Greedy Algorithms

1. Assuming there's a minimum weight spanning tree. It must have at least 1 edge in C, as the vertices in V_1 and V_2 have to be connected. If the edge e^* is already in the spanning tree, we're done, so assume it's not there. Now add e^*, since we have too many edges, we'll have a cycle. We can also say that the cycle contains e^*, as that's the edge which when added, created a cycle. There must be some other edge in C which is part of the cycle. This is because is we walk through the cycle, we traverse e^*, moving from V_1 to V_2, so we must come back from V_2 to V_1, and that requires another edge in the cut. So now we can delete this other edge in the cut, and we get a spanning tree again, and this tree will have weight $\leq$ the old tree, as e^* was the minimum weight edge in the cut.

2. Again, let's assume that we have a minimum spanning tree without e^*, and now add e^*. There will now be a cycle, and this cycle will contain e^*. There must be another edge of this cycle containing v_1, as every vertex in a cycle must have 2 edges in the cycle. Now if we delete this other edge, we will have a spanning tree, and this spanning tree will be minimum.

3. Strategy 1 fails in cases where you actually have less time for larger assignments. Take an example there $l_1, l_2 = 1, 2$ and $d_1, d_2 = 3, 2$. If you do 1 before 2, you finish 2 late, but if you do 1 after 2, you finish both on time.

Strategy 3 seems logical. The reason we are using $d_i - l_i$ as a factor, is because that's the time remaining after we finish the assignment. So it's an increasing order of how much extra time we will have for an assignment if we start now. But it can prioritize very long assignments over short ones which need to be finished early. Take $l_1, l_2 = 1, 3$ and $d_1, d_2 = 3, 4$. Our greedy approach will make us do assignment 2 first, which makes us finish assignment 1 late. But if we did assignment 1 first, we would have enough time for assignment 2 after it.

Now to prove the claim in strategy 2. Let's say A_j is immediately done before A_i, but A_j has a closer deadline compared to A_i. What happens if we swap A_i and A_j? Firstly, it's clear that the lateness of any assignment other than A_i and A_j don't change. Now let's just look at A_i and A_j. We can take cases, but I'll provide some sort of general reasoning as to why the swap won't decrease the max lateness.

Before the swap, A_j was finished later, but had an earlier deadline. So no matter what, the lateness will be more than or equal to A_i. So after swapping, we just have to make sure the lateness of both A_i, A_j are less than the lateness of A_j before swapping. Firstly after swapping the lateness of A_j obviously decreases, it's done earlier. But the lateness of A_i has increased. But think about it like this, A_i's finishing time after swapping is same as A_j's finishing time before swapping. And also, A_i's deadline is further away than A_j's deadline. So since the finishing times are equal and we have more time in 1 case, A_i's lateness after swapping is $\leq$ A_j's lateness before swapping.

Now we can use this swapping trick to do the whole question. Assume there's an optimal solution which isn't increasing order of deadlines. We can just do an insertion sort like algorithm. And keep doing swaps to bring the array to increasing order of deadlines. We have also shown that in each swap, the max lateness isn't increasing, and it can't really decrease as we started with an optimal solution. So at the end of our insertion sort, the max lateness is same as that of the optimal solution, so increasing order of deadlines is always optimal.

4. Since each assignment takes unit time, there's a simple way to check if an assignments is done on time. If you do the assignments in order, the k^{th} assignment is done on time if and only if its deadline is greater than or equal to k. We just now have to order the n assignments, maximizing the total reward.

Here's how our greedy algorithm is going to work. We first sort the assignments in decreasing order of rewards. After that, assume the assignments are $A_1, \ldots, A_n$. Now we iterate through this array. We check if A_i can be completed in time i.e. there's some empty spot from 1 to deadline of A_i. If there is, we place in the last spot that we could, such that A_i is completed in time. If not, we just place A_i at the end of the array, or as far as possible to the end.

The algorithm is simple enough, but why does this work? We'll prove by induction that there is an optimal solution where $A_1, \ldots, A_k$ are placed at the same place as our algorithm. Let's explain how the induction works. Assume there's a solution where $A_1, \ldots, A_{k-1}$ have been placed how they have in our algorithm, now think about A_k.

- Case 1: There's a spot for A_k such that it can be completed on time. Can we show that it's also completed on time in the optimal solution? If not, in the optimal solution, there's some A_l in the spot where $l > k$. But then just swap A_l and A_k, now we get an extra reward due to A_k, we might lose the reward A_l. But our net gain is positive as reward of A_k is more than reward of A_l. Now we have to show there's a solution where it's placed as late as possible, but this is intuitive right. If it's placed eariler, just swap it with A_l (whatever's in the latest spot), A_k is still completed, you still get its reward, and A_l is completed even earlier, so it won't lose its reward either.
- Case 2: There's no spot for A_k such that it can be completed on time. This would mean even in our optimal solution, A_k isn't completed as there is literally no place for it to go. There is obviously some optimal solution where it's placed as late as possible, right. If it's placed earlier than that, just move it later, anyway it wasn't going to be completed on time, and you are moving some other assignment earlier, which definitely won't hurt.

5. This is actually a hard problem :/

It's not really mentioned whether we can split up the time when we do a particular assignment, or that we have to do it in 1 whole stretch. But we can see that it doesn't matter, actually splitting up assignments doesn't really help us. Let's say we are finishing some set of assignments A_1 to A_n, by splitting them up. WLOG let's also assume the final finishing times of A_1 to A_n are in ascending order. We claim that if we just did A_1 fully, then A_2 fully, ..., then A_n fully, we would have accomplished the same thing. We can prove this by induction. We know A_n is finished at the end, so it is finally finished at time $l(A_1) + l(A_2) + \cdots + l(A_n)$ (where $l(A_i)$ is total length/time taken for assignment A_i). But why not rearrange the time intervals, and do A_n at the end? To be clear, we are picking out times whenever A_n is done, and placing them at the end. The final time A_n is finishing at is still the same. And time intervals of the other assignments are moved earlier anyway, by this swapping. Now just do this for $A_{n-1}, An - 2$, so on until all of them are just done at a stretch.

Now that we've proved this, let's just assume we can split up assignments. Now our greedy algorithm is going to work like this: first sort in increasing order of assignment lengths. Now we iterate over the array, and check if the assignment A_i can be completed in time i.e. there is at least $l(A_i)$ space before the deadline of A_i. If so, we do A_i as late as possible, and split it up to achieve this, if required. If it's not possible to do it on time, we just skip it.

We now prove by induction: there exists an optimal solution where $A_1, \ldots, A_k$ are finally finished at the same time as whenever our algo decides when they should finish.

Our inductive assumption: assume there's an optimal solution where $A_1, \ldots, A_{k-1}$ are placed where they are. Now if there's enough empty space before deadline of A_k, we claim that there's an optimal solution where A_k is done before the deadline. As most greedy questions, we assume there's an optimal solution without A_k, and try to modify it such that A_k is done too. Ok, our optimal solution doesn't actually do A_k but it most likely does something else in the time when it could have done A_k. Maybe it spent 1 unit of time on many assignments, maybe it did half of A_{k+1}, maybe it did nothing, we don't really know. But our idea is to replace something with A_k right, but we can't really do this if only partial assignments are done. If somehow a full assignment A_l was done, we could have replaced it with A_k, as A_k takes lesser time than A_l for all $l > k$. So what do we do? The insight now is to reorder intervals such that assignments are done one by one, but we reorder only A_{k+1}, A_{k+2}, all the way to A_n.

Let me explain what this reordering is exactly. Say A_i was done in $[0, 1]$, nothing was done in $[1, 2]$, A_j was done in $[2, 3]$ and A_i was done in $[3, 4]$. We reorder such that no assignment is done between the starting and completion of another assignment, but importantly, we keep empty intervals empty. So if we want to reorder in the example given, we do A_j from $[0, 1]$, nothing from $[1, 2]$, A_i from $[2, 3]$ and A_i from $[3, 4]$.

Ok, now after this reordering, now what do we see done before A_k's deadline (other than A_1 to A_{k-1}). If suppose one of the assignments from A_{k+1} to A_n was finished completely before A_k's deadline, great. We just delete it and insert A_k. Now if none of these assignments were finished before A_k's deadline, we can say only at most one of them was started before A_k's deadline. Because, if one of them was started, it has to be completed before the next one is started, and none of them finish before A_k's deadline. So what we can do is just delete this assignment. We'll now have enough empty intervals before the deadline of A_k to fit the length of A_k, move them to the end, and fit A_k.

Now that we've proven there is an optimal solution with A_k, we have to show that A_k is done as late as possible (just like how it's done in our algo). But this is easy right, if not, we can swap such that it is done as late as possible, and move everything else earlier. Clearly this doesn't make any assignment completed late, so we're done.

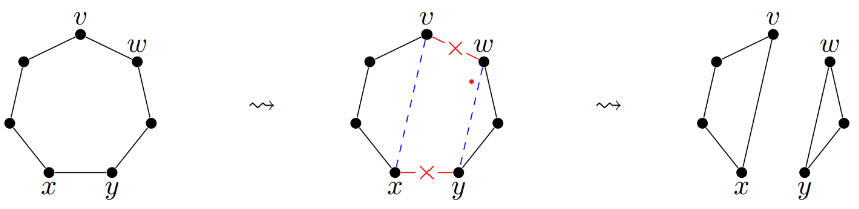
6. We have to first sort all the intervals in a meaningful order. Let's say they are just sorted according to starting time (breaking ties arbitrarily). Our greedy approaches will be as follows: when we get a new interval, use a new station only if we really have to. And if some old stations can be used, use a random one of them. We just have to prove that this algorithm is optimal.

Suppose at a point in our algorithm we have used stations 1 to $k-1$ at least once, and then for the next interval $[s_i, e_i]$ we use station k. This means that none of stations 1 to $k-1$ have trains that have departed. So at time $t = s_i$, there are $k-1$ different intervals that contain it, these correspond to the trains in each station. Then there's also $[s_i, e_i]$ which contains s_i, so there are at least k trains which need a station at this common time. This would mean the optimal (minimum) number of stations is at least k. So at every point in our algo, when we require an extra station, we really do need that many stations in any optimal solution, which means we can't do better than our algorithm.

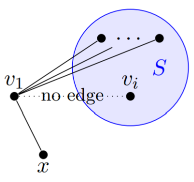
7. The algorithm is actually simple enough to describe, like most of Greedy. Sort the vertices in decreasing order of degrees. Now we just connect the vertex with the largest degree to the next few vertices, until it's degree is satisfied. That is, if v_1 has degree d, we connect it to $v_2, v_3, \dots, v_{d+1}$. We have to now decrease the degree of these vertices, and discard v_1, and continue the algo for the rest of the vertices (we also have to remove vertices which become degree 0). Our algo gives up whenever the largest degree is greater than or equal to number of vertices left.

It's easy to see that whenever our algo terminates without giving up, there's a solution. But we have to prove that our algo will always give a solution whenever a graph is possible. So our claim is basically this: if there's a graph with degree sequence $\{d_1, d_2, \dots, d_n\}$ (where degrees are in descending order), there's also a graph with degree sequence $\{d_2 - 1, d_3 - 1, \dots, d_{d_1+1} - 1, d_{d_1+2}, \dots, d_n\}$. But how do we even go about proving this? The idea like most greedy algos is assume a solution where this isn't true, and do some modifications to our graph to convert it to what our greedy algo produces.

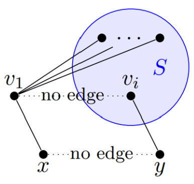
What's an operation that changes our graph, but doesn't really modify our degree sequence? Think of 4 vertices v, w, x, y. There's an edge between v, w and one between x, y, but no edge between v, x and w, y. Our operation is to basically delete the 2 edges that are there, and add the 2 edges that aren't there. This actually preserves the degrees of all 4 vertices involved.



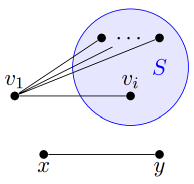
Now, to prove our claim. Say the vertices in decreasing order of degree are $v_1, \dots, v_n$ and v_1 has a degree d. We need to prove that there's a graph with the given degree sequence, where v_1 is connected to every vertex in the set $S = \{v_2, v_3, \dots, v_{d+1}\}$. So let's assume this isn't the case v_1 isn't connected to $v_i \in S$. Since v_1 needs to satisfy its degree requirements, it has to be connected with at least 1 vertex outside S, let's say it's connected to $x \notin S$. If you think about our goal operation, we already have 3 vertices ready, v_1, v_i, x. We just need a fourth vertex, that v_i is connected to and x isn't connected to. But why will some vertex exist? Ok, assume the contrary, there's no such vertex. Every vertex that v_i is connected to, x is also connected to. This would mean that the degree of x is at least as much as the degree of v_i. But x is also connected to v_1, that v_i isn't connected to, so its degree is strictly more than v_i. But that's a contradiction, if degree of x is more than v_i, it must be in the set S right, as S has all the highest degree vertices. So we can conclude there is some vertex y such that v_i is connected to y, but x isn't. And then we can perform our operation, to keep the degree sequence as same, and get a solution where v_1 is connected to v_i. This operation can be done as many times as we want, to make v_1 connected to every vertex of S. So we're done.



(a) Vertices v_i and x



(b) Goal: find y



(c) The key edge swap

8. Our idea is to sort intervals based on increasing order of end time. The intuition behind this is that, there will be very few people who can show the first person the ad, after this sorting, and it is easier to think of some greedy approach. While checking who all arrive before the first person leaves, isn't it best to select the one who leaves the latest? This seems intuitive right, as we want most number of people to see the ad.

Let's prove this claim. Suppose there's some optimal solution where among the people who overlap with the first person, the one with the latest leaving time (call this person L) isn't selected. So some other person who overlaps with the first guy is selected K. Can we always swap this person with the person we claim? One thing to observe is that, L is going to overlap with everyone before L in our array. This is because L arrives before anyone leaves, and leaves only after every single person before L leaves. So everyone will see L before leaving. Now for the elements after L, is there any person there who K sees but L doesn't see? That's not possible. If they are after L in the array they leave way after K, so in order for K to see them, they have to arrive before K leaves, but this would mean they also arrive before L leaves, so basically they see L too.

So what have we proven? Anyone K sees, L is going to see, so we might as well pay L instead of K. And if we pay L, we know for sure we aren't going to pay anyone before L (same proof as the claim) if we're being optimal. So we continue the same algorithm for the array, starting from after L. Note that some intervals after L might have already seen the ad, but that doesn't mean we shouldn't pay them in the optimal solution. Our new first person is going to be the first person who L doesn't see. But the one we select is going to be the one who overlaps with the first person and leaves the latest, this could potentially be someone before the first person, we have to start checking from after L itself.

If we just apply our claim inductively, we'll be able to prove that our greedy strategy works.

9. We have $\binom{n}{2}$ pairwise distances. We want to make sure that as many of the small distances as possible aren't from one set to the other, so we try the following greedy approach. First we sort all the pairwise distances in ascending order, and since we don't know which numbers in the same set, assume they are all in different singleton sets for now. Now we iterate through the pairwise distances. If suppose the 2 elements are in different sets, we merge them to 1 cluster. We stop our algorithm whenever we have done so much merging that we have only k sets, and we can't do any more merging. Note that there'll be exactly $n - k$ merges.

We have to prove by induction that every merge we do is valid. But how to phrase this a bit better? At any point in our algorithm, when we detect that i and j are in different clusters, there exists an optimal solution where they are in the same cluster.

So how do we do this induction? So we assume all the previous merges we have done so far are valid i.e. there exists an optimal solution where elements are in the same cluster, whenever our they are in the same set in our algo. Now we detect 2 new elements which we want to merge. What if there's an optimal solution without these 2 elements in the same set? The spacing of this clustering is at least as big as the distance between these 2 elements, as they are in different sets in the optimal solution. And for the clustering produced by our greedy algo, the spacing is at most the distance between these 2 elements, as all the pairwise distances including this distance, and the ones greater than it, don't appear in the spacing term of the greedy algo. Which proves that the greedy algo also outputs some optimal clustering.

Dynamic Programming

Flavour 1: where the subproblems are on suffixes/prefixes of the input.

10. We first sort based on starting time. Now we divide by cases on whether we select the first interval or don't. If we don't, our answer is just going to be the answer for the subarray from 2 to n. If we select the first interval, we have to reject all the intervals which start before the first interval ends. But note that after rejecting everything, we are still going to be left with a contiguous subarray, from some k to n, only because the array is sorted by starting time. To find this k, we have to just binary search actually. The query is the first interval's end time, and whatever index you get from the binary search, that will decide from which point you have to solve the subproblem. In the end our reucrsion is going to be like $f(1, n) = \max(f(2, n), l(I_1) + f(k, n))$. Since by applying this recursion the subarray always ends in n, and starting index gets smaller, there are at most n distinct recursive calls. To code this up, we can iterate backwards and find $f(i, n)$ for i from n to 1. Each $f(i, n)$ takes $O(\log n)$ time due to the binary search, so overall the time complexity is still $O(n \log n)$.

11. I don't know which question they are referring to where we found optimal cost 😅

12. Again, split based on whether we select the first interval or don't. If you take the first interval, the best you can do is $w_1 + f(k, n)$, where k is the first interval which starts after the first interval ends. And if you don't select the first interval, the answer is just $f(2, n)$. So our recursion changes like: $f(1, n) = \max(f(2, n), w_1 + f(k, n))$. We can iterate backwards to calculate $f(i, n)$ for all i, in $O(n \log n)$ time.

13. It's the same case split again. If we don't select the first interval, the answer is just $f(2, n)$. If we select the first interval, the number of subsets is $f(k, n)$. So the recursion is $f(1, n) = f(2, n) + f(k, n)$, where k is the first interval which starts after the first interval ends.

14. A natural first idea would be to case split based on which hotel we are staying in for the first night. Let $dp[i]$ be the minimum cost to travel, starting from city i. We basically need $dp[0]$, treating A as the 0^{th} city, and our base case is $dp[n+1] = 0$, treating B as the $(n+1)^{th}$ city. Our recursion is something like this:

$$dp[i] = \min_{j \in S_i}(dp[j] + c_j),\ S_i \text{ is the set of cities which can be reached from city } i$$

We need some efficient way to calculate S_i for all i, while iterating in our algorithm. It's enough to see which is the last city, and its distance which can be reached from every i.

We can solve this itself using subproblems. Let's say for city i, we have already calculated j, the furthest city we can reach, and its distance which is $d_{i+1} + d_{i+2} + \cdots + d_j$. Now we want to find the furthest city to reach from city $i - 1$, so what we do is just add d_i to our distance, assuming we can still reach j. If our distance is still not greater than d, great, j is still the furthest city. But if it exceeds i, keep decrementing j, and subtracting d_j from our distance, until the distance doesn't exceed d. This is basically a 2 pointer method, so it completes in $O(n)$ time overall, for every i.

Now back to our original question. It will be useful if we could just store all the $dp[j] + c[j]$'s in a multiset right, so that we can just find the minimum in $O(\log n)$ time. The idea is to find $dp[i]$ iterating backwards from n to 1. We concurrently do our subproblem, whenever we decrement i, we add $dp[i] + c[i]$ to our multiset. Whenever we decrement j, we delete $dp[j] + c[j]$ from our multiset. After doing all the decrementing to j for the particular i, we find the minimum of the multiset, and assign that as $dp[i]$. The time complexity of this algorithm overall is $O(n \log n)$ as we have n additions and deletions to our multiset, and well as n minimum queries.

15. For each vertex i from n to 1, we find the longest path to vertex n. Base case is $dp[n] = 0$. Now to get $dp[i]$, we just do 1+ the maximum length from every vertex it connects to. This is $O(E)$, where E is the number of edges.

16. The normal dynamic programming idea is like this: for each element i, find the longest increasing subsequence in the array ending at i. Suppose this is done for i, and we want to do it for $i + 1$. We check $arr[i + 1]$, and see which all array elements before it that are smaller than it, because we can append $arr[i]$ to any increasing subsequence ending at those. Our recursion is going to be $dp[i] = 1 + \max(\{dp[j] \mid j \in S_i\})$, where S_i is the set of all j such that $j < i$ and $arr[j] < arr[i]$. The problem with this approach is that $dp[i]$ takes $O(n)$ time, so this becomes $O(n^2)$.

We flip the question a bit. For each length l, we store the smallest number a subsequence of length l can end in. If in the first i elements there's a subsequence of length 3 ending with 5, there's no point of keeping track of a subsequence of length 3 ending with 10. Further, our array is going to be in increasing order. This is intuitive right, if you have a sequence of length 5 ending with 10, subsequence with length 4 is going to have a smaller ending number. The fact that the array is sorted could help us do something like binary search.

Let's say you are in the middle of your iteration, and what you have stored so far is $[2, 3, 5, 9, 10, 12]$ i.e. the smallest ending for a subsequence of length 1 is 2, smallest ending for a subsequence of length 2 is 3, and so on. Now let's say the next element of the array you process is 7, how will you update what you are storing? Firstly, 7 can only be appended to sequences of length 1,2,3. There's no way we could append to a sequence of length 4,5, or 6, as the smallest ending of a subsequence that long, is already larger than 7. Is there any point appending 7 to a sequence of length 1,2? That creates sequence of length 2,3 that end in 7, but we have sequences of that length ending in smaller numbers, so this is useless. So let's append it to a sequence of length 3, this gives a sequence of length 4 ending in 7. This is finally something useful, we have to update the 9 to a 7. From this example, it's clear that only one number has to be updated. For each entry in the input array, find between which 2 numbers in our dp array that it lies in, and update the larger number to the input array element. Two edge cases are there though, When our input array element is smaller than our whole dp array, just update the first element. If it's greater than our whole dp array, append the array element to our dp array, as we have a new length of increasing subsequence we can achieve. This algorithm is $O(n \log n)$, as for each element of our input array we take $O(\log n)$ time in binary search.

17. Let $dp[i]$ represent the solution in the subarray from i to n. We iterate backwards, and let's say $dp[i + 1]$ to $dp[n]$ are calculated. Now for $dp[i]$, either we select $arr[i]$ or not. If we don't select $arr[i]$, the optimal sum is just $dp[i+1]$. If we select $arr[i]$ then we can't select $arr[i + 1]$, so we just have to select elements from index $i + 2$ to n such that none of them are consecutive. This is just $dp[i + 2]$. Finally our recursion is $dp[i] = \max(dp[i + 1], arr[i] + dp[i + 2])$. This takes $O(n)$ time overall. As we have n $O(1)$ calculations.

Flavour 2: where the subproblem is on a substring of the input.

18. We case split based on what't the last multiplication we should do. If we multiply $A_1A_2 \ldots A_i$ with A_{i+1} $A_{i+2} \ldots A_n$ last, what's our cost? The first term has dimension $p_1 \times p_{i+1}$ and the second term has dimensions $p_{i+1} \times p_{n+1}$ so the cost is $p_1p_{i+1}p_{n+1}$. We take cases for all i, and also, we have the subproblems to solve, how get the 2 terms most efficiently. Overall our recursion is like this:

$$dp[1..n] = \min_i(dp[1..i] + p_1p_{i+1}p_{n+1} + dp[i + 1..n]), i \in \{1, 2, \ldots, n - 1\}$$

In the recursion, we are always having some contiguous subarray, so there are at most $O(n^2)$ recursive calls. Our base case is that $dp[i..i] = 0$. Also one more thing in the recursion is that, the size of the subarray we call recursively is strictly decreasing. So to do this algo iteratively, we first store all the solutions for subarray of length 1, length 2, and so on, so that we can directly access solutions of smaller subarrays whenever using the recursive function. Also, each recursive call takes $O(n)$ time, as we have to find the minimum of $O(n)$ number of terms. So overall this algo takes $O(n^3)$ time.

19. We partition cases based on which element is the root of the tree. If a_k is the root, a_1 to a_{k-1} have to be in the left subtree, and a_{k+1} to a_n have to be in the right subtree. The left and right subtree have to separately be constructed optimally inorder for them for the total tree to be optimal. If the optimal sum for the left subtree alone is $dp[1..k] = \sum f_i d_i$, the contributing factor in the final tree will be $\sum f_i(d_i + 1)$ as each node has extra depth by 1. This simplifies to $\sum f_i d_i + \sum f_i$. A similar story holds for the right subtree. So the final recursion is:

$$dp[1..n] = \min_k \left((dp[1..k-1]) + \sum_{i=1}^{k-1} f_i + (dp[k+1..n]) + \sum_{k+1}^{n} f_i \right), k \in \{1, 2, \ldots, n\}$$

it might look like the $\sum f_i$ term will make the time compexity extremely bad, but this is just a sum of frequencies of some contiguous subarray. If we just do a precomputation of prefix sums, it becomes very easy to get these values ($O(1)$ time in fact). Again, our recursion only calls contiguous subarrays, so there are $O(n^2)$ recursive calls. For each recursive call, we have to find the maximum among $O(n)$ values, which takes $O(n)$ time. So overall, just like the previous question, the time complexity is $O(n^3)$.