Tutorial 1

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1. Determine if the following formulae are tautology.

   a) $(p \to q) \wedge (q \to r) \to (p \to r)$

   b) $(p \to q) \wedge (q \to r) \to (r \to p)$

   c) $(p_1 \wedge p_2 \dots p_n) \to (p_1 \vee p_2 \dots p_n)$

2. Consider a set S of size n. Recall that for a relation over S to be a partial order we require the following to hold:

   i Reflexive: $x \preceq x$ for all $x \in S$

   ii Transitive: $x \preceq y$ and $y \preceq z$ implies $x \preceq z$ for all $x, y, z \in S$

   iii Antisymmetric: $x \preceq y$ and $y \preceq x$ implies $x = y$ for all $x, y \in S$.

   Moreover, $x \in S$ is called a *maximal* element in $\preceq$ if $x \preceq y$ holds only for $y = x$.
   Consider n^2 propositional variables $\{p_{ij}\}_{1 \leq i,j \leq n}$ for an enumeration of $S = (x_1, x_2, \dots, x_n)$ where p_{ij} is set to 1 iff $x_i \preceq x_j$.
   Give a formula over these variables which evaluates to $\top$ iff $\preceq$ is a partial order. Also give a formula which evaluates to $\top$ only when $\preceq$ has a maximal element.

3. *Let n and k be integers so that $n > 0$, $k \geq 0$ and $k \leq n$. You are given n booleans x_1 through x_n and your goal is to come up with an efficient propositional encoding of the following constraint:*

$$\sum_{i=1}^{n} x_i \leq k \tag{1}$$

   In the above equation, assume the booleans x_i behaves as 0 when false and 1 when true. We can arrive at the encoding by adding O(n·k) auxiliary variables and only need O(n·k) clauses. We try to solve this problem iteratively.

   - *Let $s_{i,j}$ denote that at least j variables among $x_1, \dots, x_i$ are assigned 1 (true). One can deduce that $s_{i,j}$ makes sense only when $i \geq j$. We now try to represent this constraint in propositional logic.*
   - *Now given that you have represented $s_{i,j}$ for all appropriate $1 \leq i \leq n$ and $0 \leq j \leq k$; come up with a propositional formula that represents the condition of the equation (1)*

4. Two friends find themselves trapped in a room. There are three doors coloured - red, blue and green respectively. Behind exactly one of the doors is a path to their home :-). The other two doors lead to horrible places. The inscriptions on the three doors are as follows:
   Red door: "Your home town is not behind blue door"
   Blue door: "Your home town is behind red door"
   Green door: "Your home town is not behind blue door"
   Given the fact that at LEAST ONE of the inscriptions is true and at LEAST ONE of them is false, which door would lead the boys home?