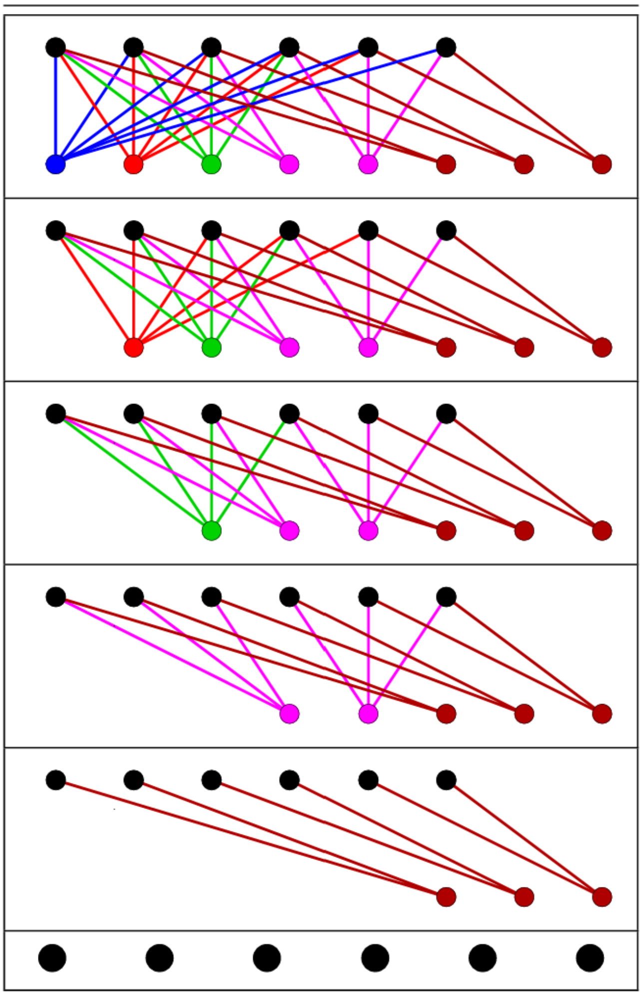
Linear Programming, Approximation, Randomized, Error correction

1. If you think about the algorithm, it selects and edge and then deletes the vertices of the edge, so it's going to go over a set of disjoint edges with no common vertex. Say it goes of k edges like this. We know that the minimum vertex cover size is at least k, as we have to select at least one vertex incident to each edge of this set, if not we won't cover the edge. And the number of vertices selected by our approximation algo is $2k$, so it is atmost twice the optimal solution.

It's easy to see that we are basically getting a matching of size k from this algorithm. We now have to prove that the maximum matching isn't bigger than $2k$, assume it's actually bigger. This would mean there's a match without having any vertex in the $2k$ vertex our algorithm returned. Why? Because each of the $2k$ vertex can be in atmost a single match (matches are vertex disjoint). But if there's an edge which doesn't have a vertex common with the $2k$ vertices, our algorithm didn't return a vertex cover, which is a contradiction.

2. Kinda complicated example 😅. It's a bipartite graph.



Let the first row of the graph be n vertices, for the figure above it is $n = 6$. For the bottom row, for every i from 2 to n, we will have $\lfloor n/i \rfloor$ of degree i, each of them connected to distinct vertices in the top row. So in the above figure we have 3 brown vertices of degree 2, 2 pink vertices of degree 3, 1 green, blue, red vertex of degree 4, 5, 6 respectively.

Now let's see what vertices our greedy algorithm chooses. Firstly all vertices in the top row have degree at most $n - 1$, as they can be connected to at most 1 degree 2 vertex, degree 3 vertex, ..., degree n vertex. But the bottom row has a degree n vertex right, so it will be chosen first. Now after deleting this vertex and all its edges, the top row all have degree at most $n - 2$, but the bottom row has a degree $n-1$ vertex, so it will be chosen. We can actually inductively prove that only bottom row vertices will be chosen.

Let's say the higher degrees from $k+1$ to n have already been deleted from the bottom row, and you only have degree 2 to degree k. We can claim that the top row vertices all have degree at most $k-1$, as each of them is connected to atmost 1 vertex of degree 2, 1 vertex of degree 3, ..., 1 vertex of degree k. But the bottom row has vertices of degree k so they'll get selected.

Now how many vertices are we selecting? This is just

$$\begin{aligned}\sum_{i=2}^{n} \lfloor n/i \rfloor &\geq \sum_{i=2}^{n}(n/i-1) \\ &= n\sum_{i=2}^{n}(1/i) - (n-1) \\ &= n\sum_{i=1}^{n}(1/i) - n\end{aligned}$$

For the optimal vertex cover, we can just select the top row completely, which has n vertices. So the approximation ratio is going to be more than $\sum_{i=1}^{n}(1/i) - 1$, which is unbounded as the harmonic series diverges.

3. Here's the problem, we have vertices v_1 to v_n with some weights w_1 to w_n, we just want a vertex cover with minimum sum of weights for all vertices selected. Our linear program for this is going to be as follows, we'll introduce new variables x_1 to x_n which kind of denote if the vertices are chosen or not. For every edge v_iv_j, we'll add a constraint $x_i + x_j \geq 1$ which is like saying one of the vertices of each edge should be chosen. We want to minimize $\sum_i w_i x_i$ (obvious constraint is $0 \leq x_i \leq 1$ for all i). And how we get an approximate solution from this linear program is by looking at the linear program's optimal solution, and selecting all v_i's which have x_i's $\geq 1/2$. This is a valid vertex cover as if $x_i + x_j \geq 1$, one of $x_i, x_j \geq 1/2$ which means every edge has a vertex selected.

For the first line inequalities, it's obvious that the optimal solution can give a feasible solution to the linear program, by just making x_i's 1 if v_i's are selected in the optimal solution, and 0 otherwise. Since this satisfies all the constraints it's feasible and will have weighted sum at least as much as the optimal solution of the linear program. Similarly the optimal vertex cover we get from the linear program is some vertex cover, so will have weighted sum at least as much as the optimal vertex cover, as the optimal vertex cover by definition is the vertex cover with the least weighted sum.

Now for the second line inequality. The key observation is that when we modify the optimal values for the linear program to get a vertex cover, we are not more than doubling the weighted sum. Each $x_i \geq 1/2$ is increased to 1, so at most the corresponding term of the weighted sum is atmost doubled. The other x_i's are made to 0, but even assuming that decrease never happens, the overall weighted sum is atmost doubled. This means the vertex cover approximation algo is at most twice as bad as the lienar program optimal solution.

4. For each of the edges e_1 to e_n, we keep variables x_1 to x_n with the constraints $0 \leq x_i \leq 1$ for all i. For any 2 edges e_i and e_j which share a common vertex, $x_i + x_j \leq 1$. The linear program is to optimize $\sum_i w_i x_i$. Take the graph as a triangle with all the weights as 1. If we keep $x_1 = x_2 = x_3 = 1/2$, we get the maximum value of the linear program as $3/2$, but obviously the maximum weight matching is only 1.

5. Let there be n loads, and m processors, and let the optimal max load be T^*. There are 2 obvious inequalities related to T^*. One is that T^* is at least the max load, as it has to be placed somewhere. The other is that T^* is at least the average load on a processor i.e. $\left(\sum_i t_i\right)/m$ as the max load is at least the average load. Let us think about what's the load on a processor each time we place a load.

Case 1: It's the first load for that processor, in this case the load on this process will be $\leq T^*$ for sure, as T^* is at least the max load.

Case 2: There's at least one other load for that processor. Note that this has to be at least the $(m+1)^{th}$ load, as the greedy algo will place the first m loads on different processes. But if there are that many loads, out the first $m+1$ loads, 2 of them have to be on the same processor in the optimal solution meaning $T^* \geq 2t_{m+1}$. And the load we're placing right is at most t_{m+1} so current load $\leq T^*/2$. But what about the rest of the loads that were already there on the processor? Our greedy algo has chosen the processor with the minimum load, so it's gonna be at most the average load at the moment, which is at most the average load after placing all loads, which is at most T^*. So finally load on that processor $\leq T^*/2 + T^* = 3T^*/2$.

Turns out this approximation factor of 3/2 isn't tight, it can be proven that the greedy algo in descending order of loads actually has an approximation factor of atmost 4/3.

6. Our linear program will have variables $a_1, a_2, \ldots, a_d, b, E$ (E is an extra variable), and our goal is to minimize E. We will have $2n$ constraints, they are of the form $h(p_j) - l_j \leq E$, and $l_j - h(p_j) \leq E$, and we have this for all j from 1 to n. These are clearly linear equations once we expand $h(p_j)$ as $\sum_i a_i x_j^{(i)} + b$, but why does this work? The 2 constraints $h(p_j) - l_j \leq E$, and $l_j - h(p_j) \leq E$ basically say that $|h(p_j) - l_j| \leq E$, and if we want to combine the constraints for all j, it becomes $\max_j |h(p_j) - l_j| \leq E$. But in our linear program E is free i.e. independent of all other parameters, but our objective is to minimize it. So in our optimal solution equality will be attained.

7. Basically the same thing as the previous question, we'll have variables $a_{i,j}, 1 \leq i \leq j \leq d$ and E. Our goal is to minimize E. We'll have $2n$ constraints, which are $h(p_j) - l_j \leq E$, and $l_j - h(p_j) \leq E$, for all j from 1 to n. By the same logic, the equality will be attained in the optimal solution for E.

8. The obvious variables which will be in our linear program are $a_1, a_2, \ldots, a_d, b$. But we'll add some extra variables. Here for each positively labelled point p_j we introduce a variable L_j in our linear program. We add the constraints that $L_j \geq 1 - h(p_j)$, and $L_j \geq 0$. Similarly for every negatively labelled point p_k we introduce a variable L_k with the constraints $L_k \geq 1 + h(p_k)$, and $L_k \geq 0$. Our objective in the linear program to minimize $\sum_j L_j + \sum_k L_k$. The optimal solution will have some equality for every L_j and L_k i.e $L_j = \max(1 - h(p_j), 0)$ and $L_k = \max(1 + h(p_k), 0)$. Because if not, they can be decreased more. This would mean the optimal value of our linear program is exactly the hinge loss.

9. X_i is a binary random variable, $E(X_i) = 1(1/i) + 0(1 - 1/i) = 1/i$. By law of expectation, $E(X) = E(\sum_i X_i) = \sum_i E(X_i) = \sum_{i=1}^{n} 1/i$.

10. Let X be a random variable representing number of intersections. We know $E(X) \leq 2n$. From Markov's inequality, we know that $P(X \geq 10n) \leq E(X)/10n \leq 2n/10n = 1/5$. So since $P(X \geq 10n) \leq 1/5$, $P(X < 10n) \geq 4/5$.

11. No, it doesn't work, let's construct a counter example. We have the standard constraints $x \geq 0, y \geq 0$. We will have n different constraints $L_1, L_2, \ldots, L_n$ where L_i is $x/(n+1-i) + y/i \leq 1$. Say our objective function is just to maximise x.

Let's see how many times our intersection point is updated if the order of constraints we have is $i, i+1, \ldots, n, 1, 2, \ldots, i-1$. With just the first line, our optimal point is clearly $(n+1-i, 0)$. But this won't satisfy the second constraint, so our optimal point will get updated to $(n-i, 0)$. This pattern will continue till we add L_n, where our optimal point will finally become $(1, 0)$ and then stop updating. So there are totally $n-i$ optimal point updates, and each time we update we have to intersect the new line with all the previous lines. So we'll have $1 + 2 + \cdots + (n-i) =$

$(n-i)(n-i+1)/2$ intersection calculations. So let's now calculate the expected number of intersections.

$$\begin{aligned}
\frac{1}{n}\sum_{i=1}^{n}\frac{(n-i)(n-i+1)}{2} &= \frac{1}{n}\sum_{i=1}^{n}\frac{(i)(i+1)}{2}\\
&= \frac{1}{2n}\sum_{i=1}^{n} i^2 + \frac{1}{2n}\sum_{i=1}^{n} i\\
&= \frac{(n)(n+1)(2n+1)}{12n} + \frac{(n)(n+1)}{4n}\\
&= \frac{(n+1)(2n+1)}{12} + \frac{n+1}{4}\\
&= \Theta(n^2)
\end{aligned}$$

This is clearly not $O(n)$ like our previous calculations.

12. Technically it isn't true in the 2D case that only 2 lines intersect at a corner of the feasible region. However, it actually doesn't matter, as even if it was the case, only 2 lines will be non-redundant. The one with the largest slope and the one with the smallest slope are the only non-redundant lines, if their constraints are satisfied it would automatically satisfy all the constraints.

But for the 3D case, you could have mutliple non-redundant constraints that a corner would satisfy. Take the top corner of a square pyramid (since this solid has only plane faces, it can be the feasible region of a linear program). The corner is part of 4 planes, and none of them are redundant. But for this question we will move the planes a bit such that no 4 planes intersect at a corner.

After we introduce the i^{th} the optimal point is going to be part of exactly 3 of the planes, so the proability that the final plane has the optimal point is $3/i$, so the probability the optimal point doesn't change is $1-3/i$. Suppose the optimal point changes, we know that it's going to be part of the new plane, by a similar proof as in the 2D case.

We have to now find the 2 other planes which intersect with this to give the optimal point. But since we know a plane where the optimal solution lies, we can just project every other plane onto this plane. We will get $i-1$ line constraints, for which we have to find the optimal point, which is just our 2D problem. If we use our 2D algorithm for this, the expected time to get the new point is $O(i)$.