1, vv_1 and vv_2 be w_1, w_2 and w_3 respectively.

Delete v and add an edge between v_1 and v_2 with a weight of $w_1+w_2+w_3$.

This graph has $n-1$ vertices and $2(n-1)-2$ edges.

By induction hypothesis it contains an even cycle.

If the cycle does not include v_1v_2 then it is directly present in the original graph also.

If it includes v_1v_2, then in the original graph it passes through v_1, v and v_2 in that order (or reverse) $\square$

That concludes the proof.

6/11/2023

Connectivity:

A path from a vertex u to a vertex v in a graph G is a sequence of distinct vertices $u=v_0, v_1, v_2, \ldots, v_l = v$, such that $\{v_i, v_{i+1}\}$ is an edge $\forall i, 0 \le i < l$.

* Also, a sub-graph with vertex set $\{v_0, v_1, \ldots, v_l\}$ and edges $\{v_i, v_{i+1}\}$ $\forall i$, $0 \le i < l$ exists $\Leftrightarrow$ There is a path from v_0 to v_l.

* If there is a path then there is an induced path.

* A graph G is connected if for every pair of vertices u, v there exists a path from u to v.

* A connected component of a graph is a maximal sub-graph that is connected.
* If C is a connected component of G then ($\nexists H \ni C \subseteq H \subseteq G$ and (H is a connected component))
* $(\exists$ a path from u to $v) \wedge (\exists$ a path from v to $w)$

$$\Rightarrow (\exists \text{ a path from } u \text{ to } w)$$

* The existence of a path is an equivalence relation on the set of vertices.

* Proof of Transitivity:

Since $\exists$ a path from u to v $\exists$ vertices $\{v_1, v_2, \dots, v_n)$ and $\{v_i, v_{i+1}\}$ form an edge and $v_1 = u$, $\{v_n, v\} \in E(G)$. and $\exists$ vertices $(w_1, w_2, \dots, w_m)$ such that $\{w_i, w_{i+1}\}$ form an edge and $w_1 = v$ and $\{w_m, w\} \in E(G)$.

(1) If $\{v_1, v_2, \dots, v_n\} \cap \{w_1, w_2, \dots, w_m\} = \emptyset$

then $(v_1, v_2, \dots, v_n, w_1, w_2, \dots, w_m)$ are the vertices that form a path from u to w

(2) If there are common vertices, then we find the first vertex v_i that equals w_j for some j.

Now consider the vertices $(v_1, v_2, \dots, v_i, w_{j+1}, \dots, w_m)$ in order. They form a path from u to w. $\square$

* Connected components of a graph are vertex disjoint and two vertices are in the same component $\Rightarrow$ There exists a path between them.

Property:

* Every connected graph with n vertices contains atleast $n-1$ edges.

Proof:

For $n=1$, the statement is trivially true.

Induction Hypothesis: Any connected graph with k vertices contains atleast $k-1$ edges. when $k < n$.

Let G be any connected graph with n vertices and the minimum number of edges. Proving the theorem for these graphs $\Rightarrow$ other graphs contains atleast $n-1$ edges.

Deleting an edge gives a graph that is not connected.

(This is because we claimed that G has the minimum number of edges to stay connected. So removing any edge must result in the graph becoming disconnected)

After deletion there are atleast 2 connected components one containing u and another containing v when $\{v,u\}$ was the deleted edge.

Claim: There are exactly 2 connected components.

If there were more than 2 connected components then they must exist in the original graph also. This is a contradiction. Since the original graph was connected.

Let n_1 be the number of vertices in the connected component containing u and n_2 be the number of vertices in the component containing v. We know $n_1+n_2=n$

By induction, there are atleast n_1-1 and atleast n_2-1 edges in the connected components respectively

So the original graph contains atleast,

$$n_1-1+n_2-1+1 \text{ edges}$$

$\Rightarrow$ There are atleast $n-1$ edges $\square$

Q. Find the minimum number of n numbers in any array using comparison.

This cannot be done in less than $n-1$ comparisons for getting correct answers for any input array.

Consider an array $a_0, a_1, \ldots, a_{n-1}$.

Consider a set of vertices with labels $a_0, a_1, \ldots, a_{n-1}$.

If a comparison has been made between a_i, a_j then create an edge between a_i and a_j.

If an algorithm performed less than $n-1$ comparisons then the graph cannot be connected.

So there are atleast 2 connected components and for every possible input, no comparison can be made between two elements belonging to different components.

Consider an input array where the algorithm returns a_i from one of the connected components. Add a quantity 'c' to every element in this component such that the minimum is now not a_i. The algorithm will still return a_i and hence it is incorrect.

* A similar proof can be done for comparison based sorting algorithms where the time complexity cannot be better than $\Theta(n \log n)$.

* A walk is a sequence of vertices $(v_1, v_2, \ldots, v_n)$ such that $\{v_i, v_{i+1}\}$ is an edge $\forall i, 0 \leq i < n$.
* A walk is closed if the first and last vertex v_1 and v_n are the same.
* A walk with no repeated edges is called a trail.
* A trail is Eulerian if every edge of the graph appears exactly once in it.
* A graph is called Eulerian if it has a closed Eulerian trail.

* A closed Eulerian trail is also called an Eulerian Circle or Eulerian circuit.

Property:

Every connected graph with n vertices has atleast n-1 edges.

Proof. (Without using connected components)

Base Case: $n=1$, the statement is trivially true.

Induction Hypothesis: Every connected graph with $n-1$ vertices has atleast $n-2$ edges for some $n>2$.

Consider a connected graph with n vertices.

Claim: There exists no vertex of degree 0.

Proof: If it did, then this vertex will not be connected to any other vertex and this is a contradiction as the graph is connected. $\square$

Case I: There exist no vertex of degree 1.

If this were true then,

$$\sum_{v \in V(G)} \text{degree}(v) \geq 2 \times n$$

$$\Rightarrow 2|E| \geq 2n$$

$$\Rightarrow |E| \geq n$$

And we are done.

Case II: There exist a vertex of degree 1.

Let this vertex be v.

Now v has to be the last vertex (or first) of a maximal path P in G. Because if v were not at the ends of some maximal path, it would have degree 2 $\longrightarrow$ (1)

We remove this vertex v from G. This results in deletion of one edge from G.

The resulting graph G' has $n-1$ vertices and is connected because, there are no vertices in G that are connected by a path containing v (by (1)).

By induction hypothesis, G' has atleast $n-2$ edges.

$\Rightarrow$ G has atleast $n-1$ edges $\square$

*The theorem given below is basically the generalised version of the Königsberg bridge problem and is one of the first results in graph theory

Theorem:

A connected graph has an Eulerian cycle if and only if every vertex has even degree.

Proof:

7/11/2023

RECURRENCE RELATIONS

Q1 Let $T(n)$ denote the number of bit strings of length n in which every occurrence of 1 is preceded by an even number of occurrences of 0. Derive a recurrence relation for $T(n)$ and show that this number is equal to the number of bit strings of length n that do not contain 11 as a substring. Prove this also by showing an explicit bijection between the two sets.

A string can either start with 1 or 0.

$1 \; \underbrace{\_ \; \_ \; \_ \; \_ \; \_ \; \_ \; \_}_{T(n-1) \text{ possible strings}}$

$0 \; \_ \; \overbrace{\_ \; \_ \; \_ \; \_ \; \_ \; \_ \; \_}^{T(n-2) \text{ possible strings}}$

This digit has to 0. Because if it were 1, the property would be violated

$\therefore \; T(n) = T(n-1) + T(n-2)$

$T(0) = 1, \; T(1) = 2$

This recurrence is the same as that for n-bit strings not containing '11'.

If string starts with 1, then there are $F(n-2)$ strings and if it starts with 0, then there are $F(n-1)$ strings (For the '11' strings)

Bijection : f

Invert 1 to 0

$$10\ldots \longrightarrow 00\ldots$$

$$0\ldots \longrightarrow 1\ldots$$

Invert 0 to 1.

If the string starts with 1, then change 1 to 0 and retain the next 0. Now recursively apply f on the $n-2$ sized string

If the string starts with 0, then invert it to 1 and apply f recursively to the $n-1$ sized string.

Another Method:

Now $T(n) = T_E(n) + T_O(n)$ where $T_E(n)$ is the number of bit strings with even number of 0s and $T_O(n)$ is the number of bit strings with odd number of 0s.

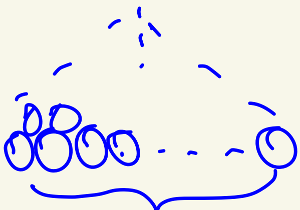
{ I am not very sure if this is how Sir partitioned the cases.

He gave a solution with the matrix method. That matrix must satisfy the characteristic equation

$$A^2 = A + I$$

If anyone can share the solution I will add it}

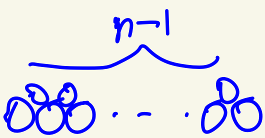
Q2 Consider an arrangement of balls in layers such that the balls in each layer are placed consecutively touching each other, and except for the bottom layer, each ball is placed between two balls in the layer below. If there are n balls in the bottom layer, how many distinct arrangements are possible? For $n = 1$, there is only one, for $n = 2$, there are two, one in which there is only one layer and the other with a single ball in layer 2. For $n = 3$, there are five, with distributions (3), (3, 1), (3, 1), (3, 2) or (3, 2, 1) of balls in layers. Prove that the number of possible arrangements is the Fibonacci number F_{2n-2}. Prove it using a recurrence and also by showing a bijection with the set of bit strings of length $2n - 3$ not containing 11, for $n > 1$.



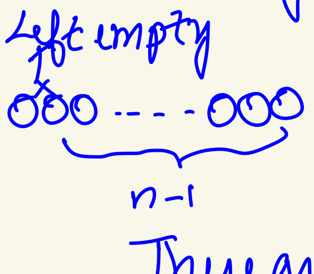
n balls are present at the bottom.

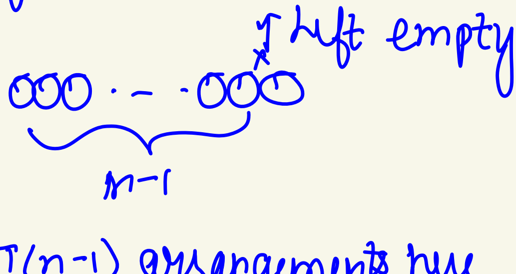
Claim: Number of such arrangements = Number of bitstrings of length 2n-3 with no occurrences of '11'.

A possibility is, every gap is filled in the first level from the bottom.



Now the remaining arrangements are T(n-1) in number





There are 2T(n-1) arrangements here

But this kind of split counts all arrangements with both corners empty twice

There are $T(n-2)$ such arrangements

$$\Rightarrow T(n) = T(n-1) + 2T(n-1) - T(n-2)$$

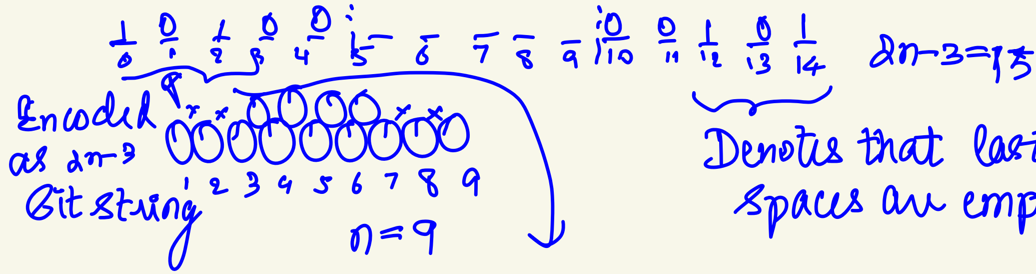
$$\Rightarrow T(n) = 3T(n-1) - T(n-2), T(1) = 1, T(2) = 2$$
$$\forall n \geq 3$$

$$F(n) = F(n-1) + F(n-2)$$
$$= 2F(n-2) + F(n-3)$$
$$= 2F(n-2) + F(n-2) - F(n-4)$$
$$F(n) = 3F(n-2) - F(n-4)$$
$$\underbrace{F(2n-2) = 3F(2n-4) - F(2n-6)}_{\text{Similar to the recurrence for } T(n)}$$

$$F(0) = 1, F(2) = 2,$$
$$F(2n-2) = 3F(2n-4) - F(2n-6), \forall n \geq 3$$

$$\therefore \boxed{T(n) = F(2n-2) \; \forall n \geq 1, F(0) = 1, F(2) = 2}$$

Now we derive a bijection between such arrangements and bit strings of length $2n-3$ not containing '11'.



Denotes that last two spaces are empty

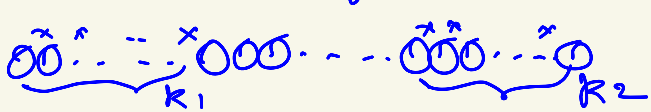
Denotes that first 2 spaces are empty.

Once the empty places are denoted by the number of '10' or '01', we place a 0 at either end to signify that between these 0's the balls are filled.
Now in between the 0's at 4 and 9, we fill the string recursively for $n=4$ (there are 5 places in the string $(2\times4-3)$)

These strings can never have any occurrence of '11'.

$\therefore$ There is a one-one function from the arrangement of balls to the bit strings.

Now we need to prove that this is a bijection ie show that given a string we know how the string is filled.
Given the string, we find the number of occurrences of '10' and '01' and let the numbers be k_1 and k_2.
If the string is of length m, we find $n = \frac{m+3}{2}$.
Now we construct an arrangement:

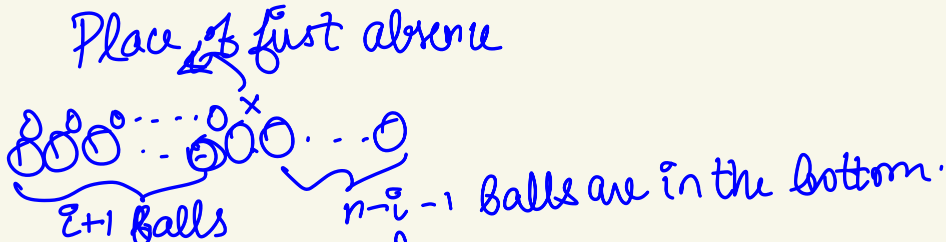


After this we recursively decode the string between the two bounding 0's and do the same for

$m' = m - 2(k_1+k_2+1)$ and $n' = n-1-k_1-k_2$ with
n', m' satisfying, $m' = 2n'-3$

$\therefore$ For every string there exists a valid arrangement $\square$

By removing the condition for balls of any layer to be in contact with neighbours, the problem becomes similar to finding the number of binary trees. So the answer would be a Catalan number.



i balls have been placed on top of the first layer with each ball in contact. So there are i+1 balls in the bottom row.

Number of such arrangements $= T(i) \cdot T(n-1-i)$

This is for the second last layer.

$$\therefore \boxed{T(n) = \sum_{i=0}^{n-1} T(i)\,T(n-1-i), \quad T(0)=1, \; T(1)=1}$$

9/11/2023

Q4. Suppose a fair coin is tossed n times. Given that two consecutive heads did not occur in the n trials, what is the expected number of heads that have occurred? Let S be the set of sequences in which each entry is either +1 or -1 and the sum of entries in any prefix of the sequence is at least -2 and at most 2. How many such sequences of length n are possible? If each entry in the sequence is generated randomly with equal probability, and the sequence terminates as soon as the prefix condition is violated, what is the expected length of the sequence?

The number of valid strings is F_n where $F_n = F_{n-1} + F_{n-2}$ and $F_1 = 2, F_2 = 3$

S_n = Sum of number of 1's in every string of the sample space.

$$\text{Expectation} = \frac{S_n}{F_n}$$

(We consider every string to be equally probable)

We know $F_n = F_{n-1} + F_{n-2} \rightarrow$ (_ _ _ 0 1)

(_ _ _ .... _ 0)

$$\Rightarrow S_n = S_{n-1} + S_{n-2} + F_{n-2}$$

(The extra 1)

$S_0 = 0,\ S_1 = 1$

The generating function for F_n is $F(x) = \dfrac{1+x}{1-x-x^2}$

$$S_n = S_{n-1} + S_{n-2} + F_{n-2}$$

$$\Rightarrow S(x) - x = xS(x) + x^2 S(x) + x^2 F(x)$$

$$\Rightarrow S(x)(1-x-x^2) = x + \frac{x^2(1+x)}{1-x-x^2}$$

$$\Rightarrow S(x) = \frac{x}{1-x-x^2} + \frac{x^2(1+x)}{(1-x-x^2)^2}$$

$$= \frac{x}{1-x-x^2}\,\frac{(1-x-x^2+x+x^2)}{(1-x-x^2)}$$

$$\Rightarrow S(x) = \frac{x}{(1-x-x^2)^2}$$

$$\approx \frac{C_1}{\left(1-\frac{x}{\left(\frac{\sqrt{5}+1}{2}\right)}\right)^2}$$ (This is the dominant term for large n)

$\therefore S_n \approx C_1 \cdot \left(\frac{\sqrt{5}+1}{2}\right)^n \cdot n$ for large n.

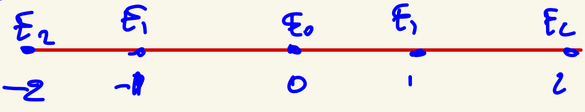
$F_n \approx \left(\frac{\sqrt{5}+1}{2}\right)^n$ for large n

C_1 comes out to be $\frac{5-\sqrt{5}}{10}$

$\therefore E_n = \frac{S_n}{F_n} \approx \left(\frac{5-\sqrt{5}}{10}\right) n < n/2$

Consider a particle constrained to move on the x axis. It starts at the origin and it is suppose to move only by one unit at a time and it's displacement from origin can never be more than +2 or less than −2. It ceases to exist if the displacement violates the given condition.

E_i = Expected number of steps when starting from $|x| = i$ before ceasing to exist.



$E_2 = \frac{1}{2}(1) + \frac{1}{2}(1+E_1) = 1 + \frac{1}{2}E_1$

Goes to ±1 with probability $\frac{1}{2}$

Already at $|x| = 2$ and goes outside range with probability $\frac{1}{2}$.

$E_1 = \frac{1}{2}(1+E_2) + \frac{1}{2}(1+E_0)$

$E_0 = 1 + E_1$

Solving the three equations gives, $E_0 = 9$, $E_1 = 8$, $E_2 = 5$

This is true for $n = 2$

Now for $n=1$, $\left.\begin{array}{l} E_0 = 1+E_1 \\ E_1 = \frac{1}{2}(1+E_0) \end{array}\right\}$ $E_0 = 3$, $E_1 = 4$

For a general n, finding E_0 is the question when the bounds are $\pm n$

9/11/2023

GRAPHS

Q1 Prove that every graph with n vertices and more than $n^2/4$ edges contains a triangle. Show that for all even $n \geq 2$, there exists a graph with n vertices and $n^2/4$ edges that does not contain a triangle. More generally, show that every graph with n vertices and more than $\frac{(r-2)n^2}{2(r-1)}$ edges contains a complete subgraph with r vertices, for all $r \geq 3$. Also show that if n is a multiple of $r-1$, there exists a graph with n vertices and $\frac{(r-2)n^2}{2(r-1)}$ edges that does not contain a complete subgraph with r vertices. This is known as Turan's theorem.

First we need to show that any graph with n vertices and more than $n^2/4$ edges contains a triangle.

The largest graph not containing a triangle has $n^2/4$ edges (when n is even) and is a complete Bipartite graph $K_{n/2, n/2}$.

If n is odd, then the largest such graph is a complete Bipartite graph $K_{\frac{n-1}{2}, \frac{n+1}{2}}$.

Let $n > 2$.

Since the graph has $> n^2/4$ edge, it must contain atleast one edge.

Let the edge be uv.

If any vertex w is adjacent to both u, v there is a triangle.

Then we consider the remaining vertices to be connected to atmost one of them.

Deleting u and v will result in the number of edges being $> \frac{n^2}{4} - (1+n-2)$

$$= \frac{n^2-4n+4}{4}$$

$$= \frac{(n-2)^2}{4}$$

By induction hypothesis this graph has a triangle.

$\therefore$ The original graph has a triangle. $\square$

For general case if the graph has more than $\frac{r-2}{2(r-1)} n^2$ edges and is not a complete graph (ie not K_n) then, $r \leq n$.

So we only need to care about cases where $r \leq n$

$P(r)$: $\forall n \geq r \geq 3$, a graph with n vertices and more than $\frac{r-2}{2(r-1)} n^2$ edges contains a sub-graph isomorphic to K_r.

We have proven that $P(3)$ is true in the previous part of the question. $\longrightarrow (1)$

Inductive hypothesis: For some r, $P(m)$ is true $\forall m < r$ $\rightarrow (2)$

Consider a graph with n vertices and more than $\frac{r-2}{2(r-1)} n^2$ edges.

$$\frac{r-2}{2(r-1)} > \frac{r-3}{2(r-2)}$$

$$\Rightarrow \frac{r-2}{2(r-1)} n^2 > \frac{r-1-2}{2(r-1-1)} n^2$$

$$\Rightarrow \text{Number of edges} > \frac{(r-1-2)}{2(r-1-1)} n^2$$

$\therefore$ This graph contains K_{r-1}. (By induction hypothesis)

Now let the vertices $v_1, v_2, \ldots, v_{r-1}$ be the vertex set of K_{r-1}.

Consider the remaining $n-(r-1)$ vertices. Let their set be V'.

Case 1: There exists a vertex $v \in V'$ such vv_i is an edge $\forall i \in \{1, 2, \ldots, r-1\}$.

$\Rightarrow$ $v, v_1, v_2, \ldots, v_{r-1}$ form a complete graph K_r and we are done.

Case 2: No vertex in V' is adjacent to all vertices in $\{v_1, v_2, \dots, v_{r-1}\}$ $\Rightarrow$ Every vertex in V' can be adjacent to atmost $r-2$ vertices in $\{v_1, v_2, \dots, v_{r-1}\}$.

We delete the set of vertices $\{v_1, v_2, \dots, v_{r-1}\}$

This results in a loss of $\frac{(r-1)(r-2)}{2}$ edges among themselves and atmost $(n-(r-1))(r-2)$ edges among this set and vertices from V'.

$\Rightarrow$ The resulting graph has $n-(r-1)$ vertices and more than

$$\frac{n^2(r-2)}{2(r-1)} - \frac{(r-1)(r-2)}{2} - (n-(r-1))(r-2)$$

$$= \frac{r-2}{2(r-1)}\left[n^2 - (r-1)^2 - 2(r-1)(n-(r-1))\right]$$

$$= \frac{r-2}{2(r-1)}\left[n^2 - (r-1)^2 - 2n(r-1) + 2(r-1)^2\right]$$

$$= \frac{r-2}{2(r-1)}\left[n^2 - 2n(r-1) + (r-1)^2\right]$$

$$= \frac{r-2}{2(r-1)}(n-(r-1))^2$$

We are now left with a graph containing $n-(r-1)$ vertices and more than $\frac{r-2}{2(r-1)}(n-(r-1))^2$ edges.

Now we induct on n for a fixed r.

$Q(n)$: $\forall n$, for some $r \geq 3$, a graph having n vertices and more than $\frac{r-2}{2(r-1)} n^2$ edges, there exists a sub-graph isomorphic to K_r.

Let $Q(l)$ be true $\forall$ $l < n$ for some n. $\longrightarrow$ (3)

Since $n-(r-1) < n$, $Q(n-(r-1))$ is true

$\Rightarrow$ A graph with $n-(r-1)$ vertices and more than $\frac{(n-(r-1))^2 (r-2)}{2(r-1)}$ edges contains a sub-graph isomorphic to K_r.

$\Rightarrow$ $Q(n)$ is true since this graph was obtained by removing some vertices and edges. $\longrightarrow$ (4)

(3) $\Rightarrow$ (4)

$\therefore$ By induction $Q(n)$ is true $\forall n$.

$P(r)$ is true $\longrightarrow$ (5)

(1) $\wedge$ (2) $\Rightarrow$ (5)

$\therefore$ By induction $P(r)$ is true $\forall r$. $\square$

Now we prove Turan's theorem:

If n is a multiple of $r-1$, then there exists a graph with n vertices and $\frac{r-2}{2(r-1)}n^2$ edges that does not contain a subgraph isomorphic to K_r.

For $r=3$, this can be proven quite easily.

Let n be divisible by 2. The graph has $n^2/4$ edges.

We partition the vertex set into 2 equal sized sets X and Y such that

$xy \in E(G) \iff (x \in X \wedge y \in Y) \vee (x \in Y \wedge y \in X)$

ie two vertices are adjacent if and only if they belong to different sets.

Such a graph is called a bipartite graph and the sets X and Y are called the partite sets of G.

Now say G contains a triangle. Let v_1, v_2, v_3 be the vertices forming the triangle.

By pigeon hole principle, atleast 2 of these vertices must belong to the same partite set.

$\Rightarrow$ There cannot be an edge between atleast 2 of them

$\Rightarrow$ K_3 cannot exist.

A contradiction.

$\therefore$ G cannot contain a triangle.

This concludes the proof for $r=3$.

Now we prove the theorem for a general r.

We have graph with n vertices and $\frac{(r-2)n^2}{2(r-1)}$ edges such that $r-1$ divides n.

$r-1 \mid n \Rightarrow$ The vertex set can be partitioned into $r-1$ sets given by $V_1, V_2, \ldots, V_{r-1}$ each of size $\frac{n}{r-1}$ and the property that $uv \in E(G) \Longleftrightarrow u \in V_i \wedge v \in V_j, i \neq j$

$\Rightarrow$ Every vertex in V_i is connected to every other vertex in V_j $\forall j \neq i$.

$\Rightarrow$ degree of every vertex $v = \frac{(r-2)n}{r-1}$

$\Rightarrow$ Number of edges $= \frac{1}{2} \sum_{i=1}^{n} \frac{n(r-2)}{r-1}$

$$= \frac{n^2(r-2)}{2(r-1)}$$

Thus we have constructed a graph satisfying the sufficient conditions.

Now say G has a subgraph isomorphic to K_r.

Let $v_1, v_2, \ldots, v_r$ be the vertices that are the part of K_r.

By Pigeon Hole Principle, atleast two of them must belong to V_i for some i.

But then they cannot have an edge between them.

A contradiction.

$\therefore$ There is no subgraph isomorphic to K_r $\square$