Lec 02: Linear and convex programming

Linear Program

$\max\ c^T x$

s.t. $Ax \le b$

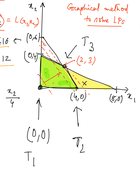
$x \ge 0$

$\max\ (7x_1 + 6x_2) = 6(a,x_2)$

s.t. $2x_1 + 4x_2 \le 16$ ← (8,4)

$3x_1 + 2x_2 \le 12$ (0,6)

$x_1, x_2 \ge 0$



$\frac{x_1}{8} + \frac{x_2}{4}$

Graphical method is slow LPs

① Not Algorithmic

② Not scalable for > 3 dim.

Simplex method

$\max\ 7x_1 + 6x_2 + 0s_1 + 0s_2$

s.t. $2x_1 + 4x_2 \le 16 \rightarrow 2x_1 + 4x_2 + s_1 = 16$

$3x_1 + 2x_2 \le 12 \quad 3x_1 + 2x_2 + s_2 = 12 \rightarrow s_2$ out

Step1: Slack the constraints $\quad s_1, s_2 \ge 0$

Step2: Set basic and non-basic

non-zero $\quad$ zero

$s_1 = 16 \quad x_1 = x_2 = 0$

$s_2 = 12$

$s_1 = 16 - 2x_1 - 4x_2$

$s_2 = 12 - 3x_1 - 2x_2$

$x_1 = 4 - \frac{1}{2}s_1 - \frac{2}{3}x_2$ non-basic

$s_2 = 8 + \frac{1}{3}s_1 - \frac{8}{3}x_2$

obj. $28 + \frac{4}{3}x_2 - \frac{7}{3}s_1$

The process is captured using Tableau method.

	x_1	x_2	s_1	s_2	b	ratio
s_1	2	4	1	0	16	16/2 = 8
s_2	3	2	0	1	12	12/3 = 4
P	7	6	0	0		

$s_1 \quad x_1 = 0$

$s_2 \quad x_2 = 0$

	x_1	x_2	s_1	s_2	b	ratio
s_1	0	8/3	1	-2/3	8	8/(8/3) = 3
x_1	1	2/3	0	1/3	4	
P	0	4/3	0	-7/3		

$x_1 = 4$

$x_2 = 0$

	x_1	x_2	s_1	s_2	b
x_2	0	1	3/8	-1/4	3
x_1	1	0	-1/4	1/2	2
P	0	0	-1/2	-2	

$x_1 = 2$

$x_2 = 3$

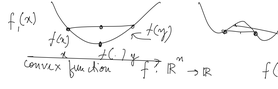
all nonpositive values STOP!

$\max\ f(x)$

s.t. $x \in S$

LP: $c^T x$, $Ax \le b$, $x \ge 0$

Convex Optimization

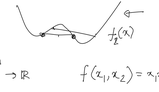


$\forall x, y \in \mathbb{R}^n$

$\forall \lambda \in [0,1]$

$f(\lambda x + (1-\lambda) y) \le \lambda f(x) + (1-\lambda) f(y)$

$f(x) = x^2$



$f(x_1, x_2) = x_1 x_2$

C is not convex

A set $C \subseteq \mathbb{R}^n$ is convex if $\forall x, y \in C$ $\forall \lambda \in [0,1]$

$\lambda x + (1-\lambda) y \in C$

$\max\ f(x)$ ← convex function

s.t. $x \in Q$ ← convex set

Convex optimization problem.

Are these convex?

- $C = C_1 \cup C_2$? C_1, C_2 are convex
- set $\{x \in \mathbb{R}^2 : x \ge 0,\ x_1 x_2 \ge 1\}$

- $f: \mathbb{R}^2_{\ge 0} \to \mathbb{R}$, $f(x) = x_1 x_2$.
- $f: \mathbb{R}^2 \to \mathbb{R}$, $f(x) = x_1^2 + x_2^2 + x_1 x_2$

- Ex:

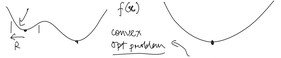
$f(x) = e^{ax}, \quad a \in \mathbb{R}$

$f(x) = -\log(x), \quad x > 0$

$f(x) = x^T x = \|x\|_2^2 = \sum_{i=1}^{n} x_i^2$

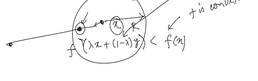
$[x_1\ x_2 \cdots x_n] \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$

$f(x) = \|x\|_1 = \sum_{i=1}^{n} |x_i|$



Def: A point x is globally optimal if x is feasible and $\nexists$ any other feasible y s.t. $f(y) < f(x)$

Def: A point x is locally optimal if x is feasible and $\exists R > 0$ s.t. $\forall$ feasible y with $\|y - x\|_2 \le R$ $f(x) \le f(y)$

Theorem: For a convex opt problem all locally opt points are globally optimal.

Proof: