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Lattices

Let $(P, \leq)$ be a poset. An element g is the greatest lower bound (glb) of elements $a, b \in P$ if $g \leq a, g \leq b$ and for all d such that $d \leq a, d \leq b$, $d \leq g$ holds.

(Similar to GCD of 2 numbers when $\leq$ is defined to be the divides relation)

This denoted by $glb(a,b)$ or $a \wedge b$

An element l is the least upper bound (lub) of a, b if $a \leq l$, $b \leq l$ and for any m such that $a \leq m, b \leq m$ we have $l \leq m$

$lub(a,b)$ is also written as $a \vee b$

(This can be thought of as LCM of 2 numbers)

* A poset is called a lattice if for any 2 elements a, b $\exists$ a glb and lub

* Set of all subsets of a set is a poset with $\leq$ being defined as $\subseteq$. → This structure is called a Boolean lattice

$glb(A,B) = A \cap B$ and $lub(A,B) = A \cup B$

* Divisor lattice.

Set of all natural numbers with the divides relation.

glb = GCD, lub = LCM

* Subsequence of an array / substrings of a string

* Subarrays of an array / Sub-arrays query

Subarrays of an array are equivalent to intervals with endpoints in $[1, n]$

This also includes the empty subarray.

glb = Intersection of two sub-arrays

The relation is not "subset of" as we need contiguous arrays.

So lub is not union of two subarrays. This happens only when the two subarrays overlap.

However there is still a lub.

$\{1, 2, \ldots, n\}$, $[i_1, j_1]$, $[i_2, j_2]$

lub : $[i_1, j_2]$

This lattice is however not a Boolean structure.

* Set of partitions / Set of equivalence relations on a set

$E_1 \leq E_2 \iff E_1 \subseteq E_2$

(or) $a E_1 b \implies a E_2 b$

$glb(E_1, E_2) = E_1 \cap E_2$

(An equivalence relation contained in both E_1 and E_2)

$lub(E_1, E_2)$ = Transitive closure of $E_1 \cup E_2$

(or) $(E_1 \cup E_2)^*$

Lattice : A poset in which glb and lub exist for every pair of elements.

pair of elements.

Algebraic View:

Think of $\vee$ and $\wedge$ as operations on a set.

Properties of these operations are defined so that they correspond to glb and lub in some poset or equivalently we can define the properties of glb and lub to get a poset

(1) $\wedge$ and $\vee$ are commutative and associative.

(2) Idempotent : $a \wedge a = a$, $a \vee a = a$ $\forall a$

(3) Absorption property : $a \wedge (a \vee b) = a$
$a \vee (a \wedge b) = a$ $\forall a, b$

If there are 2 operations $\vee, \wedge$ satisfying these properties on a set, then they exactly define the existence of a glb and lub for a poset made from this set That is we get a lattice from this set.

Given a poset with glb and lub existing for every pair of elements, operations $\wedge$ and $\vee$ can be determined which satisfying these properties.

Converse: Given operations satisfying these properties on a set, a partial order can be defined on the set such $\wedge$ corresponds to glb and $\vee$ corresponds to lub.

(The operations $\vee$ and $\wedge$ are symmetric)

Define a relation $\leq$ on the set by

$a \leq b \iff a \wedge b = a$, (equivalently)

$a \leq b \iff a \vee b = b$

We claim that such a relation $\leq$ is a partial order on the set with $\vee$ giving the lub and $\wedge$ giving the glb. Since this is true for any a, b, we get a lattice.

We first show that this a partial order.

Is $a \leq a$?

By definition $a \wedge a = a$ so $a \leq a$ ✓ (Reflexivity)
(Idempotent property)

Is $a \leq b$ and $b \leq a \Rightarrow a = b$?

$$\left.\begin{array}{l} a \leq b \Rightarrow a \wedge b = a \\ b \leq a \Rightarrow b \wedge a = b \end{array}\right\} \Rightarrow a = b \checkmark \text{ (Anti-symmetric)}$$

(By commutativity of $\wedge$)

Is $a \leq b$ and $b \leq c \Rightarrow a \leq c$?

$a \leq b \Rightarrow a \wedge b = a$

$b \leq c \Rightarrow b \wedge c = b$

$$\begin{aligned} a \wedge c &= (a \wedge b) \wedge c = a \wedge (b \wedge c) \\ &= a \wedge b \quad \text{(By associativity of } \wedge\text{)} \\ &= a \end{aligned}$$

$\therefore a \wedge c = a \Rightarrow a \leq c$ ✓ (Transitivity)

$\therefore \leq$ is a partial order on the set

This poset is a lattice with $\wedge$ giving the glb and $\vee$ giving the lub.

First show that $a \wedge b$ satisfies the properties of glb.

$a \wedge b \leq b$

Because $(a \wedge b) \wedge b = a \wedge b$ (By idempotent property)

Similarly $a \wedge b \leq a$

$\Rightarrow a \wedge b \leq$ both a, b

Let $d \leq a$ and $d \leq b$. If we show $d \leq a \wedge b$ then we are done

$d \leq a \Rightarrow d \wedge a = d$

$d \leq b \Rightarrow d \wedge b = d$

$$d \wedge (a \wedge b) = (d \wedge a) \wedge b$$
$$= d \wedge b$$
$$= d$$

$\therefore d \wedge (a \wedge b) = d$

$\Rightarrow d \leq a \wedge b$

and we are done.

$a \wedge b$ gives the glb of 2 elements in this poset.

So far the relation has only been defined in terms of $\wedge$.

Now without the absorption property $\vee$ may or may not give the lub of two elements.

$a \wedge b = a \Leftrightarrow a \vee b = b$

This uses the absorption property and essentially makes sure lub is given by $\vee$ when glb is given by $\wedge$.

Let $a \wedge b = a$.

$$a \vee b = (a \wedge b) \vee b = b \text{ (Absorption property)}$$

Similarly for reverse implication.

Both the statements given under absorption are not required and can be done with one of them only.