Binary Search and Variants

1. We first need to find a range before performing binary search. For this we can drop the egg at floors 1,2,4,8,...until it breaks. If $2^n \leq h < 2^{n+1}$, the egg breaks at the $(n+1)^{th}$ drop and $n+1 = \lfloor \log_2 h \rfloor + 1$ which is $O(\log h)$. Now we can binary search in the range from 2^n to $2^{n+1} - 1$. The range is $O(2^n)$ so the binary search will take $O(\log 2^n) = O(n) = O(\log h)$ egg droppings, so overall it's still $O(\log h)$.

2. Whenever we want to find subarray sums, it is useful to precompute a prefix sum array, which is has the sums of the first i elements i.e. $[0, a_1, a_1 + a_2, \ldots, a_1 + a_2 + \cdots + a_n]$. This takes $O(n)$ time to calculate as the i^{th} prefix sum is obtained by adding an array element to the $(i-1)^{th}$ prefix sum. With these prefix sums, any subarray sum from a_i to a_j is obtained by subtracting the prefix sum to a_{i-1} from the prefix sum to a_j. Now for every i, we can binary search the subarray sums starting from a_i, to get the least j such that sum from a_i to a_j is at least S. This is possible as sum from a_i to a_j is an increasing function w.r.t j. So for every i, we will take $O(\log n)$ time and hence the final time complexity is $O(n \log n)$.

Another method is to binary search for the minimum length directly. For fixed subarray length, finding the maximum subarray sum takes $O(n)$ time. Just find the sum for the first subarray, and for the next subarray, add the new element and subtract the first element. So this is just a 1 time traversal of the array. Now we need $O(\log n)$ queries to narrow down the fixed length by binary searching for first length where the subarray sum is at least S. Each query takes $O(n)$ time so our time complexity is $O(n \log n)$.

Turns out this question can be solved in $O(n)$ time itself. The idea is to have 2 pointers, one for the start of the subarray and one for the end of the subarray. For every start index, keep moving the end value until the sum of the subarray is at least S. Once that's reached, move the start pointer 1 step forward and repeat. Note that this would give the smallest subarray for every start index, as the end index will never need to decrease whenever the start index moves forward. Updating subarray sum is also easy whenever a pointer moves, just correct for the new element added/deleted. This algorithm is $O(n)$ as each pointer moves at most n times, they are initialized as the beginning of the array.

3. We just need to find the n^{th} and $(n+1)^{th}$ element of the merged array. For now, let's just assume that the n^{th} element is in the first array and try to find it. How to check if the i^{th} element of the first array is the n^{th} element of the merged array? We know it's greater than exactly $i - 1$ elements in the first array, so it has to be greater than $(n-1) - (i-1)$ elements in the second array, as it overall has to be greater than $n - 1$ elements. So we just check that, it should be greater than the $(n-i)^{th}$ index and smaller than $(n-i+1)^{th}$ index of the second array, if so we have found it. If it's too small i.e lesser than the $(n-i)^{th}$ element of the second array. We need to check from $(i+1)$ to n in the first array. If it's too big, we need to search from 1 to $i - 1$ in the first array. So we can binary search, dividing the search space by half in each time. Suppose we don't find the n^{th} element in the first array, we check the second array for it. Similarly we can find the $(n+1)^{th}$ element by checking both arrays.

4. We first sort the array taking $O(n \log n)$ time. Now for each element x in the array, binary search for $S - x$. This also takes $O(n \log n)$ as we binary search n times.

After sorting, there is also an $O(n)$ approach. Keep 2 pointers at start and end. If the first and n^{th} element sum greater than S, we know that the n^{th} element can never be part of a pair, it's too big even when paired with the first element. So we discard it from our search by decrementing our end pointer. If the sum was smaller than S, similar logic follows and we increment our start pointer. We keep doing this until we either get a sum S or our pointers clash. Since the distance between them is always decreasing by 1 step this is $O(n)$

5. Since the derivative is an increasing function, we can find where it has a root by doing something like bisection method. We first find an interval where the root is, suppose we check a random point a and $f'(a)$ is negative. We then check $f'(a+1), f'(a+2), f'(a+4), \ldots$ until we get a point where the derivative is positive, to locate our interval. Once we have a start and endpoint, we check the derivative at the midpoint and based on its sign, we half the interval search each iteration, increasing the number of bits of precision each time by 1.

6. The smallest base is 2 so the highest exponent is $\log_2 a$. For each exponent k, we binary search for b to get b^k to be n. So we need $O(\log a)$ queries to get b for each k, if it exists. But how much time does each query take. Assume the naive human multiplication method, multiplication take $O(d^2)$ time where d is the number of digits. Since we only multiply till a size of a, we have $O(\log a)$ digits. We multiply almost $O(\log a)$ times to do exponentiation so each query is $O(\log^3 a)$. This makes binary searching for a fixed exponent as $O(\log^4 a)$. So our final time complexity if $O(\log^5 a)$. This algorithm can be make a lot faster by using more efficient methods for multiplication and exponentiation, but eh.

7. The highest value of f is just $\left\lfloor \frac{a_1+\cdots+a_n}{n} \right\rfloor$. This is clearly achievable by making everyone having anything above this value give off their land one by one. This process will terminate as everyone can't have land below the average. Anything above this isn't achievable as if everyone has above the average, the total area of land will not be sufficient.

For the second part, we first sort the array, and calculate prefix and suffix sums (sums of the first i elements on either sides). This overall takes $O(n \log n)$ time. Now for a fixed value of f, we first binary search for f in our array to find how many elements are less than f. The extra land we need is summation of [$f-$ element lower than f]. This is just $f\times$ (no of elements lower than f) $-$ the corresponding prefix sum. Now for any value of c, we can also get the extra land we get, it's equal to corresponding suffix sum - $c\times$ (no of elements higher than c). Let us first find between which 2 numbers in the array does c lie. c doesn't have to be a number in the array, but we can binary search assuming that for now. Choose an element in the array as c, and calculate the land we get, if it's too much, choose a larger element as c, and if it's too large, choose a smaller element as c. At the end of the binary search, we can tell between which 2 numbers in the array c is. Once we have this, we can actually solve for c, as we have the equation:

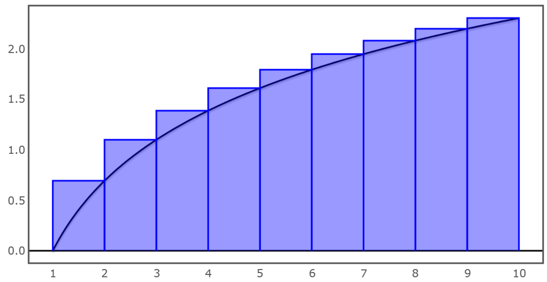
$$\text{suffix sum} - c \times \text{no of elements higher than } c = f$$

We know all the quantities except c, so we can get c.

8. Walk to 2^k and back to 0, -2^k and back to 0. Do this for every k from 0 until you find the treasure. If the treasure is at N and $2^{n-1} < N \leq 2^n$, we walk a distance of $4(1+2+4+\cdots+2^n) = 4(2^{n+1}-1) < 16(2^{n-1}) < 16N$, so we walk $O(N)$ distance.

9. $\log(n!) = \sum_{i=1}^{n} \log(i) \geq \log(n/2+1) + \log(n/2+2) + \cdots + \log(n) \geq \log(n/2) + \log(n/2) + \cdots + \log(n/2) = (n/2)\log(n/2) = (n/2)(\log n - 1) = (1/2)n\log n - n/2$

10. The inequality can be formed by comparing a riemann sum (area of rectangles) with the integral of $\log(x)$



So $\log(n!) = \sum_{i=1}^{n} \log i \geq \int_1^n \log x \; dx = [x\log x - x]_1^n = n\log n - n + 1$

11.
- True as $2n+3 \leq n^2$ for large n
- False, $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} = \Theta(n^3) > O(n^2)$
- True, will be proved in Q15
- False, $n^n = 2^{n\log n} > O(2^n)$
- False, $2^{3n} = (2^n)^3 > O(2^n)$
- True, $(n+1)^3 \leq (n+n)^3 = 8n^3 = O(n^3)$
- True, $(n+\sqrt{n})^2 \leq (n+n)^2 = 4n^2 = O(n^2)$
- True, $\log(n^3) = 3\log n = O(\log n)$

12. Out of syllabus

13. Out of syllabus

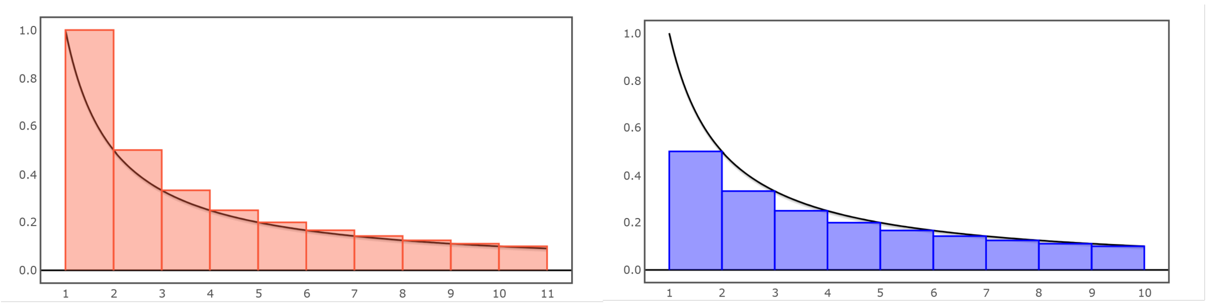
14. Out of syllabus

Reducing to a Subproblem

15. The same rectangle trick can work for both bounds

Solving integrals, we get $\log(n+1) \leq \sum_{i=1}^{n} 1/i \leq 1 + \log(n)$

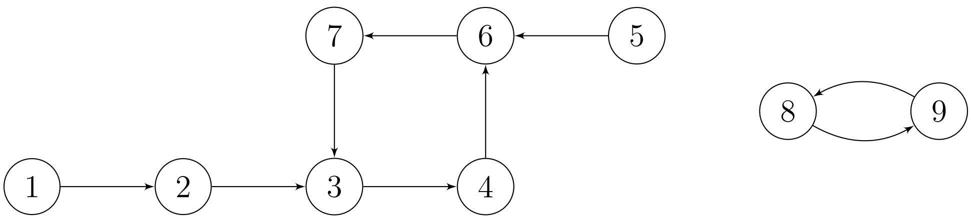
16. If we are selling it on day i, in order to maximize profit, we have to have bought it at the cheapest price that came so far. So what we can do is maintain the cheapest price (minimum) we have seen so far. Now we just assume we sell on day i to get the best profit for that day, and compare it with the best profit we have seen so far. So an extra subproblem we are solving is the minimum price we have seen so far.



17. With each question, we can eliminate 1 person from our search group. Let's say we ask i if she knows j. If i doesn't know j, j is not the celebrity as the celebrity is known by everyone. If i knows j, i isn't the celebrity as the celebrity doesn't know anyone. So we need just $n-1$ queries as we eleminate 1 person each time.

18. We can do this by thinking backwards. Our last step is probably adding a_0 to $a_1\alpha + \cdots + a_{n-1}\alpha^{n-1}$. And to get the second term, our last step is probably multiplying α to $a_1 + a_2\alpha + \cdots + a_{n-1}\alpha^{n-2}$ as α factors out. But this is great, as now this term is the same question but with $n-1$ terms, we have got the relation $T(n) = T(n-1) + 2$. Here's how the computation looks for 3 terms: $a_0 + (\alpha \times (a_1 + (\alpha \times (a_2 + (\alpha \times a_3)))))$

19. Let's convert this to some sort of a graph problem. For each i in the domain, let's draw a directed edge to $f(i)$. Since each number has 1 and only 1 outgoing edge, our graph only has cycles, and chains which merge into a cycle.



Example Graph Picture for $n = 9$

Take the example above as our question. Clearly numbers like $1, 5$ can't be in our set, as nothing maps to them. Can 2 be in our set? Also no, as since 1 isn't in our set, nothing maps to 2. We can apply this reasoning many times, all linear chains, which aren't part of any cycle can't be in our set. And the remaining elements obviously satisfy our condition, as in every cycle there is 1 ingoing and outgoing edge for every number.

The algorithm to get the set can be like this: first create an frequency array holding how many times each number is an image. Then add all the numbers which have no image (value 0 in the array) into a queue. Now we dequeue i, and basically mark it as not in solution set. We also have to decrease the frequency of $f(i)$, as i is deleted. If $f(i)$'s frequency decreases to 0, we add $f(i)$ to our queue, it also has to be deleted (this corresponds to the case where $f(i)$ is also in a chain). We keep doing this until our queue is empty, and the elements we never visited are the ones in our final answer.

20. We can first sort all the intervals in ascending order of left endpoint. If left endpoints are equal, sort in descending order of right endpoint. This ordering has the following objective. No interval can be contained by an interval after it, so it becomes easier to deal with (assuming all intervals are distinct). Now we can go one by one in this interval list. If we are at interval i, we don't have to care about later intervals, it won't be contained in them. In the intervals before i, they will contain i on the left side for sure, if any of them have the right endpoint greater than or equal to i's, i is contained, else it is not. So all we have to do is maintain the greatest right endpoint found so far. If it's $\geq i$'s endpoint, i is contained, else it isn't. And then update the greatest endpoint found so far. So overall the algorithm is $O(n \log n)$ (sorting) + $O(n)$ (single traversal of the array).

21. Let's say we wanted to calculate for $n = 4, k = 2$ which is $ab+ac+ad+bc+bd+cd$. We can actually group this based on whether d is a term or not, so it becomes $d(a+b+c)+(ab+ac+bc)$. But these are subproblems, $n = 3, k = 1$ and $n = 3, k = 2$. We can generalize this recursive idea, $f(n, k) = a_n(f(n-1, k-1)) + f(n-1, k)$ by choosing whether a_n is part of the terms or not. Our base cases are $f(1, 1) = a_1$, $f(n, 1) = a_n + f(n-1, 1)$. We can just create an $n \times k$ grid and calculate $f(i, j)$ and fill it in each cell, going in increasing order of i and increasing order of j (there are more efficient ways space complexity wise but eh).

We can also achieve this recursion logically, not only just by mere observation. As per the hint, the given entity is the coefficient of x^{n-k} in the polynomial $\prod_{i=1}^{n}(x+a_i)$. Rewriting as $[\prod_{i=1}^{n-1}(x+a_i)] * (x+a_n) = x * [\prod_{i=1}^{n-1}(x+a_i)] + a_n * [\prod_{i=1}^{n-1}(x+a_i)]$, we see that coefficient of x^{n-k} in $\prod_{i=1}^{n}(x+a_i)$ in LHS = sum of coefficients of x^{n-k-1} in $\prod_{i=1}^{n-1}(x+a_i)$ and x^{n-k} in $a_n * \prod_{i=1}^{n-1}(x+a_i)$ in RHS. Hence we get the recursion equation above.

Divide and Conquer

22. This question is very similar to the interval containment problem. We consider (a, b) to contain (c, d) if $a \leq c$ and $b \geq d$, but dominate (c, d) if $a \geq c$ and $b \geq d$. So the solution is again like this: sort in decreasing order of first coordinate, if it's equal decreasing order of 2nd coordinate. Then it's a single traversal of the array, by maintaining the max of the 2nd coordinate seen so far. If it's $\geq$ the current 2nd coordinate, the current point is dominated, else update the max of 2nd coordinate.

23. It's been given that the majority fingerprint occurs more than $n/2$ times. So if we split the array into 2, we can say in at least one of the array, the fingerprint occurs more than $n/4$ times, and hence is the majority fingerprint for that subarray too. This observation is useful for a divide and conquer idea.

Let's say we have solved the subproblems for the two $n/2$ size arrays, and have kept the majority fingerprint of each subarray at the beginning of them respectively. One of these fingerprints is going to be our final solution itself. We can just check how many times each one appears. Traverse both subarrays, and manually count how many times you see each of our candidate solutions. Whichever you see more is our final solution, and keep it at the beginning of the whole array. This method will satisfy $T(n) = 2T(n/2) + O(n)$ so $T(n)$ is $O(n \log n)$.

24. We use a balanced BST for this. But we also store one extra thing in each node, the number of elements in the subtree with root as that node. How do our functions for insert, delete, rotate change in order to maintain this data? While inserting, we just add 1 to every node on the path from the root to the new node. For deleting, we just subtract 1 from every node on the path to the node to delete. Balancing is also not that hard; as we exchange subtrees between nodes, we also have to just exchange the data which maintains the number of elements. But what's the point of maintaining this data? It helps us to find the number of elements smaller than a query in $O(\log n)$ time. Take the same path as if you are inserting the query. If you take a right turn, you know the root and the left subtree are all smaller than the query. If you take a left turn, the root and the right subtree are all larger than the query, so you have to add 1+ size of right subtree to your answer. Just keep doing this until you get to a leaf.

How do we use this data structure? We can iterate through the array and maintain this special BST of all elements we have seen, seen so far. Say we are at $A[j]$ we have to find number of $i < j$ and $A[i] > 2A[j]$. This is basically number of elements in the BST which are greater than the query $2A[j]$ which can be found in $O(\log n)$ time. And then we have to insert $A[j]$ which also takes $O(\log n)$ time. So this algorithm will take $O(n \log n)$ time.

25. Think about shading the visible parts of each line. If you move from left to right, the lines will be in increasing order of slopes, and form a sort of U shape. So we can start off by first sorting in increasing order of slope. If 2 lines have same slope, the one with smaller 'c' is redundant. Now we're going to traverse this array one by one, and maintain a list of lines which are visible. Note that in this list, except for the first and last line, only line segments are visible. Now a crucial observation is that the next line will intersect only 1 of the visible parts, as it's slope is greater than the whole thing. And which part it insersects can be found by binary search.

Let's check if line l intersects $vis[i]$ (in the visible part of it). For this, it should intersect between the endpoints of the visible part, $vis[i-1]$ and $vis[i]$'s intersection, $vis[i]$ and $vis[i+1]$'s intersection. Let's call these endpoints, a and b, and l's insersection point with $vis[i]$ as p. If the order of points of line $vis[i]$ is p, a, b, then line l intersects with something before $vis[i]$. If the order is a, b, p, then the line insersects with something after $vis[i]$. So in this way we can binary search, by dividing our search space. Once we found out which line it intersects with, all the lines in the array after it are to be neglected, they will be hidden. We don't have to spend time deleting them, just store an endpointer to vis to save time. Once we insert a new line in some location, move the endpointer to that location, and treat that as if it's the end of the array. If the next line has to be appended to the array, it will be inserted in the location just after the endpointer.

Overall the time complexity is $O(n \log n) + n \times O(\log n) = O(n \log n)$.

26. This question is very similar to the previous one, just turning it upside down. So we can do the same process, sort the lines in decreasing order of slopes, and iterate through this array. However here, we also have the additional constraint that $y \geq 0$, so the starting and ending lines also terminate. But stil, we follow the same process, and in the end, finding the vertices from the lines is easy. This is overall $O(n \log n) + O(n) = O(n \log n)$.

27. Out of syllabus

28. Let's divide our array into 2 subarrays, left half and the right half. Let's say we have already solved the subproblems within the 2 halves, now all that's left is to count number of significant inversions where i is in the first half and j is in the second half. In order to do this is $O(n)$ let's assume some preprocessing is already done for us, the left subarray and the right subarray are already sorted. This will make our counting easier, and it's now like we are doing our counting while doing a mergesort.

Keep pointer p_1 at the start of the first array, and p_2 at the start of the second array. We just want to find for every i, how many j are there such that $A[i] > 2A[j]$. So for p_1 pointing to $i = 0$, keep moving p_2 until the condition isn't satisfied. The number of j's for $i = 0$ are the number of times p_2 moved. Now increment p_1, and continue moving p_2 whenever $A[i] > 2A[j]$, for the new i, the total number of significant inversions is total number of times p_2 moved. We can keep doing this for all values, until both pointers reach the end. This takes $O(n)$ time. But our subroutine isn't done, we have to now merge these 2 subarrays to give a sorted full array, but this also takes only $O(n)$ time. So we have the recursion $T(n) = 2T(n/2) + O(n)$ so $T(n) = O(n \log n)$.

29. The iterative algorithm has already been explained at each question, let's do the divide and conquer approach.

Question 22: Suppose in our subroutine, we split the array into 2, and have 2 sets of non-dominated points, assuming the sets are represented as an array, and are sorted with first coordinate in descending order (they can't be equal, else one would dominate the other). A crucial observation is that now the second coordinate will now be sorted in ascending order, because none of the points dominate any other. Now we just do a merge operation based on the first coordinate, while adding a new point to the array, check if the 2nd coordinate is also bigger than the previous one, if not discard it.

Question 24: Done in Q27

Question 25: Check last year's quiz 1 solutions on their course website, I don't really understand it how it works

Question 26: Similar to question 25

30.
- Yes, while inserting, add 1 to each node's additional info for every node in the path from the root to the new node. While deleting, subtract 1 to every node's info for every node in the path from the root to the node to be deleted. This takes $O(\log n)$ time. Balancing will only involve swapping this info between nodes.
- No, suppose you add a node larger than every other node. All nodes need to be updated, which will take $O(n)$ time.
- No, if you delete a node with a single child, you may have to update $O(n)$ nodes (all its children)
- Yes. Say you insert a node x. Find its predecessor, and make x point to its successor. And make the predecessor point to x. For deletion of x, find the

predecessor of x and make it point to its successor before deletion. And since predecessor can be found in $O(h)$, we're done.

- Yes. After inserting a node, the new node will have additional info as 0. Then keep travelling from the node to the root, reassigning info of each node you visit as $info(node) = 1 + max(info(node \rightarrow left), info(node \rightarrow right))$. For deletion it will be the same assignment, travel from the node you deleted to the root. For balancing operations, the info will just swap between nodes.
- No. Suppose you have a node smaller than every other node. Then you will have to make the rank of every other node increase, and we'll have to update $O(n)$ nodes.

31. Fibonacci numbers are generated by $F_{n+1} = F_n + F_{n-1}$, so we need the last 2 numbers. And we'll need F_n, F_{n+1} to generate F_{n+2}. So let's observe the relation between 2 consecutive Fibonacci numbers and the next 2 consecutive ones.

$$\begin{aligned} F_{n+1} &= 1 \times F_n + 1 \times F_{n-1} \\ F_n &= 1 \times F_n + 0 \times F_{n-1} \end{aligned}$$

Since this is a linear transformation, we can write it as:

$$\begin{pmatrix} F_{n+1} \\ F_n \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} F_n \\ F_{n-1} \end{pmatrix}$$

And by repeatedly applying this property, we can get:

$$\begin{pmatrix} F_{n+1} \\ F_n \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n \begin{pmatrix} F_1 \\ F_0 \end{pmatrix}$$

We have seen how to do x^n in $O(\log n)$ time by repeated squaring, and the same trick can be applied for matrices.