7 Statistical Intervals Based on a Single Sample

Part I

Introduction

- Point estimation, because it is a single number, by itself provides no information about the precision and reliability of estimation.
- Consider, for example, using statistic $\bar{X}$ to calculate a point estimation for the true average breaking strength (g) of paper towels of a certain brand, and suppose that $\bar{X} = 9322.7$.
- Because of sampling variability, it is virtually never the case that $\bar{X} = \mu$.
- The point estimation says nothing about how close it might be to μ.

Introduction

- An alternative to reporting a single sensible value for the parameter being estimated is to calculate and report an entire interval of plausible values – interval estimate or confidence interval (CI).
- Confidence interval is always calculated by first selecting a confidence level, which is a measure of the degree of reliability to the interval.
- A confidence interval with a 95% confidence level for the true average breaking strength might have a lower limit of 9162.5 and an upper limit to 9482.9.
- Then at the 95% confidence level, any value of μ between 9162.5 and 9482.9 is plausible.

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Introduction

- A confidence level of 95% implies that 95% of all sample would give an interval that include μ ,or whatever other parameter is being estimated, and only 5% of all samples would yield an erroneous interval.
- The most frequently used confidence levels are 95%, 99%, and 90%.
- The higher of confidence level, the more strongly we believe that the value of the parameter being estimated lies within the interval (an interpretation of any particular confidence level will be given shortly).

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Introduction

- Information about the precision of an interval estimate is conveyed by the width of the interval.
- If the confidence level is high and the resulting interval is quite narrow, our knowledge of the value of the parameter is reasonably precise.
- A very wide confidence interval, however, gives the message that there is a great deal of uncertainty concerning the value of what we are estimating.
- Figure 7.1 shows 95% confidence intervals for true average breaking strengths of two different brands of paper towels.
- One of these intervals suggests precise knowledge about μ ,whereas the other suggests a very wide range of plausible values.

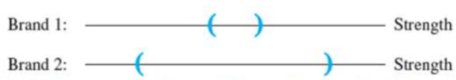

Figure 7.1 CIs indicating precise (brand 1) and imprecise (brand 2) information about μ

7.1 Basic Properties of Confidence Intervals

Basic Properties of Confidence Intervals

- The basic concepts and properties of confidence intervals (CIs) are most easily introduced by first focusing on a simple, albeit somewhat unrealistic, problem situation.

- Suppose that the parameter of interest is a population mean μ and that
  1) The population distribution is normal
  2) The value of the population standard deviation σ is known
- Normality of the population distribution is often a reasonable assumption.

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Basic Properties of Confid

Example 7.1

- Industrial engineers who specialize in ergonomics are concerned with designing workspace and worker-operated devices so as to achieve high productivity and comfort.
- The article "Studies on Ergonomically Designed Alphanumeric Keyboards" (Human Factors, 1985:175-187) reports on a study of preferred height for an experiment keyboard with large forearm-wrist support.
- A sample of $n = 31$ trained typists was selected, and the preferred height was determined for each typist.
- The resulting sample average preferred height was $\bar{x} = 80.0$ cm.
- Assuming that the preferred height is normally distributed with $\sigma = 20$ cm. (a value suggested by data in the article), obtain a confidence interval (interval of plausible values) for μ , the true average preferred height for the population of all experienced typists.

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Random Samples

Basic Propert

The rv's $X_1, X_2, \ldots, X_n$ are said to form a (simple) **random sample** of size n if

1. The X_i's are independent rv's.
2. Every X_i has the same probability distribution.

- The actual sample observations $x_1, x_2, \ldots, x_n$ are assumed to be the result of a **random sample** $X_1, \ldots, X_n$ from a normal distribution with mean value μ and standard deviation σ.
- The results described in Chapter 5 then imply that, irrespective of the sample size n, the sample mean $\bar{X}$ is normally distributed with expected value μ and standard deviation $\sigma/\sqrt{n}$.
- Standardizing $\bar{X}$ by first subtracting its expected value and then dividing by its standard deviation yield the standard normal variable

5.4 The Distribution of the Sample Mean

Let $X_1, X_2, \ldots, X_n$ be a random sample from a distribution with mean value μ and standard deviation σ. Then

1. $E(\bar{X}) = \mu_{\bar{X}} = \mu$
2. $V(\bar{X}) = \sigma^2_{\bar{X}} = \sigma^2/n$ and $\sigma_{\bar{X}} = \sigma/\sqrt{n}$

In addition, with $T_o = X_1 + \cdots + X_n$ (the sample total), $E(T_o) = n\mu$, $V(T_o) = n\sigma^2$, and $\sigma_{T_o} = \sqrt{n}\sigma$.

$$Z = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \qquad \textbf{(7.1)}$$

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Basic Properties of Confidence Intervals

- Because the area under the standard normal curve between –1.96 and 1.96 is 0.95, ones.

$$P\left(-1.96 < \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} < 1.96\right) = .95 \qquad \textbf{(7.2)}$$

- Now let's manipulate the inequalities inside the parentheses in (7.2) so that they appear in the equivalent form $l < \mu < u$, where the endpoints l and u involve $\bar{X}$ and $\sigma/\sqrt{n}$.
- This is achieved through the following sequence of operations, each yielding inequalities equivalent to the original ones.

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Basic Properties of Confidence Intervals

1. Multiply through by $\sigma/\sqrt{n}$

$$-1.96 \cdot \frac{\sigma}{\sqrt{n}} < \overline{X} - \mu < 1.96 \cdot \frac{\sigma}{\sqrt{n}}$$

2. Subtract $\overline{X}$ from each term:

$$-\overline{X} - 1.96 \cdot \frac{\sigma}{\sqrt{n}} < -\mu < -\overline{X} + 1.96 \cdot \frac{\sigma}{\sqrt{n}}$$

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Basic Properties of Confidence Intervals

3. Multiply through by –1 to eliminate the minus sign in front of μ (which reverses the direction of each inequality):

$$\overline{X} + 1.96 \cdot \frac{\sigma}{\sqrt{n}} > \mu > \overline{X} - 1.96 \cdot \frac{\sigma}{\sqrt{n}}$$

that is,

$$\overline{X} - 1.96 \cdot \frac{\sigma}{\sqrt{n}} < \mu < \overline{X} + 1.96 \cdot \frac{\sigma}{\sqrt{n}}$$

The equivalence of each set of inequalities to the original set implies that

$$P\left(\overline{X} - 1.96 \frac{\sigma}{\sqrt{n}} < \mu < \overline{X} + 1.96 \frac{\sigma}{\sqrt{n}}\right) = .95 \qquad \textbf{(7.3)}$$

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Basic Properties of Confidence Intervals

The event inside the parentheses in (7.3) has a somewhat unfamiliar appearance; previously, the random quantity has appeared in the middle with constants on both ends, as in $a \le Y \le b$.

In (7.3) the random quantity appears on the two ends, whereas the unknown constant μ appears in the middle.

To interpret (7.3), think of a **random interval** having left endpoint $\bar{X} - 1.96 \cdot \sigma/\sqrt{n}$ and right endpoint $\bar{X} + 1.96 \cdot \sigma/\sqrt{n}$.
In interval notation, this becomes

$$\left(\bar{X} - 1.96 \cdot \frac{\sigma}{\sqrt{n}},\quad \bar{X} + 1.96 \cdot \frac{\sigma}{\sqrt{n}}\right) \qquad \textbf{(7.4)}$$

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$$\left(\bar{X} - 1.96 \cdot \frac{\sigma}{\sqrt{n}},\quad \bar{X} + 1.96 \cdot \frac{\sigma}{\sqrt{n}}\right) \qquad \textbf{(7.4)}$$

Basic Properties of Confidence Intervals

- The interval (7.4) is random because the two endpoints of the interval involve a random variable.
- It is centered at sample mean $\bar{X}$ and extends $1.96 \cdot \sigma/\sqrt{n}$ to each side of $\bar{X}$.
- Thus the interval's width is $2 \cdot (1.96) \cdot \sigma/\sqrt{n}$, a fixed number; only the location of the interval (its midpoint $\bar{X}$) is random (Figure 7.2).

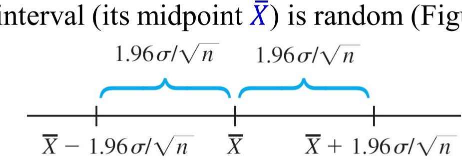

Figure 7.2 The random interval (7.4) centered at $\bar{X}$

- Now (7.3) can be paraphrased as "*the probability is 0.95 that the random interval* (7.4) *includes or covers the true value of* μ."
- Before any data is gathered, it is quite likely that μ will lie inside the interval (7.4).

$$P\left(\bar{X} - 1.96\frac{\sigma}{\sqrt{n}} < \mu < \bar{X} + 1.96\frac{\sigma}{\sqrt{n}}\right) = .95 \qquad \textbf{(7.3)}$$

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Basic Properties of Confidence Intervals

Definition

- If, after observing $X_1 = x_1, X_2 = x_2, \dots, X_n = x_n$, we compute the observed sample mean $\bar{x}$ and then substitute $\bar{x}$ into (7.4) in place of $\bar{X}$, the resulting fixed interval is called a **95% confidence interval for** μ.
- This CI can be expressed either as

$$\left(\bar{x} - 1.96 \cdot \frac{\sigma}{\sqrt{n}}, \bar{x} + 1.96 \cdot \frac{\sigma}{\sqrt{n}}\right) \text{ is a 95\% CI for } \mu$$

or as

$$\bar{x} - 1.96 \cdot \frac{\sigma}{\sqrt{n}} < \mu < \bar{x} + 1.96 \cdot \frac{\sigma}{\sqrt{n}} \text{ with 95\% confidence}$$

- A concise expression for the interval is $\bar{x} \pm 1.96 \cdot \sigma/\sqrt{n}$, where – gives the left endpoint (lower limit) and + gives the right endpoint (upper limit).

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Example 7.2

- The quantities needed for computation of the 95% CI for true average preferred height are $\sigma = 2.0, n = 31$, and $\bar{x} = 80.0$.
- The resulting interval is

$$\bar{x} \pm 1.96 \cdot \frac{\sigma}{\sqrt{n}} = 80.0 \pm (1.96)\frac{2.0}{\sqrt{31}} = 80.0 \pm .7 = (79.3, 80.7)$$

- That is, we can be highly confident, at the 95% confidence level, that 79.3 < ⍰ < 80.7.

- This interval is relatively narrow, indicating that μ has been rather precisely estimated.

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Interpreting a Confidence Level

$$\left(\overline{X} - 1.96 \cdot \frac{\sigma}{\sqrt{n}},\ \ \overline{X} + 1.96 \cdot \frac{\sigma}{\sqrt{n}}\right) \qquad \textbf{(7.4)}$$

Interpreting a Confidence Level

- The confidence level 95% for the interval just defined was inherited from the probability 0.95 for the random interval (7.4).
- Intervals having other levels of confidence will be introduced shortly.
- For now, though, consider how 95% confidence can be interpreted.
- We started with an event whose probability was 0.95—that the random interval (7.4) would capture the true value of μ—and then used the data in Example 7.1 to compute the CI (79.3, 80.7).
- It is therefore tempting to conclude that μ is within this fixed interval with probability 0.95.

$$\left(\overline{X} - 1.96 \cdot \frac{\sigma}{\sqrt{n}},\ \ \overline{X} + 1.96 \cdot \frac{\sigma}{\sqrt{n}}\right) \qquad \textbf{(7.4)}$$

Interpreting a Confidence Level

- But by substituting $\bar{x} = 80.0$ for $\bar{X}$, all randomness disappears; the interval (79.3, 80.7) is not a random interval, and μ is a constant (unfortunately unknown to us).
- Thus it is *incorrect* to write the statement $P(\mu \textit{ lies in } (79.3, 80.7)) = .95$.

- A correct interpretation of "95% confidence" relies on the long-run relative frequency interpretation of probability :
- To say that an event A has probability 0.95 is to say that if the experiment on which A is defined is performed over and over again, in the long run A will occur 95% of the time.

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For example, let A be the event that a package sent within the state of California for 2nd day delivery actually arrives within one day. The results from sending 10 such packages (the first 10 replications) are as follows:

Package #	1	2	3	4	5	6	7	8	9	10
Did A occur?	N	Y	Y	Y	N	N	Y	Y	N	N
Relative frequency of A	0	.5	.667	.75	.6	.5	.571	.625	.556	.5

the ratio $n(A)/n$ is called the *relative frequency* of occurrence of the event A in the sequence of n replications.

Interpreting Probability

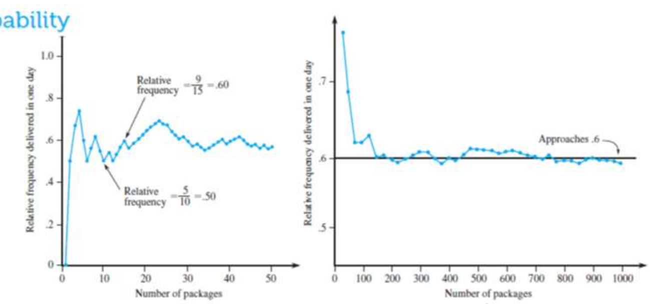

Figure 2.2 Behavior of relative frequency (a) Initial fluctuation (b) Long-run stabilization

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Interpreting a Confidence Level

- Suppose we obtain another sample of typists' preferred heights and compute another 95% interval.
- Now consider repeating this for a third sample, a fourth sample, a fifth sample, and so on.

Let A be the event that

$$\bar{X} - 1.96 \cdot \frac{\sigma}{\sqrt{n}} < \mu < \bar{X} + 1.96 \cdot \frac{\sigma}{\sqrt{n}}$$

Since $P(A) = .95$, in the long run 95% of our computed CIs will contain μ.

Interpreting a Confidence Level

- This is illustrated in Figure 7.3, where the vertical line cuts the measurement axis at the true (but unknown) value of μ.

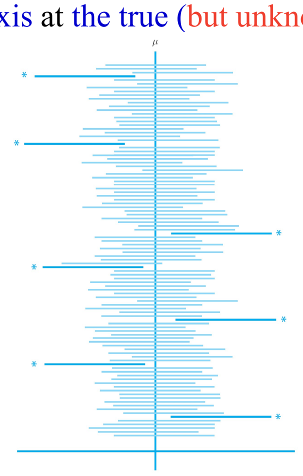

Figure 7.3 One hundred 95% CIs (asterisks identify intervals that do not include μ).

Interpreting a Confidence Level

- Notice that 7 of the 100 intervals shown fail to contain μ. In the long run, only 5% of the intervals so constructed would fail to contain μ.
- According to this interpretation, the confidence level 95% is not so much a statement about any particular interval such as (79.3, 80.7).
- Instead it pertains to what would happen if a very large number of like intervals were to be constructed using the same CI formula.

Other Levels of Confidence

Other Le

$$P\left(-1.96 < \frac{\overline{X} - \mu}{\sigma/\sqrt{n}} < 1.96\right) = .95 \quad \textbf{(7.2)}$$

- The confidence level of 95% was inherited from the probability .95 for the initial inequalities in (7.2).
- If a confidence level of 99% is desired, the initial probability of .95 must be replaced by .99, which necessitates changing the z critical value from 1.96 to 2.58.
- A 99% CI then results from using 2.58 in place of 1.96 in the formula for the 95% CI.
- In fact, any desired level of confidence can be achieved by replacing 1.96 or 2.58 with the appropriate standard normal critical value.

Other Levels of Confidence

- Recall from chapter 4 the notation for a z critical value: z_{α} is the number on the horizontal z scale that captures **upper tail area α.**

Other Levels of Confidence

- As Figure 7.4 shows, a probability (i.e., central z curve area) of $1 - \alpha$ is achieved by using $z_{\alpha/2}$ in place of 1.96.

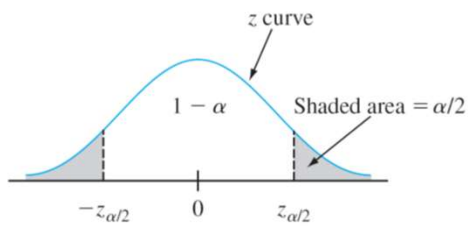

Figure 7.4 $P(-z_{\alpha/2} \le Z < z_{\alpha/2}) = 1 - \alpha$

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Other Levels of Confidence

$\bar{x} \pm z_{\alpha/2} \cdot \sigma/\sqrt{n}.$

Definition

A **100(1 – α)% confidence interval** for the mean μ of a normal population when the value of σ is known is given by

$$\left(\bar{x} - z_{\alpha/2} \cdot \frac{\sigma}{\sqrt{n}}, \bar{x} + z_{\alpha/2} \cdot \frac{\sigma}{\sqrt{n}}\right) \qquad \textbf{(7.5)}$$

or, equivalently, by $\bar{x} \pm z_{\alpha/2} \cdot \sigma/\sqrt{n}$

The formula (7.5) for the CI can also be expressed in words as point estimate of

$\mu \pm (z\ critical\ value)\ (standard\ error\ of\ the\ mean).$

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Example 7.3

- The production process for engine control housing units of a particular type has recently been modified.
- Prior to this modification, historical data had suggested that the distribution of hole diameters for bushings on the housings was normal with a standard deviation of 0.100 mm.
- It is believed that the modification has not affected the shape of the distribution or the standard deviation, but that the value of the mean diameter may have changed.
- A sample of 40 housing units is selected and hole diameter is determined for each one, resulting in a sample mean diameter of 5.426 mm.
- Let's calculate a confidence interval for true average hole diameter using a confidence level of 90%.

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Example 7.3

cont'd

- This requires that $100(1 - \alpha) = 90$, from which $\alpha = 0.10$ and $z_{\alpha/2} = z_{0.05} = 1.645$ (corresponding to a cumulative z-curve area of 0.9500).
- The desired interval is then

$$5.426 \pm (1.645)\frac{.100}{\sqrt{40}} = 5.426 \pm .026 = (5.400, 5.452)$$

- With a reasonably high degree of confidence, we can say that $5.400 < \mu < 5.452$.
- This interval is rather narrow because of the small amount of variability in hole diameter ($\sigma = 0.100$).

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