1 Lecture 01

This course deals with *abstract mathematical objects*, which are defined by the properties they satisfy.

Properties: defined by propositions/statements which are either true or false. Here are a few examples of propositions:

1. 7 is a prime number.
2. All natural numbers are even.
3. All even numbers greater than 2 can be written as the sum of 2 primes.

We shall try to define the natural numbers themselves using the properties they satisfy. Let's start with these 2 axioms:

1. 0 is a natural number. [1]
2. For every natural number n, there exists a natural number $n+1$.

The first axiom tells us that there is a starting number (which we call 0), and the second axiom tells us that for every natural number there is a *next* natural number.

It might be a bit weird to use the addition symbol in our axioms when we haven't even defined numbers yet. Note that this is just a notation; to make it clear we can write $next(n)$ instead of $n+1$ to indicate the next natural number. It's best to think of $next(n)$ as a function which just spits out a new natural number for each input n.

Predicates: a statement which involves variables, which can take any value in some domain. Think of a predicate $P(x)$ as a function which assign true or false to each value x. For example, $P(x)$ could denote *x is the square of an integer*.

There are 3 ways to make a predicate into a proposition:

1. Substitute a constant for x, for example $P(18)$ is a proposition.
2. $\exists x\ P(x)$: this proposition is true if there is some object a for which $P(a)$ is true.
3. $\forall x\ P(x)$: this proposition is true if $P(x)$ is true for all objects x.

Using this notation, we can precisely write our previous 2 axioms for natural numbers:

1. $\exists n\ n = 0$
2. $\forall n\ \exists m\ (m = next(n))$

Let's think more about the second axiom. We need to place more restrictions on this *next* function to get our natural numbers. For example, if we allow $next(0) = 0$, our natural

[1] Whether we add 0 or not to the set of natural numbers is simply a matter of convention. For this course, it is convenient to add it to the set.

numbers just becomes the set $\{0\}$, and it satisfies the axioms we have so far. We could also have $next(0) = 1, next(1) = 0$. So one restriction we could think of to avoid this is to keep $next(n) \neq 0$ for all n.

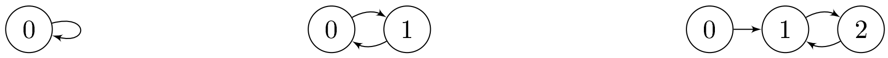

Figure 1: Valid number systems[2] without any condition on *next*

Is this enough? Not really, as we can still think of counterexamples, like $next(0) = 1, next(1) = 2, next(2) = 1$. Basically we have ensured that *next* doesn't loop back to 0. But we must ensure that it doesn't loop back at all (or even to the same number). How we shall do this is to add the restriction that *next* should not point to a number which has already been mapped to i.e. we make it a one-one function. Let's now add these conditions to our axioms:

1. $\exists n \; n = 0$
2. $\forall n \; \exists m \; (m = next(n))$
   (a) $\forall n \; next(n) \neq 0$
   (b) $\forall m \; \forall n \; next(m) = next(n) \implies m = n$

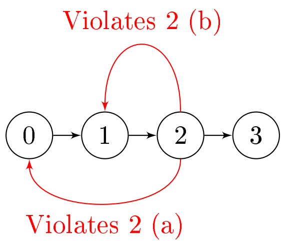

Figure 2: Diagrammatic explanation of why *next* always points to a new number

It turns out our axioms are still not complete. We have ensured that *next* always points to a new number, but we haven't really ensured that every natural number can be formed by applying *next* to 0 a finite number of times. Here are some counterexamples:

1. $\{0, \frac{1}{3}, \frac{2}{3}, \ldots\}$ where $next(n) = n + 1$
2. $[0, \infty)$ where $next(n) = n + 1$

By repeatedly applying *next* to our growing chain, we should end up with the set of all natural numbers. A neat way of stating this is to just keep an axiom that induction itself works i.e. if a statement is true for $0, next(0), next(next(0)), \ldots$ it must be true for all natural numbers. So here is our final set of axioms, which does lead only to our natural numbers:

[2]It's important to keep in mind what makes one number system different from another is how the nodes are linked, it's not about what symbol we keep for each node like 0, 1, 2

1. $\exists n\ n = 0$
2. $\forall n\ \exists m\ (m = next(n))$
   (a) $\forall n\ next(n) \neq 0$
   (b) $\forall m\ \forall n\ next(m) = next(n) \implies m = n$
3. $[P(0)] \wedge [\forall n\ \{P(n) \implies P(next(n))\}] \implies [\forall n\ P(n)]$

Exercise 1.1. *Prove that $\forall n\ next(n) \neq n$. Can we have this statement instead of 2 (b) to define natural numbers?*

Solution. Proof by induction
Define $P(n)$ to be $next(n) \neq n$. $P(0)$ is true from 2 (a).
Also, $next(n) \neq n \implies next(next(n)) \neq next(n)$ as $next$ is one- one (or contrapositive of 2 (b)). This is basically $P(n) \implies P(next(n))$.
From this we conclude $P(n)$ i.e. $next(n) \neq n$ for all n.
This can't be used instead of 2 (b). Counterexample: $next(0) = 1, next(1) = 2, next(2) = 1$.

Exercise 1.2. *Instead of keeping induction as an axiom, we could ensure that there are no other starting points for a chain other than 0. This might ensure that all numbers are part of the chain starting from 0.*

Can we replace axiom 3 with the following:
$\forall n\ n \neq 0 \iff \exists m\ next(m) = n$

Solution. No, we have a counterexample, take the set
$\{0, 1, 2, \dots\} \cup \{\dots, -1.5, -0.5, 0.5, 1.5, \dots\}$ where $next(n)$ is the standard $n + 1$.
It satisfies the new set of 3 axioms but aren't equivalent to natural numbers.

Exercise 1.3. *Is there a more concrete way to show that from axiom 3 that all natural numbers can be obtained composing next to 0 a finite (including 0) number of times?*

Solution. Let $P(n)$ denote n obtained composing *(*next) to 0 a finite (including 0) number of times. $P(0)$ is obviously true. It's also clear that $P(n) \implies P(next(n))$, as if n can be written as $next(next(\dots(next(0))\dots))$, $next(n)$ can also be written that way by just composing one more $next$ to the expression. This completes our proof.