Graph Theory

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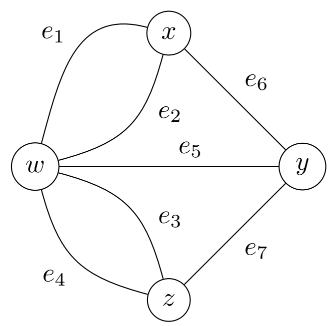
1 Fundamental Concepts

1.1 Basic Definitions, Propositions and Theorems

Definition 1.1.1. *A graph is a triple consisting of a vertex set **V(G)**, an edge set **E(G)** and a **relation** that associates with each edge two vertices (not necessarily distinct) called its endpoints.*

Here is an example,

Ex 1.1.1. $E(G) = \{e_1, e_2, \ldots, e_7\}$ *and* $V(G) = \{x, y, z, w\}$



Graph 1.1.1: The Konisberg Bridge Problem - The Birth Of Graph Theory

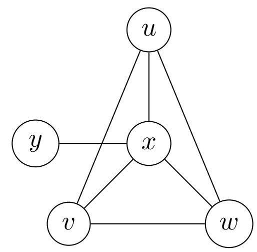
The question was whether the citizens of Konisberg could cross at one location denoted by x, y, z, and w and go to every other location at least once while using each bridge denoted by $e_1, e_2, e_3, e_4, e_5, e_6$, and e_7 exactly once and be back.
The answer to this problem is no. As can be seen from Graph 1.1.1, clearly to enter and exit each location by using every bridge exactly once, there should be an even number of bridges coming out of the location. However, from location y, there are three edges/bridges. Therefore, this situation is not possible.

Definition 1.1.2. *A **loop** is an edge whose endpoints are equal. **Multiple edges** are edges having same pair of endpoints.*

Definition 1.1.3. *A **Simple graph** is a graph having no loops or multiple edges. See defintion 1.1.2.*

A simple graph is specified by a vertex set and treating the edge set as a set of unordered pairs of vertices $e = uv$ (or vu) for an edge 'e' with vertices u and v.
When u and v are the endpoints of an edge, they are **adjacent** and are **neighbours**.
Notation: $u \Leftrightarrow v$

Ex 1.1.2. $V(G)$=$\{u, v, w, x, y\}$ *and* $E(G)$=$\{uv, uw, ux, vw, xw, xy\}$. *The graph can be drawn like this :*



Definition 1.1.4. *A graph is* ***finite*** *if the edge set and the vertex set are finite.*

Definition 1.1.5. *The* ***null graph*** *is the graph whose vertex set and edge set are empty.*

Definition 1.1.6. *The* ***complement*** $\overline{G}$ *of a simple graph* G *is the simple graph with vertex set* $V(G)$ *defined by* $uv \in E(\overline{G})$ *if and only if* $uv \notin E(\overline{G})$.

Definition 1.1.7. *A* ***clique*** *in a graph is a set of pairwise adjacent vertices. An* ***independent set*** *(or* ***stable set****) in a graph is a set of pairwise non adjacent vertices.*

Definition 1.1.8. *A graph* G *is called* ***bipartite*** *if* $V(G)$ *is the union of two disjoint (possibly empty) independent sets called* ***partite sets*** *of* G.

Definition 1.1.9. *The* ***chromatic number*** *of a graph* G*, written* $\chi(G)$*, is the* ***minimum*** *number of colours needed to label the vertices so that adjacent vertices receive different colours. A graph* G *is called* k***-partite*** *if* $V(G)$ *can be expressed as the union of* k *(possibly empty) independent sets.*

Defintion 1.1.9 is the generalisation of bipartite graphs. Clearly $\chi(G)$ is the minimum number of independent sets needed to partition $V(G)$. Therefore a graph is k-partite if and only if its chromatic number is atmost k. The term **partite set** is used when referring to a set in a partition into independent sets.

Definition 1.1.10. *A* ***path*** *is a simple graph whose vertices can be ordered in so that two vertices are adjacent if and only if they are consecutive in the list. A* ***cycle*** *is a graph with an equal number of vertices and edges whose vertices can be placed around a circle so that tw vertices are adjacent if and only if they appear consecutively along the circle.*

Definition 1.1.11. *A* ***subgraph*** *of a graph* G *is a graph* H *such that* $V(H) \subseteq V(G)$ *and* $E(H) \subseteq E(G)$ *and the assignment of endpoints to edges in* H *is the same as in* G*. Then* $H \subseteq G$ *and "*G **contains** H*". A graph* G *is said to be* ***connected*** *if each pair of vertices in* G *belongs to a path; otherwise,* G *is* ***disconnected****.*

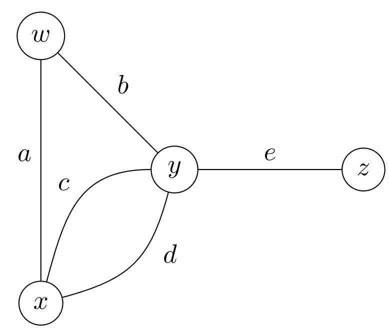
Definition 1.1.12. *If vertex* v *is an endpoint of edge* e*, then* v *and* e *are* ***incident****. The degree of a vertex* v *(in a loopless graph) is the number of indcident edges.*

Definition 1.1.13. *Let G be a loopless graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G) = \{e_1, e_2, \ldots, e_n\}$. The adjacency matrix $A(G)$ of G is the n-by-n matrix in which entry a_{ij} is the number of edges in G with endpoints $\{v_i, v_j\}$. The incidence matrix $M(G)$ is the n-by-m matrix in which entry m_{ij} is 1 if v_i is an endpoint of e_j and otherwise is 0.*

An adjacency matrix is determined by a vertex ordering. Every adjacency matrix is symmetric and that of a simple graph is symmetric with all entries 1 or 0 with diagonal entries as 0. If the simple graph is also connected then the adjacency matrix is defined by

$$a_{ij} = \begin{cases} 1, & \textit{if} \quad i \neq j \\ 0, & \textit{if} \quad i = j \end{cases}$$

Ex 1.1.3. *Consider the following graph :*



The adjacency and incidence matrices are given by :

$$A(G) = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 2 & 0 \\ 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$M(G) = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

where the rows and columns of $A(G)$ are (w, x, y, z) corresponding to indices $(1, 2, 3, 4)$ and the rows and columns of $M(G)$ are (w, x, y, z) and (a, b, c, d, e) respectively.

Describing a graph using adjacency and and incidence matrices require explicit ordering of the vertices and edges. However several graphs which are just renamed versions of each other form a set of structurally similar graphs. These can be studied by defining a bijection from the vertex set of one graph to that of another *i.e.* a **bijection** can be formalised.

Definition 1.1.14. *An* ***isomorphism*** *from a simple graph G to a simple graph H is a bijection $f : V(G) \rightarrow V(H)$ such that $uv \in E(G)$* ***if and only if*** *$f(u)f(v) \in E(H)$. G is said to be* ***isomorphic*** *to H and this is written as $G \cong H$*

Let there be two simple graphs G and H such that the adjacency matrices $A(G)$ and $A(H)$ are permutations of each other then they are isomorphic to each other.

Proposition 1.1.1. *An isomorphism in an* ***equivalence relation*** *on the set of simple graphs.*

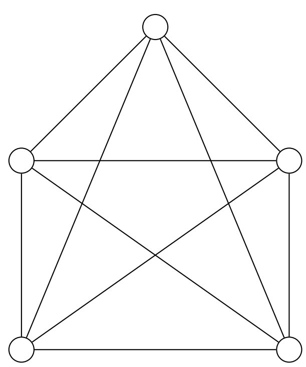
An equivalence relation partitions a set into **equivalence classes**. Two elements are related if and only if they both lie in the same equivalence class.

Definition 1.1.15. *An* ***isomorphism class*** *of graphs is an equivalence class of graphs under the isomorphism relation.*

Paths with n vertices are pairwise isomorphic; the set of all n-vertex paths form an isomorphism class.

Definition 1.1.16 (Cycles,Paths and Complete Graphs)**.** *n-vertex paths are denoted by P_n and n-vertex cyles are denoted by C_n. A* ***complete graph*** *is a simple graph whose vertices are* ***pair-wise adjacent*** *and a n-vertex complete graph is denoted by K_n. A* ***complete bipartite graph*** *or* ***biclique*** *is a simple bipartite graph such that two vertices are adjacent if and only if they are in different partite sets. When the indepenedent sets have sizes r and s then such a graph is denoted by $K_{r,s}$.*

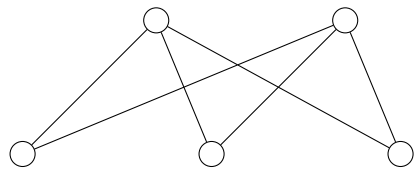
Ex 1.1.4. *This graph is an example of K_5 :*



Graph 1.1.2: A complete graph : K_5

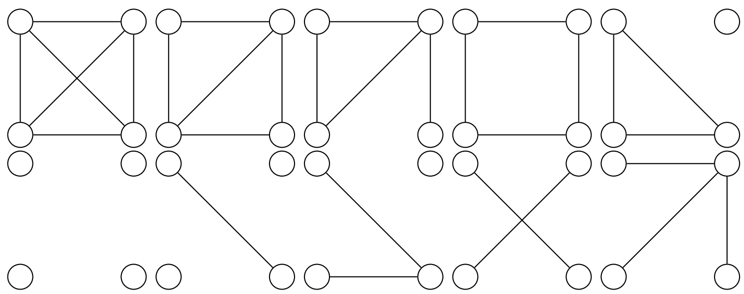
The follwing graph is an example of $K_{2,3}$ or $K_{3,2}$:

For n fixed (*i.e.* labelled) vertices there are a total of $\frac{n(n-1)}{2}$ *i.e.* $\binom{n}{2}$ vertex pairs and each pair can either have an edge connecting to them or not. Therefore there are $2^{\binom{n}{2}}$ graphs possible. Of these some would belong to the same

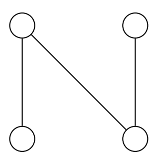


Graph 1.1.3: A complete bipartite graph : $K_{2,3}$

isomorphism class and thus can all be studied using a class representative. For example for $n = 4$ there are $2^6 = 64$ distinct graphs (with labelled vertices) but there are only 9 isomorphic classes covering all 64 graphs. They are :



and then finally there is P_4



The graphs directly below the first row are the complementary graphs. This can be easily checked. P_4 is isomorphic to it's complement graph *i.e.* $P_4 \cong \widetilde{P_4}$.

Definition 1.1.17. *A graph is* ***self-complementary*** *if it is isomorphic to its complement. A* ***decomposition*** *of a graph is a list of sub-graphs such that each edge appears in exactly one sub-graph in the list.*

Clearly P_4 is self-complementary. Also any n-vertex graph and it's complement decompose K_n. K_{n-1} and $K_{1,n-1}$ also decompose K_n.

Definition 1.1.18 (Petersen Graph)**.** *The Petersen graph is the simple graph whose vertices are the* 2*-element subsets of a* 5*-element set and whose adjacent vertices are the pairs of disjoint* 2*-element subsets.*

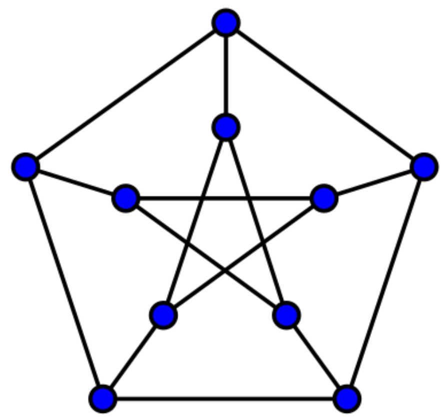


Figure 1: Petersen Graph

The degree of every vertex in the Petersen graph is 3 because for every 2-element subset of a 5-element set there are three ways to pick a disjoint 2-element subset from the remaining 3 elements. Let the 5-element set be $\{a, b, c, d, e\}$. The vertex set is given by $V(G) = \{ab, cd, ea, bc, de, ce, eb, bd, da, ac\}$. These 10 vertices form 2 5-cycles given. They are given by $\{ab, cd, ea, bc, de\}$ in that order and $\{ce, eb, bd, da, ac\}$ in that order.

Proposition 1.1.2. *A connected bipartite graph has a unique partition.*

Proposition 1.1.3. *All cycles of the form C_{2n} are bipartite. If the vertices are denoted by $\{v_1, v_2, \ldots, v_{2n}\}$ and are cyclic in that order, then the partite sets are given by $\{v_1, v_3, \ldots, v_{2n-1}\}$ and $\{v_2, v_4, \ldots, v_{2n}\}$*

Proposition 1.1.4. *If two vertices are non-adjacent in the Petersen graph then they have exactly one common neighbour.*

Definition 1.1.19. ***Girth*** *of a graph with a cycle is the length of its shortest cycle. A graph with no cycle has infinite girth.*

Corollary 1.1.1. *The Petersen graph has girth 5.*

Proof. As the graph is simple there are no multiple edges nor are there any loops.
Therefore girth ≥ 3.
There are no triangles in the graph. This is because for a triangle to be exist, there must be three disjoint 2-element subsets of a 5-element set which is clearly not possible.
Therefore girth ≥ 4.
Since a two non-adjacent vertices cannot have more than one common neighbour (from proposition 1.1.4) there are no 4-cycles as that would require two common neighbours for each pair of non-adjacent vertices.

Therefore girth ≥ 5.
Clearly the set of vertices labelled $\{ab, cd, ea, bc, de\}$ form a 5-cycle.
Therefore girth of the Petersen graph is 5. $\square$

Definition 1.1.20. *An **automorphism** is an **isomorphism** from a graph G to itself. A graph is **vertex-transitive** if $\forall u, v \in V(G)$ $\exists$ an automorphism that maps u to v.*

Some points :

1. Automorphisms of G are essentially permutations of $V(G)$ that can be applied to both the rows and columns of $A(G)$ without changing $A(G)$. Let σ be a permutation matrix. If $\sigma \cdot A(G) = A(G)$ then σ leads to an autmorphism.

2. A biclique $K_{r,s}$ is such that any permutation applied on one partite set leads to no change in the adjacency matrix and produces the same biclique. Thus there are a total of $r!s!$ automorphisms when $r \neq s$ and $2(r!)^2$ automorphisms when $r = s$ as the partite sets can also be interchanged.

3. A biclique is vertex-transitive only if $r = s$.

4. Every cycle is vertex-transitive.

5. For $n \geq 3$ no path P_n is vertex-transitive.

6. The Petersen Graph is vertex transitive.

Lemma 1.1.1. *The complement of a simple disconnected graph must be a simple connected graph.*

Proof. Let G be a simple disconnected graph having connected sub-graphs G_1,G_2,...,G_k such that no two sub-graphs are connected.
Let $a \in V(G_i)$ and $b \in V(G_j)$ where $i \neq j$, $i, j \in \{1, 2, \dots, k\}$. Clearly a and b are disconnected in G as they belong to different sub-graphs. Therefore they will be connected in $\overline{G}$ by the definition of complement. Hence any two vertices that are disconnected in G will be connected in $\overline{G}$.
Let x,$y \in V(G_r)$ for some $r \in \{1, 2, \dots, k\}$. Clearly, x and y are connected in G. x and y have no common neighbours in any of other sub-graphs. Therefore $\overline{G}$ x and y will have a common vertex z and thus be connected. Hence any two vertices that are connected in G will remain connected in $\overline{G}$. $\square$

An interesting result from this proof is that for a given set of labelled vertices, there are always more connected graphs than there are disconnected graphs because while the complement of a disconnected graph is indeed a connected simple graph, there is no need for the complement of a simple connected graph to be disconnected. Take P_4 as an example. It is self-complentary and a simple connected graph.

1.2 Paths,Cycles and Trails

A lot of theorems in graph theory can be proved using induction.

Theorem 1.2.1 (Strong Principle Of Induction)**.** *Let $P(n)$ be a statement for an integer parameter n. If the following two conditions hold, then $P(n)$ is true for each positive integer n.*

1. *$P(1)$ is true.*
2. *For all $n > 1$, "$P(k)$ is true for $1 \leq k < n$" implies "$P(n)$ is true".*

The proof of this theorem assumes the **Well Ordering Property** for the positive integers to be true: "Every nonempty set of positive integers has a least element" .

Definition 1.2.1. *A **walk** is a list $v_0, e_1, v_1, \dots, e_k, v_k$ of vertices and edges such that, for $1 \leq i \leq k$, the edge e_i has endpoints v_{i-1} and v_i. A **trail** is a walk with no repeated edge. A u, v-**walk** or u, v-**trail** has first vertex u and last vertex v; these are it's endpoints. A u, v-**path** is a path whose vertices of degree 1 are u and v. The **length** of a walk, trail, path or cycle is its number of edges. A walk or trail is **closed** if the endpoints are the same.*

If the graph in question is simple then a walk or a trail is completely specified by listing the vertices in order as there are no loops or multiple edges.

Definition 1.2.2. *A walk W is said to **contain** a path P if the vertices and edges of P occur as a sublist of the vertices and edges of W in order but not necessarily consecutive.*

Lemma 1.2.1. *Every u, v-walk contains a u, v-path.*

Proof. This can be proved by strong induction on the length l of the walk. Let W be a u, v-walk of length l.
Basis Step: A walk with $l = 0$ is a single vertex with both u, v coinciding. A u, v-path of length 0 is contained in this walk. Clearly the lemma is true for walks of zero length.
Induction Step: Let the lemma be true for all walks with length less than some positive integer k *i.e.*, the lemma is true for all walks with $l < k$ for some $k \geq 1$. Now consider a walk W of length $l = k$. If W has no repeated vertices then it is a u, v-path and we are done as that would imply the lemma is true for $l = k$. Else, let W have a repeated vertex w. Removing all the vertices and edges between every consecutive appearence of w and leaving the final list with a single copy of w results in a walk W' of shorter length. From the induction hypothesis this u, v-walk contains a u, v-path and since W' is contained in W, the path is also contained in W. The induction step is thus complete.
Hence by induction(strong induction) we have,
"Every u, v-walk contains a u, v-path." □

Note that the phrase u and v are joined implies that they are adjacent whereas, the phrase u and v are connected implies they lie on a path.

Definition 1.2.3. *The **connection relation** on $V(G)$ consists of ordered pairs (u, v) such that u is connected to v. The connection relation is an equivalence relation.*

Definition 1.2.4. *A **maximal connected** sub-graph of G is a sub-graph that is connected and is not contained in any other connected sub-graph of G.*

Definition 1.2.5. *The **components** of a graph G are its maximal connected sub-graphs. A component (or graph) is **trivial** if it has no edges; otherwise it is **nontrivial**. An **isolated vertex** is a vertex of degree 0.*

The equivalence classes of the connection relation on $V(G)$ are the vertex sets of the components of G. An isolated vertex forms a trival component of one vertex and no edge.