Midterm Exam: CS 215

1. Consider a permutation of the first n positive integers, generated uniformly randomly (i.e. each of the $n!$ different permutations are equally likely). The ordered pair (i, j) in the permutation is called an inversion if $i < j$ but j precedes i (i.e. occurs earlier than i) in the permutation. Determine the expected number of inversions in a uniformly randomly generated permutation of the first n positive integers. [10 points]

2. Consider independently drawn sample values $x_1, x_2, ..., x_n$, each from Poisson(λ/n) where n is known. What is the maximum likelihood estimate for λ? Derive the bias, variance, MSE of this estimator. Is this a consistent estimator? Why (not)? [10 points]
(There is a physical significance to this question, even though one needn't understand it to answer the question. The noise in an image pixel is typically Poisson in nature. The values $x_1, x_2, ..., x_n$ correspond to n images of the same scene acquired in quick succession with acquisition time T/n per image, instead of acquiring one image in time T.)

3. If $X \sim \mathcal{N}(0, 1)$, then prove that $P(|X| \geq u) \leq \sqrt{2/\pi}\dfrac{e^{-u^2/2}}{u}$ for all $u > 0$. How does this bound compare with that given by Chebyshev's inequality? [10+5 = 15 points]

4. Consider n values $\{x_i\}_{i=1}^n$ drawn independently from a Laplacian distribution with mean 0 and parameter σ. The probability density for a Laplacian random variable X is given by $f_X(x) = \dfrac{1}{2\sigma}e^{-|x|/\sigma}$ (note the absolute value in the exponent). Given $\{x_i\}_{i=1}^n$, derive the maximum likelihood estimate for σ, as well as its bias, variance, MSE. [15 points]

5. In this problem, we will derive higher order moments of specific random variables in a new way.
   (a) Consider $X \sim \mathcal{N}(\mu, \sigma^2)$. Then prove that $E[g(X)(X - \mu)] = \sigma^2 E[g'(X)]$ where g is a differentiable function such that $E[|g'(X)|] < \infty, |g(x)| < \infty$. Use this to derive an expression for $E[X^3]$ in terms of μ and σ^2. Do not use any other method (eg: MGFs) to derive $E[X^3]$. [5+5=10 points]
   (b) Consider $X \sim$ Poisson(λ). Then prove that $E[\lambda g(X)] = E[Xg(X - 1)]$ where g is a function such that $-\infty < E[g(X)] < \infty, -\infty < g(-1) < \infty$. Use this to derive an expression for $E[X^3]$ assuming known expressions for $E[X], E[X^2]$. Do not use any other method (eg: MGFs) to derive $E[X^3]$. [5+5=10 points]