

# CS 208 : Automata Theory and Logic

Spring 2024

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# Chapter 1

## Propositional Logic

In this course we look at two ways of computation: a state transition view and a logic centric view. In this chapter we begin with logic centered view with the discussion of propositional logic.

**Example.** Suppose there are five courses  $C_1, \dots, C_5$ , four slots  $S_1, \dots, S_4$ , and five days  $D_1, \dots, D_5$ . We plan to schedule these courses in three slots each, but we have also have the following requirements:

- For every course  $C_i$ , the three slots should be on three different days.
- Every course  $C_i$  should be scheduled in at most one of  $S_1, \dots, S_4$ .
- For every day  $D_i$  of the week, have at least one slot free.

	$D_1$	$D_2$	$D_3$	$D_4$	$D_5$
$S_1$					
$S_2$					
$S_3$					
$S_4$					

Propositional logic is used in many real-world problems like timetables scheduling, train scheduling, airline scheduling, and so on. One can capture a problem in a propositional logic formula. This is called as encoding. After encoding the problem, one can use various software tools to systematically reason about the formula and draw some conclusions about the problem.

### 1.1 Syntax

We can think of logic as a language which allows us to very precisely describe problems and then reason about them. In this language, we will write sentences in a specific way. The symbols used in propositional logic are given in Table 1.1. Apart from the symbols in the table we also use variables usually denoted by small letters  $p, q, r, x, y, z, \dots$  etc. Here is a short description of propositional logic symbols:

- **Variables:** They are usually denoted by smalls ( $p, q, r, x, y, z, \dots$  etc). The variables can take up only true or false values. We use them to denote propositions.
- **Constants:** The constants are represented by  $\top$  and  $\perp$ . These represent truth values true and false.

- **Operators:**  $\wedge$  is the conjunction operator (also called AND),  $\vee$  is the disjunction operator (also called OR),  $\neg$  is the negation operator (also called NOT),  $\rightarrow$  is implication, and  $\leftrightarrow$  is bi-implication (equivalence).

Name	Symbol	Read as
true	$\top$	top
false	$\perp$	bot
negation	$\neg$	not
conjunction	$\wedge$	and
disjunction	$\vee$	or
implication	$\rightarrow$	implies
equivalence	$\leftrightarrow$	if and only if
open parenthesis	(	
close parenthesis	)	

Table 1.1: Logical connectives.

For the timetable example, we can have propositional variables of the form  $p_{ijk}$  with  $i \in [5]$ ,  $j \in [5]$  and  $k \in [4]$  (Note that  $[n] = \{1, \dots, n\}$ ) with  $p_{ijk}$  representing the proposition ‘course  $C_i$  is scheduled in slot  $S_k$  of day  $D_j$ ’.

#### Rules for formulating a formula:

- Every variable constitutes a formula.
- The constants  $\top$  and  $\perp$  are formulae.
- If  $\varphi$  is a formula, so are  $\neg\varphi$  and  $(\varphi)$ .
- If  $\varphi_1$  and  $\varphi_2$  are formulas, so are  $\varphi_1 \wedge \varphi_2$ ,  $\varphi_1 \vee \varphi_2$ ,  $\varphi_1 \rightarrow \varphi_2$ , and  $\varphi_1 \leftrightarrow \varphi_2$ .

#### Propositional formulae as strings and trees:

Formulae can be expressed as strings over the alphabet  $\mathbf{Vars} \cup \{\top, \perp, \neg, \wedge, \vee, \rightarrow, \leftrightarrow, (, )\}$ .  $\mathbf{Vars}$  is the set of symbols for variables. Not all words formed using the alphabet qualify as propositional formulae. A string constitutes a well-formed formula (wff) if it was constructed while following the rules. Examples:  $(p_1 \vee \neg q_2) \wedge (\neg p_2 \rightarrow (q_1 \leftrightarrow \neg p_1))$  and  $p_1 \rightarrow (p_2 \rightarrow (p_3 \rightarrow p_4))$ .

Well-formed formulas can be represented using trees. Consider the formula  $p_1 \rightarrow (p_2 \rightarrow (p_3 \rightarrow p_4))$ . This can be represented using the parse tree in figure Figure 1.1a. Notice that while strings require parentheses for disambiguation, trees don’t, as can be seen in Figure 1.1b and Figure 1.1c.

## 1.2 Semantics

Semantics give a meaning to a formula in propositional logic. The semantics is a function that takes in the truth values of all the variables that appear in a formula and gives the truth value of the formula. Let 0 represent “false” and 1 represent “true”. The semantics of a formula  $\varphi$  of  $n$  variables is a function

$$\llbracket \varphi \rrbracket : \{0, 1\}^n \rightarrow \{0, 1\}$$



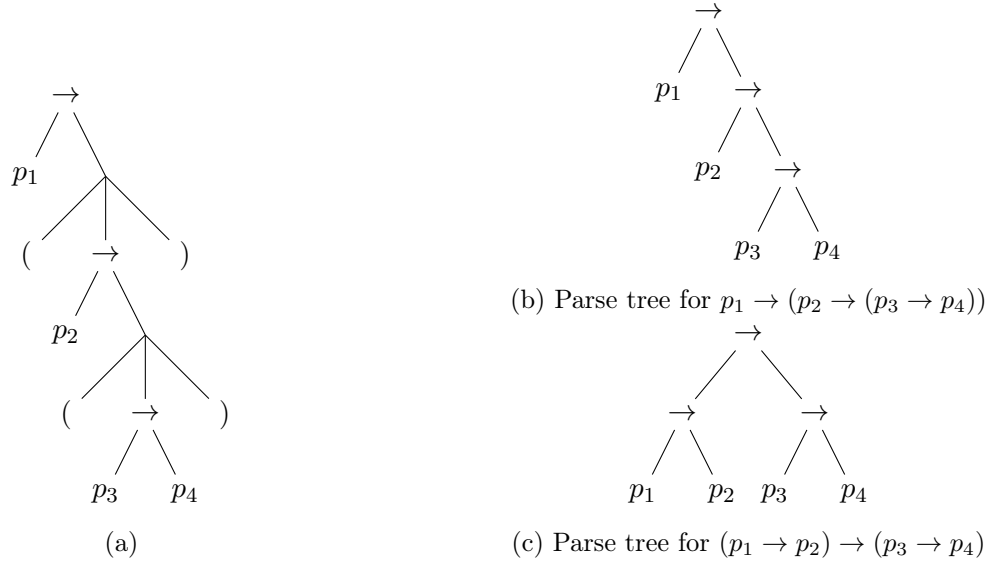


Figure 1.1: Parse trees obviate the need for parentheses.

It is often presented in the form of a truth table. Truth tables of operators can be found in table Table 1.2.

$\varphi$	$\neg\varphi$	$\varphi_1$	$\varphi_2$	$\varphi_1 \wedge \varphi_2$	$\varphi_1$	$\varphi_2$	$\varphi_1 \vee \varphi_2$
0	1	0	0	0	0	0	0
1	0	0	1	0	0	1	1
		1	0	0	1	0	1
		1	1	1	1	1	1

(a) Truth table for  $\neg\varphi$ . (b) Truth table for  $\varphi_1 \wedge \varphi_2$ . (c) Truth table for  $\varphi_1 \vee \varphi_2$ .

$\varphi_1$	$\varphi_2$	$\varphi_1 \rightarrow \varphi_2$	$\varphi_1$	$\varphi_2$	$\varphi_1 \leftrightarrow \varphi_2$
0	0	1	0	0	1
0	1	1	0	1	0
1	0	0	1	0	0
1	1	1	1	1	1

(d) Truth table for  $\varphi_1 \rightarrow \varphi_2$ . (e) Truth table for  $\varphi_1 \leftrightarrow \varphi_2$ .

Table 1.2: Truth tables of operators.

**Remark.** Do not confuse 0 and 1 with  $\top$  and  $\perp$ : 0 (false) and 1 (true) are meanings, while  $\top$  and  $\perp$  are symbols.

#### Rules of semantics:

- $\llbracket \neg\varphi \rrbracket = 1$  iff  $\llbracket \varphi \rrbracket = 0$ .
- $\llbracket \varphi_1 \wedge \varphi_2 \rrbracket = 1$  iff  $\llbracket \varphi_1 \rrbracket = \llbracket \varphi_2 \rrbracket = 1$ .
- $\llbracket \varphi_1 \vee \varphi_2 \rrbracket = 1$  iff at least one of  $\llbracket \varphi_1 \rrbracket$  or  $\llbracket \varphi_2 \rrbracket$  evaluates to 1.

- $\llbracket \varphi_1 \rightarrow \varphi_2 \rrbracket = 1$  iff at least one of  $\llbracket \varphi_1 \rrbracket = 0$  or  $\llbracket \varphi_2 \rrbracket = 1$ .
- $\llbracket \varphi_1 \leftrightarrow \varphi_2 \rrbracket = 1$  iff at both  $\llbracket \varphi_1 \rightarrow \varphi_2 \rrbracket = 1$  and  $\llbracket \varphi_2 \rightarrow \varphi_1 \rrbracket = 1$ .

**Truth Table:** A truth table in propositional logic enumerates all possible truth values of logical expressions. It lists combinations of truths for individual propositions and the compound statement's truth.

**Example.** Let us construct a truth table for  $\llbracket (p \vee s) \rightarrow (\neg q \leftrightarrow r) \rrbracket$  (see Table 1.3).

$p$	$q$	$r$	$s$	$p \vee s$	$\neg q$	$\neg q \leftrightarrow r$	$(p \vee s) \rightarrow (\neg q \leftrightarrow r)$
0	0	0	0	0	1	0	1
0	0	0	1	1	1	0	0
0	0	1	0	0	1	1	1
0	0	1	1	1	1	1	1
0	1	0	0	0	0	1	1
0	1	0	1	1	0	1	1
0	1	1	0	0	0	0	1
0	1	1	1	1	0	0	0
1	0	0	0	1	1	0	0
1	0	0	1	1	1	0	0
1	0	1	0	1	1	1	1
1	0	1	1	1	1	1	1
1	1	0	0	1	0	1	1
1	1	0	1	1	0	1	1
1	1	1	0	1	0	0	0
1	1	1	1	1	0	0	0

Table 1.3: Truth table of  $(p \vee s) \rightarrow (\neg q \leftrightarrow r)$ .

### 1.2.1 Important Terminology

A formula  $\varphi$  is said to (be)

- **satisfiable** or **consistent** or SAT iff  $\llbracket \varphi \rrbracket = 1$  for some assignment of variables. That is, there is at least one way to assign truth values to the variables that makes the entire formula true. Both a formula and its negation may be SAT at the same time ( $\varphi$  and  $\neg\varphi$  may both be SAT).
- **unsatisfiable** or **contradiction** or UNSAT iff  $\llbracket \varphi \rrbracket = 0$  for all assignments of variables. That is, there is no way to assign truth values to the variables that makes the formula true. If a formula  $\varphi$  is UNSAT then  $\neg\varphi$  must be SAT (it is in fact valid).
- **valid** or **tautology**:  $\llbracket \varphi \rrbracket = 1$  for all assignments of variables. That is, the formula is always true, no matter how the variables are assigned. If a formula  $\varphi$  is valid then  $\neg\varphi$  is UNSAT.
- **semantically entail**  $\varphi_1$  iff  $\llbracket \varphi \rrbracket \preceq \llbracket \varphi_1 \rrbracket$  for all assignments of variables, where  $0$  (false)  $\preceq 1$  (true). This is denoted by  $\varphi \models \varphi_1$ . If  $\varphi \models \varphi_1$ , then for every assignment, if  $\varphi$  evaluates to 1 then  $\varphi_1$  will evaluate to 1. Equivalently  $\varphi \rightarrow \varphi_1$  is valid.

- **semantically equivalent** to  $\varphi_1$  iff  $\varphi \models \varphi_1$  and  $\varphi_1 \models \varphi$ . Basically  $\varphi$  and  $\varphi_1$  have identical truth tables. Equivalently,  $\varphi \leftrightarrow \varphi_1$  is valid.
- **equisatisfiable** to  $\varphi_1$  iff either both are SAT or both are UNSAT. Also note that, semantic equivalence implies equisatisfiability but **not** vice-versa.

Term	Example
SAT	$p \vee q$
UNSAT	$p \wedge \neg p$
valid	$p \vee \neg p$
semantically entails	$\neg p \models p \rightarrow q$
semantically equivalent	$p \rightarrow q, \neg p \vee q$
equisatisfiable	$p \wedge q, r \vee s$

Table 1.4: Some examples for the definitions.

**Example.** Consider the formulas  $\varphi_1 : p \rightarrow (q \rightarrow r)$ ,  $\varphi_2 : (p \wedge q) \rightarrow r$  and  $\varphi_3 : (q \wedge \neg r) \rightarrow \neg p$ . The three formulas  $\varphi_1$ ,  $\varphi_2$  and  $\varphi_3$  are semantically equivalent. One way to check this is to construct the truth table.

On drawing the truth table for the above example, one would realise that it is laborious. Indeed, for a formula with  $n$  variables, the truth table has  $2^n$  entries! So truth tables don't work for large formulas. We need a more systematic way to reason about the formulae. That leads us to proof rules...

But before that let us get a closure on the example at the beginning of the chapter. Let  $p_{ijk}$  represent the proposition 'course  $C_i$  is scheduled in slot  $S_k$  of day  $D_j$ '. We can encode the constraints using the encoding strategy used in tutorial 1 - problem 3. That is, by introducing extra variables that bound the sum for first few variables (sum of  $i$  is atmost  $j$ ). Using this we can encode the constraints as :  $\sum_{k=1}^4 p_{ijk} \leq 1$ ,  $\sum_{j=1}^5 p_{ijk} \leq 1$ ,  $\sum_{i=1}^5 p_{ijk} \leq 1$ ,  $\sum_{k=1}^4 \sum_{i=1}^5 p_{ijk} \leq 3$ ,  $\sum_{k=1}^4 \sum_{j=1}^5 p_{ijk} \leq 3$  and  $\neg(\sum_{k=1}^4 \sum_{j=1}^5 p_{ijk} \leq 2)$ .

### 1.3 Proof Rules

After encoding a problem into propositional formula we would like to reason about the formula. Some of the properties of a formula that we are usually interested in are whether it is SAT, UNSAT or valid. We have already seen that truth tables do not scale well for large formulae. It is also not humanly possible to reason about large formulae modelling real-world systems. We need to delegate the task to computers. Hence, we need to make systematic rules that a computer can use to reason about the formulae. These are called as proof rules.

The overall idea is to convert a formula to a normal form (basically a standard form that will make reasoning easier - more about this later in the chapter) and use proof rules to check SAT etc.

Rules are represented as

$$\frac{\text{Premises}}{\text{Inferences}} \text{Connector}_{i/e}$$

- **Premise:** A premise is a formula that is assumed or is known to be true.

- **Inference:** The conclusion that is drawn from the premise(s).
- **Connector:** It is the logical operator over which the rule works. We use the subscript  $i$  (for introduction) if the connector and the premises are combined to get the inference. The subscript  $e$  (for elimination) is used when we eliminate the connector present in the premises to draw inference.

**Example.** Look at the following rule

$$\frac{\varphi_1 \wedge \varphi_2}{\varphi_1} \wedge_{e1}$$

In the rule above  $\varphi_1 \wedge \varphi_2$  is assumed (is premise). Informally, looking at  $\wedge$ 's truth table, we can infer that both  $\varphi_1$  and  $\varphi_2$  are true if  $\varphi_1 \wedge \varphi_2$  is true, so  $\varphi_1$  is an inference. Also, in this process we eliminate (remove)  $\wedge$  so we call this AND-ELIMINATION or  $\wedge_e$ . For better clarity we call this rule  $\wedge_{e1}$  as  $\varphi_1$  is kept in the inference even when both  $\varphi_1$  and  $\varphi_2$  could be kept in inference. If we use  $\varphi_2$  in inference then the rule becomes  $\wedge_{e2}$ .

Table 1.5 summarises the basic proof rules that we would like to include in our proof system.

Connector	Introduction	Elimination
$\wedge$	$\frac{\varphi_1 \quad \varphi_2}{\varphi_1 \wedge \varphi_2} \wedge_i$	$\frac{\varphi_1 \wedge \varphi_2}{\varphi_1} \wedge_{e1} \quad \frac{\varphi_1 \wedge \varphi_2}{\varphi_2} \wedge_{e2}$
$\vee$	$\frac{\varphi_1}{\varphi_1 \vee \varphi_2} \vee_{i1} \quad \frac{\varphi_2}{\varphi_1 \vee \varphi_2} \vee_{i2}$	$\frac{\varphi_1 \vee \varphi_2 \quad \varphi_1 \rightarrow \varphi_3 \quad \varphi_2 \rightarrow \varphi_3}{\varphi_3} \vee_e$
$\rightarrow$	$\frac{\boxed{\begin{array}{c} \varphi_1 \\ \vdots \\ \varphi_2 \end{array}}}{\varphi_1 \rightarrow \varphi_2} \rightarrow_i$	$\frac{\varphi_1 \quad \varphi_1 \rightarrow \varphi_2}{\varphi_2} \rightarrow_e$
$\neg$	$\frac{\boxed{\begin{array}{c} \varphi \\ \vdots \\ \perp \end{array}}}{\neg \varphi} \neg_i$	$\frac{\varphi \quad \neg \varphi}{\perp} \neg_e$
$\perp$		$\frac{\perp}{\varphi} \perp_e$
$\neg\neg$		$\frac{\neg\neg\varphi}{\varphi} \neg\neg_e$

Table 1.5: Proof rules.

In the  $\rightarrow_i$  rule, the box indicates that we can *temporarily* assume  $\varphi_1$  and conclude  $\varphi_2$  using no extra non-trivial information. The  $\rightarrow_e$  is referred to by its Latin name, *modus ponens*.

**Example 1.** We can now use these proof rules along with  $\varphi_1 \wedge (\varphi_2 \wedge \varphi_3)$  as the premise to conclude  $(\varphi_1 \wedge \varphi_2) \wedge \varphi_3$ .

$$\begin{array}{c}
 \frac{\varphi_1 \wedge (\varphi_2 \wedge \varphi_3)}{\varphi_2 \wedge \varphi_3} \wedge_{e2} \quad \frac{\varphi_1 \wedge (\varphi_2 \wedge \varphi_3)}{\varphi_1} \wedge_{e1} \\
 \frac{\varphi_2 \wedge \varphi_3}{\varphi_2} \wedge_{e1} \quad \frac{\varphi_2 \wedge \varphi_3}{\varphi_3} \wedge_{e2} \\
 \frac{\varphi_1 \quad \varphi_2}{\varphi_1 \wedge \varphi_2} \wedge_i \\
 \frac{\varphi_1 \wedge \varphi_2 \quad \varphi_3}{(\varphi_1 \wedge \varphi_2) \wedge \varphi_3} \wedge_i
 \end{array}$$

## 1.4 Natural Deduction

If we can begin with some formulas  $\phi_1, \phi_2, \dots, \phi_n$  as our premises and then conclude  $\varphi$  by applying the proof rules established we say that  $\phi_1, \phi_2, \dots, \phi_n$  syntactically entail  $\varphi$  which is denoted by the following expression, also called a *sequent*:

$$\phi_1, \phi_2, \dots, \phi_n \vdash \varphi.$$

We can also infer some formula using no premises, in which case the sequent is  $\vdash \varphi$ . Applying these proof rules involves the following general rule:

We can only use a formula  $\varphi$  at a point if it occurs prior to it in the proof and if **no box enclosing that occurrence of  $\varphi$  has been closed already**.

**Example.** Consider the following proof of the sequent  $\vdash p \vee \neg p$ :

1.	$\neg(p \vee \neg p)$	assumption
2.	$p$	assumption
3.	$p \vee \neg p$	$\vee_{i1} 2$
4.	$\perp$	$\neg_e 3, 1$
5.	$\neg p$	$\neg_i 2-4$
6.	$p \vee \neg p$	$\vee_{i2} 5$
7.	$\perp$	$\neg_e 6, 1$
8.	$\neg\neg(p \vee \neg p)$	$\neg_i 1-7$
9.	$p \vee \neg p$	$\neg\neg_e 8$

**Example.** Proof for  $p \vdash \neg\neg p$  which is  $\neg\neg_i$ , a derived rule

1.	$p$	premise
2.	$\neg p$	assumption
3.	$\perp$	$\neg_e 1, 2$
4.	$\neg\neg p$	$\neg_i 2-3$

**Example.** A useful derived rule is *modus tollens* which is  $p \rightarrow q, \neg q \vdash \neg p$ :

1.	$p \rightarrow q$	premise
2.	$\neg q$	premise
3.	$p$	assumption
4.	$q$	$\rightarrow_e$ 3,1
5.	$\perp$	$\neg_e$ 4,2
6.	$\neg p$	$\neg_i$ 3-5

**Example.**  $\neg p \wedge \neg q \vdash \neg(p \vee q)$ :

1.	$\neg p \wedge \neg q$	premise
2.	$p \vee q$	assumption
3.	$p$	assumption
4.	$\neg p$	$\wedge_{e1}$ 1
5.	$\perp$	$\neg_e$ 3,4
6.	$p \rightarrow \perp$	$\rightarrow_i$ 3-5
7.	$q$	assumption
8.	$\neg q$	$\wedge_{e2}$ 1
9.	$\perp$	$\neg_e$ 7,8
10.	$q \rightarrow \perp$	$\rightarrow_i$ 7-9
11.	$\perp$	$\vee_e$ 2,6,10
12.	$\neg(p \vee q)$	$\neg_i$ 2-11

## 1.5 Soundness and Completeness of our proof system

A proof system is said to be sound if everything that can be derived using it matches the semantics.

**Soundness:**  $\Sigma \vdash \varphi$  implies  $\Sigma \models \varphi$

The rules that we have chosen are indeed individually sound since they ensure that if for some assignment the premises evaluate to 1, so does the inference. Otherwise they rely on the notion of contradiction and assumption. Hence, soundness for any proof can be shown by inducting on the length of the proof.

A complete proof system is one which allows the inference of *every* valid semantic entailment:

**Completeness:**  $\Sigma \models \varphi$  implies  $\Sigma \vdash \varphi$

Let's take some example of semantic entailment.  $\Sigma = \{p \rightarrow q, \neg q\} \models \neg p$ .

$p$	$q$	$p \rightarrow q$	$\neg q$	$\neg p$
0	0	1	1	1
0	1	1	0	1
1	0	0	1	0
1	1	1	0	0

As we can see, whenever both  $p \rightarrow q$  and  $\neg q$  are true,  $\neg p$  is true. The question now is how do we derive this using proof rules? The idea is to ‘mimic’ each row of the truth table. This means that we assume the values for  $p$ ,  $q$  and try to prove that the formulae in  $\Sigma$  imply  $\varphi$ <sup>1</sup>. And to prove an implication, we can use the  $\rightarrow_i$  rule. Here’s an example of how we can prove our claim for the first row:

1.	$\neg p$	given
2.	$\neg q$	given
3.	$(p \rightarrow q) \wedge \neg q$	assumption
4.	$\neg p$	1
5.	$((p \rightarrow q) \wedge \neg q) \rightarrow \neg p$	$\rightarrow_i$ 3,4

Similarly to mimic the second row, we would like to show  $\neg p, q \vdash ((p \rightarrow q) \wedge \neg q) \rightarrow \neg p$ . Actually for every row, we’d like to start with the assumptions about the values of each variable, and then try to prove the property that we want.

1.	$\neg p$	given
2.	$\neg q$	given
3.	$(p \rightarrow q) \wedge \neg q$	assumption
4.	$\neg p$	1
5.	$((p \rightarrow q) \wedge \neg q) \rightarrow \neg p$	$\rightarrow_i$ 3,4

1.	$\neg p$	given
2.	$q$	given
3.	$(p \rightarrow q) \wedge \neg q$	assumption
4.	$\neg p$	1
5.	$((p \rightarrow q) \wedge \neg q) \rightarrow \neg p$	$\rightarrow_i$ 3,4

1.	$p$	given
2.	$\neg q$	given
3.	$(p \rightarrow q) \wedge \neg q$	assumption
4.	$p \rightarrow q$	$\wedge_e$ 3
5.	$q$	$\rightarrow_e$ 1,4
6.	$\perp$	$\neg_e$ 2,5
7.	$\neg p$	$\perp_e$ 6
8.	$((p \rightarrow q) \wedge \neg q) \rightarrow \neg p$	$\rightarrow_i$ 3,4,5,6,7

1.	$p$	given
2.	$q$	given
3.	$(p \rightarrow q) \wedge \neg q$	assumption
4.	$p \rightarrow q$	$\wedge_e$ 3
5.	$\neg q$	$\wedge_e$ 3
6.	$q$	$\rightarrow_e$ 1,4
7.	$\perp$	$\neg_e$ 2,6
8.	$\neg p$	$\perp_e$ 7
9.	$((p \rightarrow q) \wedge \neg q) \rightarrow \neg p$	$\rightarrow_i$ 3,4,5,6,7,8

Figure 1.2: Mimicking all 4 rows of the truth table

This looks promising, but we aren’t done, we have only proven our formula under all possible assumptions, but we haven’t exactly proven our formula from nothing given. But note that the reasoning we are doing looks a lot like case work, and we can think of the  $\vee_e$  rule. In words, this rule states that if a formula is true under 2 different assumptions, and one of the assumptions is always true, then our formula is true. So if we just somehow rigorously show at least one of our row assumptions is always true, we will be able to clean up our proof using the  $\vee_e$  rule.

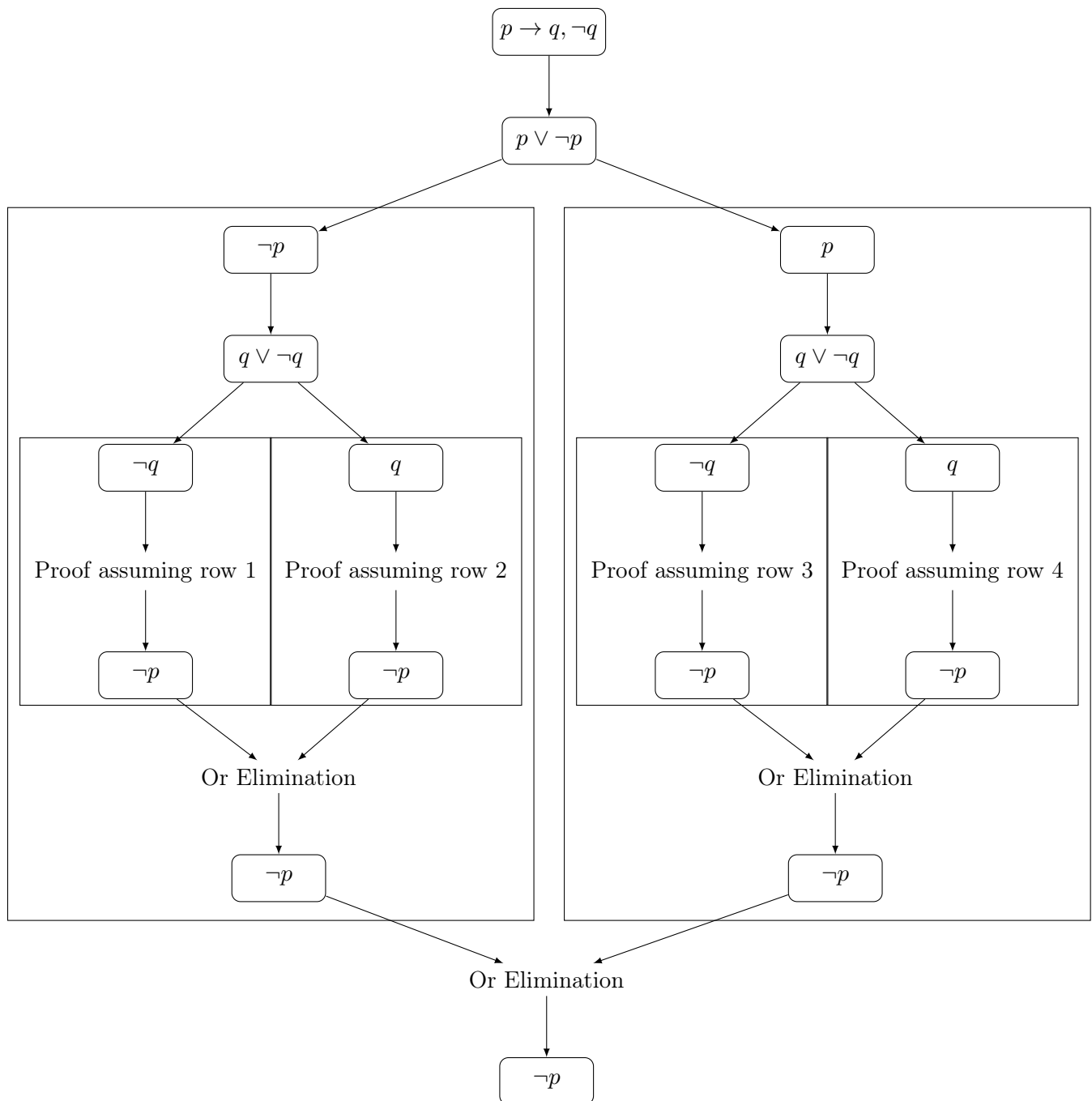
But as seen above, we were able to show a proof for the sequent  $\vdash \varphi \vee \neg \varphi$ . If we just recursively apply this property for all the variables we have, we should be able to capture every row of truth table. So combining this result, our proofs for each row of the truth table, and the  $\vee_e$  rule, the whole proof is constructed as below. The only thing we need now is the ability to construct proofs for each row given the general valid formula  $\bigwedge_{\phi \in \Sigma} \phi \rightarrow \varphi$ .

This can be done using **structural induction** to prove the following:

Let  $\varphi$  be a formula using the propositional variables  $p_1, p_2, \dots, p_n$ . For any assignment to these variables define  $\hat{p}_i = p_i$  if  $p_i$  is set to 1 and  $\hat{p}_i = \neg p_i$  otherwise, then:

$$\begin{aligned} \hat{p}_1, \hat{p}_2, \dots, \hat{p}_n \vdash \varphi & \text{ is provable if } \varphi \text{ evaluates to 1 for the assignment} \\ \hat{p}_1, \hat{p}_2, \dots, \hat{p}_n \vdash \neg \varphi & \text{ is provable if } \varphi \text{ evaluates to 0 for the assignment.} \end{aligned}$$

<sup>1</sup> $\Sigma$  semantically entails  $\varphi$  is equivalent to saying intersection of formulae in  $\Sigma$  implies  $\varphi$  is valid



## 1.6 What about Satisfiability?

Using Natural Deduction, we can only talk about the formulas that are a contradiction or valid. But there are formulas that are neither i.e. they are satisfiable for some assignment of variables but



not all. Example, for some  $p$  and  $q$ ,

$$\not\models p \wedge q$$

But clearly,  $p \wedge q$  is satisfiable when both  $p$  and  $q$  are true. Natural deduction can only claim statements like,

$$\begin{aligned} &\vdash \neg p \rightarrow \neg(p \wedge q) \\ &\vdash (p \rightarrow (q \rightarrow (p \wedge q))) \end{aligned}$$

An important link between the two situations is that,

A formula  $\phi$  is valid **iff**  $\neg\phi$  is not satisfiable

## 1.7 Algebraic Laws and Some Redundancy

### 1.7.1 Distributive Laws

Here are some identities that help complex formulas to some required forms discussed later. These formulas are easily derived using the natural deduction proof rules as discussed above.

$$\begin{aligned} \phi_1 \wedge (\phi_2 \vee \phi_3) &\models (\phi_1 \wedge \phi_2) \vee (\phi_1 \wedge \phi_3) \\ \phi_1 \vee (\phi_2 \wedge \phi_3) &\models (\phi_1 \vee \phi_2) \wedge (\phi_1 \vee \phi_3) \end{aligned}$$

### 1.7.2 Reduction of bi-implication and implication

We also see that bi-implication and implication can be reduced to  $\vee, \wedge$  and  $\neg$  and are therefore redundant in the alphabet of our string.

$$\begin{aligned} \phi_1 \longleftrightarrow \phi_2 &\models (\phi_1 \rightarrow \phi_2) \wedge (\phi_2 \rightarrow \phi_1) \\ (\phi_1 \rightarrow \phi_2) &\models ((\neg\phi_1) \vee \phi_2) \end{aligned}$$

### 1.7.3 DeMorgan's Laws

Similar to distributive laws, the following laws (again easily provable via natural deduction) help reduce any normal string to a suitable form (discussed in the next section).

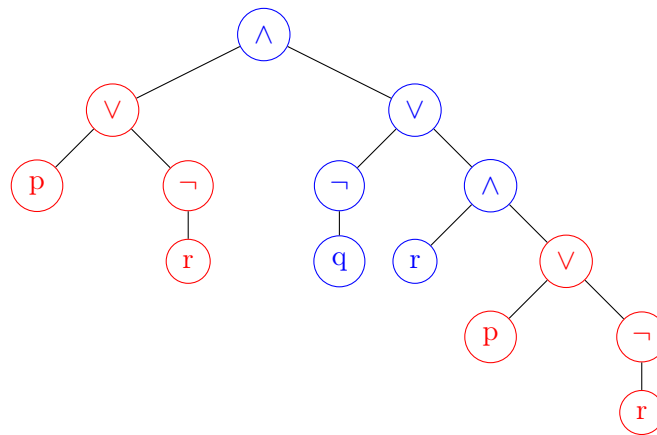
$$\begin{aligned} \neg(\phi_1 \wedge \phi_2) &\models (\neg\phi_1 \vee \neg\phi_2) \\ \neg(\phi_1 \vee \phi_2) &\models (\neg\phi_1 \wedge \neg\phi_2) \end{aligned}$$

## 1.8 Negation Normal Forms

In mathematical logic, a formula is in negation normal form (NNF) if the negation operator ( $\neg$ ) is only applied to variables and the only other allowed Boolean operators are conjunction ( $\wedge$ ) and disjunction ( $\vee$ ). One can convert any formula to a NNF by repeatedly applying DeMorgan's Laws to any clause that may have a  $\neg$ , until only the variables have the  $\neg$  operator. For example

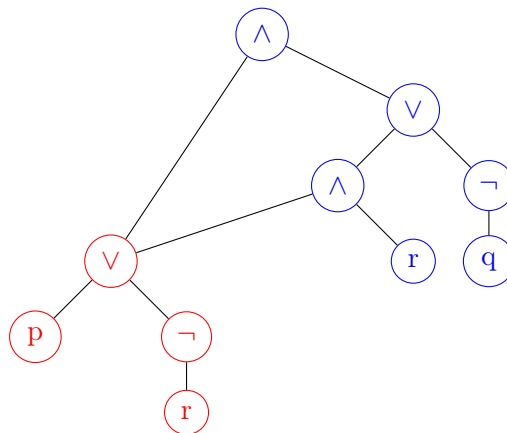
A NNF formula may be represented using its parse tree, which doesn't have any negation nodes except at the leaves, consider  $(p \vee \neg r) \wedge (\neg q \vee (r \wedge (p \vee \neg r)))$

**Parse Tree Representation** of the Above Example:



Since , the red part of the tree is repeating twice , we can make a **DAG(Directed Acyclic Graph)** instead of the parse tree.

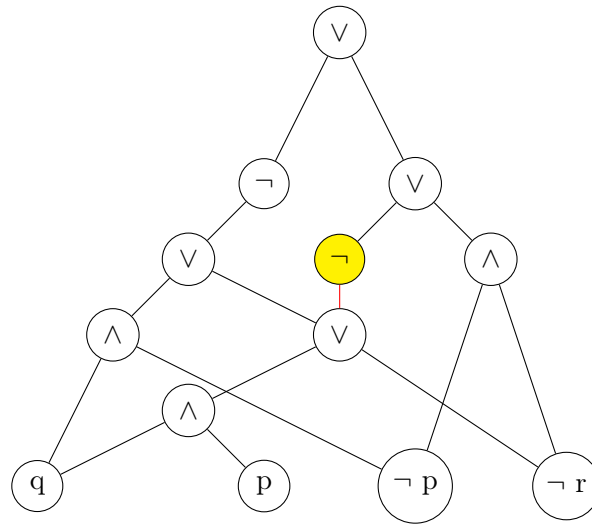
**DAG Representation**



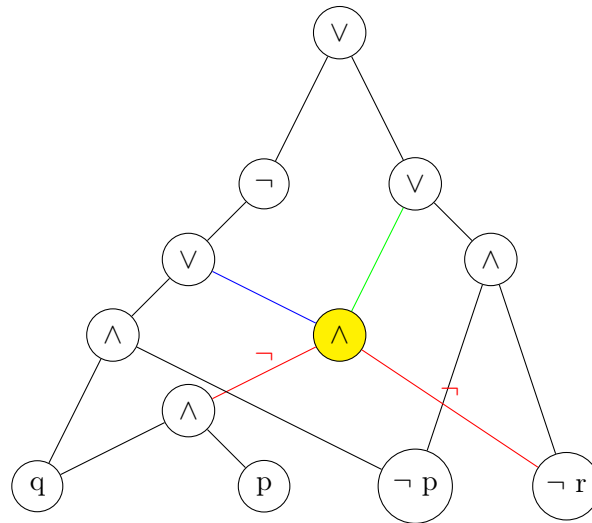
## 1.9 From DAG to NNF-DAG

Given a DAG of propositional logic formula with only  $\vee$  ,  $\wedge$  and  $\neg$  nodes , can we efficiently get a DAG representing a semantically equivalent NNF formula ?

Idea 1: Let's push the " $\neg$ " downwards by applying the De Morgans Law and see what happens.  
 Lets Consider the following example and the highlighted  $\neg$ .



Pushing down the highlighted  $\neg$  across the red edge.



Now, we have an issue in the blue edge. The blue edge wanted the non-negated tree node but due to the above mentioned change, it is getting the negated node. So, this idea won't work. We want to preserve the non-negated nodes as well.

Modification : Make two copies of the DAG and negate (i.e.,  $\neg$  pushing) only 1 of the copies and if a node wants non-negated node then take that node from the copied tree.

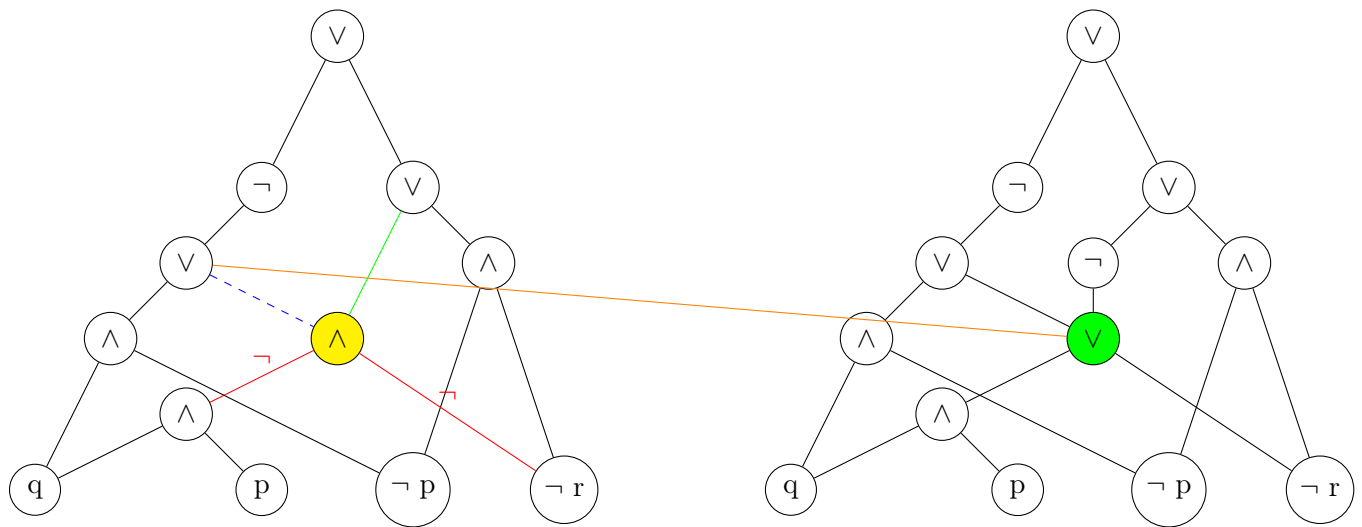


Figure 1.3: Step 1

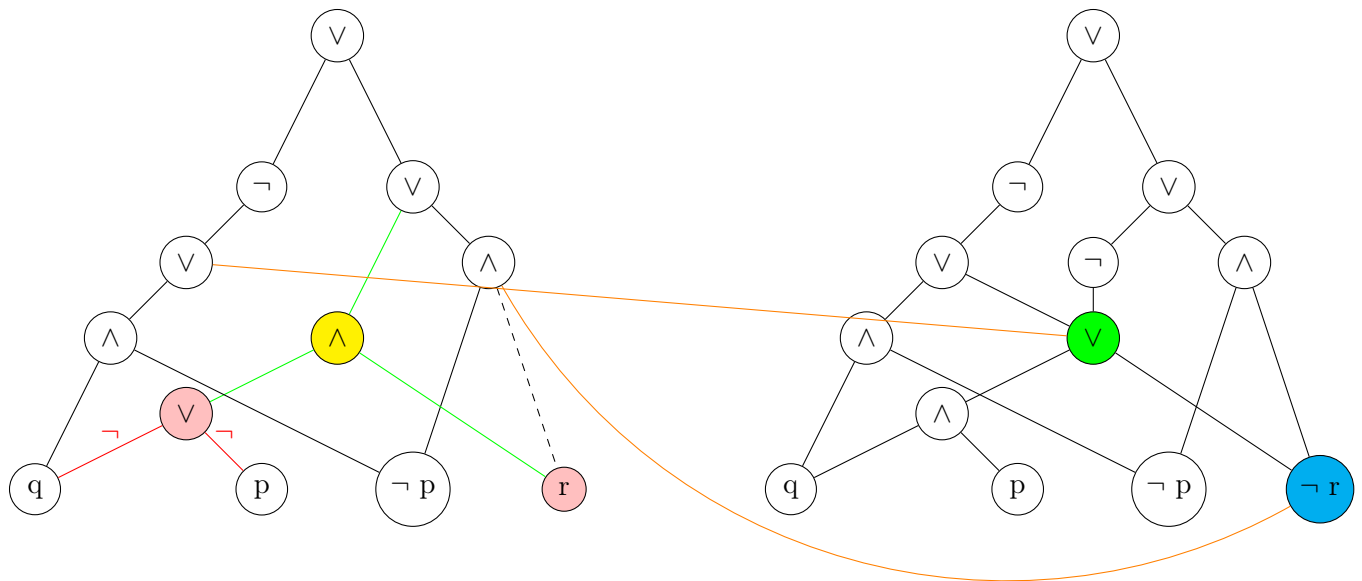


Figure 1.4: Step 2

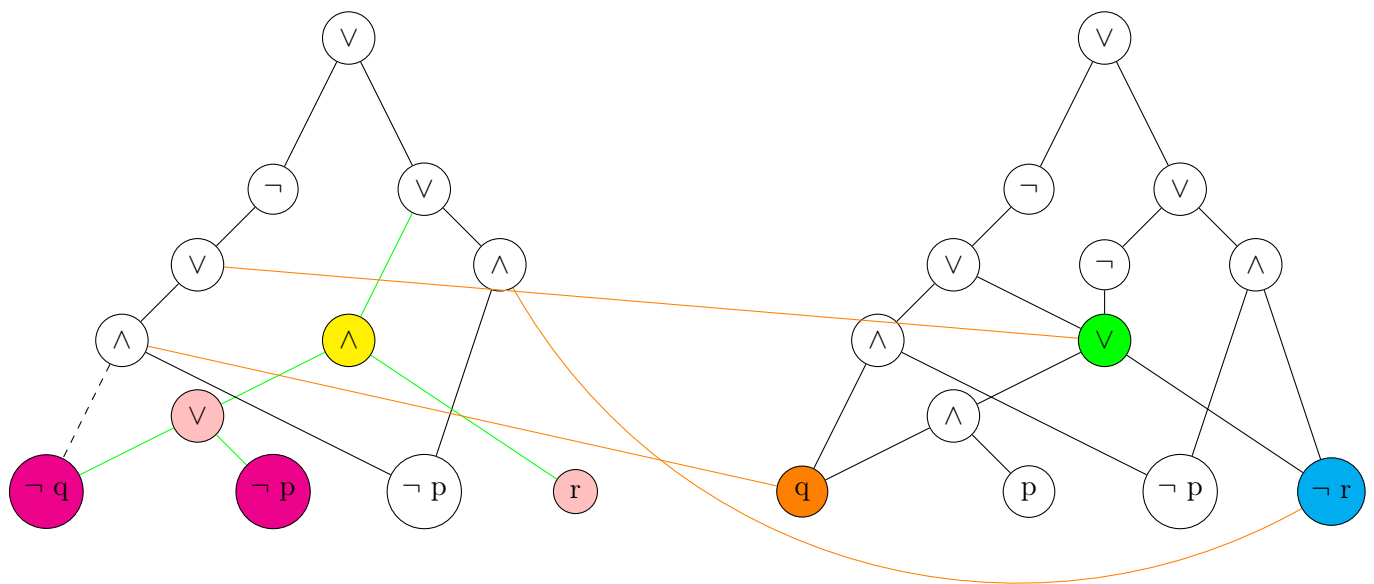


Figure 1.5: Step 3

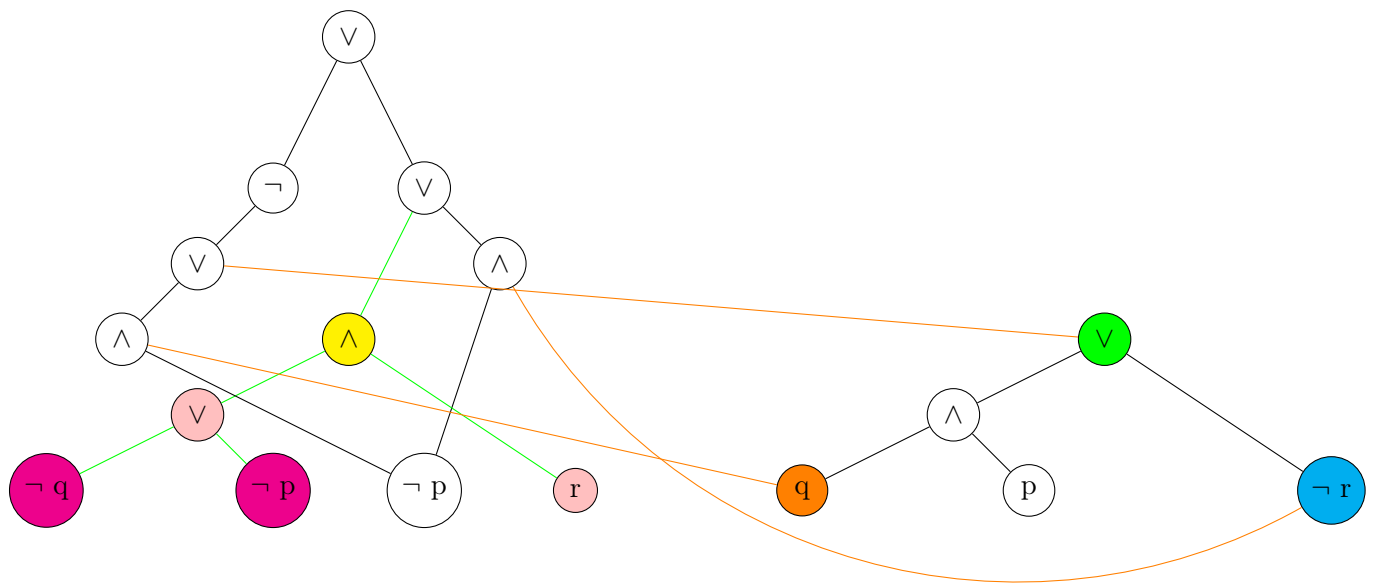


Figure 1.6: Step 4 : Remove all redundant nodes



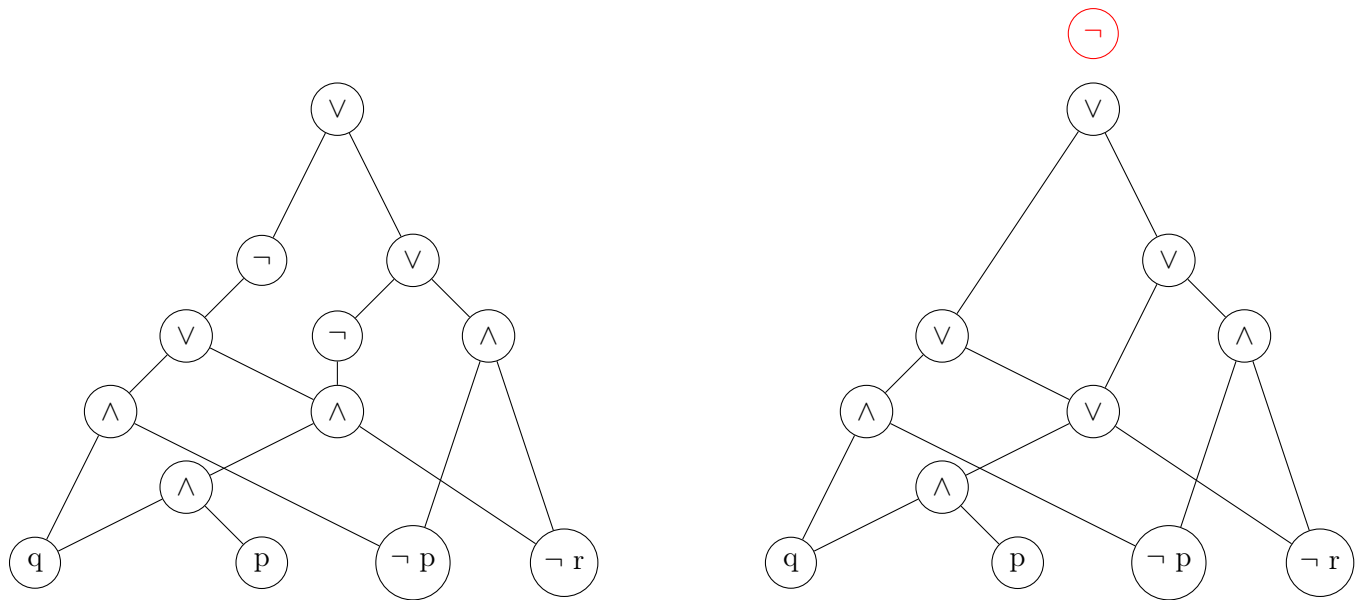


Figure 1.8: Step 2: Negate the entire DAG obtained in step 1

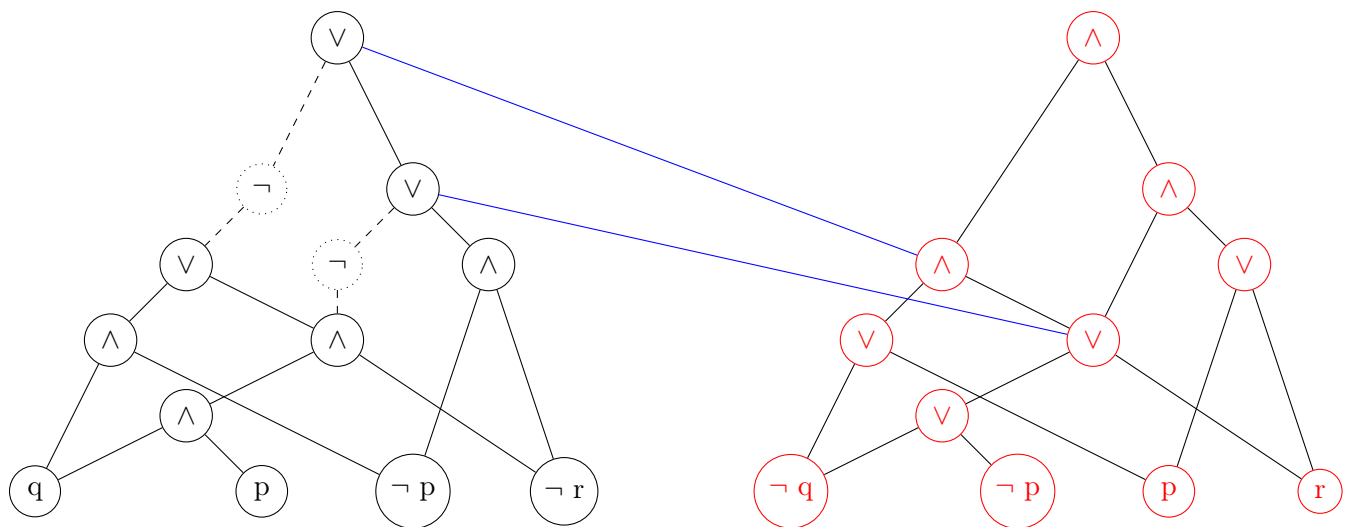
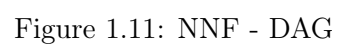
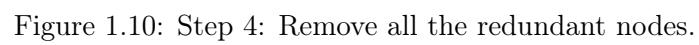


Figure 1.9: Step 3: Remove the " $\neg$ " nodes from the first DAG by connecting them to the corresponding node in the negated DAG.







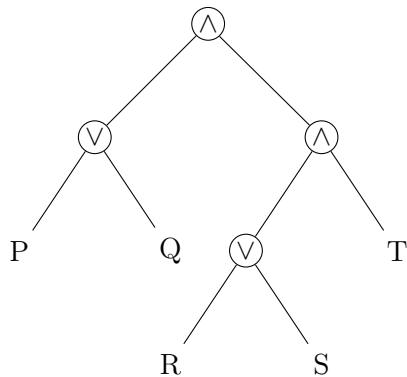
**NOTE :** The size of the NNF - DAG obtained using the above algorithm is atmost two times the size of the given DAG. Hence we have an  $O(N)$  formula for converting any arbitrary DAG to a semantically equivalent NNF - DAG.

## 1.11 Conjunctive Normal Forms

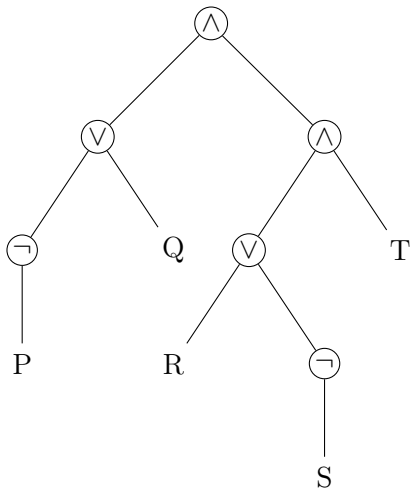
A formula is in conjunctive normal form or clausal normal form if it is a conjunction of one or more clauses, where a clause is a disjunction of variables.

**Examples:**

- Parse Tree for a formula in CNF



- Parse Tree for the formula  $(\neg(p \wedge \neg q) \wedge (\neg(\neg r \wedge s)) \wedge t)$  in NNF



## Some Important Terms

- **LITERAL :**
  - Variable or it's complement.
  - Example :  $p$  ,  $\neg p$  ,  $r$  ,  $\neg q$

- **CLAUSE :**

- A clause is a disjunction of literals such that a literal and its negation are not both in the same clause, for any literal
- Example :  $p \vee q \vee (\neg r)$  ,  $p \vee (\neg p) \vee (\neg r)$  -> Not Allowed

- **CUBE :**

- a Cube is a conjunction of literals such that a literal and its negation are not both in the same cube, for any literal
- Example  $p \wedge q \wedge (\neg r)$  ,  $p \wedge (\neg p) \wedge (\neg r)$  -> Not Allowed

- **CONJUNCTIVE NORMAL FORM(CNF)**

- A propositional formula is said to be in Conjunctive Normal Form (CNF) if it is a conjunction of clauses
- Product of Sums
- Example  $(p \vee q \vee (\neg r)) \wedge (q \vee r \vee (\neg s) \vee t)$

- **DISJUNCTIVE NORMAL FORM(DNF)**

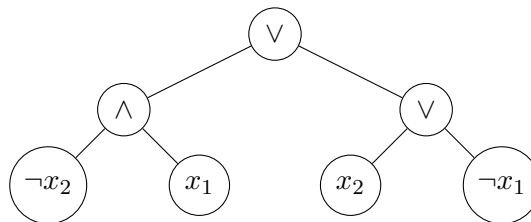
- A formula is said to be in Disjunctive Normal Form (DNF) if it is a disjunction of cubes.
- Sum of Products
- Example  $(p \wedge q \wedge (\neg r)) \vee (q \wedge r \wedge (\neg s) \wedge t)$

**Given a DAG of propositional logic formula with only  $\vee$  ,  $\wedge$  and  $\neg$  nodes , can we efficiently get a DAG representing a semantically equivalent CNF/DNF formula ?**

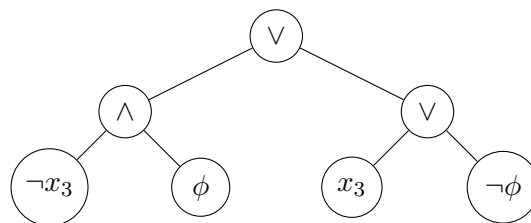
Tutorial 2 Question 2:

The Parity Function can be expressed as  $(( ( \dots (x_1 \oplus x_2 ) \oplus x_3 ) \dots \oplus x_n )$

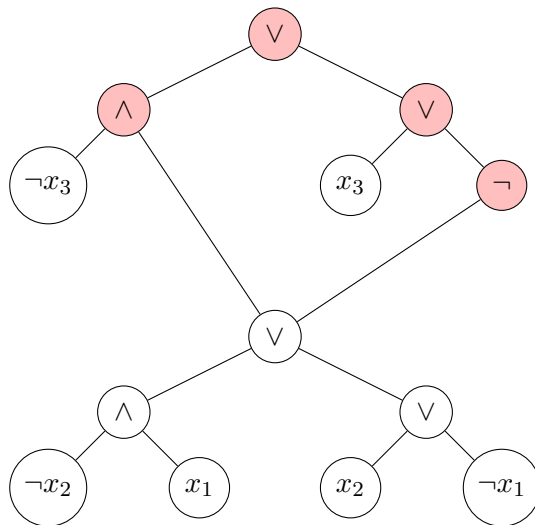
The Parse Tree for  $x_1 \oplus x_2$  is



Now consider  $\phi = x_1 \oplus x_2$  then Parse tree for  $(x_1 \oplus x_2) \oplus x_3$  will be



Now putting  $\phi = x_1 \oplus x_2$  in the tree, we get the following DAG representation



We notice that on adding  $x_i$  we are adding 4 nodes. Hence, the **size of DAG of the parity function is atmost  $4n$** . And we have already shown that size of NNF-DAG is atmost 2 times the size of DAG. So, the **size of the semantically equivalent NNF-DAG is atmost  $8n$** .

Also , in the Tutorial Question we have proved that the DAG **size of the semantically equivalent CNF/DNF formula is atleast  $2^{n-1}$** .

**NNF  $\rightarrow$  CNF/DNF (Exponential Growth in size of the DAG)**

## 1.12 Satisfiability and Validity Checking

It is **easy to check validity of CNF**. Check for every clause that it has some  $p$  ,  $\neg p$ . If **there is a clause which does not have both( $p$  ,  $\neg p$ )**, then the **Formula is not valid** because we can always assign variables in such a way that makes this clause false.

Example - If we have a clause  $p \vee q \vee (\neg r)$  , we can make it 0 with assignments  $p = 0$ ,  $q = 0$  and  $r = 1$ .

And if both a variable and its conjugate are present in a clause then since  $p \vee \neg p$  is valid and  $1 \vee \phi = 1$  for any  $\phi$ . So, the clause will be always valid.

Hence , we have an  **$O(N)$  algorithm to check the validity of CNF** but the price we pay is conversion to it(which is exponential).

It is hard to check validity of DNF because we will have to find an assignment which falsifies all the cubes.

What is meant by satisfiability?

Given a Formula , is there an assignment which makes the formula valid.

It is **easy to check satisfiability of DNF**. Check for every cube that it has some  $p$  ,  $\neg p$ . If there is a cube which does not have both( $p$  ,  $\neg p$ ), then the Formula is satisfiable because we can always assign variables in such a way that makes this cube true and hence the entire formula true. Overall, **We just need to find a satisfying assignment for any one cube.**

Hard to check satisfiability using CNF. We will have to find an assignment which simultaneously

satisfy all the cubes.

**NOTE :** A formula is valid if it's negation is not satisfiable. Therefore , we can convert every validity problem to a satisfiability problem. Thus, it suffices to worry only about satisfiability problem.

### 1.13 DAG to Equisatisfiable CNF

**Claim:** Given any formula, we can get an equisatisfiable formula in CNF of linear size efficiently.

*Proof:* Let's consider the DAG  $\phi(p, q, r)$  given below:

1. Introduce new variables  $t_1, t_2, t_3, \dots, t_n$  for each of the nodes. We will get an equisatisfiable formula  $\phi'(p, q, r, t_1, t_2, t_3, \dots, t_n)$  which is in CNF.
2. Write the formula for  $\phi'$  as a conjunction of subformulas for each node of form given below:

$$\begin{aligned}\phi' = & (t_1 \iff (p \wedge q)) \wedge \\ & (t_2 \iff (t_1 \vee \neg r)) \wedge \\ & (t_3 \iff (\neg t_2)) \wedge \\ & (t_4 \iff (\neg p \wedge \neg r)) \wedge \\ & (t_5 \iff (t_3 \vee t_4)) \wedge \\ & t_5\end{aligned}$$

3. Convert each of the subformula to CNF. For the first node it is shown below:

$$\begin{aligned}(t_1 \iff (p \wedge q)) &= (\neg t_1 \wedge (p \wedge q)) \wedge (\neg(p \wedge q) \vee t_1) \\ &= (\neg t_1 \vee p) \wedge (\neg t_1 \vee q) \wedge (\neg p \vee \neg q \vee t_1)\end{aligned}$$

4. For checking the equisatisfiability of  $\phi$  and  $\phi'$ : Think about an assignment which makes  $\phi$  true then apply that assignment to  $\phi'$ .

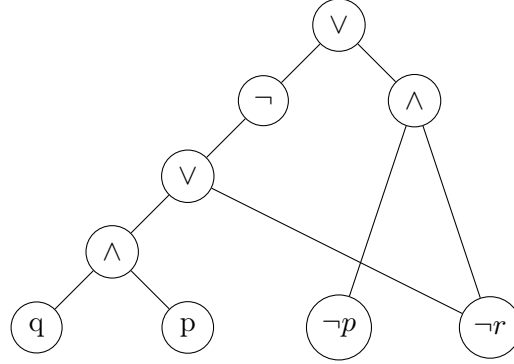
### 1.14 Tseitin Encoding

The Tseitin encoding technique is commonly employed in the context of Boolean satisfiability (SAT) problems. SAT solvers are tools designed to determine the satisfiability of a given logical formula, i.e., whether there exists an assignment of truth values to the variables that makes the entire formula true.

The basic idea behind Tseitin encoding is to introduce additional auxiliary variables to represent complex subformulas or logical connectives within the original formula. By doing this, the formula can be transformed into an equivalent CNF representation.

Say you have a formula  $Q(p, q, r, \dots)$ . Now we can use and introduce auxiliary variables  $t_1, t_2, \dots$  to make a new formula  $Q'(p, q, r, \dots, t_1, t_2, \dots)$  using Tseitin encoding which is equisatisfiable as  $Q$ .  $Q'$  is equisatisfiable as  $Q$  but not semantically equivalent. Size of  $Q'$  is linear in size of  $Q$ .

Lets take an example to understand better. Consider the formula  $(\neg((q \wedge p) \vee \neg r)) \vee (\neg p \wedge \neg r)$



We define a new equisatisfiable formula with auxiliary variables  $t_1, t_2, t_3, t_4$  and  $t_5$  as follows:

$$(t_1 \iff p \wedge q) \wedge (t_2 \iff t_1 \vee \neg r) \wedge (t_3 \iff \neg t_2) \wedge (t_4 \iff \neg p \wedge \neg r) \wedge (t_5 \iff t_3 \vee t_4) \wedge (t_5)$$

## 1.15 Towards Checking Satisfiability of CNF and Horn Clauses

A Horn clause is a disjunctive clause (a disjunction of literals) with at most one positive literal.

A Horn Formula is a conjunction of horn clauses, for example:

$$(\neg x_1 \vee \neg x_2 \vee x_3) \wedge (\neg x_4 \vee x_5 \vee \neg x_3) \wedge (\neg x_1 \vee \neg x_5) \wedge (x_5) \wedge (\neg x_5 \vee x_3) \wedge (\neg x_5 \vee \neg x_1)$$

Now we can convert any horn clause to an implication by using disjunction of the literals that were in negation form in the horn clause on left side of the implication and the unnegated variable on the other side of the implication. So all the variables in all the implications will be unnegated.

So the above equation can be translated as follows.

$$x_1 \wedge x_2 \implies x_3$$

$$x_4 \wedge x_3 \implies x_5$$

$$x_1 \wedge x_5 \implies \perp$$

$$x_5 \implies x_3$$

$$\top \implies x_5$$

Now we try to find a satisfying assignment for the above formula.

From the last clause we get  $x_5 = 1$ , now the fourth clause is  $\top \implies x_3$ .

Then from the fourth clause we get that  $x_3 = 1$ .

Now in the remaining clauses none of the left hand sides are reduced to  $\top$ .

Hence, we set all remaining variables to 0 to get a satisfying assignment.

**Algorithm 1:** HORN Algorithm

---

```

1 Function HORN( $\phi$ ):
2   foreach occurrence of  $\top$  in  $\phi$  do
3     | mark the occurrence
4   while there is a conjunct  $P_1 \wedge P_2 \wedge \dots \wedge P_{k_i} \rightarrow P'$  in  $\phi$  do
5     | // such that all  $P_j$  are marked but  $P'$  isn't
6     | if all  $P_j$  are marked and  $P'$  isn't then
7       | | mark  $P'$ 
8   if  $\perp$  is marked then
9     | return 'unsatisfiable'
10  else
11    | return 'satisfiable'

```

---

**Complexity:**

If we have  $n$  variables and  $k$  clauses then the solving complexity will be  $O(nk)$  as in worst case in each clause you search for each variable.

**1.16 Counter example for Horn Formula**

In our previous lectures, we delved into the Horn Formula, a valuable tool for assessing the satisfiability of logical formulas.

**1.16.1 Example**

We are presented with a example involving conditions that determine when an alarm ( $a$ ) should ring. Let's outline the given conditions:

1. If there is a burglary ( $b$ ) in the night ( $n$ ), then the alarm should ring ( $a$ ).
2. If there is a fire ( $f$ ) in the day ( $d$ ), then the alarm should ring ( $a$ ).
3. If there is an earthquake ( $e$ ), it may occur in the day ( $d$ ) or night ( $n$ ), and in either case, the alarm should ring ( $a$ ).
4. If there is a prank ( $p$ ) in the night ( $n$ ), then the alarm should ring ( $a$ ).
5. Also it is known that prank ( $p$ ) does not happen during day ( $d$ ) and burglary ( $b$ ) does not takes place when there is fire ( $f$ ).

Let us write down these implications

$$\begin{aligned}
 b \wedge n &\Rightarrow a & f \wedge d &\Rightarrow a & e \wedge d &\Rightarrow a \\
 e \wedge n &\Rightarrow a & p \wedge n &\Rightarrow a & d \wedge n &\Rightarrow \perp \\
 b \wedge f &\Rightarrow a & p \wedge d &\Rightarrow \perp
 \end{aligned}$$

Now we want to examine the possible behaviour of this system under the assumption that alarm rings during day. For this we add two more clauses:

$$\top \Rightarrow a \quad \top \Rightarrow d$$

This directly gives us that  $a, d$  have to be true, what about the rest? We can see that setting all the remaining variables to false is a satisfying assignment for this set of formulae.

Hence we have none of prank, earthquake, burglary or fire and hence alarm should not ring.

This means that our formulae system is incomplete.

To achieve this, we try to introduce new variables  $Na$  (no alarm),  $Nf$  (no fire),  $Nb$  (no burglary),  $Ne$  (no earthquake), and  $Np$  (no prank).

We extend the above set of implications in a natural way using these formulae:

$$a \wedge Na \Rightarrow \perp \quad b \wedge Nb \Rightarrow \perp \quad f \wedge Nf \Rightarrow \perp \quad e \wedge Ne \Rightarrow \perp \quad p \wedge Np \Rightarrow \perp$$

$$Nb \wedge Nf \wedge Ne \wedge Np \Rightarrow Na$$

All the implications will hold true for the values  $b = p = e = f = Nb = Ne = Nf = Np = 0$ .

Here, we are getting  $b = Nb$ , which is not possible, hence, we need the additional constraint that  $b \iff Nb$ . But on careful examination we see that this cannot be represented as a horn clause.

Therefore, it becomes necessary to devise an alternative algorithm suited for evaluating satisfiability.

## 1.17 Davis Putnam Logemann Loveland (DPLL) Algorithm

This works for more general cases of CNF formulas where it need not be a Horn formula. Let us first discuss techniques and terms required for our Algorithm.

- **Partial Assignment (PA) :** It is any assignment of some of the propositional variables.  
Ex.  $PA = \{x_1 = 1, x_2 = 0\}$  ;  $PA = \{\}$ , etc.
- **Unit Clause :** It is any clause which only has one literal in it. Ex.  $\dots \wedge (\neg x_5) \wedge \dots$   
Note: If any Formula has a unit clause then the literal in it has to be set to true.
- **Pure Literal :** A literal which doesn't appear negated in any clause. Say a propositional variable  $x$  appear only as  $\neg x$  in every clause it appears in., or say  $y$  appears only as  $y$  in every clause.  
Note: If there is a pure literal in the formula, it does not hurt any clause to set it to true. All the clauses in which this literal is present will become true immediately.

We will now utilize every techniques we learnt to simplify our formula. First we check if our formula has unit clause or not. If yes then we assign the literal in that clause to be 1. Note,  $\varphi[l = 1]$  is the formula obtained after setting  $l = 1$  everywhere in the formula. We also search for pure literals. If we find a pure literal then we can simply assign it 1 (or 0 if it always appears in negated form) and proceed, this cannot harm us (cause future conflicts) due to the definition of Pure Literal. If we do not have any of these then we have only one option left at the moment which is try and error.

We assign any one of the variable in the formula a value which we choose by some way (not described here). Then we go on with the usual algorithm until we either get the whole formula to



be true or false. At this step we might have to backtrack if the formula turns out to be false. If it is true we can terminate the algorithm.

**Note:** Our algorithm can be as worse as a **Truth Table** as we are trying every assignment. But as we are applying additional steps, after making a decision there are high chances that we get a unit clause or a pure literal.

Now as we have done all the prerequisites let us state the algorithm.

---

**Algorithm 2:** SAT( $\varphi$ , PA)

---

```

{ // These are the base cases for our recursion }
if  $\varphi = \top$  then
    return SAT(sat, PA)
else if  $\varphi = \perp$  then
    return SAT(unsat, PA)
else if  $C_i$  is a unit clause (literal  $l$ ) and  $C_i \in \varphi$  then
    { // This step is called Unit Propagation }
    return SAT( $\varphi[l = 1]$ ,  $PA \cup \{l = 1\}$ ) { // Here recursively call the algorithm on the simplified }
                                                    // formula  $\varphi[l = 1]$ 

else if  $l$  is a pure literal and  $l \in \varphi$  then
    { // This step is called Pure Literal Elimination }
    return SAT( $\varphi[l = 1]$ ,  $PA \cup \{l = 1\}$ )
else
    { // This step is called Decision Step }
     $x \leftarrow \text{choose\_a\_var}(\varphi)$ 
     $v \leftarrow \text{choose\_a\_value}(\{0, 1\})$ 
    if SAT( $\varphi[x = v]$ ,  $PA \cup \{x = v\}$ ).status = sat then
        return SAT(sat,  $PA \cup \{x = v\}$ )
    else if SAT( $\varphi[x = 1 - v]$ ,  $PA \cup \{x = 1 - v\}$ ).status = sat then
        return SAT(sat,  $PA \cup \{x = 1 - v\}$ )
    else
        return SAT(unsat, PA)
    end if
end if

```

---

**Question** Can the formula be a horn formula after steps 1 and 2 can't be applied anymore? i.e. If our formula does not have any unit clause or pure literal can it be a horn formula?

**ANS.** Yes. Here is an example

$$(a \vee \neg b) \wedge (\neg a \vee b)$$

## 1.18 DPLL in action

### 1.18.1 Example

Consider the following clauses, for which we have to find whether all can be satisfied for a variable mapping or not using DPLL algorithm-

$$\begin{array}{llll} C_1 : (\neg P_1 \vee P_2) & C_2 : (\neg P_1 \vee P_3 \vee P_5) & C_3 : (\neg P_2 \vee P_4) & C_4 : (\neg P_3 \vee P_4) \\ C_5 : (P_1 \vee P_5 \vee \neg P_2) & C_6 : (P_2 \vee P_3) & C_7 : (P_1 \vee P_3 \vee P_7) & C_8 : (P_6 \vee \neg P_5) \end{array}$$

Lets make two possible decision trees for these clauses.

PLE - Pure literal elimination      UP - Unit Propagation      D - Decision

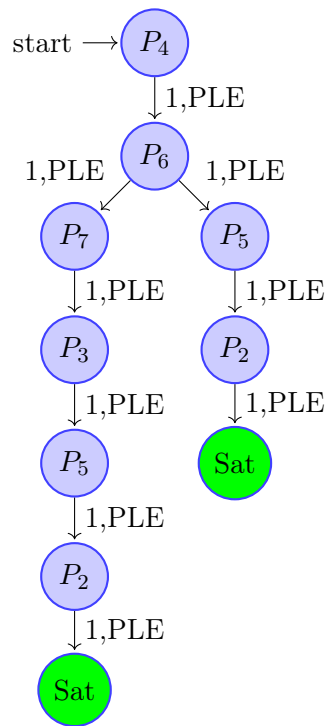


Figure 1.13: DPLL

The following is the decision tree if we remove the point 2 of DPLL. Note the increase in number of operations.

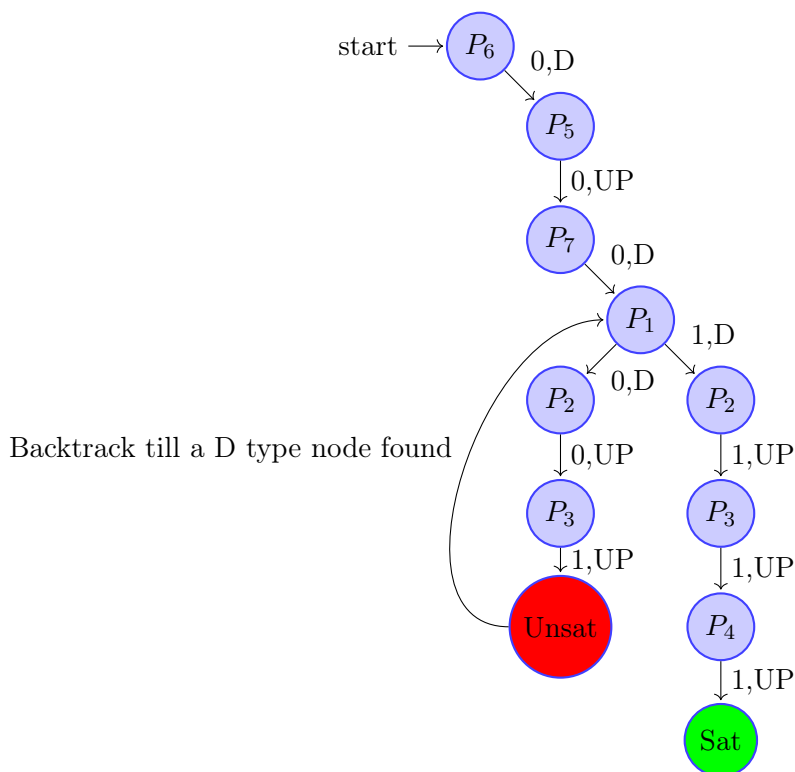


Figure 1.14: DPLL without point 2

## 1.19 Applying DPLL Algorithm to Horn Formulas

Let us apply DPLL algorithm to Horn Formula

If there are no variables on LHS, it becomes a Unit Clause *i.e.*  $\top \rightarrow x_i$  and is equivalent to  $(x_i)$ .

Horn Method can be viewed in terms of DPLL algorithm as following:

- Apply Unit Propagation until you can't apply.
- After that, set all remaining variables to 0.

Advantage of Horn's method is after all possible Unit Propagations are done, it sets all remaining variables to 0, but in DPLL we need to go step by step for each remaining variable.

But Horn's method can only be applied in a special case, moreover, in Horn's method we only figure out which variables to set true as opposed to DPLL which can figure out whether variable needs to be set to true or false via the Pure Literal Elimination.

## 1.20 DPLL on Horn Clauses

We shall quickly investigate what happens when we feed in **Horn clauses** to the **DPLL** (Davis-Putman-Logemann-Loveland) algorithm.

Consider the following,

- Consider the first step in solving for the satisfiability of a given set of Horn clauses in implication form where,
  - If the LHS of an implication is true we set the literal on the RHS of the implication to be true in all its implications.
  - Repeating the above step sets all essential variable which are to be set to 1, true.

This step is equivalent to the first two steps of the DPLL algorithm,

- Satisfy unit literal clauses by assignment.
- Recomputing the formula for the above bullet.
- Repeating this procedure until no unit clause is left.

The above steps in the two different schemes do the same are essentially doing the same thing. Now if the given clauses were Horn, we know that putting all the remaining variables false is a satisfying assignment. This means if our DPLL algorithm preferentially assigns 0 to each decision, the procedure thus converges to the method for checking the satisfiability for Horn formulae.

## 1.21 Rule of Resolution

This is yet another powerful rule for inference. Let us first jot down the rule here:

$$\frac{(a_1 \vee a_2 \vee a_3 \cdots \vee a_n \vee x) \wedge (b_1 \vee b_2 \vee b_3 \cdots \vee b_m \vee \neg x)}{(a_1 \vee a_2 \vee a_3 \cdots \vee a_n \vee b_1 \vee b_2 \vee b_3 \cdots \vee b_m)} \text{ resolution}$$

However intuitive it may look this rule poses as a powerful tool to check the satisfiability of logical formulae, we can reason out an algorithm to check the satisfiability of a formula (CNF) as follows: Let us first define a formula to be **unresolved** if there exists a literal and its negation in the formula (they cannot be in the same clause by the definition of a clause). If a formula is **resolved** (i.e., not unresolved) then it is satisfiable (**SAT**), as the variables which appear in their negated form we assign false, and the other variables to true.

Let  $\mathfrak{C}$  be the set of clauses for a give CNF.

1. If  $\mathfrak{C}$  contains tautologies we can drop them, if  $\mathfrak{C}$  becomes empty upon dropping the tautologies, we mark the given CNF **SAT**. (However by definition, clauses by themselves cannot be tautologies.)
2. As the formula is unresolved, we can apply the resolution rule, this gives us a new clause.
3. If the formula so formed is the empty clause, we deem the formula to be **UNSAT** otherwise check if the formula is resolved, if not from repeat step 1.

Before rationalizing the soundness of the above sequence of steps let us first see an example.

**An Example:** Consider  $\mathfrak{C} = \{C_1, C_2, C_3\}$  as given below:

- $C_1 := \neg p_1 \vee p_2 \ (p_1 \implies p_2)$
- $C_2 := p_1 \vee \neg p_2 \ (p_2 \implies p_1)$
- $C_3 := \neg p_1 \vee \neg p_2 \ (p_1 \wedge p_2 \implies \perp)$

Then, a dry run of the above method would look like:

1. Since both  $p_1$  and  $p_2$  appear in negated and un-negated form, we apply resolution on  $C_1$  and  $C_2$ , which generates  $C_4$ , as follows:

$$\frac{(\neg p_1 \vee p_2) \ (p_1 \vee \neg p_2)}{(\neg p_1 \vee p_1)} \text{ resolution}$$

2. Once again we apply resolution on  $C_3$  and  $C_4$  (this is not really a clause by definition, one can choose to drop the tautologies as soon as encountered),

$$\frac{(\neg p_1 \vee \neg p_2) \ (\neg p_1 \vee p_1)}{(\neg p_2 \vee \neg p_1)} \text{ resolution}$$

3. The clause that we have got is now resolved and thus, our formula is satisfiable.

### 1.21.1 Completeness of Resolution for Unsatisfiability of CNFs

Here as we claimed above, given any CNF, if it is unsatisfiable the remainder of continuous resolutions is the empty clause which we deem **UNSAT**. We here prove the consistency of the claim.

For this we employ the method of mathematical induction, we induct on the number of propositional variables in our CNF. Let  $p_1, p_2 \dots p_m$  be our propositional variables.

**Base:**  $n = 1$  An unsatisfiable CNF in a single literal **must** contain the clauses  $(p_1)$  and  $(\neg p_1)$  which upon resolution gives us  $(\ )$  the empty clause hence we raise **UNSAT**.

**Inductive Hypothesis:** Assume that our claim holds  $\forall m \leq n - 1$  we now show that it holds from  $m = n$  as follows,

- Remove tautologies from  $\mathfrak{C}$ , the set of all clauses.
- Choose a literal  $p_i$  such that both  $p_i$  and  $\neg p_i$  both appear in the CNF. (If no such literal exists the formula is resolved as defined earlier and has a satisfying assignment). Apply resolution repeatedly as long as the same  $p_i$  satisfies this condition.

- If the CNF contains ( ), in which case we raise **UNSAT**, otherwise
  - If  $p_i$  **vanishes** from the CNF, then calling our hypothesis, we can raise **UNSAT** as the equivalent form that we have got must be unsatisfiable independent of the value of the vanished literal as the initial formula was unsatisfiable.
  - If  $p_i$  **exists** in one of negated or un-negated forms. In which case we repeat the procedure. This time the number of available **pairs** has reduced by 1 as  $p_i$  cannot be selected again.
- As the selection step can take place at most  $n$  times, (as a new **pair**(as in bullet 2) cannot be generated in the CNF by resolution operations), Consider the case with a **pair** available for every literal then the procedure must conclude **UNSAT** in  $n$  steps otherwise at the end of  $n$  steps we have no **pairs**, which ensures a satisfying assignment for the formula.

Broadly speaking, what we are showing is that upon repeated resolution of an unsatisfiable CNF, if ( ) has not been encountered, the number of propositional variables must decrease.

## Chapter 2

# DFAs and Regular Languages

Consider a set of formulas made up of propositional variables  $\{x_1, x_2, \dots, x_n\}$ ,  $\phi(x_1, x_2, \dots, x_n)$ . We defined the set  $L \subseteq \{0, 1\}^n$ , the **language** defined by the formula as the set of strings which form a **satisfying assignment** for the formula  $\phi$ .

Basically using **Propositional Logic**, we were able to represent a large set of **finite length** strings having some properties in a **compact form**. This leads us to a question what about string of arbitrary length having some properties. How do we formulate them?

The answer to this is **Automata**: A way to formulate arbitrary length strings in a compact form.

### 2.1 Definitions

- **Alphabet**: A **finite, non-empty** set of symbols called **characters**. We usually represent an alphabet with  $\Sigma$ . For example  $\Sigma = \{a, b, c, d\}$ .
- **String**: A **finite** sequence of letters form an alphabet. An important thing to note here is that even though the alphabet may contain just 1 character, it can form **countably infinite** number of strings, each of which are **finite**. In this course we only deal with **finite strings** over a **finite alphabet**.
- **Concatenation Operation** ( $\cdot$ ) We can start from a string , take another string an as the name suggests concatenate them to form another string:

$$\begin{aligned} a \cdot b &\neq b \cdot a && \text{Not Commutative} \\ (a \cdot b) \cdot c &= a \cdot (b \cdot c) && \text{Associative} \end{aligned}$$

- **Identity Element** The algebra of the strings defined over the concatenation operator has the identity element :  $\varepsilon$  : **empty string**:

$$\sigma \cdot \varepsilon = \varepsilon \cdot \sigma = \sigma$$

Note the the empty strings remains same for strings of all alphabets.

- **Language** A subset of all finite strings on  $\Sigma$ . This set doesn't have to be finite even though the strings are of finite length.  
Note that a set of all finite strings of  $\Sigma$  is countably infinite (cardinality:  $\mathbb{N}$ ), so the number of languages of  $\Sigma$  is uncountably infinite (cardinality:  $2^{\mathbb{N}}$ ).

- $\Sigma^*$  is defined to be the set of all finite strings on  $\Sigma$ , including  $\varepsilon$ . Note that  $\Sigma^* = \bigcup_{k \geq 0} \Sigma^k$ , where  $\Sigma^k$  is the set of all strings on  $\Sigma$  with exactly  $k$  letters. Note that we can prove that the number of strings for  $\Sigma^*$  are countably finite by representing each string as a unique number in base  $(n + 1)$  system, where  $|\Sigma| = n$ , we can get an injection to natural numbers.

## 2.2 Deterministic Finite Automata

**Generalization of Parity function:** Lets go back to the question we asked first. Suppose we are given a string of arbitrary length and don't know the length of the string. This can be done by propositional logic. So we need a new formalism to represent sets of strings with any length. we want to develop a mechanism where we are given the bits of the string one by one, and I don't know when it will stop. So I must be ready with the answer each time a new bit arrives.

The solution to this lies in our discussion during the first lecture. I will record just one bit of information: whether I have received an even or odd number of 1s till now. Every time I receive a new bit, I will update this information: if it's a 0, I won't do anything, and if it's a 1, I will change my answer from even to odd, or vice-versa.

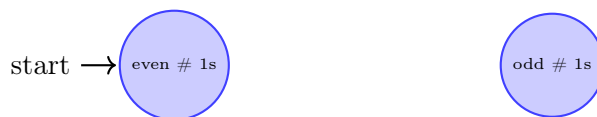
**States:** These are nodes which contain relevant summary of what we have seen so far



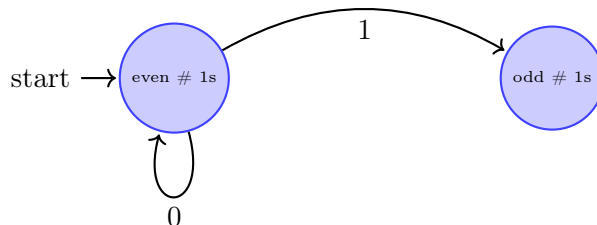
In our case we want to know whether there were even or odd number of 1s.



Where do we start from ? When I have seen nothing there are even number of 1s.

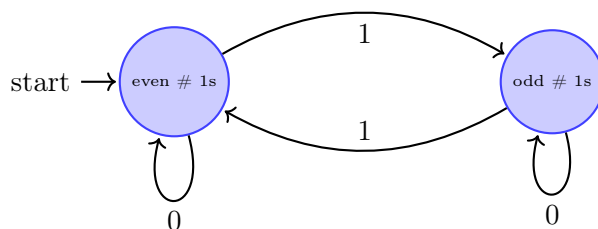


Now, suppose I receive a 0, I would remain in the same state, but if I get a 1, the parity changes.

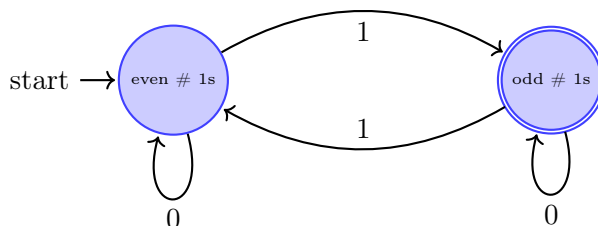


Now, if I am in the second state, if I get a 1 I will change states, and if I get a 0, parity is unchanged to I remain in the same state:





So when do we know that our string we have seen till now belongs to some language or not, we know that by marking some states as **accepting states**: usually represented by double circles, if we end up on this state, the string recieved till now belongs to our language: *is accepted*.

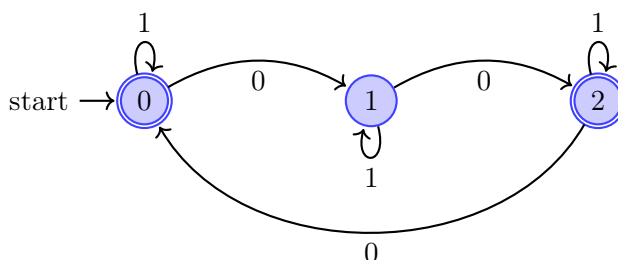


Such a formalism with **finite** states is known as **Finite Automata**.

Further, if for every string in  $\Sigma^*$ , there exists a unique path we will follow in the automata, such automata are also known **Deterministic Finite Automata (DFA)**.

#### Example

$\Sigma = \{0, 1\}$ , let  $L = \{w | w \in \Sigma^*, \text{ no. of 0's is a } 0 \bmod 3 \text{ or } 2 \bmod 3\}$



## 2.3 DFA Design - Example 1

Draw a Deterministic Finite Automaton (DFA) for  $L := \{w \in \{a, b\}^* : 2 \text{ divides } n_a(w) \text{ and } 3 \text{ divides } n_b(w)\}$ . Here  $n_a(w)$  stands for the number of  $a$ 's in  $w$ , and  $n_b(w)$  stands for the number of  $b$ 's in  $w$ . For example,  $n_a(abbaab) = n_b(abbaab) = 3$ . Therefore,  $ababbaa \in L$  but  $aabababa \notin L$ .

In certain scenarios, expressing a language solely through propositional logic becomes impractical, particularly when the length of the strings is unknown or variable. For instance, consider above example. In this case, the length  $n$  of the string is not explicitly provided, making it challenging to construct a propositional logic expression directly. Propositional logic typically operates on fixed, predetermined conditions or patterns within strings, which cannot accommodate variable lengths. However, deterministic finite automata (DFAs) offer a suitable alternative for such situations. DFAs

are well-suited for languages where the structure and properties depend on the characters within the string rather than on fixed string lengths. By employing states and transitions based on input characters, DFAs can effectively recognize languages with variable-length strings and complex patterns, making them a more appropriate choice when string length is not predetermined.

We will denote  $\Sigma$  as the set of alphabets.

$$\Sigma = \{a, b\}$$

$$L = \{\omega \in \Sigma^* : n_a(\omega) \text{ is divisible by 2 and } n_b(\omega) \text{ is divisible by 3}\}$$

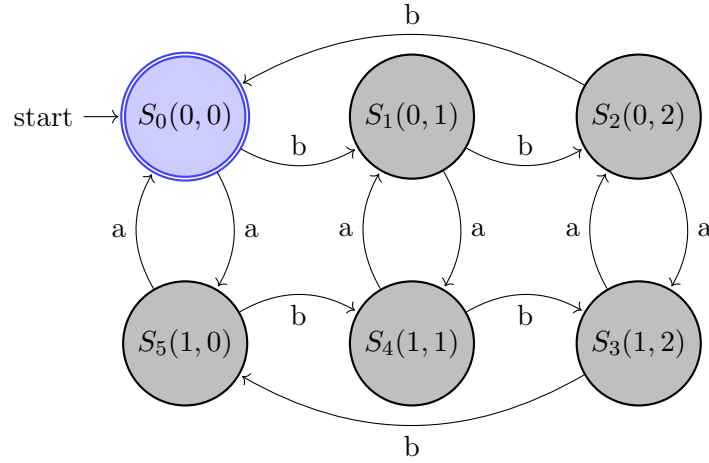


Figure 2.1: DFA for above task

$$S_0 : n_a(\mathbf{w}) \% 2 = 0 \text{ and } n_b(\mathbf{w}) \% 3 = 0$$

$$S_1 : n_a(\mathbf{w}) \% 2 = 0 \text{ and } n_b(\mathbf{w}) \% 3 = 1$$

$$S_2 : n_a(\mathbf{w}) \% 2 = 0 \text{ and } n_b(\mathbf{w}) \% 3 = 2$$

$$S_3 : n_a(\mathbf{w}) \% 2 = 1 \text{ and } n_b(\mathbf{w}) \% 3 = 2$$

$$S_4 : n_a(\mathbf{w}) \% 2 = 1 \text{ and } n_b(\mathbf{w}) \% 3 = 1$$

$$S_5 : n_a(\mathbf{w}) \% 2 = 1 \text{ and } n_b(\mathbf{w}) \% 3 = 0$$

## 2.4 DFA Design - Example 2

We will look at another example now.

$$\Sigma = \{a, b\}$$

$$L = \{\mathbf{w} \in \Sigma^* \mid n_{ab}(\mathbf{w}) = n_{ba}(\mathbf{w})\}$$

$$w = \underline{a} \overline{b} a \overline{a} b \overline{a} b \quad (n_{ab}(\mathbf{w}) = 3 \text{ \& } n_{ba}(\mathbf{w}) = 2)$$

If we try to cleverly convert the problem into a simpler one, we will observe that  $n_{ab}(\mathbf{w}) = n_{ba}(\mathbf{w})$  will always be true if the start and end alphabets are same (be it a or b).

So the above problem simplifies to:

$$L = \{\mathbf{w} \in \Sigma^* \mid \text{First and Last alphabet are same}\}$$

Forming an automation for this task can be done as:

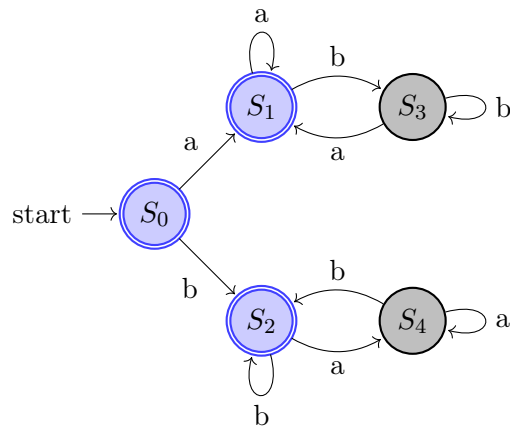


Figure 2.2: DFA: Last and First alphabets are same

## Chapter 3

# Non-Deterministic Finite Automata (NFA)

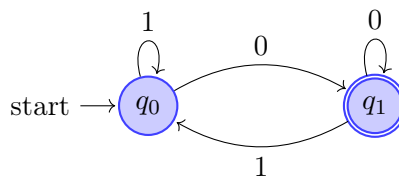
In the last few lectures, we covered the formalization of deterministic finite automata (DFA) where the transition function outputs a single state for a given input and current state. In this lecture, we will discuss non-deterministic finite automata (NFA) where the transition function can output multiple states (or a set of states) instead.

### 3.1 NFA Representation

As discussed earlier, a DFA is a 5-tuple  $(Q, \Sigma, q_0, \delta, F)$  where:

- $Q$  is a finite set of all states
- $\Sigma$  is the alphabet, a finite set of input symbols
- $q_0 \in Q$  is the initial state
- $\delta : Q \times \Sigma \rightarrow Q$  is the transition function
- $F \subseteq Q$  is the set of final/accepting states

For example, consider the following automaton:



It can be represented as:

$$\text{DFA} ( \{q_0, q_1\}, \{0, 1\}, q_0, \delta, \{q_1\} )$$

where  $\delta$  is the transition function:

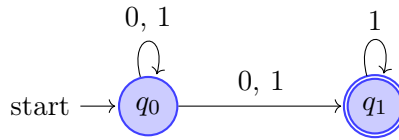
Q	$\Sigma$	Q'
$q_0$	0	$q_1$
$q_0$	1	$q_0$
$q_1$	0	$q_1$
$q_1$	1	$q_0$

### 3.1.1 Formalization of Non-Deterministic Finite Automata

However, in NDFA,

- $q_0 \subseteq Q$  is the set of initial states
- $\delta : Q \times \Sigma \rightarrow 2^Q$  is the transition function

Hence, consider the following non-deterministic finite automaton (A):



It can be represented as:

$$\text{NDFA} ( \{q_0, q_1\}, \{0, 1\}, \{q_0\}, \delta', \{q_1\} )$$

where  $\delta'$  is the transition function: As we can see,  $\delta'$  is a partial function whose output is a set of

Q	$\Sigma$	$2^Q$
$q_0$	0	$\{q_0, q_1\}$
$q_0$	1	$\{q_0, q_1\}$
$q_1$	1	$\{q_1\}$

states instead of a single state.

### 3.1.2 NFA into action

Due to its transition function, an NDFA gives **choices for path taken** at some states for a given input string. Any string for which there exists a path from the initial state to a final state is considered to be accepted by the NDFA.

Hence, the NDFA shown above has the language  $L(A) = \Sigma^* \setminus \{\epsilon\}$ , i.e., the set of all strings over the alphabet  $\Sigma$  except the empty string.

For example, the string 011 is accepted by  $A$  as it has the following path  $q_0 \xrightarrow{0} q_0 \xrightarrow{1} q_0 \xrightarrow{1} q_1$ .

To determine if a string is accepted by an NDFA, we can check if the set of states reachable from the initial state by reading the string contains any final state.

Here we have: Here we have:

Initial State	$\Sigma^*$	Reachable States
$q_0$	0	$\{q_0, q_1\}$
$q_0$	01	$\{q_0, q_1\}$
$q_0$	011	$\{q_0, q_1\}$

As  $\{q_0, q_1\}$  contains  $q_1 \in F$ , the string 011 is accepted by  $A$ . By construction, there will always be a set of choices which reach  $q_1$  from  $q_0$  for the input 011.

## 3.2 Equal expressiveness of DFA or NFA

We claim that regular languages, ie, the set of languages which can be defined by DFAs, is exactly the set of languages which can be defined by NFAs. In other words, for every NFA, there exists a DFA that accepts the same language, and vice versa. Even though NFA gives choices at some states, it can still be represented as some equivalent DFA. Even though NFA gives choices at some states, it can still be represented as some equivalent DFA. This implies that **NDFAs have no more expressive power than DFAs** in terms of string acceptance. However, representation in form of NFA is much more succinct than DFA.

We can convert a NFA to a DFA by considering the set of reachable states as states of the equivalent DFA.

### 3.2.1 Construction of DFA from NFA

For an NFA  $(Q, \Sigma, Q_0, \delta, F)$ , construct a DFA  $(\mathcal{P}(Q), \Sigma, Q_0, \delta', F')$  with:

$$\delta'(G, \sigma) = \bigcup_{q \in G} \delta(q, \sigma)$$

$$F' = \{q : q \in \mathcal{P}(Q), q \cap F \neq \emptyset\}$$

Notice that in our new DFA, the states are labelled as subsets of the states of the NFA. This means that we have  $2^n$  states in our DFA if the NFA had  $n$ . It is left to the reader to verify that the DFA we have defined satisfies all the requirements of a DFA.

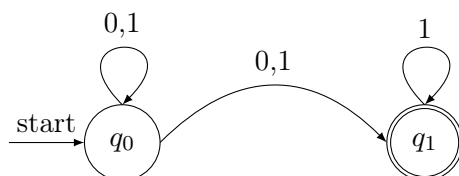
We claim that after the same characters are inputted into both the NFA and DFA, the state of the DFA is labelled the same as the set of current states of the NFA. This claim is easy to check using the definition of the  $\delta'$  function.

Now, in an NFA, a string is accepted if any one of the active states at the end of the string is in the set of accepting states. Clearly, with our interpretation of the DFA, this is equivalent to being in a state that belongs to the  $F'$  we have defined.

### 3.2.2 Step Wise Conversion from NFA to DFA

Now we'll see how to convert a NFA to a DFA through an example.

We call the following NFA 'A'



The language depicted by this NFA is all the strings formed using  $\{0,1\}$  excluding the empty string( $\epsilon$ ). We represent this as:

$$L(A) = \{w \in \{0,1\}^* | w \text{ is accepted by } A\}$$

that is,

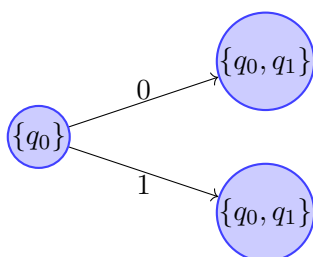
$$L(A) = \Sigma^* \setminus \{\epsilon\}$$

The transition function( $\delta$ ) table looks as follows:

$Q$	$\Sigma$	$2^Q$
$q_0$	0	$\{q_0, q_1\}$
$q_0$	00	$\{q_0, q_1\}$
$q_0$	01	$\{q_0, q_1\}$

To convert this NFA to a DFA, we need to track the states that can be reached after  $n$  choices which will be a subset of  $2^Q$ .

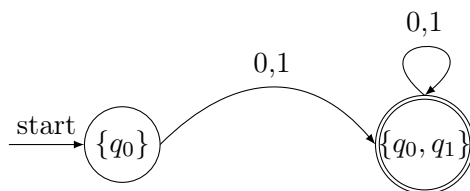
Step 1



Step 2



Final DFA



Call this DFA ‘ $A'$ ’.

One property we’ve extensively used in the conversion is:

$$\text{If } S \subseteq Q, \text{ then } \delta(S, 0) = \bigcup_{q \in S} \delta(q, 0)$$

Now, to show that the languages represented by the NFA( $A$ ) and the DFA( $A'$ ) are the same, i.e,  $L(A) = L(A')$ , we need to show the following:

1.  $L(A) \subseteq L(A')$
2.  $L(A') \subseteq L(A)$

### 3.2.3 Proof of Equivalence

To show that the languages represented by the NFA( $A$ ) and the DFA( $A'$ ) are the same, i.e,  $L(A) = L(A')$ , we need to show the following:

- $L(D) \subseteq L(N)$
- $L(N) \subseteq L(D)$

We can prove that the equivalent DFA  $D$  accepts exactly the same language as the original NFA  $N$ , i.e,  $L(D) = L(N)$ .

## 3.3 Reflective Insights

Even with all the extra "powers" and behaviours NFAs can have on top of those of DFAs, they are equally expressive. This shows us that the limitation in expressiveness is not in the behaviour of state transitions but in the finiteness of states. However, due to the exponential blowup in the number of states, it is often more human-readable to express certain automata/languages through NFAs, making it a convenient representation tool.

## 3.4 Proof of Equivalence – Correctness

In the last lecture, we discussed the conversion of Nondeterministic Finite Automata (NFA) to its equivalent Deterministic Finite Automata (DFA)

### 3.4.1 Claim

We aim to demonstrate the equivalence of the languages accepted by NFA  $A$  and DFA  $A'$ , denoted as  $L(A)$  and  $L(A')$  respectively. In other words, we want to show that  $L(A) = L(A')$ , where  $A$  represents the original NFA and  $A'$  represents the DFA obtained through subset construction from NFA  $A$ . Alternatively, we can establish that  $L(A) \subseteq L(A')$  and  $L(A') \subseteq L(A)$ , which implies  $L(A) = L(A')$ .



### 3.4.2 Proof

We'll prove this claim by showing that for all  $n \geq 0$  and for every word  $w \in \Sigma^*$  with  $|w| = n$ , NFA  $A$  can reach state  $q \in Q$ <sup>1</sup> upon reading  $w$  if and only if DFA  $A'$  reaches state  $S \subseteq Q$  such that  $q \in S$ .

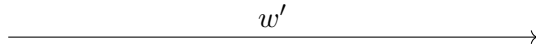
Given the condition that for all  $n \geq 0$ , and for every word  $w$ , where  $w \in \Sigma^*$  such that  $|w| = n$ , we aim to demonstrate the equivalence between the NFA  $A$ 's ability to reach state  $q \in Q$  upon reading word  $w$  and the DFA  $A'$ 's capability to reach state  $S \subseteq Q$ , where  $q \in S$ . In other words, we want to show that words in the language recognized by NFA  $A$  reach a certain state (say, a final state  $q$ ), **if and only if** words in the language recognized by DFA  $A'$  reach state  $S$  (where  $S \subseteq Q$ ,  $q \in S$ , and  $S$  is a final state in the equivalent DFA  $A'$ ).

Thus, we can establish that the set of words accepted by NFA  $A$  is equivalent to the set of words accepted by DFA  $A'$ , implying  $L(A) = L(A')$ , thereby validating our claim.

We will demonstrate this by induction on  $n$ .

1. Base case: When  $n = 0$ , i.e.,  $|w| = 0$ , it essentially means that we are at the initial state of the automaton. The initial state of the DFA  $A'$  is represented by the singleton set containing the initial states of the NFA  $A$ . So, definition of initial state of DFA  $A'$  satisfies the claim.
2. Induction hypothesis: Assume the claim holds for all  $0 \leq n < k$  for some  $k > 0$ .
3. Inductive step: We'll show that the claim holds for  $n = k$ .  
Since  $|w| = k$ , let  $w = w'.a$  ( $w'$  concatenate  $a$ ), where  $w \in \Sigma^*$ ,  $w' \in \Sigma^*$ ,  $a \in \Sigma$ , and  $|w'| = k-1$ .

By hypothesis, there exists the following path in both automata :



This basically means that there exists some path in NFA and equivalent DFA which is as follows :

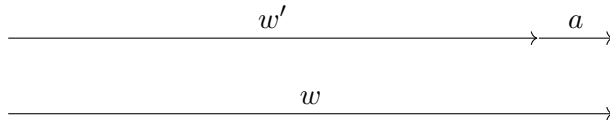
In NFA  $A$  :



In DFA  $A'$ <sup>2</sup>:



Now, if a symbol  $a$ ,  $a \in \Sigma$  comes, word  $w$  is formed :



<sup>1</sup> $Q$  is the set of states in our original NFA  $A$ .

<sup>2</sup> $\dots q_0 \Rightarrow$  set containing initial states of the original NFA  $A$

$\dots q \Rightarrow \subseteq Q$  containing the state  $q$ , where  $Q$  is the set of states of the original NFA  $A$

$\dots \hat{q} \Rightarrow \subseteq Q$  containing the state  $\hat{q}$ , where  $Q$  is the set of states of the original NFA  $A$

NFA  $A$  will reach some state  $\hat{q}$



DFA  $A'$  will reach some state  $\dots \hat{q}$  ( $\dots \hat{q} \subseteq Q$  and contains  $\hat{q}$ )



This is because state  $\dots q$  of DFA  $A'$  contains  $q$ , which transitions DFA  $A'$  to state  $\dots \hat{q}$  (a set containing  $\hat{q}$ ) when symbol  $a$  is encountered.

Thus, we have shown that for all  $n \geq 0$ , and for every word  $w$  where  $w \in \Sigma^*$  such that  $|w| = n$ , NFA  $A$  can reach state  $q \in Q$  on reading word  $w$  **AND** DFA  $A'$  reaches state  $S \subseteq Q$  such that  $q \in S$ .

Hence, proved the condition and the claim.

**Remarks:**

- (a) As the length of the word increases, the number of choices for state transitions in the NFA grows exponentially.
- (b) If an NFA has  $N$  states, the equivalent DFA can have an exponential number of states in the worst case.

### 3.5 Compiler Special Case – Lexical Analyzer

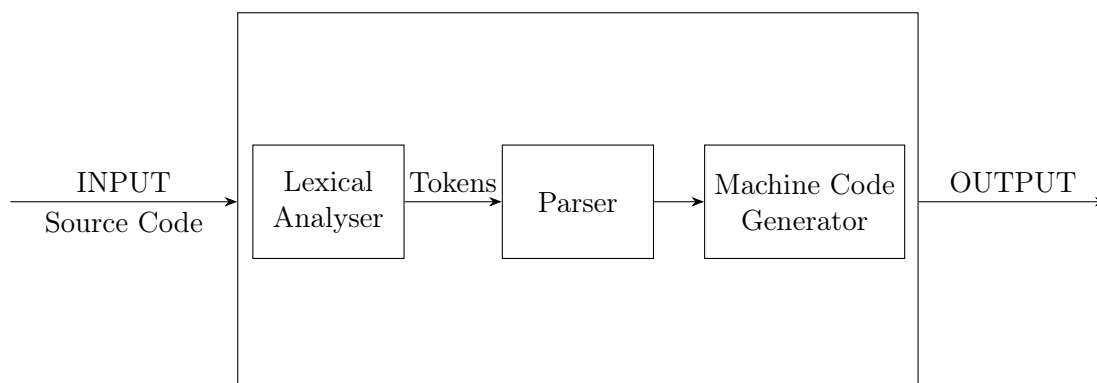


Figure 3.1: A Simple Compiler

In simpler terms, a compiler is composed of three main components: a lexical analyzer, a parser, and a code generator. Its primary function is to process a sequence of characters as input. The lexical analyzer breaks down a sequence of characters into tokens, which are then analyzed by a parser. To accomplish this, the lexical analyzer employs a NFA to determine whether a particular state can be reached in the corresponding DFA and traces a path accordingly. When faced with a long string, finding the final state can be challenging. At each step, there are multiple choices to

explore. However, as the length of the input increases, exhaustively exploring each choice becomes increasingly difficult. Converting an NFA to a DFA may result in an exponential increase in states. However, it's important to note that not all states are necessary to reach the final state. This excessive expansion of states may lead to unnecessary complexity, creating the entire DFA when it's not actually required.

So, the lexical analyzer determines whether there exists a path in the automaton that leads to an accepting state for a given word without constructing the equivalent DFA.

Suppose the Lexical Analyser of Compiler have to check whether a GIVEN WORD is accepted by the following NFA.

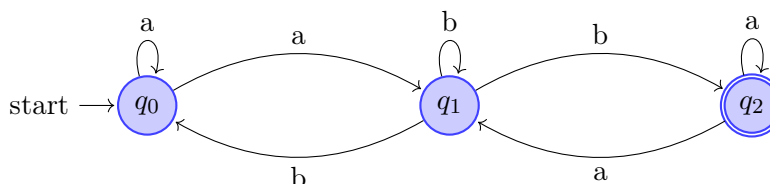
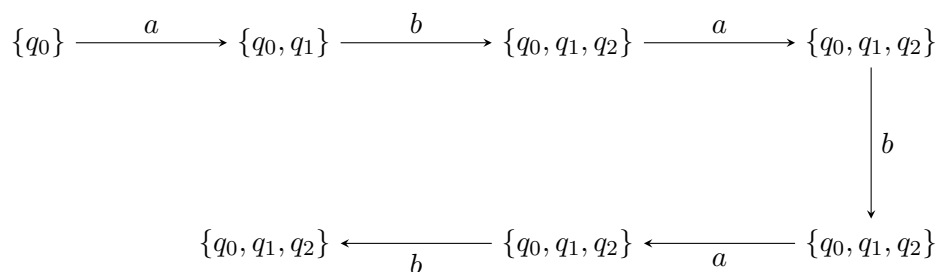


Figure 3.2: Lexical analyser Automaton

**Question:** How can a lexical analyzer determine whether a given word leads to an accepting state without constructing the equivalent DFA?

**Answer:**



Lexical Analyser can track the set of states the NFA could be in after reading each symbol of the GIVEN WORD, updating the state based on the transitions specified by the GIVEN NFA.

- If the final set of states contains at least one accepting state of the original NFA, then there exists a successful path for the given word to reach an accepting state, indicating acceptance by the given automaton.
- The time complexity for this process is  $O(n \cdot k)$ , where  $n$  represents the size of the automaton and  $k$  denotes the length of the word. This complexity is significantly lower than the usual exponential time complexity observed in similar processes.<sup>3</sup>
- Here the GIVEN WORD is accepted by the GIVEN NFA<sup>4</sup>

<sup>3</sup>Size of automaton = No. of states in Automaton + No. of Transition Arrows in Automaton

<sup>4</sup> $q_2$  is contained in the last set

### 3.6 NFA with $\epsilon$ -edges

A variation of NFA is NFA with  $\epsilon$  edges, it may expand the language by making words accepted that were otherwise unaccepted.

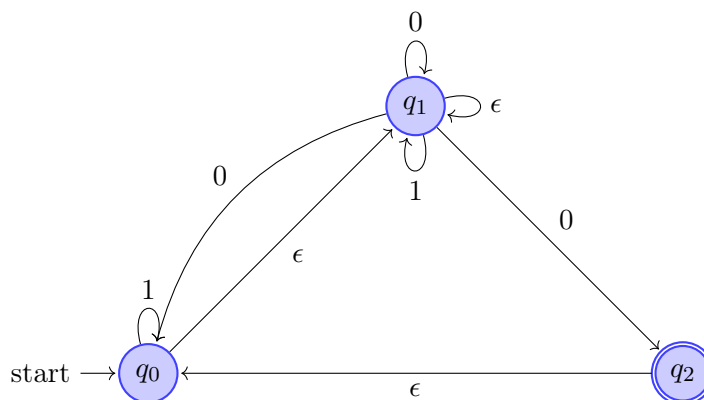


Figure 3.3: Automaton with Epsilon Edges

$\epsilon$ -edges bring non-determinism to the NFA, as you can sit on a node and take one of the  $\epsilon$ -edges possible from that node to jump to another node without consuming any letter of the input.

Figure 3.3 shows how  $\epsilon$  edges are used to connect states of an automaton for free (without consuming any letter from input), we can see that  $10 \notin L$  without the  $\epsilon$  edge between  $q_0$  and  $q_1$ , but with the presence of this  $\epsilon$  edge  $10 \in L$ .

$\epsilon$ -edge also allows us to connect two automata. In Figure 3.4, the accepting states of  $L_1$  are connected to the start states of  $L_2$  automaton, which generates an automaton for accepting  $L_1 \cdot L_2$ . Here  $\cdot$  is the concatenation operator. So if  $\omega_1 \in L_1$  and  $\omega_2 \in L_2$ , then  $\omega_1 \cdot \omega_2 \in L_1 \cdot L_2$  will be accepted by this new automaton.

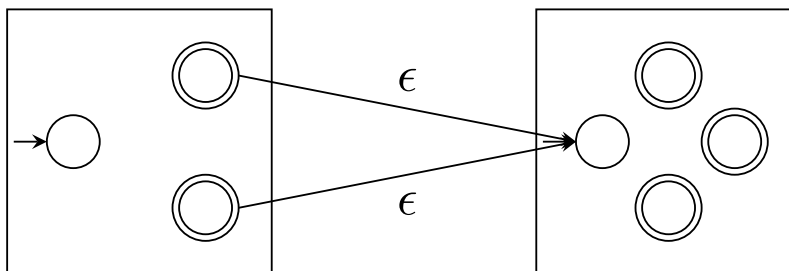


Figure 3.4:  $L_1$  accepting automaton and  $L_2$  accepting automaton connected

Now, we will try to find an equivalent DFA of this NFA having  $\epsilon$  edges. For that, we will first find an NFA without  $\epsilon$  edges which will preserve the original NFA, and this obtained NFA can then be constructed into a DFA.

Initially, just look at the  $\epsilon$ -edges only and find for each state its  $\epsilon$ -closure, which is the set of the states that we can reach from it by taking only  $\epsilon$ -edges.

From each node, we can go to every node that is present in the  $\epsilon$ -closure of that node for free. So wherever non- $\epsilon$  edges take us from these nodes in  $\epsilon$ -closure, we can reach from the node itself whose

closure it was. So all these states will be connected to this node, in the new NFA.

The starting states and the final states of this new NFA will be the  $\epsilon$ -closures of the start and final states respectively of the original NFA.

So NFA without  $\epsilon$ -edges for the NFA in Figure 3.3 will look like as shown in Figure 3.5

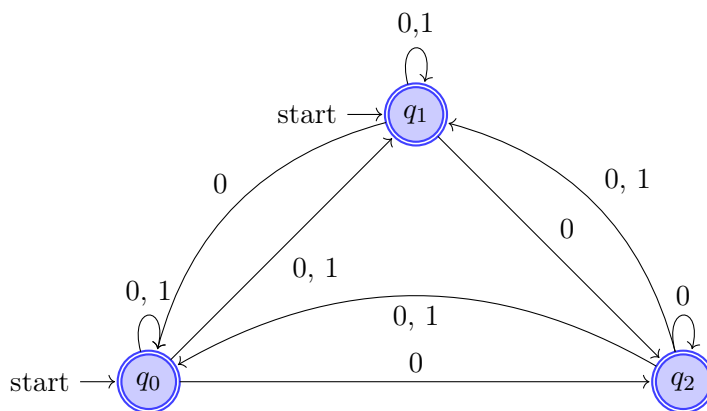


Figure 3.5: Automaton without Epsilon Edges

### 3.6.1 Equivalence with DFA

- If we can show that  $\epsilon$ -NFA have an equivalent NFA, then the equivalence of  $\epsilon$ -NFA and DFA is proved because every NFA has an equivalent DFA<sup>5</sup>.

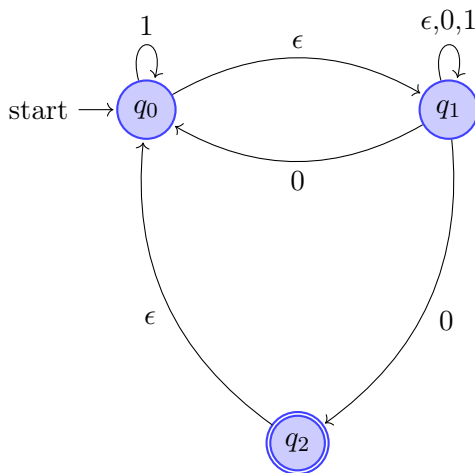


Figure 3.6: A Simple  $\epsilon$ -NFA

Focus on  $\epsilon$ -Edges of the Automaton to get the Epsilon Closure of the states in an Automaton

For Figure-3.6 Automaton:

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<sup>5</sup>equivalence of NFA and DFA is already proved

$$q_0 \rightsquigarrow \{q_0, q_1\} \quad \epsilon\text{-closure}_{q_0}$$

$$q_1 \rightsquigarrow \{q_1\} \quad \epsilon\text{-closure}_{q_1}$$

$$q_2 \rightsquigarrow \{q_0, q_1, q_2\} \quad \epsilon\text{-closure}_{q_2}$$

Rules for converting  $\epsilon$ -NFA to equivalent NFA<sup>6</sup>:

1. Non-epsilon transitions for a symbol, denoted as  $a \in \Sigma$ , originating from any state  $q'$  within the epsilon closure of a state, say  $q$ , will also be present as non-epsilon transitions for state  $q$  in the equivalent NFA. These transitions leads to the same destination states for  $q$  in new NFA as they were for the state  $q'$  in the original epsilon-NFA.
2. The states in the epsilon closure of the initial state in the epsilon-NFA can serve as the initial state in the new NFA. However, this is not necessary, we can make an equivalent NFA where  $\epsilon$ -NFA and the new NFA have same initial state(s).
3. All states within the epsilon-NFA that include the accepting state in their epsilon closure will also act as final states in the new/equivalent NFA.

For the Figure-3.6 epsilon-NFA, after applying the aforementioned rules, we obtain the following equivalent NFA:

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<sup>6</sup>These rules will become more clearer in the next class, when we will learn about leading epsilon transitions and trailing epsilon transitions within the states of  $\epsilon$ -NFA.

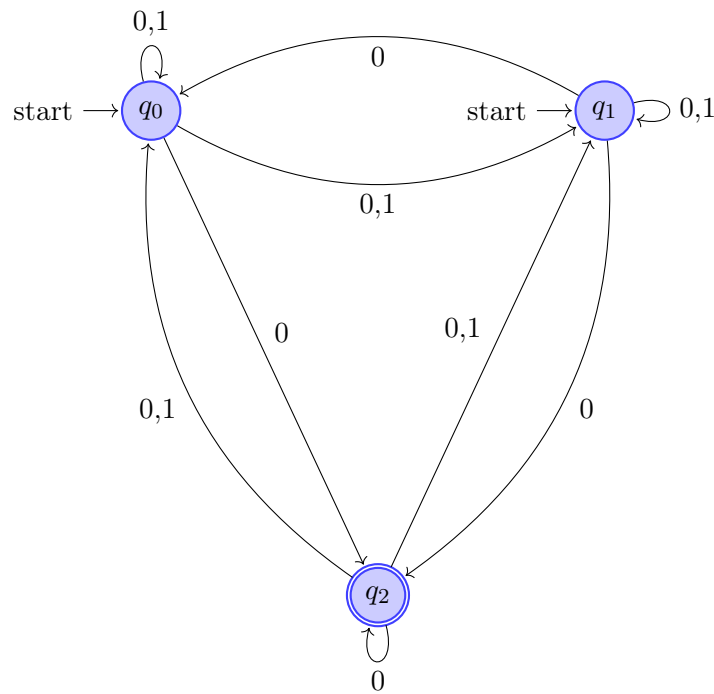


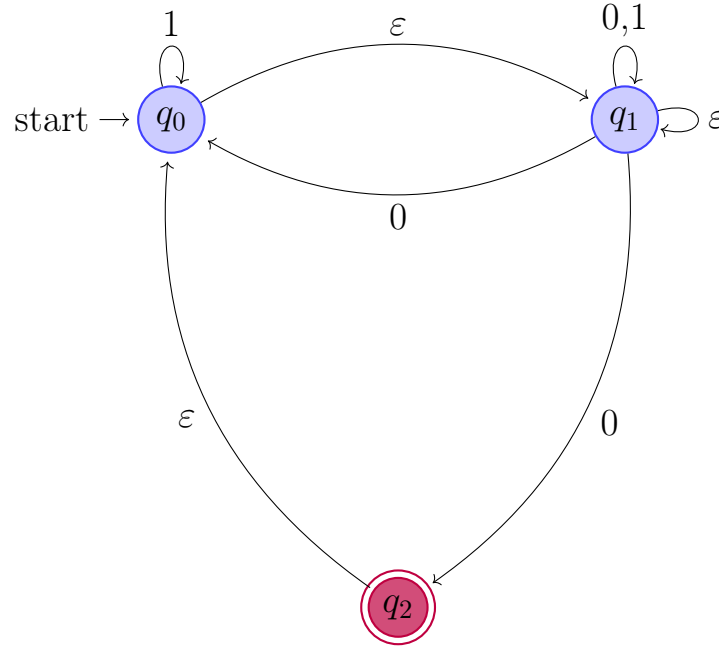
Figure 3.7: Equivalent NFA for Figure-3.6

## 3.7 Recap

We were trying to convert an NFA with  $\varepsilon$  edges to an equivalent NFA without  $\varepsilon$  edges in the previous lecture. We'll do that in more detail in this lecture.

### 3.7.1 Converting $\varepsilon$ -NFA to non- $\varepsilon$ -NFA

- **$\varepsilon$ -closure** of a node is defined as set of those nodes which can be reached from that node by traversing over  $\varepsilon$ -edges, i.e. without consuming any character from the alphabet  $\Sigma$ . Also, the node itself is trivially a part of its  $\varepsilon$ -closure.
- **$\varepsilon$ -edges** are those edges which can be traversed without consuming any character from the alphabet  $\Sigma$ , i.e. by consuming an empty string. Observe that the string "10"  $\in L$  with  $\varepsilon$ -edges but without  $\varepsilon$ -edges, string "10"  $\notin L$  where  $L$  is the Language of the NFA.

Figure 3.8: A Simple  $\varepsilon$ -NFA with alphabet  $\Sigma = \{0, 1\}$ 

For example in this NFA,

$$\varepsilon\text{-closure}(q_0) = \{q_0, q_1\}$$

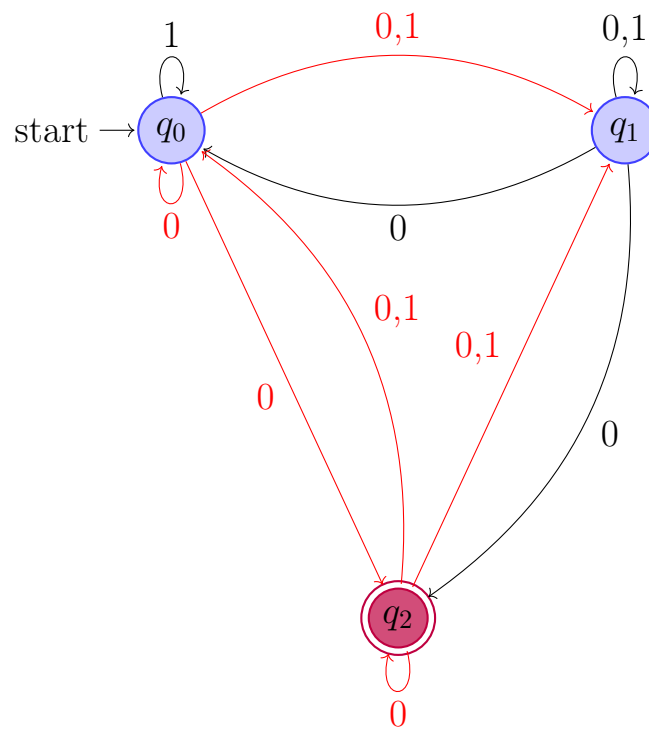
$$\varepsilon\text{-closure}(q_1) = \{q_1\}$$

$$\varepsilon\text{-closure}(q_2) = \{q_0, q_1, q_2\}$$

Now for the algorithm to convert an NFA with  $\varepsilon$ -edges to an equivalent NFA without  $\varepsilon$ -edges with the same alphabet  $\Sigma = \{0, 1\}$  and with the same states, apply the following three steps:-

1. For each node  $q$  in NFA, find its  $\varepsilon$ -closure (say  $S$ ) and mark all its non- $\varepsilon$ -edges as they are. Now, for each node  $q' \neq q$  in  $S$ , mark all non- $\varepsilon$ -edges starting from  $q'$  going to  $q''$  as extra edges starting from  $q$  going to  $q''$ . For eg., for the node  $q_0$ , the only distinct node in its  $\varepsilon$ -closure is  $q_1$  so mark the edges  $\{0, 1\}$  from  $q_1$  to  $q_1$ ,  $\{0\}$  from  $q_1$  to  $q_0$  and  $\{0\}$  from  $q_1$  to  $q_2$  as extra edges  $\{0, 1\}$  going from  $q_0$  to  $q_1$ ,  $\{0\}$  from  $q_0$  to  $q_0$  and  $\{0\}$  from  $q_0$  to  $q_2$  respectively (marked in red).
2. Mark all those states as accepting whose  $\varepsilon$ -closures contain atleast one of the accepting states of the  $\varepsilon$ -NFA. For eg., here only  $q_2$  has the accepting state  $q_2$  in its  $\varepsilon$ -closure, so only  $q_2$  is marked as accepting.
3. Starting states in the new non- $\varepsilon$ -automaton will be the same as the starting states in the original  $\varepsilon$ -automaton.



Figure 3.9: Equivalent non- $\varepsilon$ -NFA

Subsequently, we'll use "original/ori" for the  $\varepsilon$ -automaton and "new" for the non- $\varepsilon$ -automaton.

- One may ask why the states which are in the  $\varepsilon$ -closures of the accepting states of the original automaton are not marked as accepting in the non- $\varepsilon$ -NFA. The answer is :- every string which reaches an accepting state  $a$  can also reach any of the states in  $\varepsilon\text{-closure}(a)$  by taking  $\varepsilon$ -edges and those strings are indeed in the language of the  $\varepsilon$ -NFA. But there are many such strings also which can reach at the states in the  $\varepsilon\text{-closure}(a)$  which are not in the language of the  $\varepsilon$ -NFA. So, if we mark those states as accepting, we are changing the language which is not our intention. For eg.,  $q_0 \in \varepsilon\text{-closure}(q_2)$  but if we mark  $q_0$  as accepting in the non- $\varepsilon$ -NFA, then the string "1" will also be included in the language of the non- $\varepsilon$ -NFA but "1"  $\notin L(\varepsilon\text{-NFA})$ .

### 3.7.2 Correctness of Algorithm

- If we say  $w \in L(ori)$  it means that either  $w$  as it is  $\in L(ori)$  or  $w$  with some  $\varepsilon$ 's inserted  $\in L(ori)$ . On the other hand, if we say  $w \in L(new)$  then it means that  $w$  as it is  $\in L(new)$ .

Now, what we mean by correctness of the algorithm is that the language of the new non- $\varepsilon$ -automaton should exactly be equal to the language of the original  $\varepsilon$ -automaton. So we need to prove that

$$L(new) = L(ori)$$

1. Let's prove  $L(new) \subseteq L(ori)$

Let's take any arbitrary string  $w = c_1c_2 \dots c_m \in L(new)$

- (a) Case 1:  $w$  is empty

It means atleast one of the starting states  $s$  in the new automaton is an accepting state. Now, according to our algorithm (step 2), all accepting states in the new automaton are either accepting states in the original automaton as well or are those states which have atleast one accepting state in the original automaton in their  $\varepsilon$ -closures. So,  $\varepsilon\text{-closure}(s)$  definitely contains an accepting state in the original automaton. So, we have a path  $\varepsilon^k$  for some  $k \geq 0$  from  $s$  to an accepting state in the original automaton and thus  $w \in L(ori)$ .

- (b) Case 2:  $w$  is non-empty

It means that we have a sequence of jumps from a starting state  $s_0$  in the new automaton to  $s_1$  upon reading  $c_1$  and then from  $s_1$  to  $s_2$  upon reading  $c_2$  and so on till the state  $s_m$  upon reading  $c_m$  such that  $s_m$  is an accepting state. Now, according to our algorithm (step 1), for  $s_0$  in the original automaton either we have a direct edge  $\{c_1\}$  from  $s_0$  to  $s_1$  or we have some  $s'_0 \in \varepsilon\text{-closure}(s_0)$  which has the edge  $\{c_1\}$  from  $s'_0$  to  $s_1$ . So, effectively we can reach  $s_1$  from  $s_0$  in the original automaton as well by following a path  $\varepsilon^k c_1$  for some  $k \geq 0$ . Similarly, we can have a path  $\varepsilon^{k_1} c_2 \varepsilon^{k_2} c_3 \dots$  for  $k_i \geq 0$  till  $s_m$  in the original automaton. Now, according to our algorithm (step 2), either  $s_m$  is an accepting state in the original automaton as well or there lies some accepting state in the  $\varepsilon\text{-closure}(s_m)$ . So, finally, we can have a sequence of  $\varepsilon^{k'}$  for some  $k' \geq 0$  to reach to an accepting state from  $s_m$  in the original automaton and thus,  $w \in L(ori)$ .

Hence, proved.

2. Let's prove  $L(ori) \subseteq L(new)$

Let's take any arbitrary string  $w = c_1c_2 \dots c_m \in L(ori)$

- (a) Case 1:  $w$  is empty

It means that there exists atleast one starting state  $s$  in the original automaton such

that  $\varepsilon\text{-closure}(s)$  contains an accepting state  $s'$ . Now, according to our algorithm (step 2), all accepting states in the new automaton are either accepting states in the original automaton as well or are those states which have atleast one accepting state in the original automaton in their  $\varepsilon$ -closures. So, the starting state  $s$  is an accepting state in the new automaton and thus  $w \in L(\text{new})$ .

(b) Case 2:  $w$  is non-empty

It means that we have a path  $\varepsilon^{k_0}c_1\varepsilon^{k_1}c_2\dots\varepsilon^{k_{m-1}}c_m\varepsilon^{k_m}$  for all  $k_i \geq 0 \forall i \in \{0, 1, 2, \dots, m\}$  from a starting state  $s_0$  in the original automaton to an accepting state  $s_{m+1}$  where taking  $\varepsilon^{k_i}$  from any state  $x$  means going to some state  $y \in \varepsilon\text{-closure}(x)$ . Here,  $s_1$  is the state reached after reading  $c_1$ ,  $s_2$  after reading  $c_2\dots s_m$  after reading  $c_m$  and  $s_{m+1}$  after taking  $\varepsilon^{k_m}$ . Now, according to our algorithm (step 1), for  $s_0$  in the new automaton, we have a direct edge  $\{c_1\}$  from  $s_0$  to  $s_1$ . Similarly, we'll have direct edges from  $s_1$  to  $s_2$  and so on. So, finally, we can reach  $s_m$  from  $s_0$  in the new automaton. Now, according to our algorithm (step 2),  $s_m$  is an accepting state in the new automaton because  $\varepsilon\text{-closure}(s_m)$  contains accepting state  $s_{m+1}$  in the original automaton. Thus,  $w \in L(\text{new})$ .

Hence, proved.

Thus, we've shown that  $L(\text{new}) = L(\text{ori})$

### 3.7.3 Intuition of the algorithm

- It should be clear from the correctness proof itself where we constructed paths in the new and original automata using the step 1 of the algorithm. Also, if the path in the original automaton had trailing  $\varepsilon$ 's we weren't able to reach the same state in the new automaton but had to make the languages of the two automata exactly same so we concluded step 2 of the algorithm.

### 3.7.4 Extras

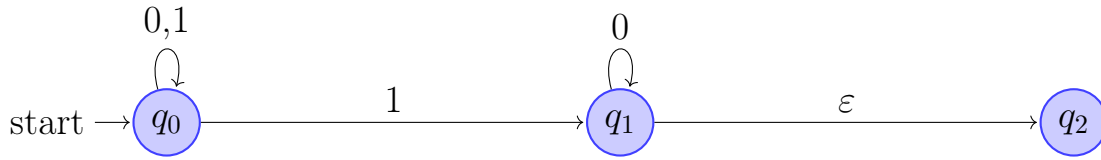
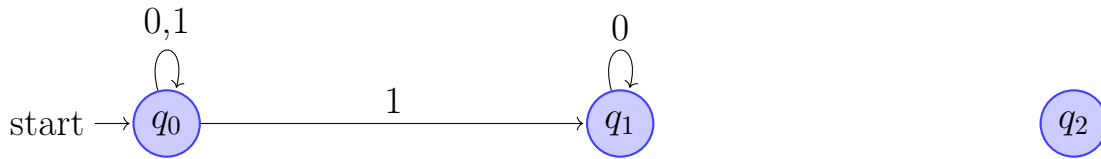
- **Language of a node  $q$**  is defined as the set of all those strings  $w \in \Sigma^*$  such that upon reading  $w$  character by character we can reach  $q$  from any of the starting states of the automaton.
- Note that in  $\varepsilon$ -automaton we could also take  $\varepsilon$ -edges in between the characters, before the first character as well as after the last character of  $w$  and reach  $q$  so such strings would also be considered in the language of  $q$ .
- And the language of an automaton is defined as the union of the languages of all its accepting nodes. So, we have

$$L(\text{new}) = \bigcup_i L(q_i) \forall \text{ accepting states } q_i \text{ of the new non-}\varepsilon\text{-automaton}$$

$$L(\text{ori}) = \bigcup_j L(q_j) \forall \text{ accepting states } q_j \text{ of the original } \varepsilon\text{-automaton}$$

- A lemma:  $L(q_i)_{\text{original}} \supseteq L(q_i)_{\text{new}}$  holds true where  $q_i$  is any node of the NFA and  $L(q_i)$  is the language of that node.

- Proof: Going by the same idea as we did in Correctness proof part 1, we can prove the lemma. Like if  $w \in L(q_i)_{new}$  and  $w$  is empty then  $q_i$  must be one of the starting states in the new automaton and also in the original automaton so  $w \in L(q_i)_{original}$  since starting states are the same in both the automata. If  $w \in L(q_i)_{new}$  and  $w$  is non-empty, then we can have a path with some  $\varepsilon$ 's in between the characters of  $w$  from some starting state  $s_0$  to  $q_i$  in the original automaton and thus  $L(q_i)_{original} \supseteq L(q_i)_{new}$  is indeed true. Where the  $\supseteq$  sign comes in is the case when the path of  $w$  has some trailing  $\varepsilon$ 's. For eg. consider this  $\varepsilon$ -NFA and its equivalent non- $\varepsilon$ -NFA,

Figure 3.10:  $\varepsilon$ -NFAFigure 3.11: Equivalent Non- $\varepsilon$ -NFA

$L(q_2)_{original}$  is non-empty and for instance contains the string "1" with the path "1 $\varepsilon$ " from  $q_0$  to  $q_1$  but  $L(q_2)_{new}$  is clearly empty, i.e., the null set  $\phi$ . This happened because the path contained a trailing  $\varepsilon$ . So,  $L(q_2)_{new} \subset L(q_2)_{original}$ .

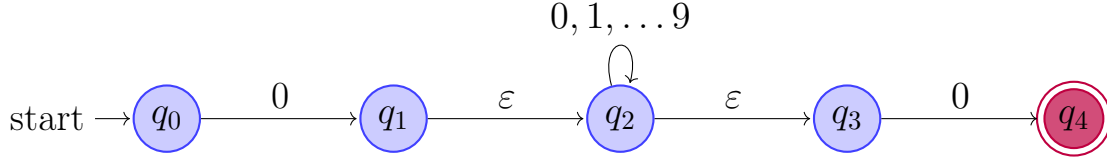
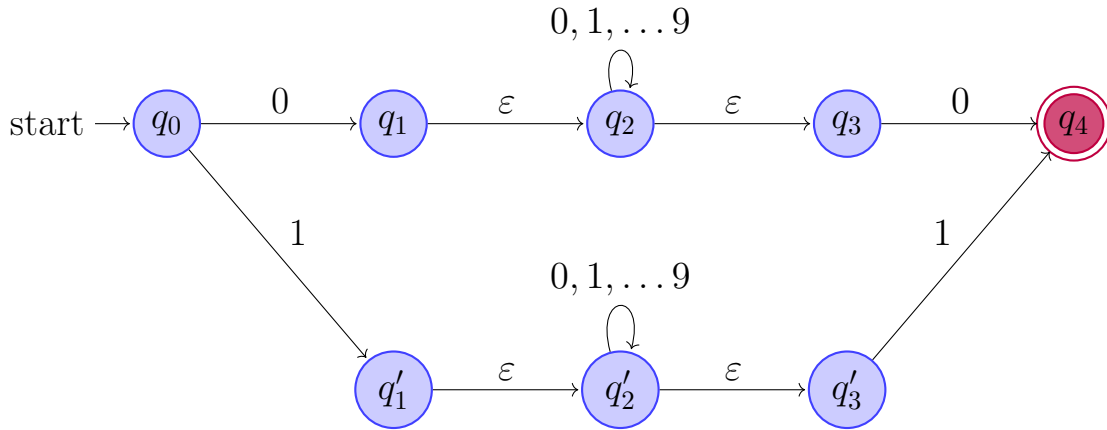
### 3.7.5 Significance of $\varepsilon$ -edges

$\varepsilon$ -edges allow us to jump from one part of the NFA to another part of the NFA without consuming any additional character. Thus, if we have to do some operations sequentially we can make our life simple by using  $\varepsilon$ -edges. For eg., suppose we have to check  $w = u.v$  such that  $u$  contains an even number of 1's and  $v$  contains  $1 \bmod 3$  number of 1's. We can accomplish the above task by constructing two NFA's wherein the first NFA will accept all the satisfying  $u$ 's and the second NFA will accept all the satisfying  $v$ 's. Now, we'll just join the accepting states of first NFA to the starting states of the second NFA using  $\varepsilon$ -edges and convert the accepting states of first NFA & the starting states of the second NFA into normal intermediate states. Thus, we will get a single NFA with starting states same as the starting states of the first NFA and accepting states same as the accepting states of the second NFA. Here,  $\varepsilon$ -edges allowed us to capture the non-determinism of the breaking point between  $u$  &  $v$ .

\* Any automaton containing  $\varepsilon$ -edges cannot be a DFA because  $\varepsilon$ -edges bring in uncertainty as we could choose to stay in that state or take the  $\varepsilon$ -edge.

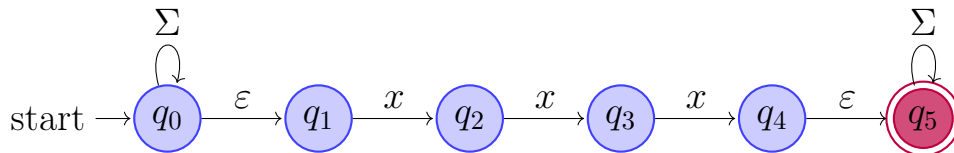
### 3.7.6 Examples

- $L = \{x \in \Sigma^* | x = u.v.w, v \in \Sigma^*, u, w \in \Sigma^+, |u| \leq 2, u=w\}$  where  $\Sigma = \{0, 1, 2 \dots 9\}$   
 For instance, "0000"  $\in L$  because either take  $u = w = 0, v = 00$  or take  $u = w = 00, v = \varepsilon$   
 Also, "1234"  $\notin L$  because we cannot have any satisfiable  $u, v, w$ .  
 It's NFA will be a combination of 110 NFA's of the following form :-

Figure 3.12:  $\varepsilon$ -NFA for  $u = w = 0$ Figure 3.13:  $\varepsilon$ -NFA for  $u = w = 0, 1$ 

One for each  $u = w = 0, 1, 2 \dots 9$  so 10 here and 100 more for  $u = w = 00$  to 99. We can have separate NFA's with different possible  $q'_1, q'_3$  states. Like instead of 0 put 1 to 9 and 00 to 99 there on the edge from  $q_0$  to  $q'_1$  and  $q'_3$  to  $q_4$  and in this way we can form the whole NFA just like in figure 6.

- $L = .*xxx.*$  where  $\Sigma = \{a, b, \dots z\}$   
 Here, we want to locate "xxx" in the text so the following NFA captures this language:-

Figure 3.14:  $\varepsilon$ -NFA for searching "xxx"

- **Takeaway task :** Think about how KMP implicitly constructs NFA for pattern matching.

### 3.8 Equivalence in Finite Automata

In our exploration of finite automata, we've encountered deterministic finite automata (DFA), non-deterministic finite automata without epsilon transitions (NFA), and non-deterministic finite automata with epsilon transitions ( $\varepsilon$ -NFA). Surprisingly, despite their apparent differences, these three models are fundamentally equivalent in terms of computational power.

DFA, characterized by their deterministic nature, are particularly useful in scenarios where determinism is crucial, such as in hardware design where predictability is paramount.

$\varepsilon$ -NFA, on the other hand, introduce a high level of abstraction by allowing transitions without consuming input symbols, which can simplify the representation of certain languages and aid in conceptual clarity.

NFA without epsilon transitions also provide a similarly high level of abstraction, allowing for flexibility in modeling complex systems and languages.

### 3.9 DFA definition

Any DFA can be completely represented by the tuple  $DFA = (Q, \Sigma, q_0, \delta, F)$  where,

- $Q$  is the set of all states
- $\Sigma$  is the alphabet
- $q_0$  is the starting state
- $\delta : (Q \times \Sigma) \rightarrow Q$  is the transition function
- $F$  is the set of final or accepting states

#### 3.9.1 Combinations of DFAs

We can represent the combination of two DFAs defined on the same alphabet  $\Sigma$ ,  $DFA_1 = (Q, \Sigma, q_0, \delta, F)$  and  $DFA_2 = (Q', \Sigma, q'_0, \delta', F')$  by another DFA,

$$DFA_3 = (Q \times Q', \Sigma, (q_0, q'_0), \hat{\delta}, \hat{F}) \quad (3.1)$$

where  $\hat{\delta}((q, r), a) = (\delta(q, a), \delta'(r, a))$  and  $\hat{F}$  can be defined according to the required operation on the DFAs.

For example, if  $Q = \{q_0, q_1\}$  and  $Q' = \{r_0, r_1, r_2\}$  and  $F = \{q_1\}$  and  $F' = \{r_1, r_2\}$  then for the **intersection** of these Automata,  $\hat{F} = \{(q_1, r_1), (q_1, r_2)\}$  and for the **union** of the Automata,  $\hat{F} = \{(q_1, r_0), (q_1, r_1), (q_1, r_2), (q_0, r_1), (q_0, r_2)\}$ . Similarly we can define the **complement** operation by taking  $\hat{F} = Q - F$ .

With these basic rules in place, we can go on to define more complicated combinations like  $(DFA_1 \cap DFA_3) \cup (DFA_2 \cap DFA_3) - (DFA_1 \cap DFA_2)$

We now have another way to tell if a language is a subset of another language. To tell if  $L_1 \subseteq L_2$ , we have to show that  $L_1 \cap L_2^c = \phi$ . The problem thus reduces to showing that in the DFA defined by  $DFA_1 \cap DFA_2^c$  there is no path from the start node to any accepting state.

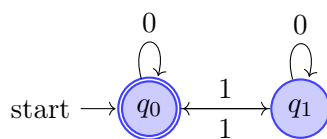


Figure 3.15: Two-State Automaton

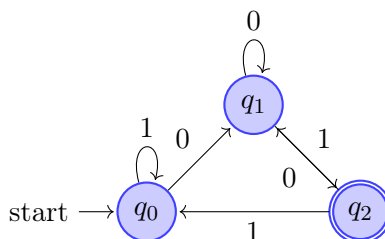


Figure 3.16: Three-State Automaton

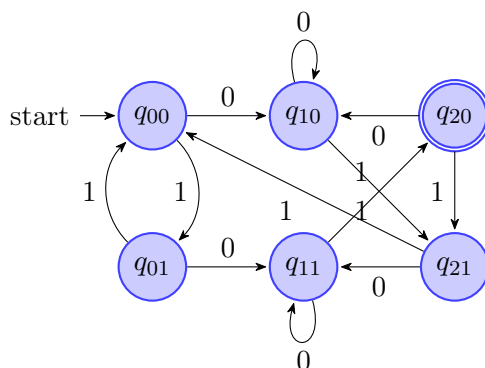


Figure 3.17: Intersection of the two automata

### 3.9.2 Combinations of NFAs

For two NFAs, the union and intersection is defined in a similar way with the only difference being the transition function since it now returns a set of states instead of a single state.

$$\hat{\delta}((q, r), a) = \delta(q, a) \times \delta'(r, a) \quad (3.2)$$

However, for NFAs, just flipping the accepting and non-accepting states won't give us the complement.

Thus to take the intersection or union of two NFAs, we can follow a similar approach to DFAs but for complementation, the NFA must first be converted to a DFA and then complemented to give the actual complement of the original NFA.

### 3.9.3 Closure Properties

Given  $L_1, L_2$  are two regular languages over the alphabet  $\Sigma = \{a, b\}$ ,  $\overline{L_1}$ ,  $L_1 \cup L_2$ ,  $L_1 \cap L_2$  are also regular languages (because their corresponding DFAs can be constructed by the combination of the original DFAs).

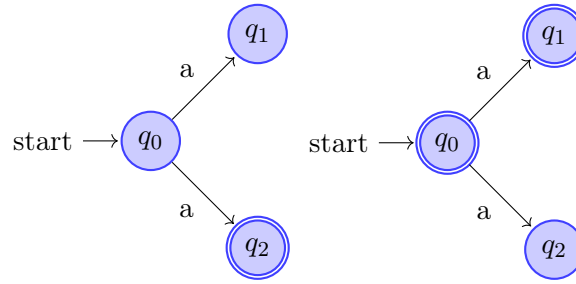


Figure 3.18: These NFAs are not complements of each other

### 3.10 Substitution

We will start with an example.

Consider two alphabets  $\Sigma_1 = \{a, b\}$  and  $\Sigma_2 = \{0, 1, 2\}$  and languages  $L_1 = a^*b^*$  defined on  $\Sigma_1^*$  and  $L_a = 0^{*(1+2)^*1^*}$  and  $L_b = 1^{*(0+2)^*}$  defined on  $\Sigma_2^*$ .

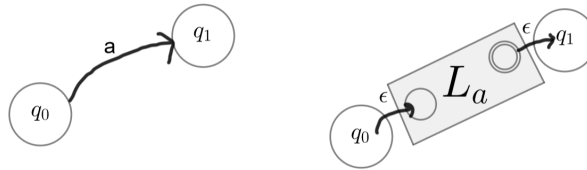
Now we define a set  $\text{subst}(L_1, L_a, L_b)$  as

$$\text{subst}(L_1, L_a, L_b) = \left\{ w \in \Sigma_2^* \mid \exists u \in L_1 = \alpha_1 \alpha_2 \dots \alpha_k \text{ such that } w \in L_{\alpha_1} L_{\alpha_2} \dots L_{\alpha_k} \right\} \quad (3.3)$$

Or equivalently,  $\text{subst}(L_1, L_a, L_b) = \bigcup_{u = \alpha_1 \alpha_2 \dots \alpha_k \in L_1} L_{\alpha_1} L_{\alpha_2} \dots L_{\alpha_k}$

Intuitively, the substitution operation is to replace each letter by a language.

Diagrammatically, it is represented as replacing each edge by an entire automaton and connecting the initial and accepting states to the original states by  $\epsilon$  edges.



Using this operation, we can easily prove that if  $L_1$  and  $L_2$  are regular languages then  $L_1 \cdot L_2$  is also regular, since  $L = \{a \cdot b\}$  is regular then  $\text{subst}(L, L_1, L_2)$  will also be regular.

#### 3.10.1 Infinite Languages

If we have a DFA with  $n$  states and there exists some string of length greater than  $n$  which is accepted by the DFA, then by the pigeon-hole principle there must exist some state  $q$  in the DFA such that there is a cycle with  $q$  in it. Suppose that the accepting string is  $u \cdot v \cdot w$ , where  $v$  is the string that starts and ends at the same state, then the strings  $u \cdot w$ ,  $u \cdot v^2 \cdot w$ ,  $u \cdot v^{*} \cdot w$  are also accepted in the DFA. Hence the language formed by the string is of infinite length.



## Chapter 4

# Regular Expressions

### 4.1 Introduction

In arithmetic, we can use the operations  $+$  and  $\times$  to build up expressions such as  $(5 + 3) \times 4$ . Similarly, we can use the regular operations to build up expressions describing languages, which are called regular expressions. An example is:  $(0 \cup 1)0^*$ . The value of the arithmetic expression is the number 32. The value of a regular expression is a language. In this case, the value is the language consisting of all strings starting with a 0 or a 1 followed by any number of 0s. We get this result by dissecting the expression into its parts. First, the symbols 0 and 1 are shorthand for the sets  $\{0\}$  and  $\{1\}$ . So  $(0 \cup 1)$  means  $(\{0\} \cup \{1\})$ . The value of this part is the language  $\{0, 1\}$ . The part  $0^*$  means  $\{0\}^*$ , and its value is the language consisting of all strings containing any number of 0s. Second, like the  $\times$  symbol in algebra, the concatenation symbol  $\circ$  often is implicit in regular expressions. Thus  $(0 \cup 1)0^*$  actually is shorthand for  $(0 \cup 1) \circ 0^*$ . The concatenation attaches the strings from the two parts to obtain the value of the entire expression. Regular expressions have an important role in computer science applications. In applications involving text, users may want to search for strings that satisfy certain patterns. Regular expressions provide a powerful method for describing such patterns. Utilities such as **awk** and **grep** in UNIX, modern programming languages such as Perl, and text editors all provide mechanisms for the description of patterns by using regular expressions.

### 4.2 Formal Definition of a Regular Expression

Let  $R$  be a regular expression if:

1.  $a$  for some  $a$  in the alphabet  $\Sigma$ ,
2.  $\varepsilon$ ,
3.  $\emptyset$ ,
4.  $(R_1 \cup R_2)$ , where  $R_1$  and  $R_2$  are regular expressions,
5.  $(R_1 \cdot R_2)$ , where  $R_1$  and  $R_2$  are regular expressions, or
6.  $(R_1^*)$ , where  $R_1$  is a regular expression.

In items 1 and 2, the regular expressions  $a$  and  $\varepsilon$  represent the languages  $\{a\}$  and  $\{\varepsilon\}$ , respectively. In item 3, the regular expression  $\emptyset$  represents the empty language. In items 4, 5, and 6, the expressions represent the languages obtained by taking the union or concatenation of the languages  $R_1$  and  $R_2$ , or the Kleene star of the language  $R_1$ , respectively.

## 4.3 Semantics of Regular Language

The semantics of regular language involves understanding the structure and meaning of expressions within the language.

### 4.3.1 Atomic Expressions

Consider an atomic expression  $[a]$ , where  $a$  represents a single letter. In this context,  $[a]$  denotes the language consisting of only the string  $a$ . However, it's important to note that the  $a$  within  $[a]$  is not considered a letter of the alphabet; rather, it signifies a language comprising a single letter string.

### 4.3.2 Union Operation

For expressions  $e_1$  and  $e_2$ ,  $[e_1 + e_2]$  represents the union of the languages denoted by  $[e_1]$  and  $[e_2]$ . In simpler terms,  $[e_1 + e_2]$  encompasses all strings that belong to either  $[e_1]$  or  $[e_2]$ .

### 4.3.3 Concatenation Operation

When considering  $e_1$  concatenated with  $e_2$ , denoted as  $[e_1.e_2]$ , it signifies the concatenation of languages represented by  $[e_1]$  and  $[e_2]$ . This operation results in a language consisting of all possible combinations of strings where the first part belongs to  $[e_1]$  and the second part belongs to  $[e_2]$ .

### 4.3.4 Example

Let's illustrate with an example:  $(ab) + a$ . This expression represents the union of the language containing the string  $ab$  and the language containing the string  $a$ , resulting in  $\{ab, a\}$ .

### 4.3.5 Order of Precedence

In the semantics of regular language, the order of precedence for operations is as follows:  $*$  (Kleene star)  $>$   $\cdot$  (concatenation)  $>$   $+$  (union).

### 4.3.6 Kleene Star

The Kleene star operation  $[e_1^*]$  denotes the union of zero or more concatenations of  $[e_1]$ . In other words,  $[e_1^*]$  encompasses all possible strings that can be formed by concatenating any number of strings from  $[e_1]$ .

### 4.3.7 Example

For an expression  $e_1 = a + b$ , the language denoted by  $[e_1]$  is  $\{a, b\}$ . Therefore,  $[(a + b)^*]$  represents the set of all possible strings comprising  $as$  and  $bs$ , including the empty string  $\varepsilon$ ,  $a$ ,  $b$ ,  $ab$ ,  $ba$ ,  $aa$ ,  $bb$ , and so on.

### 4.3.8 Further Examples

- $[a^* + b^*] = \{u \in \Sigma^* \mid u = a^n \text{ or } u = b^n \text{ for some } n \geq 0\}$
- $[a^* \cdot b^*] = \{u \in \Sigma^* \mid u = a^n b^m \text{ and } n \geq 0 \text{ and } m \geq 0\}$
- $[(a^* \cdot b^*)^*]$  represents the set of all strings containing any number of occurrences of strings composed of *as* followed by *bs*.
- $0^*10^* = \{w \mid w \text{ contains a single } 1\}$
- $\Sigma^*1\Sigma^* = \{w \mid w \text{ has at least one } 1\}$
- $\Sigma^*001\Sigma^* = \{w \mid w \text{ contains the string } 001 \text{ as a substring}\}$
- $1^*(01^+)^* = \{w \mid \text{every } 0 \text{ in } w \text{ is followed by at least one } 1\}$
- $(\Sigma\Sigma)^* = \{w \mid w \text{ is a string of even length}\}$
- $(\Sigma\Sigma\Sigma)^* = \{w \mid \text{the length of } w \text{ is a multiple of } 3\}$
- $01 \cup 10 = \{01, 10\}$
- $0\Sigma^*0 \cup 1\Sigma^*1 \cup \{0, 1\} = \{w \mid w \text{ starts and ends with the same symbol}\}$
- $(0 \cup \varepsilon)1^* = 01^* \cup 1^*$
- $(0 \cup \varepsilon)(1 \cup \varepsilon) = \{\varepsilon, 0, 1, 01\}$
- $1^*\emptyset = \emptyset$
- $\emptyset^* = \{\varepsilon\}$  The star operation puts together any number of strings from the language to get a string in the result. If the language is empty, the star operation can put together 0 strings, giving only the empty string.

## 4.4 Kleene's Theorem

In the last lecture, we introduced regular expressions and saw how they can be used to represent languages. In this lecture, we explore one of the most fundamental theorems of Automata Theory, Kleene's Theorem.

**Kleene's Theorem:** In terms of expressive power,

$$\text{Regular Expressions} \equiv \text{NFAs with } \epsilon \text{ edges} \equiv \text{NFAs without } \epsilon \text{ edges} \equiv \text{DFAs}$$

In previous lectures, we have already proved the equivalence of NFAs with  $\epsilon$  edges, NFAs without  $\epsilon$  edges and DFAs. So in order to prove Kleene's Theorem, it suffices to show the equivalence between Regular Expressions and NFAs with  $\epsilon$  edges.

**To prove:** Regular Expressions  $\equiv$  NFAs with  $\epsilon$  edges  
i.e.,  $L(\text{Reg. Ex.}) = L(\text{NFAs with } \epsilon \text{ edges})$

The proof can be divided into two subparts

1.  $L(\text{Reg. Ex.}) \subseteq L(\text{NFAs with } \epsilon \text{ edges})$
2.  $L(\text{Reg. Ex.}) \supseteq L(\text{NFAs with } \epsilon \text{ edges})$

We prove each of these parts separately and then combine them to get the required result.

#### 4.4.1 Part 1: $L(\text{Reg. Ex.}) \subseteq L(\text{NFAs with } \epsilon \text{ edges})$

Let us consider the following example.

Suppose our alphabet is  $\Sigma = \{0, 1\}$   
and our regular expression is  $((0.1)^* + (1.0)^*) \cdot (1.1 + 0.0)^*$

We wish to construct an NFA with  $\epsilon$  edges that accepts the same language as our regular expression.

To do this, we construct the parse tree of the regular expression and use a bottom up approach starting from the leaves to construct an NFA for each node of the parse tree.

The parse tree of the given regular expression is as follows.

$[\cdot [\cdot [\cdot [+ [\cdot [\cdot [0] [1] ] ] ] [\cdot [\cdot [1] [0] ] ] ] ] ] [\cdot [\cdot [+ [\cdot [1] [1] ] ] [\cdot [0] [0] ] ] ] ] ]$

We start from the leaves, in this case the nodes labelled 0 and 1. Consider the leaf 0. It represents a language with only one string, i.e.,  $L(0) = \{0\}$ . It can be represented by the NFA

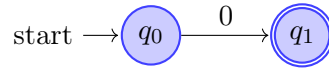


Figure 4.1: NFA that accepts only  $\{0\}$

Similarly leaf 1 can be represented by the NFA

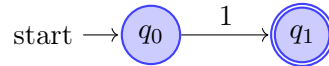


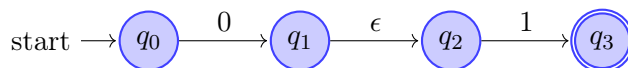
Figure 4.2: NFA that accepts only  $\{1\}$

Now we have NFAs for each of the leaf nodes. Let us see how to construct the NFAs for the rest of the nodes from these.

Suppose we have the NFAs of regular expressions **a** and **b**, and we want to obtain the NFA of **a.b**. Since  $\cdot$  is simply the concatenation operator, the new NFA should accept all strings that satisfy both the NFAs in sequence. Such an NFA can be constructed by connecting the final state of **a** and the starting state of **b** (and marking them as regular states) by an  $\epsilon$  edge. This results in

a new NFA whose starting state is the starting state of **a** and final state is the final state of **b**. The language of this NFA is simply **a.b**.

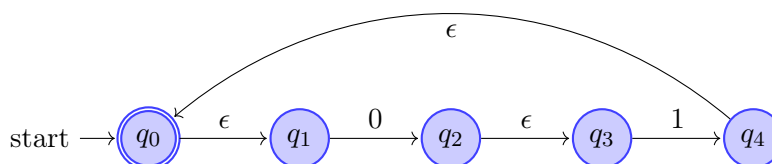
In our example, given the NFAs of **0** and **1** we want to construct the NFA of **0.1**. Following the above procedure, we obtain the NFA

Figure 4.3: NFA for **0.1**

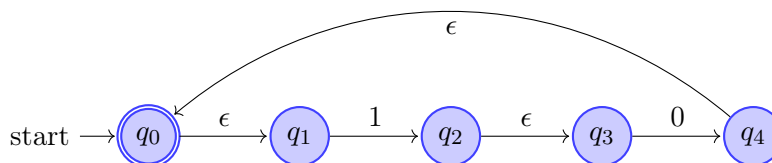
From the parse tree, the next node we need to construct an NFA for is **(0.1)\***.

The language accepted by **a\*** is simply the set of strings in which strings in the language of **a** are concatenated with themselves multiple (possibly 0) times. To obtain the NFA for **a\***, we take the NFA of **a**, introduce a new state that is both starting and accepting, and draw  $\epsilon$  edges from the original final state to the new state and from the new state to the original starting state. We also mark the original starting and accepting states as normal states. The correctness of this procedure can be proved by induction.

Applying this to **0.1**, we obtain the NFA for **(0.1)\***

Figure 4.4: NFA for **(0.1)\***

Similarly we can construct the NFA for **(1.0)\***

Figure 4.5: NFA for **(1.0)\***

Now the next step in our recursive construction of parse tree nodes is to obtain the NFA for **(0.1)\* + (1.0)\***.

The language accepted by **a+b** is simply the union of the language accepted by **a** and the language accepted by **b**. So in order for a string to be accepted by the NFA of **a+b**, it must be

accepted by either the NFA of **a** or the NFA of **b**. Hence the required NFA can be obtained by introducing a new start state, drawing  $\epsilon$  edges from it to the starting states of the original NFAs, and marking the original NFA starting states as normal states.

Applying this procedure to  $(0.1)^*$  and  $(1.0)^*$ , we get the NFA of  $(0.1)^* + (1.0)^*$

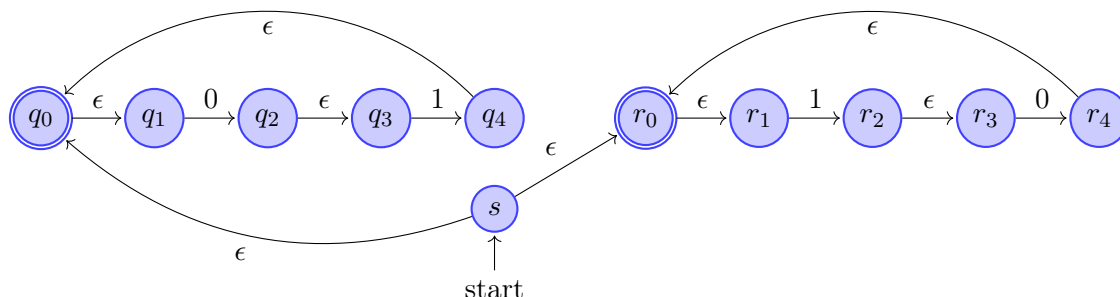
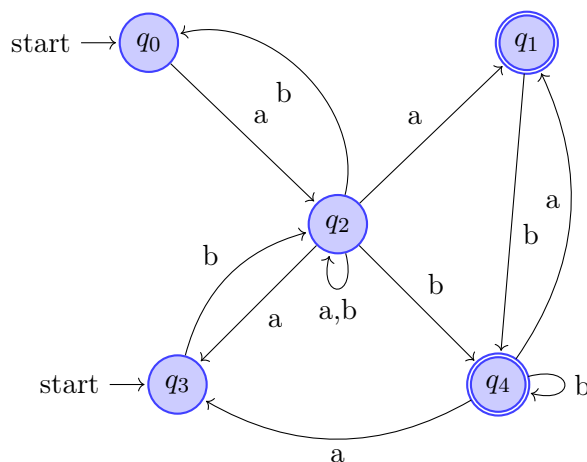


Figure 4.6: NFA for  $(1.0)^* + (0.1)^*$

Using the same procedures, we can construct NFAs for all the remaining nodes of the parse tree. We observe that the size of the NFA is linear in the size of the original regular expression.

#### 4.4.2 Part 2: $L(\text{Reg. Ex.}) \supseteq L(\text{NFAs with } \epsilon \text{ edges})$

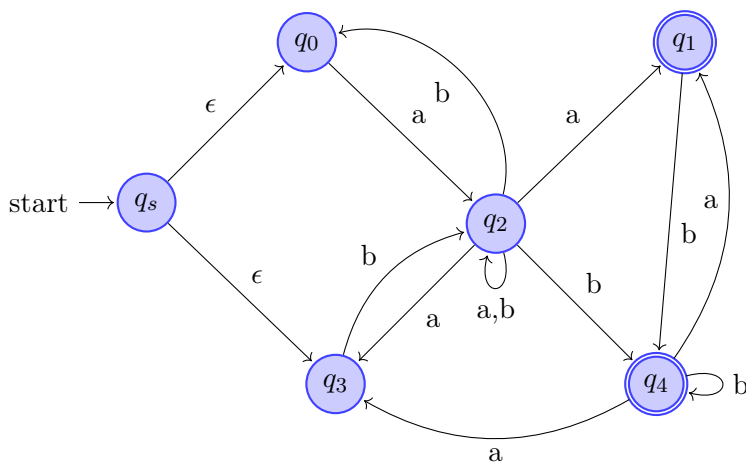
Once again we consider an example NFA with  $\epsilon$  edges and convert it to a regular expression. Now we have  $\Sigma = \{a, b\}$  and the following NFA



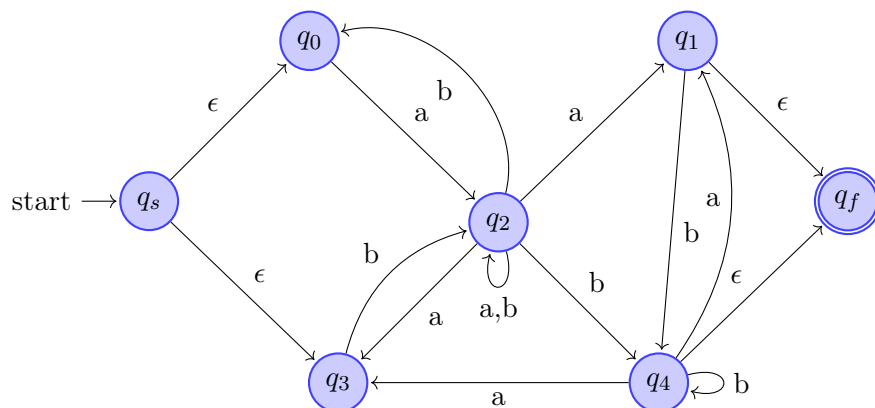
Our strategy to obtain a regular expression is to gradually reduce the 5 state automaton to one with fewer states, and keep repeating this process until we end up with a regular expression. To perform this reduction, we relax a condition on NFAs; we now allow edges to be labelled by regular expressions. An edge labelled by a regular expression simply means a path in which we consume a string satisfying the regular expression.

**Algorithm**

**Step 1:** Create a new starting state and get rid of all the old ones. Connect this new state to the old starting states using  $\epsilon$  edges.



**Step 2:** Create a new final state and get rid of the old ones. Connect all the old final states to the new state using  $\epsilon$  edges.



Observe that our NFA now has one single initial state with no edges leading back to it, and one single accepting state with no edges going out of it.

**Step 3:** Systematically remove all the states except for the initial and accepting states.

Suppose we choose state  $q_2$  to be removed first. This state was facilitating some strings' paths from the initial to the accepting state. In order to remove  $q_2$ , we must first create alternate paths not involving  $q_2$  that allow these strings to reach the final state. To do this, we follow the approach below,

1. Choose an incoming transition of  $q_2$ . In our example, suppose we choose the transition from  $q_0$  to  $q_2$  on consuming  $a$ .
2. Now consider all the outgoing edges of  $q_2$ , say from  $q_2$  to  $q_x$ . Our goal is to facilitate transitions from  $q_0$  to  $q_x$  without involving  $q_2$ . This can be done by introducing direct transitions from  $q_0$  to  $q_x$  and labelling them with the regular expressions formed by concatenating  $label(q_0 - q_2)$  and  $label(q_2 - q_x)$ , and then removing the incoming edge.

Applying step 2 to  $q_2$ ,

Observe that  $q_2$  has a self loop labelled  $a, b$ , which can be replaced by regular expression  $\mathbf{a+b}$ . This means that while concatenating  $label(q_0 - q_2)$  and  $label(q_2 - q_x)$ , we must also account for the regular expression  $\mathbf{(a+b)^*}$  in between the labels.

(a) Outgoing edge:  $q_2$  to  $q_0$  on consuming  $b$

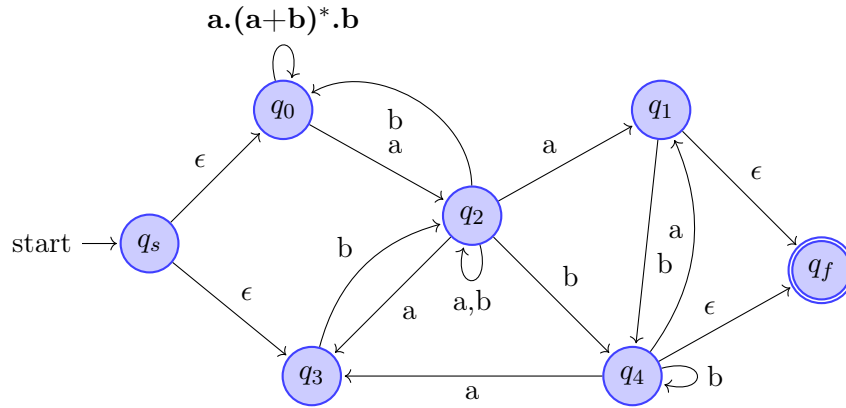


Figure 4.7: (a)

(b) Outgoing edge:  $q_2$  to  $q_1$  on consuming  $a$

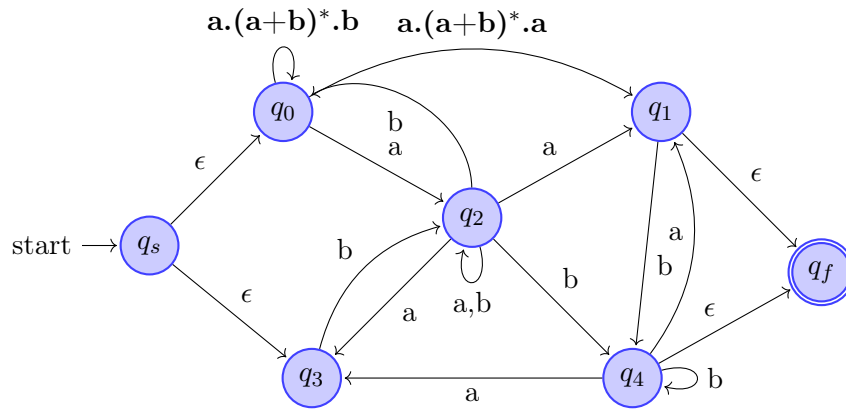


Figure 4.8: (b)



(c) Outgoing edge:  $q_2$  to  $q_3$  on consuming a

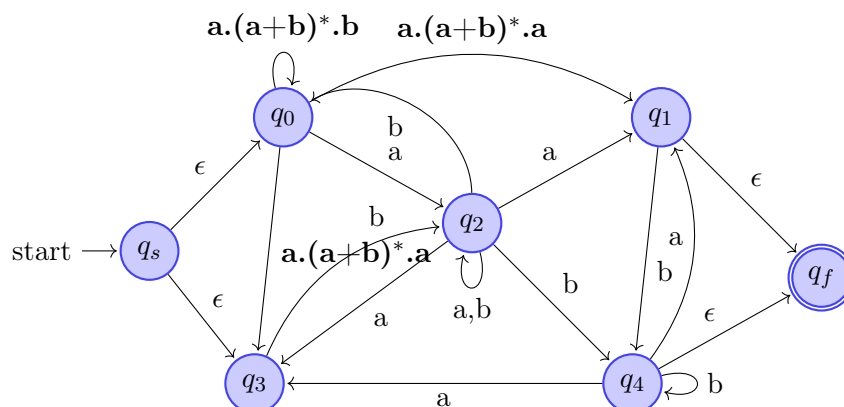


Figure 4.9: (c)

(d) Outgoing edge:  $q_2$  to  $q_4$  on consuming a

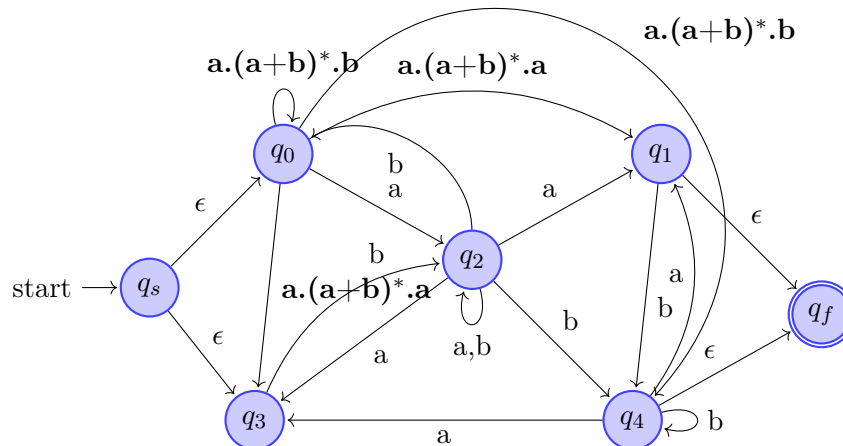


Figure 4.10: (d)

The NFAs corresponding to applying step 2 to each of the above transitions are illustrated on the next page.

Now that we have added the above four transitions, strings from  $q_0$  do not need  $q_2$  to reach the final state. So we can remove the incoming edge from  $q_0$  to  $q_2$  (see figure 11). Now we repeat the same procedure for all incoming edges of  $q_2$ . After doing this,  $q_2$  has no incoming edges and hence cannot be part of the path of any string. So it does not contribute to our NFA and can be removed along with all its outgoing edges.

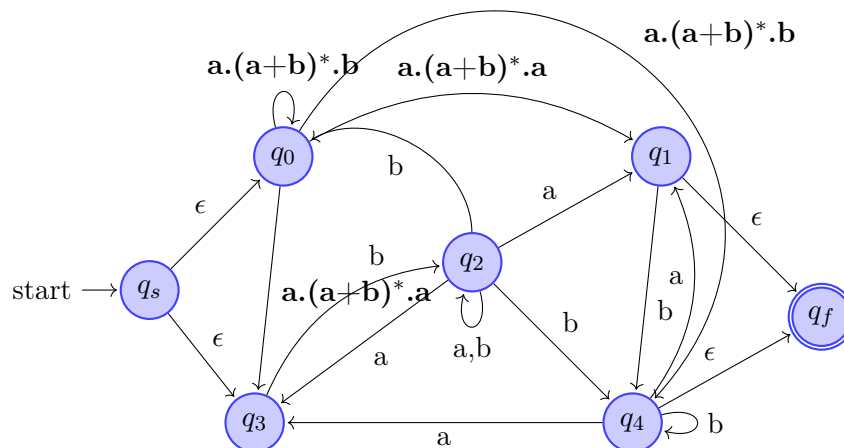
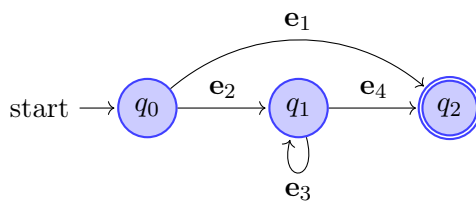


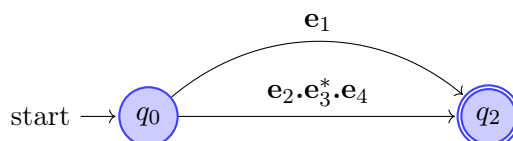
Figure 4.11: NFA after removing the incoming edge

We repeat these steps for each remaining non-initial non-accepting state of the NFA, and finally obtain the required regular expression.

To see how exactly the NFA becomes a regular expression, we illustrate the pre-final step for an arbitrary NFA.

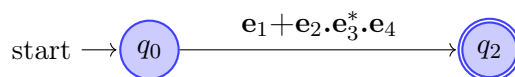


Eliminating the intermediate state, we get



Combining the two transitions,

So we obtain the regular expression  $e_1 + e_2.e_3^*.e_4$ .



### 4.4.3 Checking Subsethood of Languages

We return to the example from the previous lecture, proving the equivalence of the regular expressions  $(\mathbf{a^*b^*})^*$  and  $(\mathbf{a+b})^*$ . Now we can leverage Kleene's Theorem and convert both expressions into their equivalent DFAs, and then check if both DFAs accept the same language or not. To check this, we can check each of the following conditions separately,

1.  $L(\text{DFA}_1) \subseteq L(\text{DFA}_2)$
2.  $L(\text{DFA}_1) \supseteq L(\text{DFA}_2)$

and then combine their results to get

$$L(\text{DFA}_1) = L(\text{DFA}_2)$$

But how do we check if a language  $L_1$  is a subset of another language  $L_2$  from just their DFAs? We use the following algorithmic procedure.

We say that  $L_1 \subseteq L_2$  when every string in  $L_1$  is also present in  $L_2$ . So we can say

$$L_1 \subseteq L_2 \equiv L_1 \cap (\Sigma^* \setminus L_2) = \phi$$

The language  $\Sigma^* \setminus L_2$  is just the set of all strings not accepted by  $L_2$ . So to obtain its DFA, all we have to do is invert the acceptance status of each state (accepting states become normal states and vice versa). Also since  $L_2$  is a regular language,  $\Sigma^* \setminus L_2$  (denoted by  $L_2^c$  or  $\bar{L}_2$ ) is also a regular language.

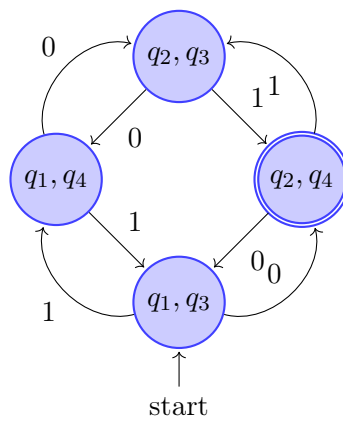
Now given the DFAs of  $L_1$  and  $L_2^c$ , we wish to construct the DFA of  $L_1 \cap L_2^c$ . This new DFA should accept only those strings that are accepted by both  $L_1$  and  $L_2^c$ .

We achieve this by running both DFAs simultaneously on the same string, and checking if we end up in a pair of accepting states. This is analogous to taking the Cartesian product of both transition functions. Consider the following example:



Then the DFA representing their intersection is given by

Since the initial condition we wanted to satisfy was  $L_1 \cap L_2^c = \phi$ , we need a way of checking this from the DFA. If the language of a DFA is  $\phi$ , this means the DFA does not accept any string, meaning there should be no path from an initial to an accepting state.



This completes the systematic approach to determine subsethood.

## Chapter 5

# DFA Minimisation

### 5.1 Minimum States in a DFA

So, far we have dealt with DFA, NFA without  $\epsilon$ , NFA with  $\epsilon$  and Regular Expressions. We have also seen that all of them are equivalent and inter-convertible and represent regular languages. Consider the language which consists of all strings which are terminated by one. The regular expression for this will be:  $(0 + 1)^*1$ . Here is a 2-state DFA for the same language:

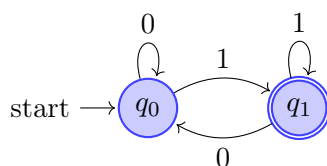


Figure 5.1: 2-state DFA for the language

The above DFA has two states  $q_0$  and  $q_1$ .  $q_0$  is reached when the last seen letter was 0 (also at the start).  $q_1$  is reached when the last seen letter was 1 (also, it is an accepting state). We can even construct a 4-state DFA for the same language. Here is an example:

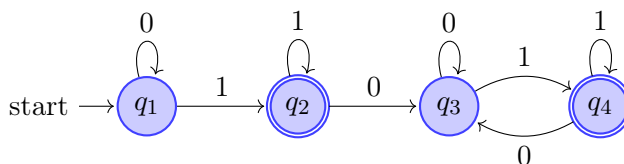


Figure 5.2: 4-state DFA for the language

The above DFA has four states  $q_1$ ,  $q_2$ ,  $q_3$ ,  $q_4$ .

- $q_1$ : Last letter 0 and no 1s so far
- $q_2$ : Last letter 1 and no 101 seen so far
- $q_3$ : Last letter 0 and more than one 1s seen so far

- $q_4$ : Last letter 1

One can construct many more 4-state DFAs for the same language. Above is another example.

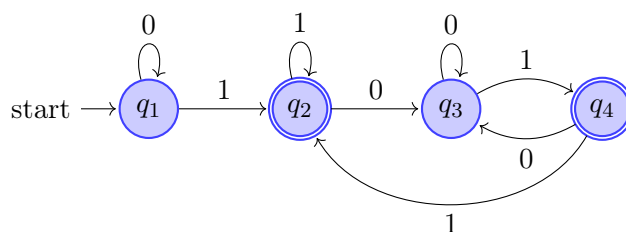


Figure 5.3: Another 4-state DFA for the same language

A natural question which comes to our mind is that can we construct another 2-state DFA for this language. One can find through trial and error, that this is not possible.

Is one state DFA possible for this language? Let us assume that this is possible. Two cases arise. If that state is accepting, then it will also accept  $\epsilon$ , which is not possible. If that state is not accepting, then the language will be empty. We have arrived at a contradiction. Hence, one state DFA is not possible for this language.

Therefore, for this language, the minimum states in any DFA can be 2. We also observe that number of such 2-state DFAs is 1. So, we will now claim that for every language, the number of minimal DFAs is 1 and try to prove this. Before we prove this, we will introduce the notion of indistinguishability.

## 5.2 Indistinguishability

Two states of a DFA  $q_i$  and  $q_j$  are considered indistinguishable, if  $\forall w \in \Sigma^*$ , we start with  $q_i$ , process  $w$  and reach  $q'_i$  and we start with  $q_j$ , process  $w$  and reach  $q'_j$ , then either  $q'_i \in F$  and  $q'_j \in F$  or  $q'_i \notin F$  and  $q'_j \notin F$ , where  $F$  is the set of all final states of the DFA. So, we are basically changing the start states and checking whether we reach the same type of states or not through the same string.

This relation is denoted by  $\equiv$ . It has the following properties:

- It is **reflexive**. It is clear to see that every state is indistinguishable to itself, as it will reach a particular state on seeing  $w$ . (Due to the nature of a DFA)
- Also, it is clear to see that this relation is **symmetric**.
- This relation is also **transitive**. We can prove this by contradiction. Let us assume that  $(q_i \equiv q_j) \wedge (q_j \equiv q_k)$  but  $q_i \not\equiv q_k$ . Then  $\exists w$  such that  $q'_i \in F$  and  $q'_k \notin F$ , where  $q'_i$  and  $q'_k$  are the states we reach from  $q_i$  and  $q_k$  respectively on seeing  $w$ . From the equivalence of  $q_i$  and  $q_j$ , we have  $q'_j \in F$  but from the equivalence of  $q_j$  and  $q_k$ , we have  $q'_j \notin F$ , where  $q'_j$  is the state that we reach from  $q_j$  on seeing  $w$ . Hence, we have arrived at a contradiction. Therefore, this relation is transitive.
- A relation which is reflexive, symmetric and transitive, is **equivalent**. Thus the states of the DFA, on which this relation is defined can be partitioned into equivalence classes.

In the above example (Figure: 5.2),  $q_1$  and  $q_3$  belong to the same equivalence class, and  $q_2$  and  $q_4$  also belong to another equivalence class. Let us now try to construct a 2-state DFA from the above 4-state DFA example (Figure: 5.2). We will choose, one element each from both of the equivalence classes. Let us take  $q_1$  from the first class and  $q_4$  from the second class.

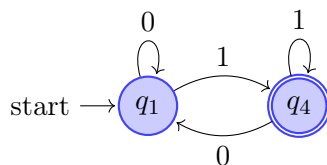


Figure 5.4: 2-state DFA constructed from the 4-state DFA

From  $q_1$ , if we see a 0, we land at  $q_1$  itself. If we see a 1, we would have landed at  $q_2$ , but since  $q_2$  and  $q_4$  are equivalent, we replace  $q_2$  by  $q_4$ . Similarly, from  $q_4$ , if we see a 1, we remain at  $q_4$ , but if we see a 0, we would have landed at  $q_3$ , but since  $q_1$  and  $q_3$  are equivalent, we replace  $q_3$  by  $q_1$ . Also,  $q_2$  and  $q_4$  both were acceptable earlier, now  $q_4$  is acceptable.

So, through these equivalence classes, we have minimized our 4-state DFA into a 2-state DFA, and also this 2-state DFA is structurally the same as the previous one (Figure: 7.10), thus again making us think that the claim that there is a single minimal DFA for every language might be correct.

Another interesting thing we observe is that an accepting state cannot be indistinguishable from a non-accepting state. We can take  $w = \epsilon$ , and observe that the states we reach from this pair of states do not satisfy the definition of indistinguishability relation. However, an initial state and a non-initial state can belong to the same equivalence class. (For example, above  $q_1$  and  $q_3$  belonged to the same class.)

Now, several important questions arise. How can we find the equivalence classes of this relation? When we will come to know that we cannot compress our DFA further (by compress, we mean reducing the number of states of the DFA)? How can we prove our claim that the minimal DFA will be unique?

We will try to answer all these questions subsequently. Let us start with the easiest one.

### 5.3 Equivalence classes of Indistinguishability relation

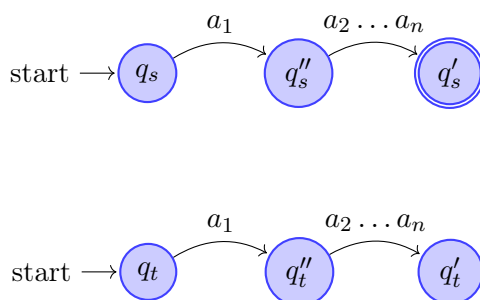
Now, we will develop an algorithm to find the equivalence classes of this relation.

Firstly, we should keep in mind our previous observation that an accepting state cannot be indistinguishable from a non-accepting state. That is  $q_i \not\equiv q_j \forall q_i \in F$  and  $q_j \in (Q \setminus F)$ , where  $Q$  is the set of all states of the DFA.

Suppose, we find two states  $q_s$  and  $q_t$  which are distinguishable. Thus there exists a string  $w$  such that  $q_s$  leads to an accepting state but  $q_t$  leads to a non-accepting state.



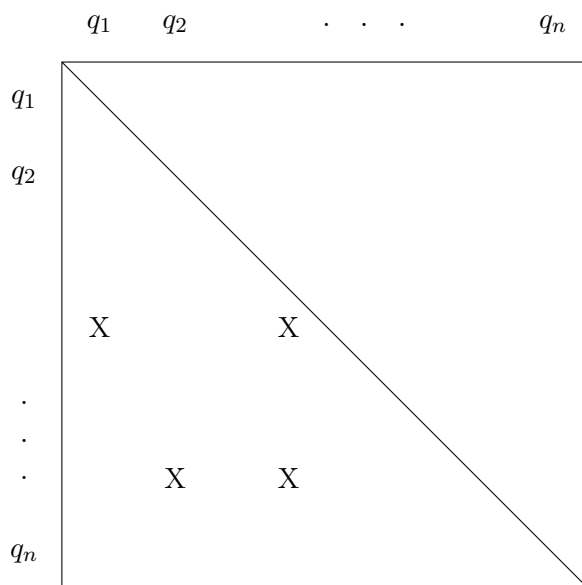
Now,  $w$  can be decomposed into  $a_1 a_2 \dots a_n$ , where  $|w| = n$ .



From, the above figures, one can observe that  $q''_s$  and  $q''_t$  are also distinguishable, where  $w' = a_2 \dots a_n$  is the string which is making them distinguishable.

Thus, we can observe that for all states  $q_i$  and  $q_j$  such that  $q_i \neq q_j$  and for all  $a \in \Sigma$ , such that  $q_s$  on seeing  $a$  lands at  $q_i$  and  $q_t$  on seeing  $a$  lands at  $q_j$  then  $q_s$  will be distinguishable with  $q_t$ , where  $q_s$  and  $q_t$  are two states of the DFA. (We are basically extending  $w' = a + w$ .)

Therefore using a distinguishable pair, we have found another one. This is the basis of our algorithm. We initialize our set with all the pairs, where one state belongs to the set of accepting states and other state does not belong to the set of non accepting states. And then through the above step, we keep on increasing the size of this set. (Note that this algorithm is not exponential, because there are only  $\binom{n}{2}$  pairs possible, and we do need to check an already visited pair.)

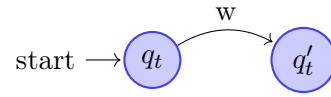
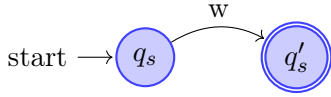


We stop the process, when no more crosses can be inserted.

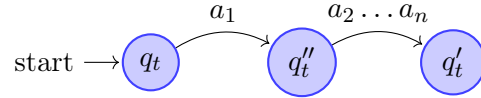
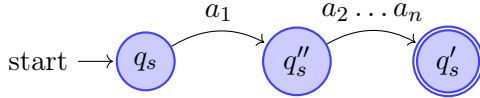
Now, it is quite natural for us to ask the question that whether our algorithm is correct or not i.e. can we still find a pair of distinguishable states, which are not detected even after our algorithm finishes?

**Proof:** Let us assume that there are two states  $q_s$  and  $q_t$  which are distinguishable but not recognized by our algorithm. By definition, there exists a string  $w$  such that  $q_s$  leads to an accepting state on seeing  $w$  and  $q_t$  reaches a non-accepting state.

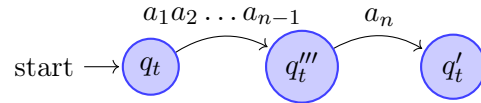
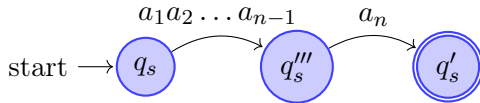




Now  $w$  can be written as  $a_1a_2 \dots a_n$ , where  $|w| = n$ . And also assume that we reach  $q_s''$  and  $q_t''$  on seeing  $a_1$  from  $q_s$  and  $q_t$  respectively. It is clear that, if our algorithm has not detected  $q_s$  and  $q_t$  as distinguishable, then it would not even have detected  $q_s''$  and  $q_t''$  as a distinguishable pair. (Because, if it would have done, then the next step would have been to make  $q_s$  and  $q_t$  as distinguishable.)



Now, we will inductively move forward our algorithm. Thus  $q_s'''$  and  $q_t'''$  would not also be detected as distinguishable after our algorithm finishes. But clearly, this is not possible, because our algorithm initializes the set of pairs of {accepting, non-accepting} states and then it's first step is to move backwards. So, it would have marked  $q_s'''$  and  $q_t'''$  as distinguishable in the first step itself.



We have achieved contradiction. Therefore, we can safely conclude that our algorithm terminates and is also correct.

Now, once our algorithm detects all pairs of distinguishable states, we can choose equivalence classes of the indistinguishability relation. (Then, we can choose one representative from each class and move forward with our proof of existence of a unique minimal DFA.)

It is worth noting that distinguishability is not an equivalent relation. It is not even reflexive. It is also not transitive. But, it is indeed symmetric. And also, it proved out to be very useful for finding these equivalence classes.

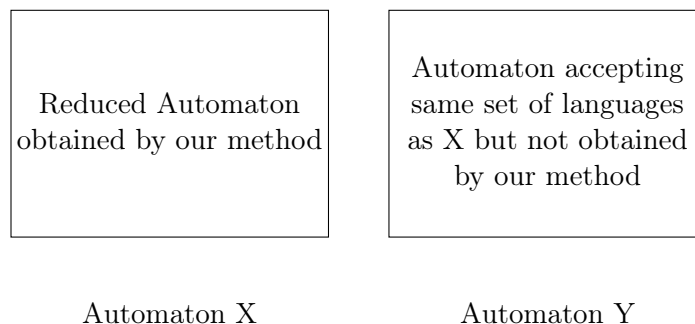
## 5.4 Further Analysis

We have developed an algorithm which can shrink the size of a given DFA. But is that the best we can do? Can the size of the DFA be further reduced? Are there many DFAs which are minimal or just one unique? The following sections will try to answer these questions.

**Note:** We are assuming that there is no redundant nodes in final DFA obtained, i.e. those nodes which can not be reached from starting state by any string. If there is one such, we can always remove it to get an equivalent DFA.

### 5.4.1 Optimality of Acquired DFA

Let's say we have an automaton X, which is the reduced automaton obtained by picking one state from each of the equivalence classes of the set of states, and Y is an automaton obtained by some other method for the same language.



We wish to show that the number of states of Y is at least as many as X.

Let's define a new term "**Language of State**". Given a DFA A and a state  $s$ , we define  $L_s^A$  as the set of all finite strings which will end in any of the accepting state if we start from  $s$ . Readers should be able to see that when we say that two states are distinguishable, we mean that there Language is different. Similarly, when we say that the two states are indistinguishable, we are actually asserting that the language of the two states is same.

Let say we started with the automaton D. After termination the resultant automaton was A. Let  $S_A$  represent the set of distinct such languages of nodes of A. Since A has all states indistinguishable from every other state, the  $|S_A| = \text{no of states}$ .

**Claim:** Given any DFA B, equivalent to A,

$$\forall L \in S_A (\exists S (L_S^B \equiv L)), \text{ where } S \text{ represent a state of } B$$

**Proof:** If  $L \in S_A$ , then by definition it must be the language of some node in A, say  $s$ . Since all the nodes of A are ir-redundant, so let's say string  $w$  is one such string through which we can reach this node starting from the starting state of A. Now run the same string  $w$  on the automaton B. Say we reach the state  $S$ . Then  $L \equiv L_s^A \equiv L_S^B$ , because if say string  $x$  is present in former and not in latter then, string  $w.x$  will be an accepting string in A and not in B, and vice-versa. Thus  $L_S^B \equiv L$ .

For every language in  $S_A$ , we must have at least one node in B. Since one node has a specific language, this gives us a lower bound to number of nodes,  $|S_A|$ .

**A achives the lower bound of states, hence it is the(?) minimum state DFA which represent the same regular language represented by D.**

### 5.4.2 Uniqueness of Acquired DFA

So far we have discussed about proving the optimality of the automaton A, which is the output of our algorithm. We defined an important notion of "Language of State" and proved a very important result which can be summarized as follow **Given two DFAs A and B where both represent some particular regular language, then for any string  $w$ , the states reached in A and B by reading it will have same language,i.e.  $L_{s_A}^A \equiv L_{s_B}^B$** . Now we will try to answer that if there are mutiple DFAs which are optimal or just one.

Let's consider two automats, one is our DFA A which has been proved to be optimal and another DFA C which is a equivalent DFA and is also optimal(given). We will show that C has to be isomorphic to A.

Since  $C$  is an optimal DFA, it can not be further shrunk. That implies two things: No redundant nodes, no pair of nodes is indistinguishable. That means that the language of the nodes of  $C$  is distinct and unique. Since  $A$  and  $C$  are both optimal, both must have same number of states, and all the nodes of both of them are reachable, and all the nodes of both of them have unique languages (unique over the domain of single DFA not collectively).

Consider any node  $s$  of  $A$ , take a string  $w$  which take us to it, run it over  $B$ , say we reach the node  $S$ , then  $s$  and  $S$  have same language. Since the nodes of  $B$  are distinguishable, so  $S$  does not depend on  $w$ .

This can be used to define an injection from the nodes of  $A$  to the nodes of  $B$ . It is injective function because of distinguishability of the nodes of  $A$  as well as of  $B$ . The function is also **bijective** because the number of nodes of  $A$  and  $B$  are both finite and equal, making it both injective and surjective, thus bijective.

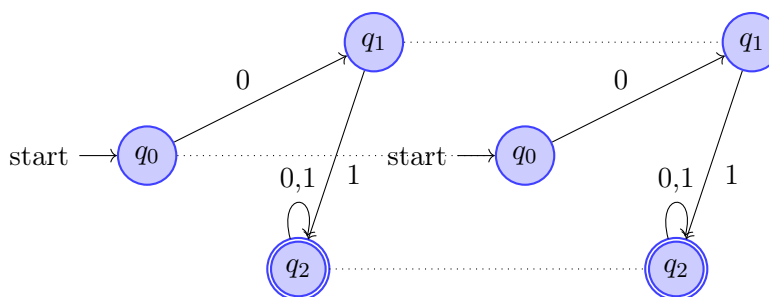


Figure 5.9: Two DFAs with corresponding states connected by dotted lines.

**Claim:**

**As per this function, starting state of  $A$  will be mapped to starting state of  $B$ .** This is because the languages of these two states is essentially the language of their respective DFA which is same (given).

**Claim:**

**An accepting state will be mapped to an accepting state of  $B$ .** This is because any state which has  $\epsilon$  in its language is an accepting state by the definition of "language of state", similarly an accepting state will have  $\epsilon$  in its language. So an accepting state of  $A$  mapped to whatever state of  $B$ , that state must have  $\epsilon$  in its language making it an acceptable state of  $B$ .

**Claim:**

**Consider two states of  $A$ ,  $S_1^A$  and  $S_2^A$ . Suppose that they are mapped to  $S_1^B$  and  $S_2^B$  respectively. Then if there is an  $\alpha$  labelled edge going from  $S_1^A$  to  $S_2^A$ , there must be an  $\alpha$  labelled edge going from  $S_1^B$  to  $S_2^B$  also.** Consider any string  $w$  that takes us to  $S_1^A$ , we can see that  $w.\alpha$  will take us to  $S_2^A$ . Now since  $S_1^A$  is mapped to  $S_1^B$  and  $S_1^B$  is mapped to  $S_2^B$  we can say that "any" string that takes us to one in  $A$  will take us to the image of it when run on  $B$  (Why? proved above:)). Thus  $w$  takes us to  $S_1^B$  and  $w.\alpha$  takes us to  $S_2^B$ . Because a DFA is deterministic, there should be  $\alpha$  labelled edge going from  $S_1^B$  to  $S_2^B$ . (Food for thought: Is mentioning determinism important here?)

**Isomorphism of labelled graphs:**

An isomorphism is a vertex bijection which is both edge-preserving and label-preserving.

**Isomorphism of DFAs:**

An isomorphism is a vertex bijection which is both edge-preserving and label-preserving, where im-

age of starting state is starting state and image of a accepting state is an accepting state. All these three claims shows that A and B are same automaton.

This tool can be used to check whether two different regular expressions represent the same language or different languages. Just convert them into DFAs, and then to the minimal DFAs. Check for the isomorphism of the two DFAs if they are isomorphic, then the regular expressions are equivalent otherwise not. ( Isomorphism  $\iff$  equivalence )

## 5.5 From states to words

Till now we were talking about languages of states and equivalence of two states. Now we will extend this notion for words.

### 5.5.1 Language of word

Consider a language L. we define Language of word  $w$ ,  $[w]$  as set of all strings  $x$  such that  $w.x \in L$ . Now we will define a relation " $\sim_L$ ". If languages of two words  $w_1$  and  $w_2$  is same for L, then we say that  $w_1$  is related to  $w_2$ . This is a reflexive, symmetric and transitive relation. (Basically, if set 1 and set 2 are same and then it is also true for the other way round that is set 2 and set 1 are same. If set 1 and set 2 are same and if set 2 and set 3 are same then set 1 and set 3 will also be same.) Thus we have defined an equivalence relation over words, where the equivalence classes depends on language L.

### 5.5.2 Relation between states of minimal DFA and equivalence classes for $\sim_L$

If a string  $w$  ends up in a state of DFA, say  $s$ , then the language of the state is equivalent to  $[w]_L$ . This can be easily proved by using definition of the "language of state" and "language of word". If a string  $w.x \in L$ , then if you run the  $w.x$  on DFA, you first reach that state using  $w$  then  $x$  takes to some accepting state. Hence  $x \in L_s^{DFA}$ . Also if  $x \in L_s^{DFA}$ , then essentially  $w.x$  will end in a accepting state because  $w$  ends in state  $s$ .

A particular equivalence class represent a particular language and a state represent a particular language. Since for a minimal DFA, all the states represent distinct language, we can define a bijection from equivalence classes to the state of minimal DFA, where a equivalence class is mapped to that state which represent the same language as that of any string in that equivalence class.

## 5.6 Setting up the Parallel

We defined the Nerode equivalence relation on a language L over the alphabet  $\Sigma$ . This relation states that  $\forall w_1, w_2 \in \Sigma^*$ ,  $w_1 \sim_L w_2$  iff  $\forall x \in \Sigma^*$ ,  $(w_1 \cdot x \in L \iff w_2 \cdot x \in L)$

If the language L is regular, then a minimized DFA  $A = (Q, \Sigma, \delta, q_0, F)$  can also be defined for the language L.

For this language  $L$ ,  $w_1$  and  $w_2$  are two words in  $\Sigma^*$  such that  $w_1 \sim_L w_2$ . Let  $q_i$  and  $q_j$  be two states  $\in Q$  such that

$$\begin{aligned} q_i &\rightarrow \text{state reached in } A \text{ after reading } w_1 \\ q_j &\rightarrow \text{state reached in } A \text{ after reading } w_2 \end{aligned}$$

Since  $w_1 \sim_L w_2$ , it follows that for all  $x \in \Sigma^*$ , the state reached in DFA  $A$  after reading  $w_1 \cdot x$  and the state reached in DFA  $A$  after reading  $w_2 \cdot x$  will either both belong to  $F$  or both belong to  $Q \setminus F$ . Otherwise, one word would be accepted by  $L$  while the other would not, leading to a contradiction in the definition of the equivalence relation.

Therefore, by the definition of indistinguishability, states  $q_i$  and  $q_j$  can be concluded to be indistinguishable. However, in a minimized DFA, all distinct pairs of states are distinguishable. Consequently, the only states in  $A$  that can be indistinguishable are those where the state is compared with itself.

This demonstrates that if two words/strings belong to the same equivalence class with respect to  $L$ , then both strings will end up in the same state  $q \in Q$  in the minimized DFA.

Till now, we have defined two equivalence relations  $\sim_L$  over the language  $L$  and  $\equiv$  over the states of a DFA characterizing a language  $L$ . We aim to define a relation between the number of equivalence classes of both these relations.

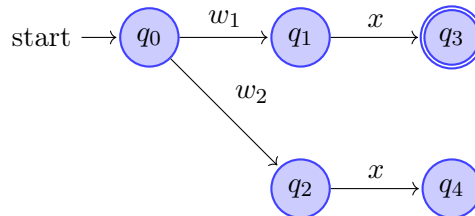
### 5.6.1 Can $|\sim_L| > |\equiv|$ ?

NO. We will in fact prove below that  $|\sim_L| \leq |\equiv|$

Let there be strings  $w_1, w_2$  s.t.

$$\begin{aligned} w_1 &\in i^{th} \text{ equivalence class of } \sim_L \\ w_2 &\in j^{th} \text{ equivalence class of } \sim_L \\ \text{where } i &\neq j \end{aligned}$$

Since they belong to different equivalence classes, we can say  $\exists x \in \Sigma^*$  s.t.  $w_1 \cdot x \in L$  and  $w_2 \cdot x \notin L$  (or can be vice-versa, does not matter).



- Here,  $q_1$  represents the equivalence class of  $w_1$  while  $q_2$  represents the equivalence class of  $w_2$ .
- By the definition of indistinguishability,  $q_1$  and  $q_2$  are distinguishable states as there exists a letter in the alphabet which leads them to another pair of distinguishable states.

- Doing this for every pair of equivalence classes, we find that the states representing the distinct equivalence classes are all distinguishable from each other leading us to the following relation.

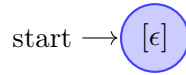
$$|\sim_L| \leq |\equiv|$$

i.e. the number of indistinguishability equivalence classes is atleast the the number of Nerode equivalence classes

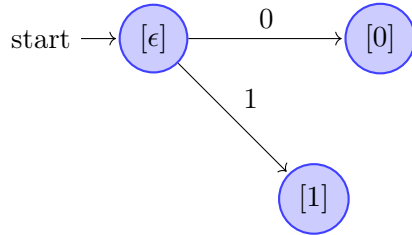
### 5.6.2 Can $|\sim_L| < |\equiv|$ ?

In order to inspect this , we are going to construct a DFA using  $\sim_L \subseteq \Sigma^* \times \Sigma^*$  relation which will then accepts the language  $L$ . Let here,  $\Sigma = 0, 1$

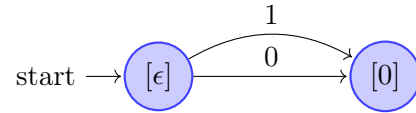
- A state denoting  $[\epsilon]$  is made and is also denoted as the start state.



- Then we pick a letter from the alphabet and look the transitions from each of the existing states. If the next word already lies in the equivalence class of one of the existing states, we draw the arrow representing the particular transition otherwise a new state is made representing the equivalence class of the newly made word.

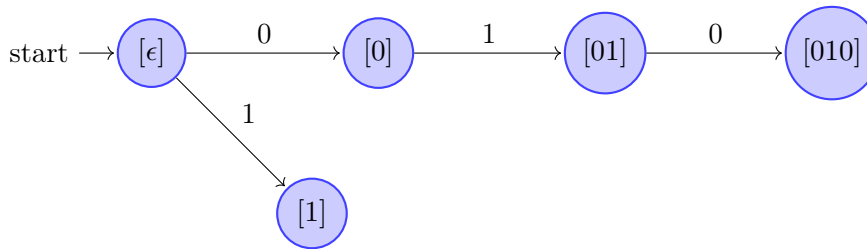


Here, 1 does not lie in the equivalence class of 0, i.e.,  $[0]$ .



If 1 lies in  $[0]$ , then the automata will look like this.

- By repeatedly applying point 2, the automata can keep on expanding like as below



- However, this construction is guaranteed to converge because we have already proved that  $|\sim_L| \leq |\equiv|$ . It is known that the cardinality of equivalence classes of the indistinguishability relation is equal to the number of states in the minimal DFA representing the language  $L$ . Since the number of such states is finite, it follows that  $|\equiv|$  is also finite. Given that  $|\sim_L| \leq |\equiv|$ , it can be deduced that  $|\sim_L|$  is finite. Consequently, this guarantees the convergence of the algorithm in question. [If the above algorithm does not converge, that means new states will keep on forming contradicting the  $|\sim_L| \leq |\equiv|$  identity.]

- Finally, all those states whose equivalence classes have words that are accepted by the language  $L$  are marked as accepting states.

Hence, the aforementioned algorithm successfully allows us to construct a finite state automata (DFA) which accepts only the words that are accepted by the language  $L$ . Note that here the number of states in the new automata is equal to  $|\sim_L|$ .

Now, let us assume that  $|\sim_L| < |\equiv|$  holds. This implies that our newly constructed automata is the minimized DFA for the language  $L$ .

However, in the last lecture we had proved that the DFA constructed by  $\equiv$  relation is minimized one and is unique. Hence, it leads to a contradiction, making our assumption wrong.

Thus proved that

$$|\sim_L| = |\equiv|$$

**Note:** The Nerode equivalence relation, unlike the indistinguishability relation, can be defined for any language  $L \subseteq \Sigma^*$ , irrespective of the fact that whether the language is regular or not.

## 5.7 Myhill-Nerode Theorem

*$L$  is regular if and only if  $\sim_L$  has a finite number of equivalence classes.*

This theorem provides an exact characterization of a regular language, unlike the Pumping Lemma. While the Pumping Lemma does not guarantee that  $L$  is regular if it holds, here, if  $\sim_L$  has a finite number of equivalence classes, the language  $L$  must be regular. The proof of this theorem can be found here.

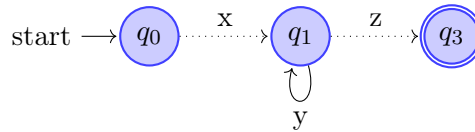
## Chapter 6

# Pumping Lemma for Regular Languages

If  $L$  is a regular language, then there exists a finite DFA with a minimum number of states (say  $p$ ) that recognizes it. Let's consider a string  $w$  of length at least  $p$  that is accepted by this DFA. By the Pigeonhole Principle, we can deduce that when the DFA processes the string, it must revisit at least one state, thereby implying the existence of a loop.

### 6.1 Example

Consider string  $w \in L$ , where  $L$  is a regular Language. Suppose the first state in DFA which is revisited again on processing string  $w$  is  $q_1$ .



Hence for any  $i \geq 0$   $xy^iz \in L$

The length of substring  $xy$  cannot exceed the size of the DFA. Hence,

$$|xy| \leq n$$

where  $n$  is the length of DFA

Also  $|y| \geq 1$  as it should be possible to go back to the same state

### 6.2 Formal Statement of Pumping Lemma

For any regular language  $L$ , the following holds

$$\exists p > 0 \forall w, (w \in L) \cap (|w| \geq p) \exists x, y, z (w = x.y.z) \forall n (n \geq 0) \Rightarrow xy^n z \in L$$

Contrapositive:

$$\forall p > 0 \exists w (w \in L) \cap (|w| \geq p) \forall x, y, z (w = x.y.z) \exists n (n \geq 0) \cap xy^n z \notin L$$

If the above formula holds true then  $L$  is not a regular language.



- If a language  $L$  is regular then Pumping Lemma holds true for  $L$  but if Pumping Lemma holds true for Language  $L$  then it does not necessarily mean that  $L$  is regular.
- But if Pumping Lemma does not hold true for Language  $L$  then  $L$  is not regular.

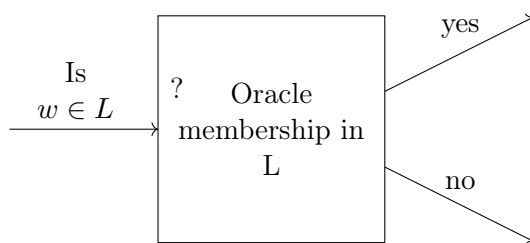
### 6.3 Pumping Lemma as Adversarial Game

Game between an adversary (who wants to show that the language  $L$  is not regular) and a believer (who believes that the language is regular) is as follows:

- The believer chooses an integer  $p > 0$  and claims this is the count of states in the DFA that she believes recognizes the language.
- Adversary chooses a string  $w \in L$  such that  $|w| > p$ .
- Believer then splits  $w$  into three parts  $w = xyz$ , where  $|xy| \leq p$  and  $|y| > 0$
- Adversary now chooses an integer  $n \geq 0$  such that  $xy^n z \notin L$  thereby winning the game. If the adversary can't do this then they lose.

### 6.4 If a language $L$ is regular and the number of states in DFA for $L$ is $n$ , is $L$ infinite?

Given the DFA for language  $L$ , if there is a path from the initial state to a state forming a cycle, and from that state there's a path to an accepting state, then  $L$  is infinite because the DFA can endlessly loop within this cycle, generating an infinite number of accepted strings. Instead if only a black box is provided, which can determine whether a given string  $w$  is in  $L$  or not.



Test all strings  $w$ , such that  $n < |w| \leq 2n$  for membership in  $L$ . If we find even one string which returns 'yes' on testing then  $L$  is infinite. (Since pumping lemma can be applied to such a string, and it can also be pumped to show that an infinite sequence of strings are in  $L$ ).

If all strings  $w$ , ( $n < |w| \leq 2n$ ) return 'no' on testing then we claim that there are no strings in  $L$  of length more than  $2n$ , and thus there are only a finite number of strings in  $L$ .

**Proof:** On applying the pumping lemma for such  $w$  repeatedly, we can obtain strings of length between  $n$  and  $2n$ , which again belong to  $L$ . This is because, for  $w = xyz$ , removing loop  $y$  every time decreases the length by at least 1. This is a contradiction to our assumption that there are no strings of length between  $n$  and  $2n$  in  $L$ .

Hence  $L$  is infinite if it contains atleast one string of length at least  $n + 1$ .

## 6.5 Example

Consider the Language  $L = \{(0+1)^*01^k \mid k \text{ is prime}\}$  with alphabet  $\Sigma = \{0,1\}$

The application of the Pumping Lemma is independent of the initial state chosen such that length of string remains  $\geq p$  (pumping length)

In the case of this language, the Pumping Lemma can indeed be applied to the substring  $1^k$ , where  $k$  is any prime number.

Now we can break  $w = 1^k$  into  $w = xyz$  such that  $y \neq \epsilon$  and  $|xy| \leq n$ . Let  $|y| = m$  ( $m > 0$ ), then  $|xz| = k - m$

Consider the string  $a = xy^{k-m}z$

$$|a| = |xz| + (k - m)|y| = (m + 1)(k - m)$$

As length of string  $a$  is not a prime (it has factors  $m + 1, k - m$ ), we have  $a \notin L$

Hence Language  $L$  is not regular.

### Reverse Language

If Language  $L$  is regular then its reverse language  $L^R$  is also regular.

To construct automaton for  $L^R$  from that of  $L$  swap initial and final states and reverse the edges.

## Chapter 7

# Pushdown Automata

### 7.1 Pushdown Automata for non-regular Languages

Thus far, the finite automata we have studied cannot be used to represent non-regular languages. One such example of a non-regular language is the language of balanced parentheses. Let it be  $L$ .

**Proving why  $L$  is not a regular language** *It is known that intersection a language  $L$  with any regular language preserves the regularity of the original language. i.e. the new language formed after intersection will have the same regularity as the original language  $L$ .*

*Therefore, considering  $L \cap ({}^n)^n$  results in  $({}^n)^n$  which by applying the Pumping Lemma can be shown that it is not a regular language, it can be thus concluded that the language of balanced parenthesis is also not a regular language.*

So if we can equip the finite state automata with more structures like a stack, it will be able to accept the non-regular languages too.

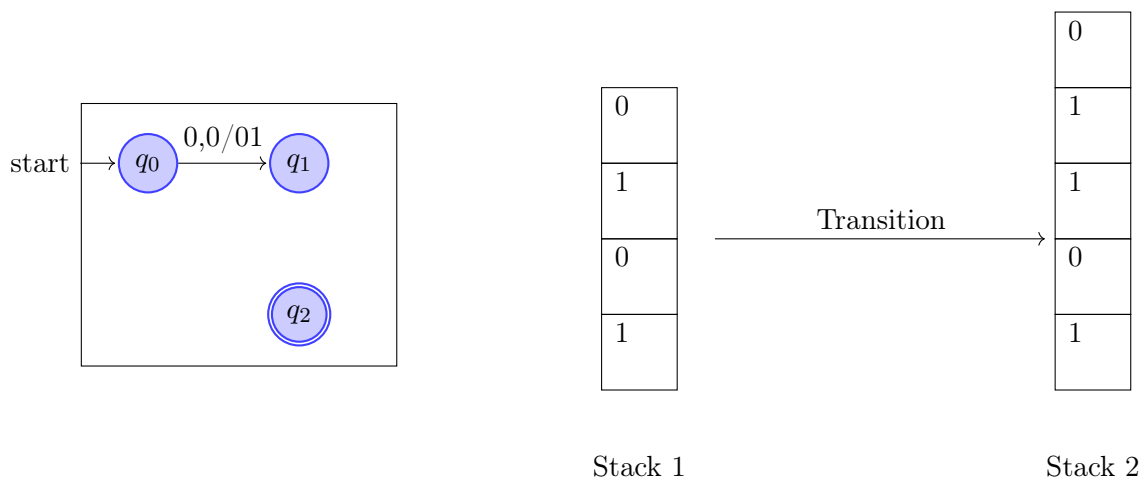


Figure 7.1: Automaton and Stacks

## 7.2 Description

**PushDown Automata** (PDA) are essentially NFAs equipped with a stack that is maintained on an alphabet that is different from the alphabet used for state transition. PDAs have greater accepting power than regular NFAs and can accept non-regular languages as well.

### 7.2.1 Conventions

Just like we described a NFA  $L$  as  $L(Q, \Sigma, q_0, \delta, F)$ , we describe PDAs as characterized by

$$L(Q, \Sigma, \Gamma, q_0, Z_0, \delta, F)$$

where-

- $Q$  is the set of all states in the PDA
- $\Sigma$  is the alphabet accepted by the PDA
- $\Gamma$  is the set of symbols that are pushed/popped from the stack associated with the PDA
- $q_0$  is the starting state of the PDA. There may be more than one starting state for the PDA.
- $Z_0$  is the starting symbol in the stack. This is so that a symbol can be popped on the first transition.
- $\delta$  is the transition function for the PDA. While the transition function for regular NFA was of the form  $\delta : Q \times \Sigma \cup \{\epsilon\} \rightarrow 2^Q$ , the transition function for PDAs is characterized by  $\delta : Q \times \Sigma \cup \{\epsilon\} \times \Gamma \rightarrow 2^{Q \times \Gamma^*}$ , where  $\Gamma^*$  is the set of strings made by the alphabet  $\Gamma$ .
- $F$  is the set of final states of the PDA

A single transition on a PDA from state  $q_1$  to  $q_2$  is represented by  $x, a/ba$  where  $x \in \Sigma$  is the original symbol and may be  $\epsilon$ ,  $a$  is the top element of the stack and is popped off (it cannot be  $\epsilon$ ), and  $ba$  is the string that is pushed onto the stack after  $a$  is popped off. The final top element of the stack is  $b$  after this transition takes place.

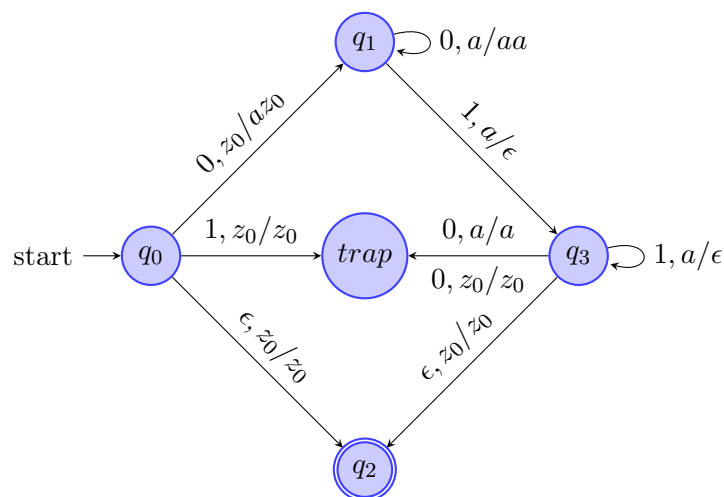
This places a restriction on PDAs as compared to NFAs, that is, for any transition to occur, the top element of the stack also has to match the top element specified in the transition.

### 7.2.2 Example

Let us characterize the non regular language  $\{0^n 1^n | n \geq 0\}$  through a PDA with a finite number of states, as shown in Figure 7.2. We define  $\Gamma = \{z_0, a\}$  as our stack alphabet with our PDA starting with only  $z_0$  in the stack.  $a$  will represent a counter for the number of zeros in our string.

Notice that if a string deviates from the form  $0^m 1^n$  the automaton goes to trap. For every 0 in the string  $a$  is pushed onto the stack and hence the  $l(S) = m + 1$  where  $l(S)$  is the length of stack. For strings following the given structure, if  $m = n$ , then the string ends at an accepting state. If  $m > n$ , the string goes to trap and if  $m < n$ , the string ends the run at a non accepting state.

We reiterate that a string is accepted if and only if the run does not halt before the string is completed and the final state is an accepting state.

Figure 7.2: Automaton accepting  $\{0^n 1^n | n \geq 0\}$ 

### 7.2.3 Example

Now let us extend our previous automaton to accept the language  $\{0^n 1^m | n \geq m \geq 1\}$  as shown in Figure 7.3. Herein, we introduce a different notion of string acceptance, that is, if the stack becomes empty upon completion of the run, we can say that the string is accepted. This is called **acceptance through empty stack**.

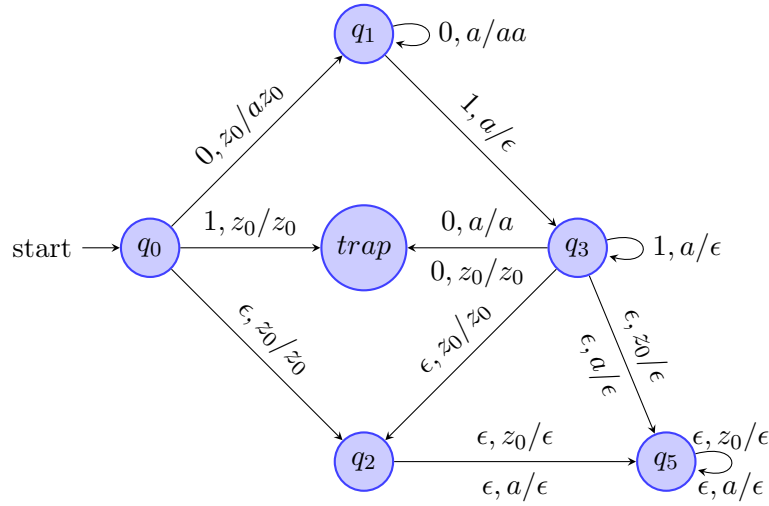
## 7.3 Acceptance through empty stack

We introduce a new way of characterizing string acceptance in PushDown Automata wherein there is no need to specify accepting states explicitly. Here, a string is accepted if it satisfies the following constraints for atleast one possible run on the PDA-

1. String should not halt in between its run, that is, if any symbol is left in the run on the string, there must exist atleast one entry in the transition function  $\delta$  corresponding to any of the possible current states, the next character and any of the possible stack tops.
2. Upon completion of the run, either the stack should be naturally empty or through a series of  $\epsilon$  transitions, should be able to make its stack empty for the string to be accepting.
3. There are no constraints on the current state and the string can be accepted through any state  $q \in Q$ .

This gives a new method of acceptance. We represent the language accepted by this method over an automaton  $A$  as  $N(A)$  and the language accepted by our original method as  $L(A)$ .

However, we do not yet know if the two methods of acceptance are equally powerful. We shall prove this claim by showing the equivalence of the two methods over PDAs by transforming one into the other and vice versa.

Figure 7.3: Automaton accepting  $\{0^n 1^m | n \geq m \geq 0\}$ 

### 7.3.1 For any automaton $A$ , $L(A)$ is not necessarily equivalent to $N(A)$

We must observe a subtlety, if a language  $L$  is accepted through PDAs through both methods, the automata that accept these need not be the same. This is easily shown through the previous two examples.

- In Figure 1,  $L(A) = \{0^n 1^n | n \geq 0\}$  but  $N(A) = \Phi$  as the stack never gets empty.
- In Figure 2,  $N(A) = \{0^n 1^m | n \geq m \geq 1\}$  but  $L(A) = \Phi$  as there are no accepting states.

So we want to prove that-

1. If  $L(A_1)$  is the language accepted by a PDA  $A_1$ , there exists another PDA  $A_2$  such that  $L(A_1) = N(A_2)$ .
2. If  $N(A_1)$  is the language accepted by a PDA  $A_1$ , there exists another PDA  $A_3$  such that  $N(A_1) = L(A_3)$ .

We will show the existence of both of these by giving a construction which when applied to  $A_1$  gives us  $A_2$  and  $A_3$  respectively.

### 7.3.2 Construction for $A_2$ such that $L(A_1) = N(A_2)$

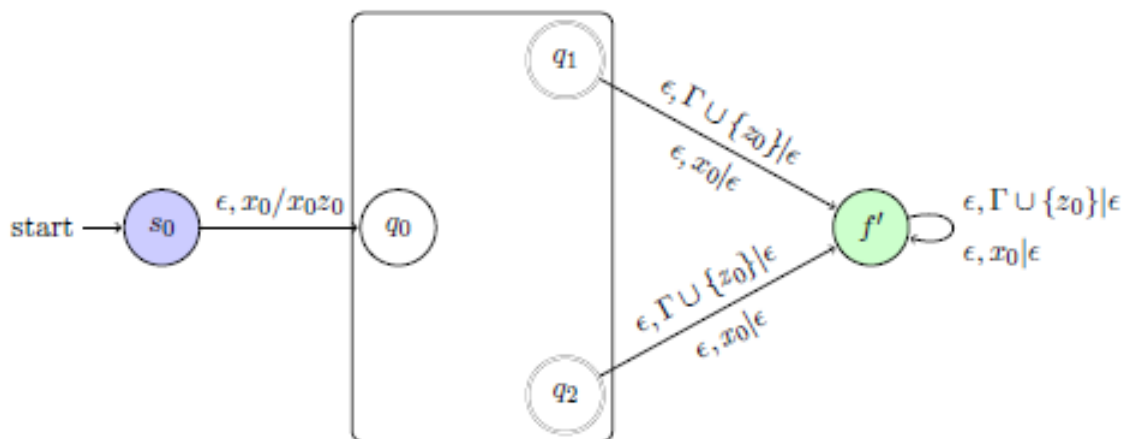
We need to modify  $A_1$  to  $A_2$  without making any assumptions about its structure except that it is a PDA, and make it such that it satisfies two conditions for the equivalence of the languages-

1. Rejection -  $N(A_2)$  should not accept any string  $s$  that is not a part of  $L(A_1)$
2. Acceptance- If a string  $s$  is a part of  $L(A_1)$  it must also be a part of  $N(A_2)$

We shall assume  $A_1$  has only one starting state, and if not we can simply add extra  $\epsilon$ -transitions to reach any start state from our original start state.

Before this start state, we append a new start state  $s_0$  and have our stack start with  $x_0 \notin \Gamma \cup \{z_0\}$  and connect it to start with an  $\epsilon$ -edge pushing  $z_0x_0$  to the stack. This is absolutely identical to the original stack with the exception of an extra  $x_0$  at the bottom of the stack.

We will convert  $A_1$  to  $A_2$  without modifying the basic structure of  $A_1$  such that for any accepting state in  $A_1$ , we add  $\epsilon$ -transitions to a new final state  $f'$ , which does not take in any input word and just pops the stack ( $x_0$  or  $z_0$  or  $\gamma \in \Gamma$ ) without performing any additions onto it.



### Acceptance

If string  $s$  is accepted in  $A_1$  it will reach  $f \in F$  on atleast one of its possible runs on  $A_1$ . If the stack has atleast one element upon reaching  $f$ , the string can go to  $f'$  and there the stack can get emptied through  $\epsilon$ -transitions from  $f'$  to itself, thus putting  $s$  in  $N(A_2)$  as well.

If the stack became empty when the string  $s$  ran on  $A_1$  we will now have  $x_0$  as the only element left in the stack (as no transition in the original automaton can pop  $x_0$  because  $x_0 \notin \Gamma \cup \{z_0\}$ ). Our automaton proceeds to  $f'$  by popping  $x_0$  and is accepted as stack is emptied.

Hence, acceptance is proved as any string in  $L(A_1)$  is also present in  $N(A_2)$ .

### Rejection

A run of string  $s$  is rejected in  $L(A_1)$  if-

1. Run successfully completes but the string is not in an accepting state
2. Run does not complete successfully because of empty stack
3. Run does not complete successfully because of no valid transitions

We show that our modifications can correctly reject all such cases.

If the run does not successfully complete on  $A_1$  due to an empty stack, upon running on  $A_2$ , the stack will only have  $x_0$  and as  $x_0 \notin \Gamma \cup \{z_0\}$ , there is no transition with  $x_0$  as top, leading to the run halting and string being rejected.

If the run does not successfully complete on  $A_1$  due to no valid transition, there will not be any valid transitions in  $A_2$  as well because the stack is the same except for an extra  $x_0$  at the bottom.

If the run successfully completes but the string is not in an accepting state, then the stack never becomes empty because  $x_0$  will always be present.

Hence, rejection is also proved as no string rejected by  $L(A_1)$  is accepted by  $N(A)$ .

## 7.4 From Empty Stack PDA to Final State PDA

We will now see how to construct a PDA with final state acceptance from a PDA with empty stack acceptance.

Consider PDA  $P_N = (Q, \Sigma, \Gamma, \delta_N, q_0, Z_0)$  and  $N(P_N)$  be the set of strings that the PDA accepts by empty stack .

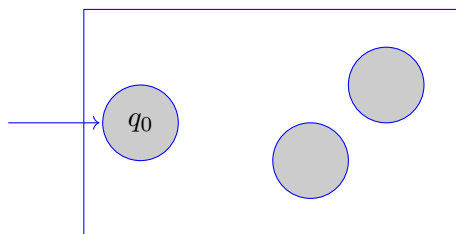
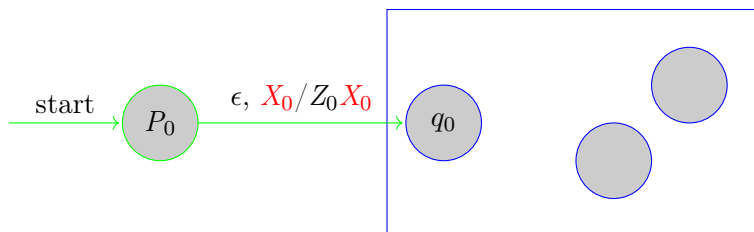


Figure 7.4:  $P_N$  - PDA with empty stack Acceptance

### 7.4.1 Construction

Now, Consider a construction to  $P$  that involves

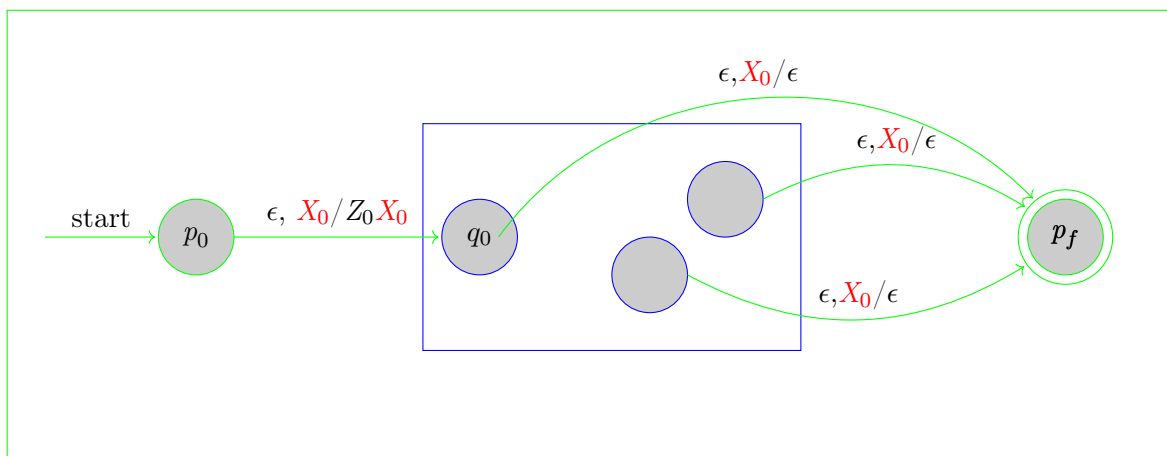
- Addition of new stack variable  $X_0 \notin \Gamma$  as the bottom of the stack.
- Addition of a new start state  $P_0$  and an  $\epsilon$  - transition from this new start state on reading  $X_0$  on the stack to  $q_0$  and  $Z_0$  as top of stack.



- Introducing a new final state  $P_f$  and  $\epsilon$ -transitions from all states in  $Q$  such that on reading  $X_0$  on the stack transition happens to state  $P_f$  and an empty stack.

After the construction, the new automaton  $P_F$  will be  $(Q \cup \{p_0, p_f\}, \Sigma, \Gamma \cup \{X_0\}, \delta_F, p_0, X_0, \{p_f\})$  whose acceptance is by final state.



Figure 7.5:  $P_F$  (After the construction)

#### 7.4.2 Required to show $N(P) = L(P_F)$

- First, Let us show that if a string  $w$  is accepted by  $P$  then it will also be present in  $L(P_F)$

If  $w$  is accepted by  $P$ , then the stack in  $P$  must have been emptied after consuming  $w$ , which implies that stack in  $P_F$  now has  $X_0$  at its top. Now, taking the  $\epsilon$ -transition by reading  $X_0$  will lead us to  $p_f$  which is an accepting state of  $P_F$ . Hence  $w$  will also be present in  $L(P_F)$ .

- Now, Let us show that if  $w$  is in  $L(P_F)$  then it will also be present in  $N(P)$

In the PDA  $L(P_F)$ , the only transitions that can be used to reach  $p_f$  state are  $\epsilon$  transitions on reading  $X_0$ . Since  $w$  is present in  $L(P_F)$ , it must have taken one of these transitions and read  $X_0$  at the top of the stack which implies that stack in  $P$  must have been emptied by  $w$ . Hence,  $w$  would also be accepted by  $P$ .

Finally one can say that,

Acceptance of PDA using final states  $\equiv$  Acceptance of PDA using empty stack.

i.e For every PDA  $A$  whose acceptance is by final states there exists a PDA  $A'$  whose acceptance is by empty stack and vice-versa.

## 7.5 Context Free Languages

Languages accepted by the PDAs are called Context Free Languages. These are strict superset of regular languages.

### 7.5.1 DPDA and NPDAs

A PDA is considered deterministic if, from a given state and upon reading a specific symbol from the top of the stack, there exists only one possible transition for a given input and a PDA is considered non-deterministic if there exists either no transition or many transitions possible for an input.

The language that can be represented using DPDAs can also be represented using the NPDAs.i.e

$$DPDA \subseteq NPDA$$

But, there are certain languages which can be represented only by NPDAs but not DPDAs.Here is an example,

$$L = \{ww^R \mid w \text{ is in } (0+1)^*\}$$

If we try to represent L using a DPDA by pushing a copy of input symbol seen on to stack, one has to make a guess every time whether input has reached end of string  $w$  or not so that popping can be done to check the palindrome nature of the string. Hence non-determinism has to be involved to represent L.

Hence,

$$DPDA \subset NPDA$$

### 7.5.2 Build-up and emptying of the stack of a PDA

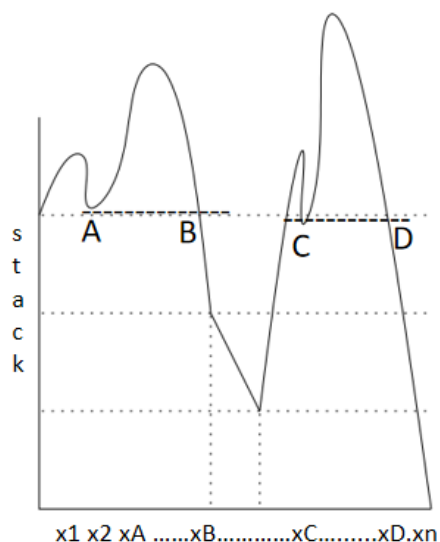


Figure 7.6: Behaviour of stack with the input being processed

In the above Figure, if we observe the part of graph where strings  $xA...xB$  and  $xC....xD$  are read, the behaviour is independent of the content of the stack before  $xA$  but just depends on the state and top of the stack when the symbol  $xA$  is being read.

Hence, a context-free language can be fragmented into some finite no.of sets (these sets need not be finite) each defined by *(top of stack, state of automaton)*

## 7.6 Acceptance by PDA

Let us begin by taking the example of the following PDA with  $\Gamma = \{Z_0, X\}$  as the stacks symbols and  $Z_0$  as the initial stack state

$$\text{The language accepted by this automata is } N(A) = \{0^n 1^n \mid n \geq 1\}$$



We took the union of languages that started at  $q_0$  emptied the stack and ended at any of the states present in the PDA A.

### 7.6.2 Recurrence Relations on Languages

Lets take a look at the individual transitions of PDA A

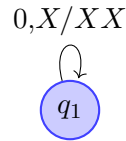


Figure 7.9: self loop on  $q_1$

Now what can we say about the language  $L_{q_1 X q_2}$

If we consider a string  $w$  such that 0 is the first letter of  $w$  then this transition is undertaken and we have added another  $X$  to the stack. So now  $w$  would have to pop two  $X$  from the stack and arrive at the state  $q_2$  for  $w$  to belong to the language  $L_{q_1 X q_2}$ . This can be written as

$$L_{q_1 X q_2} \supseteq 0.L_{q_1 X q_0}.L_{q_0 X q_2} \cup 0.L_{q_1 X q_1}.L_{q_1 X q_2} \cup 0.L_{q_1 X q_2}.L_{q_2 X q_2}$$

Now if we consider the transition

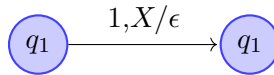


Figure 7.10: self loop on  $q_1$

We can say

$$L_{q_1 X q_2} \supseteq 1$$

We can keep doing this for all transitions from  $q_1$  and get a super set recurrence relation for  $L_{q_1 X q_2}$

## 7.7 More on recurrence relations

Talking about a general PDA with  $n$  states and  $k$  stack symbols  $n^2 k$  languages of the form  $L_{q_i S_j q_k}$  can be defined and considering individual transitions super set recurrence relations can be obtained for them. For example

$$\begin{aligned} L_1 &\supseteq 1 \cup 0.L_2.L_3 \cup \dots \\ &\vdots \\ L_{n^2 k} &\supseteq 0.1 \cup 0.L_1.L_3 \cup \dots \end{aligned}$$

### 7.7.1 Smallest and Largest Languages

Consider the relation as follows

$$L_1 \supseteq 0.1 \cup 0.L_1.1$$

Notice how both  $\Sigma^*$  and  $0^n 1^n n \geq 1$  are both languages that satisfy this relation but  $\Sigma^*$  is largest language that satisfies this relation and  $0^n 1^n n \geq 1$  is the smallest (can be proven smallest by induction if  $0^i 1^i$  belongs to  $L_1$  then so does  $0^{i+1} 1^{i+1}$ )

We will be particularly interested in finding these smallest languages satisfying the relation

**NOTE:**  $\Sigma^*$  is always a solution (largest) of the super set recurrence relation since it contains all strings

### 7.7.2 Context Free Grammar

Above equations imply that wherever we see a word of the language  $L_{q_1 X_{q_2}}$ , we can replace it with any word from the languages on the RHS. Thus we can write:

$$L_1 \rightarrow 0L_2L_3 \mid 0L_4L_1 \mid 0L_1L_5 \mid 1$$

where  $L_1 = L_{q_1 X_{q_2}}$ ,  $L_2 = L_{q_1 X_{q_0}}$ ,  $L_3 = L_{q_0 X_{q_2}}$ ,  $L_4 = L_{q_1 X_{q_1}}$  and  $L_5 = L_{q_2 X_{q_2}}$ .

Similarly, we can do the same for the languages  $L_i \forall i \in \{1, 2, \dots, n^2 k\}$ . This will form a **context free grammar**.

With proper reductions, we can say that the context free grammar generated for the language  $L = \{0^n 1^n \mid n \geq 1\}$  is:

$$S \rightarrow 0S1 \mid 01$$

We observe that the universal language  $\Sigma^* = L((0+1)^*)$  also satisfies the context free grammar, but the language  $L = \{0^n 1^n \mid n \geq 1\}$  is the minimal language that satisfies the context free grammar.

## Chapter 8

# Context Free Grammar

### 8.1 Context-free Grammar

**Definition:** A context-free grammar (CFG) is a formal grammar whose production rules can be applied to a nonterminal symbol regardless of its context. In particular, in a context-free grammar, each production rule is of the form  $V \rightarrow (V \cup T)^*$ . Where  $V$  is set of Non-Terminal and  $T$  is set of Terminals.

Formally, a context-free grammar can be represented as follows -

$$G = (V, \Sigma \cup \{\epsilon\}, P, S)$$

where

$V$  - is the set of **non-terminals**. These are symbols that can be replaced or expanded.

$\Sigma \cup \{\epsilon\}$  - is the set of **terminals**. These are symbols that cannot be replaced or expanded further.

$P$  - are the set of production rules

$S$  - is the start symbol ( $S \in V$ )

Again, a grammar is said to be the Context-free grammar only if every production is in the form of

$$G \rightarrow (V \cup T)^*, \text{ where } G \in V$$

Let's consider a context-free grammar defined as follows:

$$S \rightarrow A.S | \epsilon$$

$$A \rightarrow A1 | 0A1 | 01$$

Here,  $S$  and  $A$  are non-terminal symbols representing languages.  $\epsilon$  is a special symbol representing an empty string and is in the language  $S$ . The symbols 0 and 1 are terminal symbols and are in the language  $A$ . The set of strings that can be generated using a context-free grammar is called a *context-free language*. This language of **A** contains strings that are of the form 01 or A1 or 0A1, by just substituting all possible strings of A in the form recursively

$$A \rightarrow 01 | 011 | 0011 | 0111 | 00111 | \dots$$

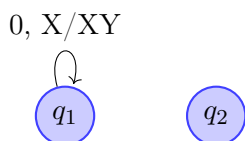
Thus, Language derived from **A** can be intuitively be told as

$$0^i 1^j, \text{ where } j \geq i \geq 1$$

### 8.1.1 PDA to CFG

In the previous lecture, we talked about the language accepted by a Pushdown Automata (PDA), starting from a state  $q_i$ , with a symbol  $X$  on the top of the stack, and reaching a state  $q_j$ , with the stack remaining the same, except the fact that the top element,  $X$ , was popped off.

Such a language is denoted by  $L_{q_i X q_j}$ , where  $q_i$  is the start state,  $X$  is the top element of the stack, and  $q_j$  is the final state. Given certain transitions in a PDA, we could also come up with recurrence relations for such languages. For eg., if the following states and transitions were given to us, and we want to find  $L_{q_1 X q_2}$



We can say that, for all states  $q_i$  in the automaton,

$$L_{q_1 X q_2} \supseteq 0 \cdot L_{q_1 X q_i} \cdot L_{q_i Y q_2}$$

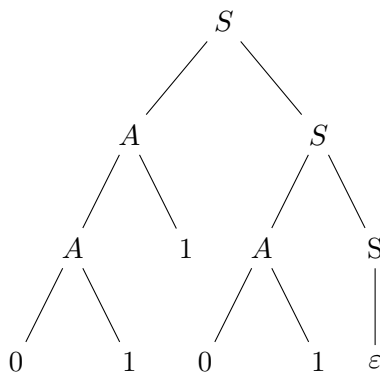
If we denote the languages by  $L_1, L_2 \dots L_{n^2 k}$ , where  $n$  is the number of states and  $k$  is the number of stack symbols, then we can write the above equation as a recurrence relation for all languages.

For example,  $L_1 \rightarrow 1 \mid 0 \cdot L_2 \cdot L_3 \mid 0 \cdot L_4 \cdot L_1 \mid \dots$   $L_2 \rightarrow \dots \vdots L_{n^2 k} \rightarrow \dots$  This set of recurrence relations is called a Context-free grammar. Hence, given a PDA, we can write the language accepted by the PDA by an empty stack as a CFG.

## 8.2 Parse Trees and Ambiguous Grammars

Suppose we have the string 01101 and we want to check whether this lies in  $L_S$ .

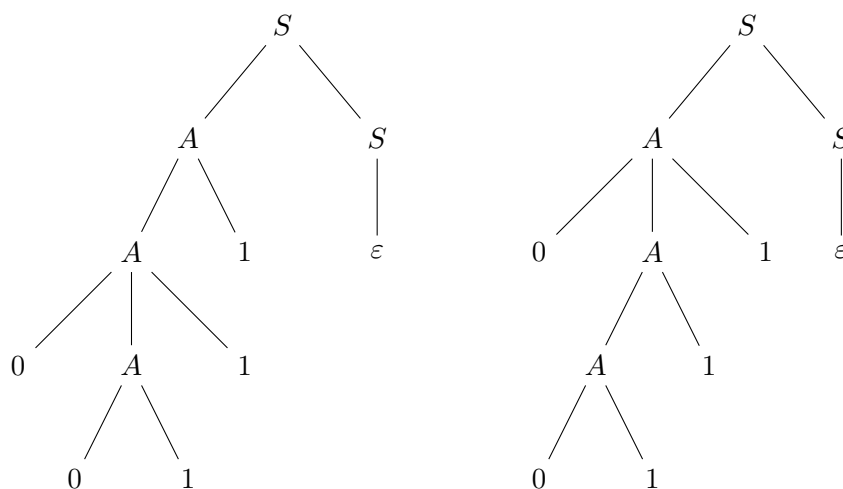
First, we will apply the rule  $S \rightarrow A \cdot S$ , since the rule  $S \rightarrow \varepsilon$  is not useful yet. Now, we further see  $A \cdot S$  recursively. For  $S$ ,  $S \rightarrow \varepsilon$  is not useful since 01101 does not lie in  $L_A$ . In this manner, we further recursively break up our string until we reach terminals and check whether the sub-parts can be further be made useful. In this manner, we can create parse/derivation trees for a string, given a grammar. The leaves in these trees are our terminals. This is shown in diagram below.



As we can see, the leaves form 01101 $\epsilon$ , from left to right, which is exactly the string we wanted. Hence, 01101  $\in L_S$ , using the production rules in the grammar.

While drawing the trees, we must ensure that the root is the start symbol, and the node at which we want to split our tree, we are using production rules with that value in the left hand side. For example, if we want to split the tree at an  $A$ , we must only use production rules of the form  $A \rightarrow \dots$ .

For this particular string and grammar, only one parse tree is possible. Let us now consider the string 00111  $\in L_S$ .



As we can see, these are two completely different, and correct parse trees for the string in the same grammar. Such grammars, where there exist multiple parse trees for some strings, are called ambiguous grammars. Hence, the grammar we have defined is indeed ambiguous.

Are there CFLs for which every CFG representing it are ambiguous grammars? Yes, there indeed are such CFLs. Such languages are called inherently ambiguous languages. If there exists even one grammar for the CFL which is unambiguous, then the language is an unambiguous language as well. Is  $L_S$  inherently ambiguous or is it an unambiguous language?

The answer is that it is not inherently ambiguous. The intuition is that, we must force our grammar to first match up all the zeros in the strings with ones, and following that, add the terminating ones. Our current grammar puts no such restriction. We could first add zeros and ones to the start and end, add ones to the end, and then go back to adding zeros and ones to the start and end. This leads to multiple parse trees. An example of a CFG representing  $L_S$ , which is unambiguous is  $S \rightarrow C \cdot S \mid \epsilon$   $C \rightarrow A \cdot B$   $A \rightarrow 0 \cdot A \cdot 1 \mid 01$   $B \rightarrow 1 \cdot B \mid \epsilon$ . As we can see,  $A$  does the job of adding zeros and ones to the start and end,  $B$  does the job of creating strings of ones, and  $C$  does the job of concatenating  $A$  and  $B$ .

This is important, because every programming language is specified using CFGs, for which we can build compilers and interpreters, and it is important for these CFGs to be unambiguous since we do not want multiple interpretations of a program.



### 8.2.1 CFG to PDA conversion

Consider the Context-free grammar defined as follows

$$\begin{aligned} S &\rightarrow AS \mid \epsilon \\ A &\rightarrow A1 \mid 0A1 \mid 01 \end{aligned}$$

Where,

**Starting Symbol:**  $S$

**Production Rules:**  $S \rightarrow AS$ ,  $S \rightarrow \epsilon$ ,  $A \rightarrow A1$ ,  $A \rightarrow 0A1$ ,  $A \rightarrow 01$

**Non-terminals:**  $\{S, A\}$

**Terminals:**  $\{0, 1\}$

Now Consider the following 2 alphabets,

$$\Sigma = \{0, 1\}, \quad \Gamma = \{S, A, X_0, X_1\}$$

where  $\Sigma$  represents the alphabet of the PDA which we are going to construct and  $\Gamma$  represents the alphabet of the stack.

**Note** that we have exactly same number of symbols as total number of terminals and non-terminals in our CFG each symbol corresponding to each terminal or non-terminal.

We will start constructing our PDA by adding some edge(s) to a single node, pushing or popping some letter from the stack while constructing derivation tree/ Parse tree by using each production rule.

#### Step 1:

We initialize our stack by pushing  $S$  as  $S$  is our starting symbol and we represent it as follows, Where left means bottom of the stack and right is its top

**Stack =  $S$ ,**



#### Step 2:

We used the rule:  $S \rightarrow AS$  here and added the edge  $\epsilon, S/AS$  into our single node as we can replace  $S$  to  $AS$  anywhere if we want to as it is defined in our CFG's production rules. Also now we pop  $S$  from our stack and push  $S, A$  (here  $A$  and  $S$  are from  $\Gamma$ ).

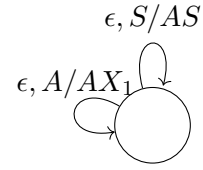
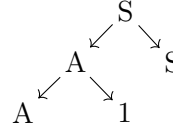
**Stack =  $S, A$ ,**



**Step 3:**

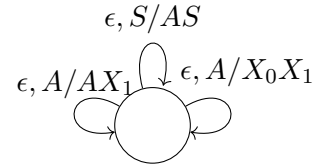
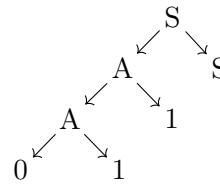
We used the rule:  $A \rightarrow A1$  here and added the edge  $\epsilon, A/AX_1$  (here  $X_1$  is just the stack symbol corresponding to the terminal '1') into our single node as we can replace A to A1 anywhere while forming a string. Also now we pop A from our stack and push  $X_1, A$  (here A and  $X_1$  are from  $\Gamma$ ).

**Stack** = S,  $X_1$ , A,

**Step 4:**

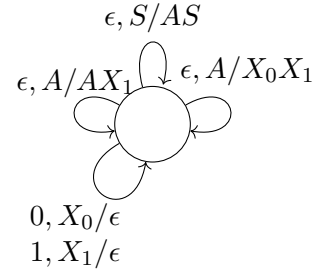
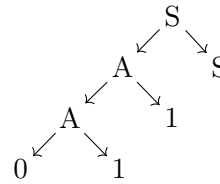
We used the rule:  $A \rightarrow 01$  here and added the edge  $\epsilon, A/X_0X_1$  (here  $X_1, X_0$  are just the stack symbol corresponding to terminals 1 and 0) into our single node. Also now we pop A from our stack and push  $X_1, X_0$  (here  $X_0$  and  $X_1$  are from  $\Gamma$ ).

**Stack** = S,  $X_1$ ,  $X_1$ ,  $X_0$ ,

**Step 5:**

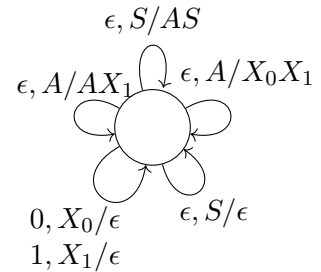
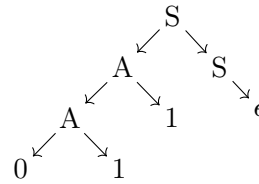
Now as we know that when terminals come on top of the stack and we read it from the top of the stack and pop it off and if a non terminal is seen then we apply any of the given production rules. So for this pop operation we now add the edges  $0, X_0/\epsilon$  and  $1, X_1/\epsilon$  into our single node. So we pop all the terminals until we see a non-terminal.

**Stack** = S,

**Step 6:**

We now use the rule:  $S \rightarrow \epsilon$  here and added the edge  $\epsilon, S/\epsilon$  into our single node. Also now we pop S from our stack and push  $\epsilon$  that is stack becomes empty again.

**Stack** =



Now this stack helps in doing the DFS of the derivation tree, we first visit all the nodes on left and then all the nodes on right.

So to conclude we can say that given any arbitrary Context-free grammar we can easily construct PDA out of it by using some simple steps as described above which recognizes the exact same language as the given CFG.

### 8.3 Cleaning Context-free Grammar

Consider the Context-free grammar defined as follows

$$\begin{aligned} S &\rightarrow ABS \mid 0A1 \\ A &\rightarrow 1A0 \mid D \mid 01 \mid \epsilon \\ B &\rightarrow 1B \mid BB0 \\ C &\rightarrow A0 \mid 01 \\ D &\rightarrow S \mid 0AD \end{aligned}$$

Where,

**Starting Symbol:**  $S$

**Non-terminals:**  $\{S, A, B, C, D\}$

**Terminals:**  $\{0, 1, \epsilon\}$

#### 8.3.1 Eliminating Useless Symbols

We can eliminate the production rules for the non-terminal symbol "C" from our context-free grammar (CFG). Here's the reasoning:

- The starting symbol "S" can generate strings using symbols from "A," "B," and itself.
- While "B" depends on itself and "A" depends on itself and "D," we ultimately find that "D" relies on "A," "S," and itself.
- Therefore, knowing the strings accepted by "C" is unnecessary for generating strings from "S." In other words, even without the production rules for "C," "S" can still generate valid strings by appropriately utilizing "A," "B," and "D."

We can further simplify the grammar by identifying and removing another redundant non-terminal symbol: "B". Analyzing the production rules for "B", we observe two key characteristics:

- **Absence of Terminal Productions:** None of "B"'s production rules generate strings consisting solely of terminal symbols.
- **No Introduction of Useful Non-Terminals:** Additionally, "B"'s rules do not introduce any other non-terminal symbols that could potentially lead to terminal strings through subsequent productions in the parse tree.
- So the language generated by the non-terminal symbol "B" is empty (as it cannot produce any finite length string by terminating at some point), we can safely remove all production rules that involve "B" from the context-free grammar.

Now that we've removed the unnecessary rules, here's the updated CFG:

$$\begin{aligned} S &\rightarrow 0A1 \\ A &\rightarrow 1A0 \mid D \mid 01 \mid \epsilon \\ D &\rightarrow S \mid 0AD \end{aligned}$$

### 8.3.2 Eliminating $\epsilon$ -Productions

For removing epsilon just see where you can apply the rule  $A \rightarrow \epsilon$ .

We can add one more rule in all that places where A appears by replacing it with  $\epsilon$ .

So after doing these changes we won't need the production rule  $A \rightarrow \epsilon$  in our CFG.

**Updated CFG:**

$$\begin{aligned} S &\rightarrow 0A1 \mid 01 \\ A &\rightarrow 1A0 \mid D \mid 01 \mid 10 \\ D &\rightarrow S \mid 0AD \mid 0D \end{aligned}$$

### 8.3.3 Eliminating Unit Productions

Similar to the above property we can do the same thing here also.

Let's say we have a production rule like this  $X \rightarrow Y$  where X, Y are non-terminals then we can add one more rule in all that places where "X" appears by replacing it with "Y".

Let's first remove the rule  $D \rightarrow S$  by doing the changes described above,

$$\begin{aligned} S &\rightarrow 0A1 \mid 01 \\ A &\rightarrow 1A0 \mid D \mid S \mid 01 \mid 10 \\ D &\rightarrow 0AD \mid 0AS \mid 0D \mid 0S \end{aligned}$$

As we can see that after doing this modification to our CFG we now have two more production rules  $A \rightarrow S$  and  $A \rightarrow D$  so let's remove them also,

$$\begin{aligned} S &\rightarrow 0A1 \mid 0D1 \mid 0S1 \mid 01 \\ A &\rightarrow 1A0 \mid 1D0 \mid 1S0 \mid 01 \mid 10 \\ D &\rightarrow 0AD \mid 0AS \mid 0DD \mid 0DS \mid 0SD \mid 0SS \mid 0D \mid 0S \end{aligned}$$

After doing this we now have at least one non-terminal symbol on the right hand side of any rule or only terminals.

### 8.3.4 Reducing further to get at least two non-terminals on the right hand side or a single terminal

Just replace the terminals 0 and 1 with their corresponding symbols " $X_0$ " and " $X_1$ " in the CFG and add corresponding new rules.

$$\begin{aligned} S &\rightarrow X_0AX_1 \mid X_0DX_1 \mid X_0SX_1 \mid X_0X_1 \\ A &\rightarrow X_1AX_0 \mid X_1DX_0 \mid X_1SX_0 \mid X_0X_1 \mid X_1X_0 \\ D &\rightarrow X_0AD \mid X_0AS \mid X_0DD \mid X_0DS \mid X_0SD \mid X_0SS \mid X_0D \mid X_0S \\ X_0 &\rightarrow 0 \\ X_1 &\rightarrow 1 \end{aligned}$$

### 8.3.5 Reducing further to get exactly two non-terminals on the RHS of any production rule which has $>2$ non-terminals on it's RHS

Let us say our production rule has 3 non-terminals on it's right hand side then we can reduce it to exactly 2 non-terminals as follows,

$$S \rightarrow X_0AX_1 \text{ can be reduced to } S \rightarrow X_0S_1 \text{ and } S_1 \rightarrow AX_1$$

Note that the CFG after doing this modification will accept exactly same strings as accepted by it initially because now in the parse tree  $S \rightarrow X_0AX_1$  will just take one step more, it first expands to  $S \rightarrow X_0S_1$  and then to  $S_1 \rightarrow AX_1$ . However, the final string generated remains unchanged.

So it turns out that for any number of non-terminal symbols we can do this inductively to get exactly 2 non-terminals on RHS of any production rule.

**Updated CFG:**

$$\begin{aligned} S &\rightarrow X_0S_1 \mid X_0X_1 \\ S_1 &\rightarrow AX_1 \mid DX_1 \mid SX_1 \\ A &\rightarrow X_1A_1 \mid X_0X_1 \mid X_1X_0 \\ A_1 &\rightarrow AX_0 \mid DX_0 \mid SX_0 \\ D &\rightarrow X_0D_1 \mid X_0D \mid X_0S \\ D_1 &\rightarrow AD \mid AS \mid DD \mid DS \mid SD \mid SS \\ X_0 &\rightarrow 0 \\ X_1 &\rightarrow 1 \end{aligned}$$

The CFG written above is said to be in it's Chomsky normal form (defined below).

### 8.3.6 Chomsky Normal form (CNF)

A Context-free grammar is said to be in it's Chomsky normal form if it does not contain any useless symbols, no epsilon productions ( $X \rightarrow \epsilon$ ), no unit productions ( $X \rightarrow Y$ ) and all production rules are either of the form  $A \rightarrow BC$ , where A, B, C are non-terminals or of the form  $D \rightarrow u$ , where D is a non-terminal and u is a terminal.

Every Context-free grammar can be expressed in it's Chomsky normal form by application of various rules as described in section 2 which accepts the same set of strings which is accepted by the original CFG.

## 8.4 Pumping Lemma for Context-free Languages

Let G be the Chomsky normal form of any Context-free grammar having n non-terminals. Consider derivation tree of a word "s" having length greater than  $2^n$  as shown in the figure (8.1) below.

Now due to the restricted nature of the CNF form of CFG (can contain maximum 2 symbols in it's RHS) the derivation tree will be a binary tree.

Now a binary tree of height n can have maximum of  $2^n$  leaves but here we are taking  $|s| > 2^n$  so the height of derivation tree must be  $> n$ . But we are having only n non-terminals in our CFG so by pigeon hole principle there must exist some non-terminal "N" which repeats atleast twice.

We split up s into 5 parts as shown in the figure (8.1),

$$s = u.v.w.x.y$$

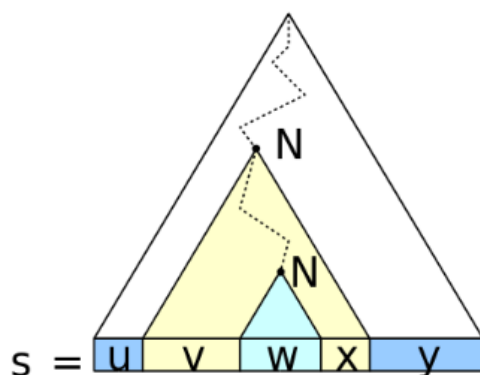
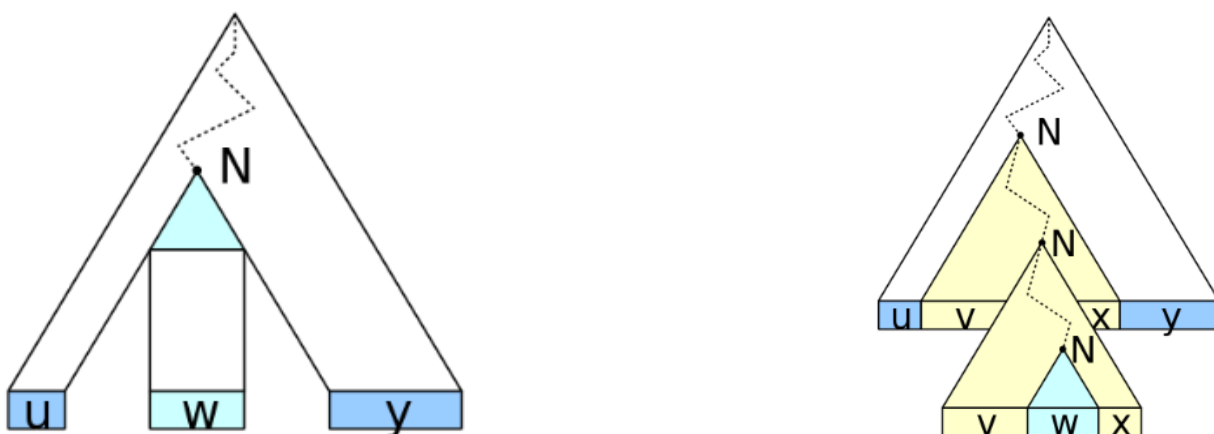
Figure 8.1: Derivation tree of a word of length  $> 2^n$ 

Figure 8.2: Examples of some possible derivation trees from figure (8.1)

- Using the property of derivation trees (of CFG) we can say that whatever I can generate from first "N" can also be generated by the second "N" and vice-versa.
- So after encountering a "N" in our derivation tree we can choose to generate a new "N" or terminate after some productions by choosing the "N" which doesn't produce another "N".
- So, to formally state pumping lemma for CFL we need to have some constraints like the sub-derivation tree of "N" should not have repetition of any non-terminal that is  $|vwx| \leq 2^n$ .
- Also as "N" is a non-terminal therefore  $|w| \geq 1$ .

#### 8.4.1 Lemma:

If a language  $L$  is context-free, then there exists some integer  $n \geq 1$  (number of non-terminals in  $G$ ) such that every string  $s$  in  $L$  that has a length of  $2^n$  or more symbols (i.e. with  $|s| \geq 2^n$ ) can be

written as

$$s = uvwxy,$$

with sub strings  $u$ ,  $v$ ,  $w$ ,  $x$ , and  $y$ , such that:

1.  $|w| \geq 1$
2.  $|vwx| \leq 2^n$ , and
3.  $u.v^i.w.x^i.y \in L$  for all  $i \geq 0$ .

**Example:** Consider a language  $L = \{0^n 1^n 2^n | n \geq 0\}$  and  $\Sigma = \{0, 1, 2\}$ . Now we need to prove that  $L$  can not be a CFL.

Suppose it was a CFL having  $n$  distinct non-terminals, we define  $k = 2^n$  and take a very large string  $w = 0^{2k} 1^{2k} 2^{2k} \in L$ .

Now as  $|vwx| \leq 2^n$  or  $|vwx| \leq k$  so  $v.w.x$  can either lie completely in  $0^{2k}$ ,  $1^{2k}$  or  $2^{2k}$  or in  $0^{2k} 1^{2k}$  or  $1^{2k} 2^{2k}$ .

In any case we can pump the string to get a word which is not in  $L$  (any case will lead to increase some but not all alphabets in that word so it won't be in  $L$ ) but it should be in  $L$ , therefore  $L$  is not a CFL.