Q1 For pair (i,j), let $I_{ij} = 1$ if (i,j) is an inversion and 0 otherwise.
I_{ij} is a random variable.
Let $X = \#$ of inversions in the permutation.
Then $X = \sum_{1 \le i < j \le n} I_{ij}$

$$E(I_{ij}) = 1 \cdot p(I_{ij} = 1) + 0 \cdot p(I_{ij} = 0)$$

Now $p(I_{ij} = 1) = p(I_{ij} = 0) = 1/2$
as there are an equal number of permutations where i precedes or succeeds j for any pair (i,j)

$\therefore E(I_{ij}) = 1/2.$

There are $C(n,2) = n(n-1)/2$ pairs.

$$\therefore E(X) = \frac{n(n-1)}{2} \times \frac{1}{2} = \frac{n(n-1)}{4}$$

using linear property of E and independence of I_{ij}.

Q3 $x_i \sim \text{Poiss}(\lambda/n)$ for all i from 1 to n

$$NLL = + \sum_{i=1}^{n} + \frac{\lambda}{n} - x_i \log(\lambda/n)$$

$$\frac{\partial NLL}{\partial \lambda} = \sum_i \frac{1}{n} - \sum_i \frac{x_i \cancel{n}}{\lambda \cancel{n}} = 0$$

$$\longrightarrow \hat{\lambda} = \sum_{i=1}^{n} x_i$$

$$E(\hat{\lambda}) = \lambda/n \times n = \lambda$$

$\therefore$ this is an unbiased estimator.

$$Var(\hat{\lambda}) = \frac{\lambda}{n} \times n = \lambda$$

$$MSE = bias^2 + var = \lambda.$$

The MSE does not decrease with n.

This is not a consistent estimator.

Q1 $\quad X \sim N(0,1).$

$$P(X \geqslant t) = \frac{1}{\sqrt{2\pi}} \int_t^\infty e^{-x^2/2}\, dx$$

$$\leq \frac{1}{\sqrt{2\pi}} \int_t^\infty \frac{x}{t} e^{-x^2/2}\, dx \quad \text{as} \quad x \geqslant t$$

$u = x^2/2$

$du = x\, dx$

$$= \frac{1}{t\sqrt{2\pi}} \int_t^\infty x\, e^{-x^2/2}\, dx$$

$$= \frac{1}{t\sqrt{2\pi}} \int_{t^2/2}^\infty e^{-u}\, du = \frac{1}{\sqrt{2\pi}\, t} \left(\frac{e^{-u}}{-1} \right)_{t^2/2}^{\infty}$$

$$= \frac{e^{-t^2/2}}{t\sqrt{2\pi}}$$

$P(|X| \geq t) = 2P(X \geq t)$ using symmetry of Gaussian

$$\therefore P(|X| \geq t) = \sqrt{\frac{2}{\pi}} \frac{e^{-t^2/2}}{t}$$

Chebyshev's inequality states that

$P(|Z| \geq t) \leq t^{-2}$.

Now $e^{-t^2/2}$ decreases faster than t^{-2}. This means the bound in this question is tighter than the one predicted by Chebyshev's inequality. This is not surprising as CI

Q5 Clearly the MLE for σ is

$$\hat{\sigma} = \frac{1}{n} \sum_{i=1}^{n} |X_i|$$

Now $$E(|X|) = \int_{-\infty}^{\infty} |x| \frac{e^{-|x|/\sigma}}{2\sigma} dx$$

$$= \cancel{2}\int_0^{\infty} \frac{x}{\cancel{2}\sigma} e^{-x/\sigma} dx \quad \text{using symmetry}$$

$$= \sigma \int_0^{\infty} \frac{x}{\sigma} e^{-x/\sigma} \frac{dx}{\sigma} = \sigma \int_0^{\infty} \underset{u}{y} \, \underbrace{e^{-y} dy}_{dv}$$

$$= \left\{ \left[y \, e^{-y} (-1) \right]_0^{\infty} - \int_0^{\infty} e^{-y} (-1) \, dy \right\} \sigma$$

$$= \sigma$$

$$E(|X|^2) = \int_{-\infty}^{\infty} x^2 \frac{1}{2\sigma} e^{-|x|/\sigma} dx$$

$$= \sigma^2 \int_0^{\infty} \frac{x^2}{\sigma^2} e^{-x/\sigma} \frac{dx}{\sigma}$$

$$= \sigma^2 \int_0^{\infty} y^2 e^{-y} dy = 2\sigma^2$$

$$\cancel{Var}\, E(\hat{\sigma}) = \frac{1}{n} \times n \, E(|X_i|) = \sigma$$

$\therefore$ unbiased estimator

$$\mathrm{Var}(\hat{\sigma}) = \frac{1}{n^2} \times n \times \mathrm{Var}(X_1)$$

$$= \frac{1}{n}(2\sigma^2 - \sigma^2) = \frac{\sigma^2}{n}$$

$$\mathrm{MSE} = \mathrm{Var} + \mathrm{bias}^2 = \sigma^2/n.$$

Q6

a)

$$E\left(g(X)(X-\mu)\right) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} \underbrace{g(x)}_{u} \underbrace{(x-\mu)\, e^{-(x-\mu)^2/2\sigma^2}\, dx}_{dv}$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \left[\left[-\sigma^2 g(x)\, e^{-(x-\mu)^2/2\sigma^2}\right]_{-\infty}^{\infty} \longrightarrow 0 \right.$$

$$\left. + \sigma^2 \int_{-\infty}^{\infty} g'(x)\, e^{-(x-\mu)^2/2\sigma^2}\, dx \right]$$

$$= \sigma^2\, E\left(g'(X)\right)$$

$$E(X^3) = E\left[X^2(X-\mu+\mu)\right]$$

$$= E\left[X^2(X-\mu)\right] + \mu\, E(X^2)$$

$$= \sigma^2 E(2X) + \mu(\mu^2+\sigma^2)$$

using $g(x) = x^2$, $g'(x) = 2x$

$$= 2\sigma^2\mu + \sigma^2\mu + \mu^2\mu$$

$$= 3\mu\sigma^2 + \mu^3$$

b) If $X \sim \text{Poiss}(\lambda)$,

$$E[\lambda g(X)] = \sum_{x=0}^{\infty} \lambda g(x) \frac{e^{-\lambda} \lambda^{x}}{x!}$$

$$= \sum_{x=0}^{\infty} g(x) \frac{e^{-\lambda} \lambda^{x+1}}{x!} \frac{x+1}{x+1}$$

$$= \sum_{x=0}^{\infty} g(x) (x+1) \frac{e^{-\lambda} \lambda^{x+1}}{(x+1)!}$$

$y \longrightarrow x+1$

$$= \sum_{y=1}^{\infty} y\, g(y-1) \frac{e^{-\lambda} \lambda^{y}}{y!}$$

$$= E[X g(X-1)]$$

(as it is equal to $\displaystyle\sum_{y=0}^{\infty} y g(y-1) \frac{e^{-\lambda} \lambda^{y}}{y!}$

Using $g(x) = x^2$, we have

$$E(\lambda \cancel{g} X^2) = E\left[X(X-1)^2\right]$$
$$= E(X^3 - 2X^2 + X)$$

$$\therefore E[X^3] = E[\lambda X^2] + 2E[X^2] - E[X]$$
$$= \lambda(\lambda + \lambda^2) + 2(\lambda + \lambda^2) - \lambda$$
$$= \lambda^3 + 3\lambda^2 + \lambda$$