Chapter 10

Error Detection and Correction

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Error

- Data can be corrupted during transmission.

- For reliable communication, errors must be detected and corrected.

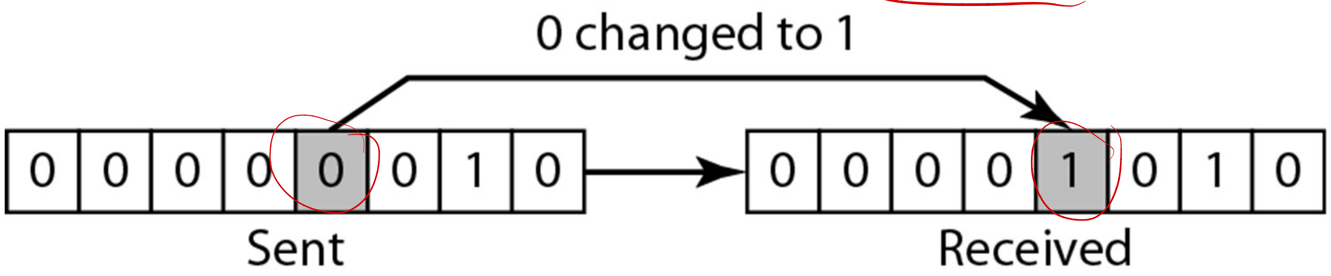
- Types of Error ? [illegible]

INTRODUCTION

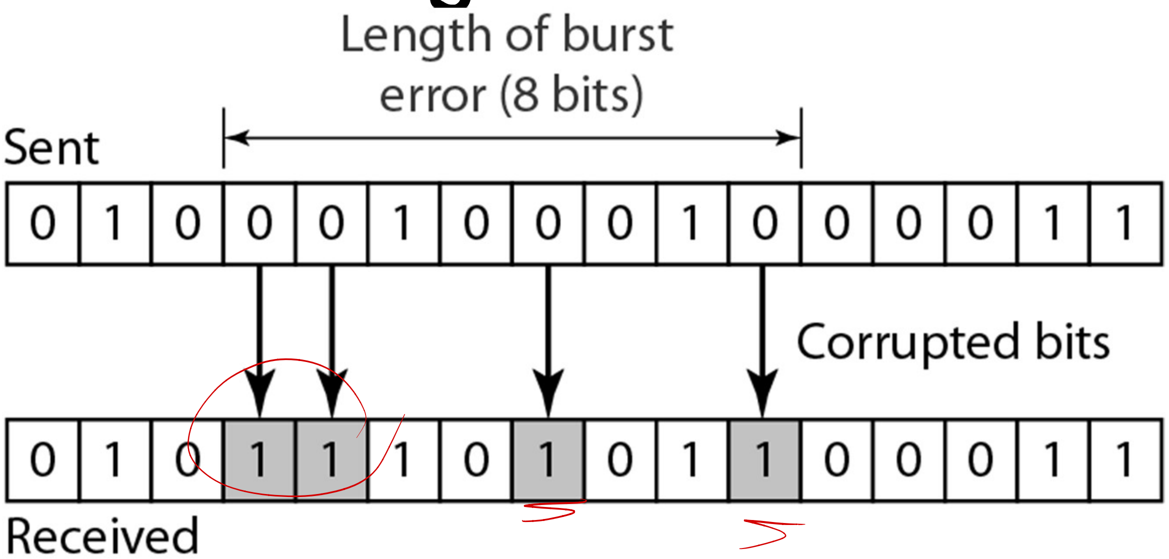
- Types of Errors
- Redundancy
- Detection Versus Correction
- Forward Error Correction Versus Retransmission
- Coding
- Modular Arithmetic

Types of Errors

- In a single-bit error, only 1 bit in the data unit has changed.

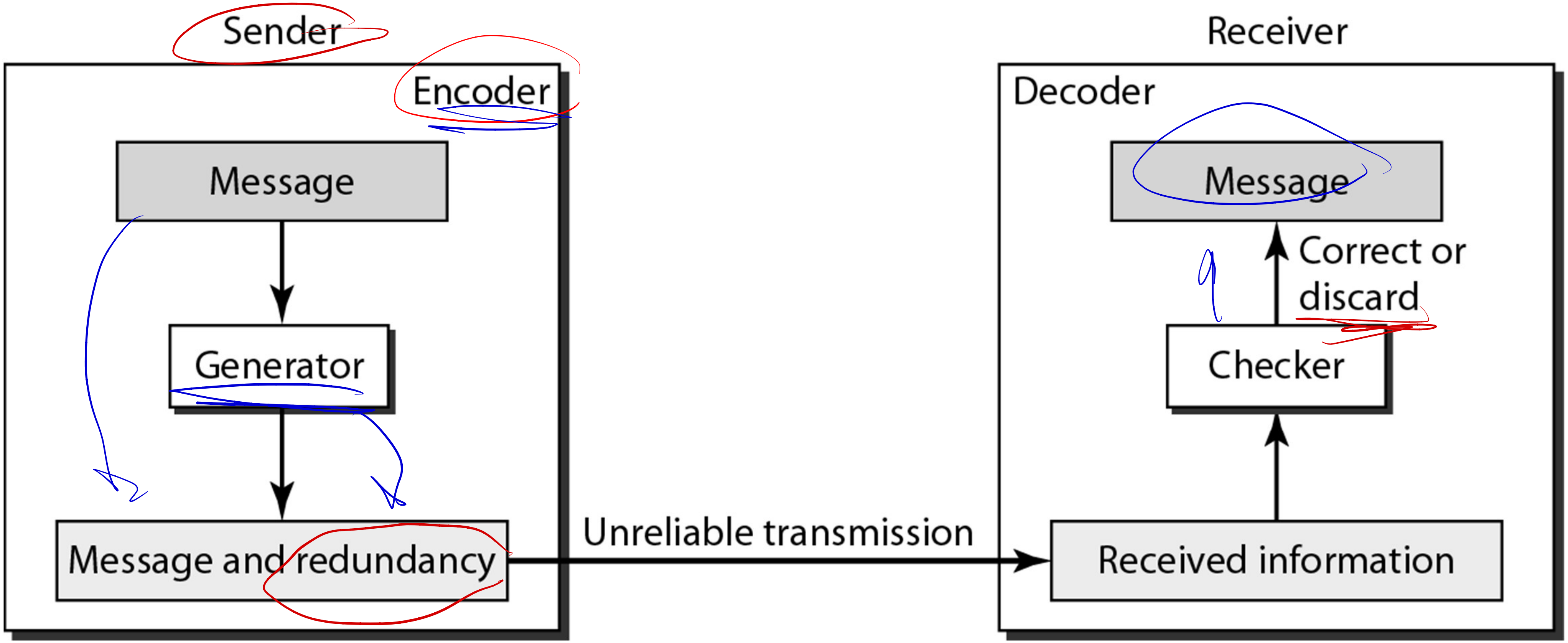


- A burst error means that 2 or more bits in the data unit have changed.



Redundancy (ส่วนที่เกิน, ซ้ำซ้อน)

- To detect or correct errors, we need to send extra (redundant) bits with data.



แค่ตรวจสอบ รู้ว่าผิดที่ไหนด้วย

Detection Versus Correction

- Detection

  แค่ตรวจสอบว่าเกิด error ขึ้น

  - Verifying only to see if any error has occurred
  - Not even interested in the number of errors

    ตรวจสอบที่ปลายทาง

  - Uses the concept of redundancy for detecting errors at the destination.
- Correction

  ข้อมูลใด

  - Necessity to know the exact number of bits that are corrupted and more importantly, their location in the message.

    ตำแหน่งไหน

  - The number of the errors and the size of the message are important factors.

Modular Arithmetic

- XORing of two single bits or two words

เหมือนกัน 0

$0 \oplus 0 = 0$ $\qquad$ $1 \oplus 1 = 0$

a. Two bits are the same, the result is 0.

$0 \oplus 1 = 1$ $\qquad$ $1 \oplus 0 = 1$

b. Two bits are different, the result is 1.

ต่างกัน 1

	1	0	1	1	0
$\oplus$	1	1	1	0	0
	0	1	0	1	0

c. Result of XORing two patterns

In modulo-N arithmetic, we use only the integers in the range 0 to N −1, inclusive.

B. A. Forouzan, Data Communications and Networking, 4th edition, McGRAW-HILL

BLOCK CODING

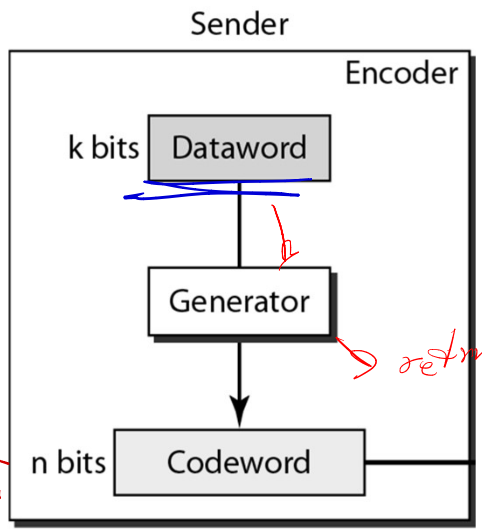
- Divide message into blocks
  - k bits : datawords
  - r bits : redundant bits
  - n bits : codewords (k + r)

Coding scheme : C(*n*, *k*)

00
01
10
11

2^k

000
001
010
011
100
101
110
111



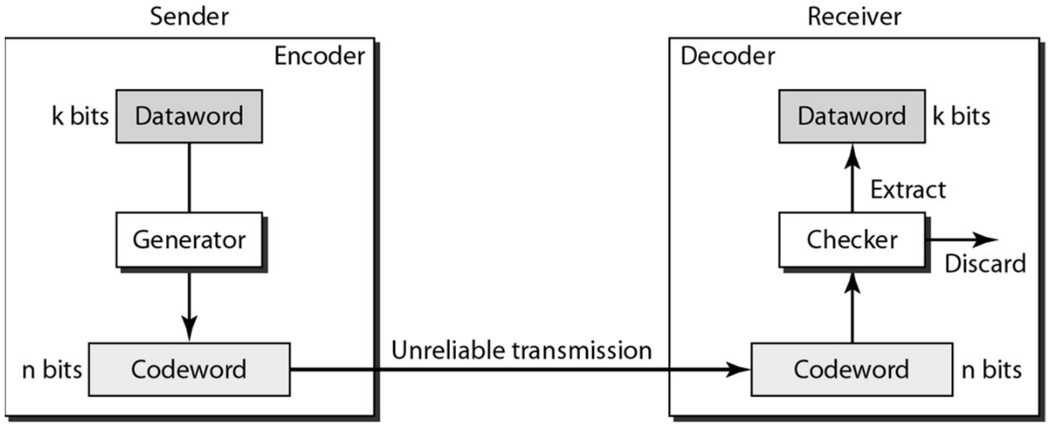
2^k Datawords, each of k bits

2^n Codewords, each of n bits (only 2^k of them are valid)

Error Detection

- Sender creates codewords out of datawords
- Received codeword
  - same as one of the valid codewords -> accepted
  - not valid codewords -> it is discarded
- if the codeword is corrupted during transmission but the received word still matches a valid codeword
  - the error remains undetected.

000 → ถ้า ได้ 011 check error ไม่เจอ



Datawords	*Codewords*
00	000
01	011
10	101
11	110

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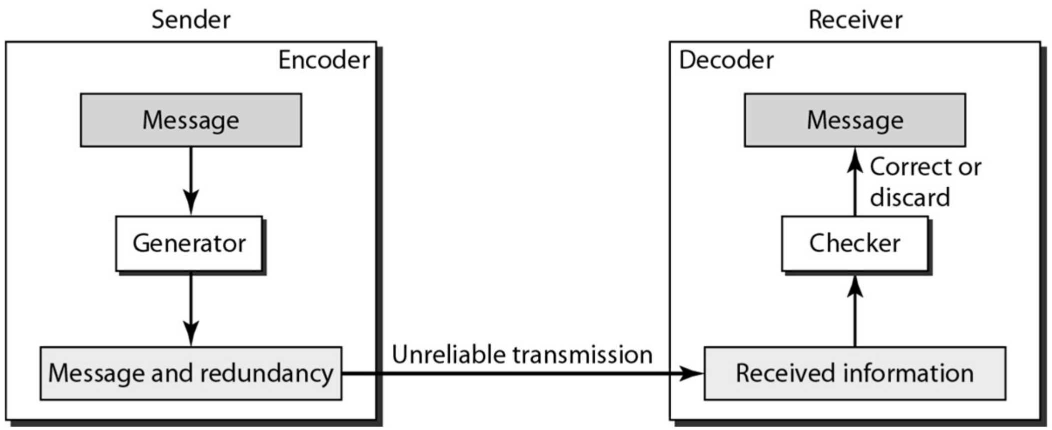
Error Correction

- Receiver needs to find (or guess) the original codeword sent
- More redundant bits than the error detection
- Idea is the same as error detection but the checker functions are much more complex

checker

ถ้า correct ได้

00000 → 00010



Dataword	*Codeword*
00	00000
01	01011
10	10101
11	11110

* The picture illustrates the idea.

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Hamming Distance

- Hamming Distance : $d(x, y)$
  - The Hamming distance between two words is the number of differences between corresponding bits

$d(0000, 0010) \rightarrow 1$

$d(1011, 1010) \rightarrow 1$

Minimum Hamming Distance

– The minimum Hamming distance is the smallest Hamming distance between all possible pairs in a set of words.

C(3,2) $d_{min} = 2$

Datawords	Codewords
00	000
01	011
10	101
11	110

$d(000, 011) = 2$

$d(000, 101) = 2$

$d(000, 110) = 2$

$d(011, 101) = 2$

$d(011, 110) = 2$

$d(101, 110) = 2$

C(5,2) $d_{min} = 3$

Dataword	Codeword
00	00000
01	01011
10	10101
11	11110

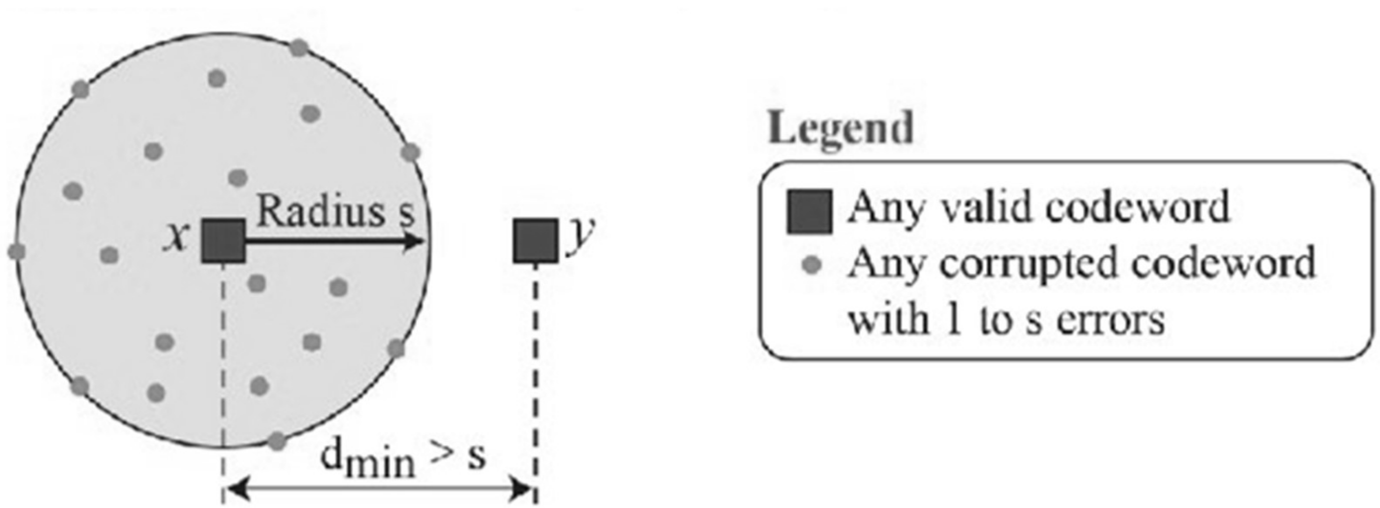
code word

Data word

Coding scheme : C(*n*, *k*)

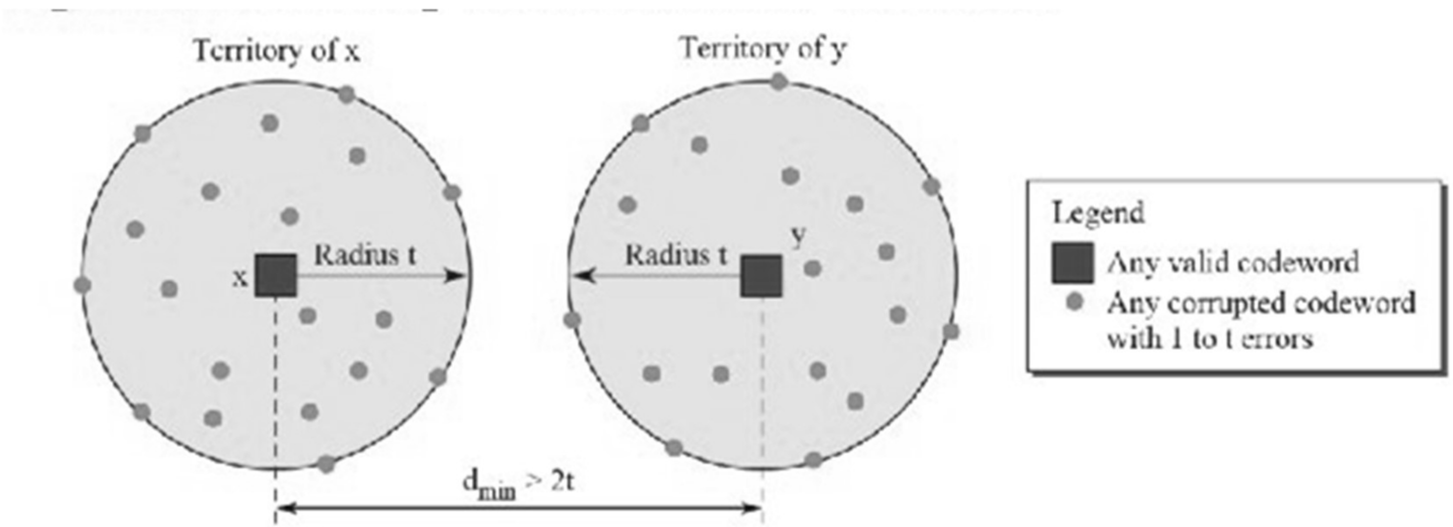
Minimum Distance for Error Detection

- To guarantee the detection of up to s errors in all cases, the minimum Hamming distance in a block code must be $d_{min} = s + 1$



Minimum Distance for Error Correction

- To guarantee correction of up to t errors in all cases, the minimum Hamming distance in a block code must be $d_{min} = 2t + 1$

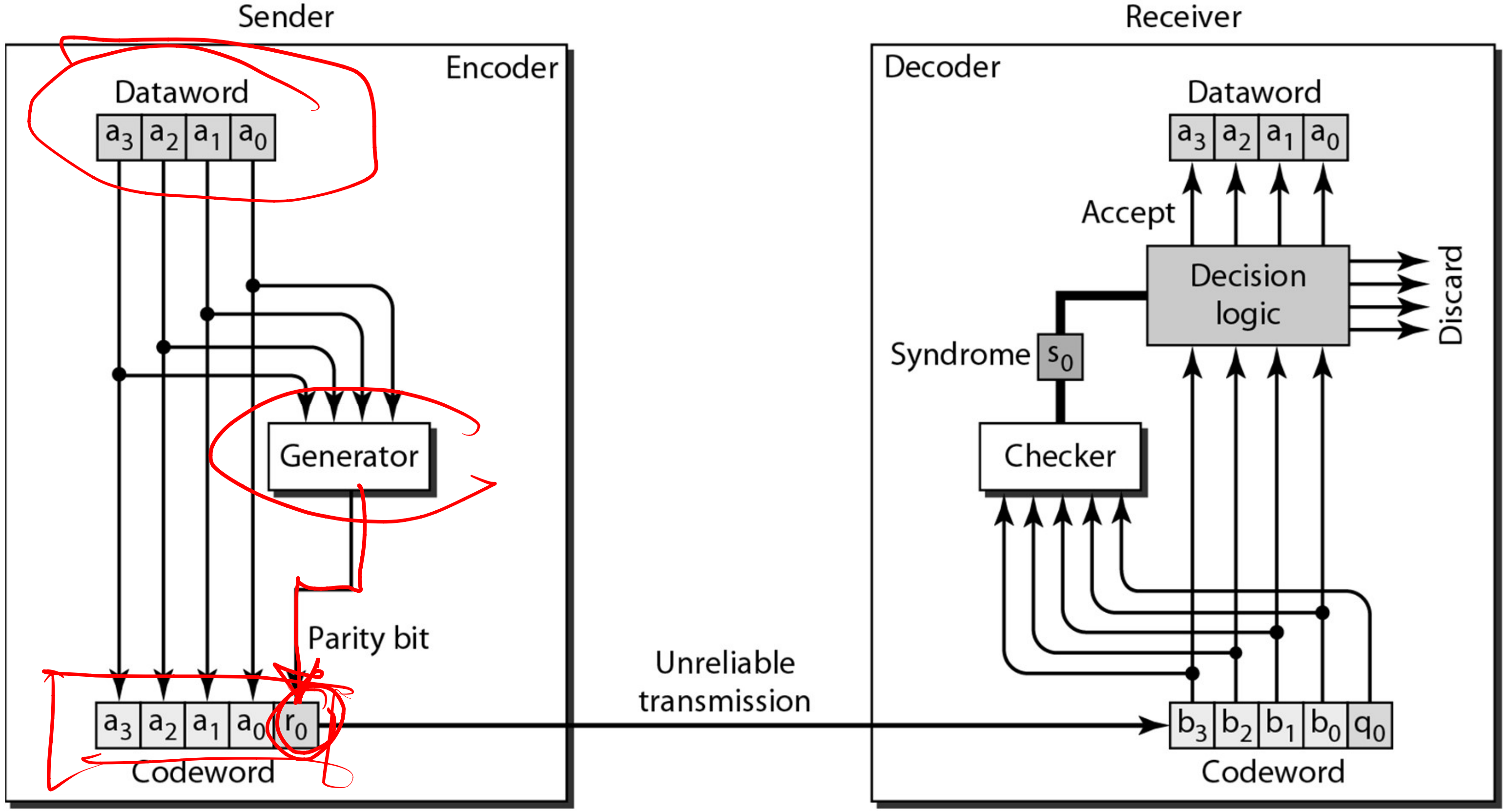


LINEAR BLOCK CODES

- Minimum Distance for Linear Block Codes
- Some Linear Block Codes
  - Simple Parity-Check Code (Even)
  - Two-Dimensional Parity-Check Code
  - Hamming Codes

Simple Parity-Check Code (Even)

- In parity check, a parity bit is added to every data unit so that the total number of 1s is even (or odd for odd-parity).



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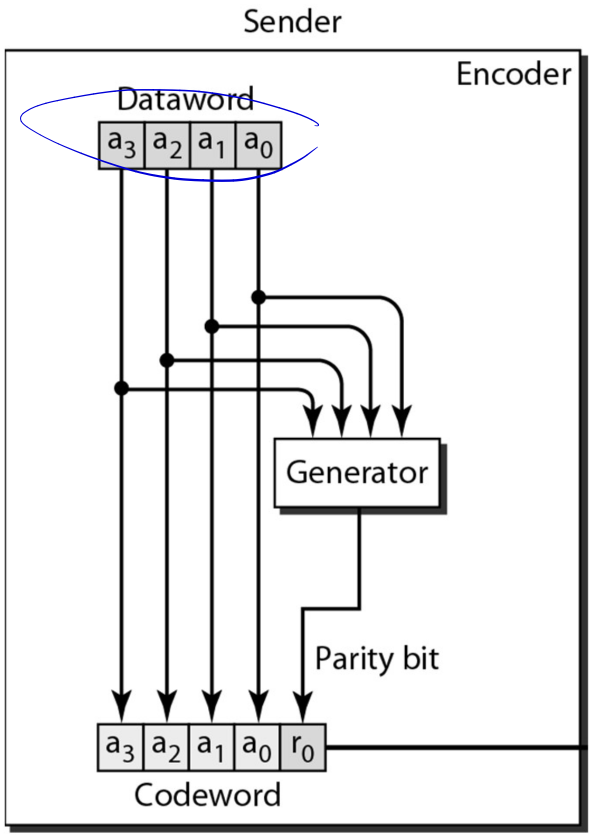
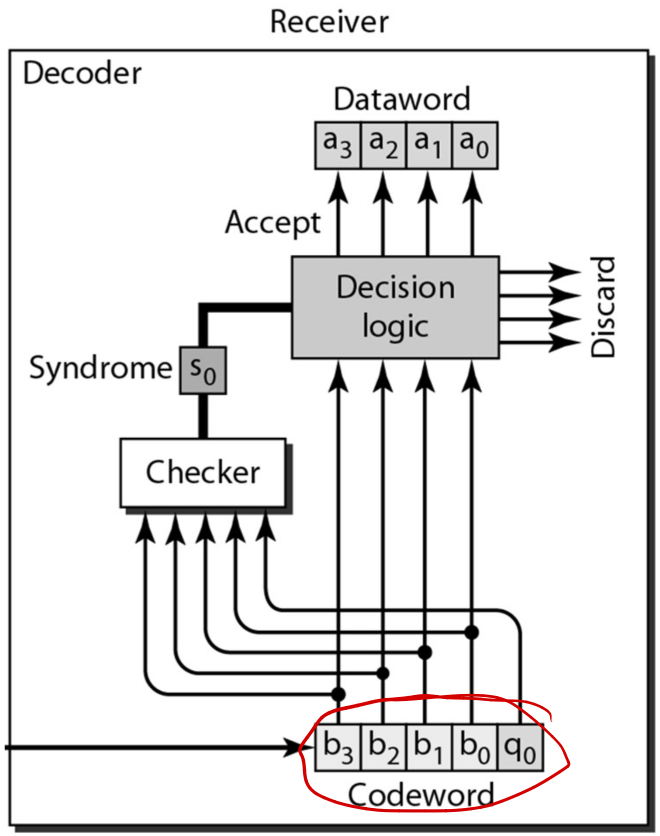
Simple Parity-Check Code (Even)

- Generator
  - $r_0 = a_3 \oplus a_2 \oplus a_1 \oplus a_0$



Simple Parity-Check Code (Even)

- Generator
  - $r_0 = \boxed{a_3 \oplus a_2 \oplus a_1 \oplus a_0}$
- Checker
  - $s_0 = b_3 \oplus b_2 \oplus b_1 \oplus b_0 \oplus q_0$



Example 1

- Suppose the sender wants to send the word world. In ASCII the five characters are coded as

1110111 0 **1101111** 0 **1110010** 0 **1101100** 0 **1100100** 1

Example 2

- Now suppose the word world in Example 1 is received by the receiver without being corrupted in transmission.

11101110 11011110 11100100 11011000 11001001

7 7 6 F 7 2 6 C 6 4

w o r l d

	0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F
0		┌	┐	└	┘	│	─									
1	┼					┴	┬	┤		├						
2		!	"	#	$	%	&	'	(	)	*	+	,	-	.	/
3	0	1	2	3	4	5	6	7	8	9	:	;	<	=	>	?
4	@	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
5	P	Q	R	S	T	U	V	W	X	Y	Z	[	\	]	^	_
6	'	a	b	c	d	e	f	g	h	i	j	k	l	m	n	o
7	p	q	r	s	t	u	v	w	x	y	z	{	|	}	~	

Example 3

[illegible]

- Now suppose the word world in Example 1 is corrupted during transmission.

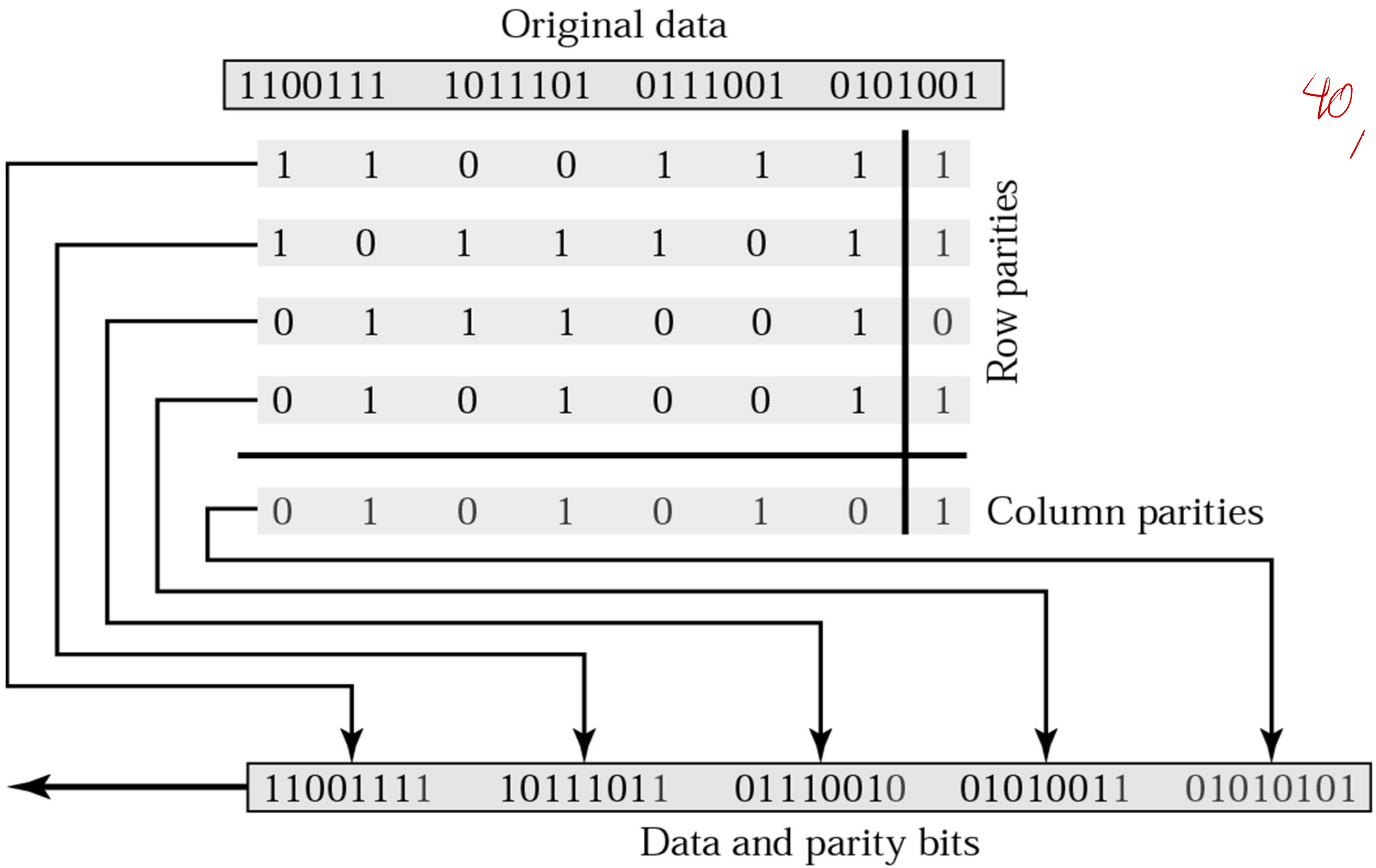
11111110 11011110 11101100 11011000 11001001

[illegible]

	0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F
0		┌	┐	└	┘	│	─									
1	┼					┴	┬	┤		├						
2		!	"	#	$	%	&	'	(	)	*	+	,	-	.	/
3	0	1	2	3	4	5	6	7	8	9	:	;	<	=	>	?
4	@	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
5	P	Q	R	S	T	U	V	W	X	Y	Z	[	\	]	^	_
6	'	a	b	c	d	e	f	g	h	i	j	k	l	m	n	o
7	p	q	r	s	t	u	v	w	x	y	z	{	|	}	~	

Two-Dimensional Parity-Check Code

40, 28
/



Example 4

- Suppose the following block is sent:

  10101001 00111001 11011101 11100111 10101010

- However, it is hit by a burst noise of length 8, and some bits are corrupted.

  10100011 10001001 11011101 11100111 10101010

1	0	1	0	0	0	1	1
1	0	0	0	1	0	0	1
1	1	0	1	1	1	0	1
1	1	1	0	0	1	1	1
1	0	1	0	1	0	1	0

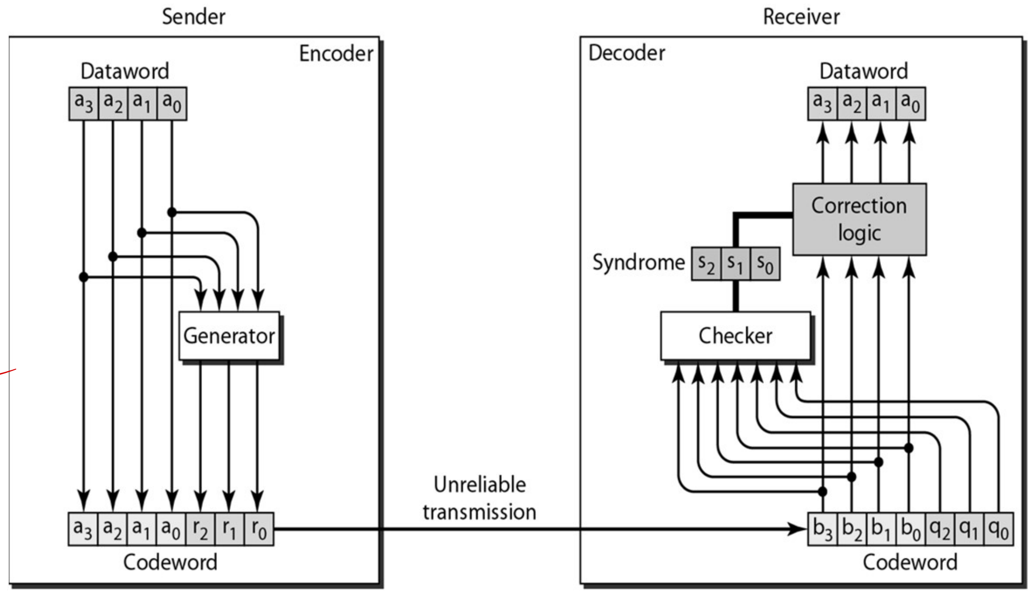
C (40, 28)

Note

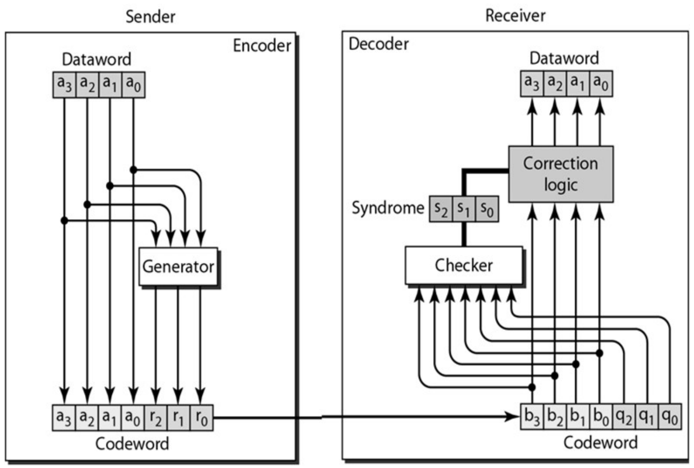
- Simple parity check can detect all single-bit errors. It can detect burst errors only if the total number
ถ้า เป็น คี่
of errors in each data unit is odd.
- In two-dimensional parity check, a block of bits is divided into rows and a redundant row of bits is added to the whole block.

Hamming Codes

- Hamming Codes C(7, 4)
  - $d_{min} = 3$
  - detect up to two errors ($d_{min} = s + 1$)
  - correct one single error ($d_{min} = 2t + 1$)
- Hamming Codes C(11, 7)



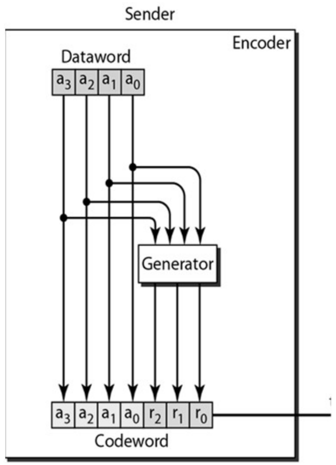
Hamming Codes C(7, 4)



- Transmitter
  - $r_0 = a_2 \oplus a_1 \oplus a_0$
  - $r_1 = a_3 \oplus a_2 \oplus a_1$
  - $r_2 = a_1 \oplus a_0 \oplus a_3$
- Receiver
  - $s_0 = b_2 \oplus b_1 \oplus b_0 \oplus q_0$
  - $s_1 = b_3 \oplus b_2 \oplus b_1 \oplus q_1$
  - $s_2 = b_1 \oplus b_0 \oplus b_3 \oplus q_2$

Syndrome	000	001	010	011	100	101	110	111
Error	None	q_0	q_1	b_2	q_2	b_0	b_3	b_1

Example for Hamming Codes C(7, 4)



- Transmitter

exclusive or

  - $r_0 = a_2 \oplus a_1 \oplus a_0$
  - $r_1 = a_3 \oplus a_2 \oplus a_1$
  - $r_2 = a_1 \oplus a_0 \oplus a_3$

เหมือนกัน 0

ต่างกัน 1

	a_3	a_2	a_1	a_0	r_2	r_1	r_0
D_0	0	0	0	1	1	0	1
D_1	0	1	1	0	1	0	0
D_2	1	0	0	0	1	1	0
D_3	1	1	1	1	1	1	1

a_3				a_2				a_1				a_0				r_2				r_1				r_0			
0	0	1	1	0	1	0	1	0	1	0	1	1	0	0	1	1	1	1	1	0	0	1	1	1	0	0	1

Example for Hamming Codes C(7, 4)

- Send by row vs Send by Columb

ถ้าข้อมูลแนวนอนจะเกิด Burst Error เกิดขึ้น

D_0							D_1							D_2							D_3						
0	0	0	1	1	0	1	0	1	1	0	1	0	0	1	0	0	0	1	1	0	1	1	1	1	1	1	1

	a_3	a_2	a_1	a_0	r_2	r_1	r_0
D_0	0	0	0	1	1	0	1
D_1	0	1	1	0	1	0	0
D_2	1	0	0	0	1	1	0
D_3	1	1	1	1	1	1	1

(แบบนี้ทำ correction ได้)

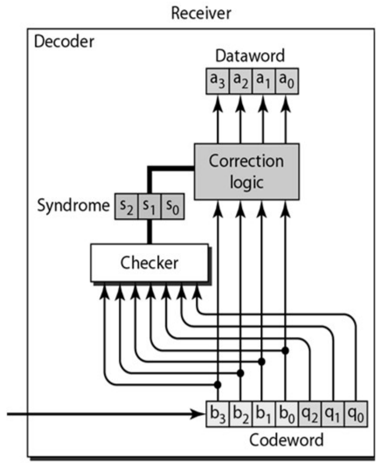
a_3				a_2				a_1				a_0				r_2				r_1				r_0			
0	0	1	1	0	1	0	1	0	1	0	1	1	0	0	1	1	1	1	1	0	0	1	1	1	0	0	1

Example for Hamming Codes C(7, 4)

b_3				b_2				b_1				b_0				q_2				q_1				q_0			
0	0	1	1	0	1	0	1	0	1	0	1	1	0	1	0	0	0	1	1	0	0	1	1	1	0	0	1

- **Receiver**
  - $s_0 = b_2 \oplus b_1 \oplus b_0 \oplus q_0$
  - $s_1 = b_3 \oplus b_2 \oplus b_1 \oplus q_1$
  - $s_2 = b_1 \oplus b_0 \oplus b_3 \oplus q_2$

		s_2	s_1	s_0
q_2	D_0	1	0	0
q_2	D_1	1	0	0
b_0	D_2	1	0	1
b_0	D_3	1	0	1



	b_3	b_2	b_1	b_0	q_2	q_1	q_0
D_0	0	0	0	1	0	0	1
D_1	0	1	1	0	0	0	0
D_2	1	0	0	1	1	1	0
D_3	1	1	1	0	1	1	1

Syndrome	000	001	010	011	100	101	110	111
Error	None	q_0	q_1	b_2	q_2	b_0	b_3	b_1

Hamming Codes C(11, 7)

- Codeword : $d_{11}\ d_{10}\ d_9\ r_8\ d_7\ d_6\ d_5\ r_4\ d_3\ r_2\ r_1$
- Transmitter
  - $r_1 = d_{11} \oplus d_9 \oplus d_7 \oplus d_5 \oplus d_3$
  - $r_2 = d_{11} \oplus d_{10} \oplus d_7 \oplus d_6 \oplus d_3$
  - $r_4 = d_7 \oplus d_6 \oplus d_5$
  - $r_8 = d_{11} \oplus d_{10} \oplus d_9$
- Receiver
  - $s_1 = d_{11} \oplus d_9 \oplus d_7 \oplus d_5 \oplus d_3 \oplus r_1$
  - $s_2 = d_{11} \oplus d_{10} \oplus d_7 \oplus d_6 \oplus d_3 \oplus r_2$
  - $s_4 = d_7 \oplus d_6 \oplus d_5 \oplus r_4$
  - $s_8 = d_{11} \oplus d_{10} \oplus d_9 \oplus r_8$

11	10	9	8	7	6	5	4	3	2	1
d	d	d	r_8	d	d	d	r_4	d	r_2	r_1

Example for Hamming Codes C(11, 7)

- Codeword : $d_{11}\ d_{10}\ d_9\ r_8\ d_7\ d_6\ d_5\ r_4\ d_3\ r_2\ r_1$
- Transmitter
  - $r_1 = d_{11} \oplus d_9 \oplus d_7 \oplus d_5 \oplus d_3$
  - $r_2 = d_{11} \oplus d_{10} \oplus d_7 \oplus d_6 \oplus d_3$
  - $r_4 = d_7 \oplus d_6 \oplus d_5$
  - $r_8 = d_{11} \oplus d_{10} \oplus d_9$

1	0	0	1	1	1	0	0	1	0	1
11	**10**	**9**	**8**	**7**	**6**	**5**	**4**	**3**	**2**	**1**

Data:
1 0 0 1 1 0 1

11	10	9	8	7	6	5	4	3	2	1
d	**d**	**d**	r_8	**d**	**d**	**d**	r_4	**d**	r_2	r_1

Example for Hamming Codes C(11, 7)

- Codeword : $d_{11}\ d_{10}\ d_9\ r_8\ d_7\ d_6\ d_5\ r_4\ d_3\ r_2\ r_1$
- Receiver
  - $s_1 = d_{11} \oplus d_9 \oplus d_7 \oplus d_5 \oplus d_3 \oplus r_1$
  - $s_2 = d_{11} \oplus d_{10} \oplus d_7 \oplus d_6 \oplus d_3 \oplus r_2$
  - $s_4 = d_7 \oplus d_6 \oplus d_5 \oplus r_4$
  - $s_8 = d_{11} \oplus d_{10} \oplus d_9 \oplus r_8$

1	0	0	1	1	1	0	0	1	0	1
11	**10**	**9**	**8**	**7**	**6**	**5**	**4**	**3**	**2**	**1**

Code:
1 0 0 1 1 1 0 0 1 0 1

S_8 S_4 S_2 S_1
0 0 0 0

-

11	10	9	8	7	6	5	4	3	2	1
d	**d**	**d**	r_8	**d**	**d**	**d**	r_4	**d**	r_2	r_1

Example for Hamming Codes C(11, 7)

- Codeword : $d_{11}\ d_{10}\ d_9\ r_8\ d_7\ d_6\ d_5\ r_4\ d_3\ r_2\ r_1$
- Receiver
  - $s_1 = d_{11} \oplus d_9 \oplus d_7 \oplus d_5 \oplus d_3 \oplus r_1$
  - $s_2 = d_{11} \oplus d_{10} \oplus d_7 \oplus d_6 \oplus d_3 \oplus r_2$
  - $s_4 = d_7 \oplus d_6 \oplus d_5 \oplus r_4$
  - $s_8 = d_{11} \oplus d_{10} \oplus d_9 \oplus r_8$

1	0	0	1	$\boxed{0}$	1	0	0	1	0	1
11	**10**	**9**	**8**	**7**	**6**	**5**	**4**	**3**	**2**	**1**

Code:
1 0 0 1 1 1 0 0 1 0 1

$S_8\ S_4\ S_2\ S_1$
$0\ \ 1\ \ 1\ \ 1 = 7$

–

11	10	9	8	7	6	5	4	3	2	1
d	**d**	**d**	r_8	**d**	**d**	**d**	r_4	**d**	r_2	r_1