$p \to q, \neg q \models \neg p$ Semantic entailment

implies implies

$p \to q, \neg q \vdash \neg p$ Syntactic entailment

Completeness

implies implies

$p \to q, \neg q \models \neg p$ Semantic entailment

Soundness

Natural Deduction Proof System } sound and complete for Prop. logic.

To prove : $p \to q, \neg q \vdash \neg p$

Step 1 : First show : $\vdash ((p \to q) \wedge \neg q) \to \neg p$

$$\boxed{\begin{array}{c} ((p \to q) \wedge \neg q) \\ \vdots \\ \neg p \end{array}}$$

$$((p \to q) \wedge \neg q) \to \neg p$$

Step 2: Construct a new proof using step 1 as sub-proof

$p \to q$

$\neg q$

$(p \to q) \wedge \neg q$ ----- $\wedge_i$ rule

$$\boxed{\begin{array}{c} ((p \to q) \wedge \neg q) \\ \vdots \\ \neg p \end{array}}$$

$$((p \to q) \wedge \neg q) \to \neg p$$

$\neg p$ ---------- $\to_e$ rule

QED

Crucial link : Step 1

Step 1: To show

$\vdash ((p \to q) \land \neg q) \to \neg p$

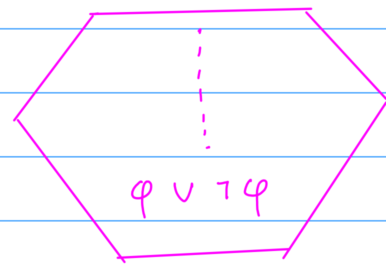
Many proofs possible

$(p \to q) \land \neg q$	
$p \to q$	$\land e_1$
$\neg q$	$\land e_2$
p	assume
q	$\to e$
$\perp$	$\neg e$
$\neg p$	$\neg i$

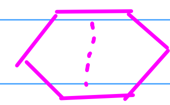
Requires ingenuity to get a short proof

Mimic each row of truth-table

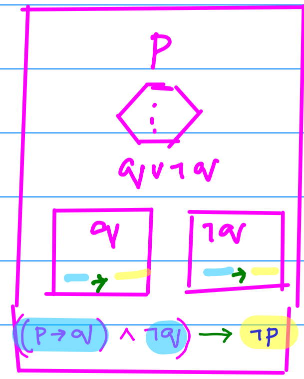
Brute force approach

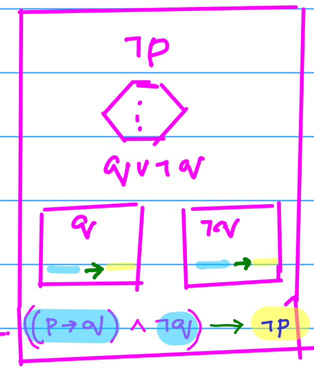


$\varphi \lor \neg \varphi$



$p \lor \neg p$





$\lor e$ ---- $\lor e$

$((p \to q) \land \neg q) \to \neg p$ ---- $\lor e$

Proofs in ND talk only of validity or contradictions

$\vdash \varphi$ $\qquad\qquad$ $\nvdash p \land q$

$\vdash \neg\varphi$ $\qquad\qquad$ $\nvdash \neg(p \land q)$

$\vdash \neg p \to \neg(p \land q)$

$\vdash (p \to (q \to (p \land q)))$

What if a formula is neither valid nor contradiction?
Can we reason directly about satisfiability?

φ is valid iff $\neg\varphi$ is not sat.

$$\varphi_1 \overset{\lor}{\cancel{\land}} (\varphi_2 \overset{\land}{\cancel{\lor}} \varphi_3) \quad =\!\!|\!\!\models \quad (\varphi_1 \overset{\lor}{\cancel{\land}} \varphi_2) \overset{\land}{\cancel{\lor}} (\varphi_1 \overset{\lor}{\cancel{\land}} \varphi_3)$$

$$\vdash \varphi_1 \land (\varphi_2 \lor \varphi_3) \to ((\varphi_1 \lor \varphi_2) \lor (\varphi_1 \land \varphi_3))$$

$$\varphi_1 \land (\varphi_2 \lor \varphi_3) \quad =\!\!|\!\!\models \quad (\varphi_1 \land \varphi_2) \lor (\varphi_1 \land \varphi_3)$$

$$\varphi_1 \lor (\varphi_2 \land \varphi_3) \quad =\!\!|\!\!\models \quad (\varphi_1 \lor \varphi_2) \land (\varphi_1 \lor \varphi_3)$$

$\land$, $\lor$, $\neg$, $\cancel{\to}$, $\leftrightarrow$ $\qquad$ $(\varphi_1 \leftrightarrow \varphi_2)$ $\qquad$ $(\varphi_1 \to \varphi_2) \land (\varphi_2 \to \varphi_1)$

$$(\varphi_1 \to \varphi_2) \quad =\!\!|\!\!\models \quad \underline{(\neg\varphi_1 \lor \varphi_2)}$$

De Morgan's

$$\neg(\varphi_1 \land \varphi_2) \quad =\!\!|\!\!\models \quad \neg\varphi_1 \lor \neg\varphi_2$$

$$\neg(\varphi_1 \lor \varphi_2) \quad =\!\!|\!\!\models \quad \neg\varphi_1 \land \neg\varphi_2$$

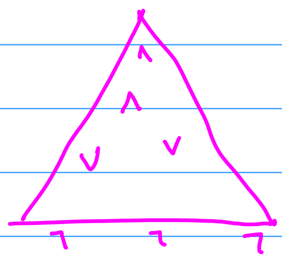
$$\neg(\varphi_1 \vee \varphi_2) \dashv\models \neg\varphi_1 \wedge \neg\varphi_2$$

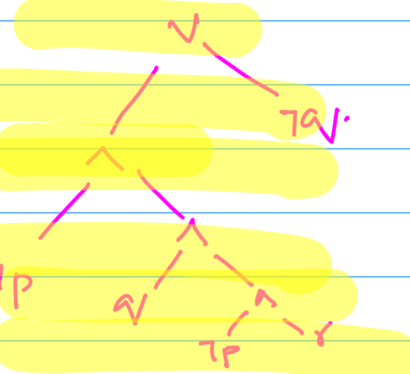
$$\neg(p \vee \neg(q \wedge \neg(p \vee \neg r))) \vee \neg q$$

$$\dashv\models (\neg p \wedge (q \wedge \neg(p \vee \neg r))) \vee \neg q$$

$$\dashv\models (\neg p \wedge (q \wedge (\neg p \wedge r))) \vee \neg q$$

Negation Normal Form (NNF)





Conjunctive Normal Form

CNF