8.2 Tests About a Population Mean

Tests About a Population Mean

- Confidence intervals for a population mean μ focused on three different cases.

- We now develop test procedures for these cases.

  Case I : Normal Population with Known σ
  Case II : Large-Sample Tests
  Case III : Normal Population Distribution

Case I: A Normal Population with Known σ

Case I: A Normal Population with Known σ

- Although assumption that value of σ is known is rarely met in practice, this case provides a good starting point because of the ease with which general procedures and their properties can be developed.
- Null hypothesis in all three cases will state that μ has a particular numerical value, the ***null value***, which we will denote by μ_0 .
- Let $X_1, \ldots, X_n$ represent random sample of size ***n*** from normal population.

Case I: A Normal Population with Known σ

- Then sample mean $\overline{X}$ has a normal distribution with expected value $\mu_{\overline{X}} = \mu$ and standard deviation $\sigma_{\overline{X}} = \dfrac{\sigma}{\sqrt{n}}$
- When H_0 is true, $\mu_{\overline{X}} = \mu_0$
- Consider now the statistic Z obtained by standardizing $\overline{X}$ under assumption that H_0 is true:

$$Z = \frac{\overline{X} - \mu_0}{\sigma / \sqrt{n}}$$

- Substitution of computed sample mean $\overline{x}$ gives z, distance between $\overline{x}$ and μ_0 expressed in "standard deviation units."

66

Case I: A Normal Population with Known σ

- For example, if null hypothesis is

$$H_0 : \mu = 100 \qquad \sigma_{\overline{X}} = \frac{\sigma}{\sqrt{n}} = \frac{10}{\sqrt{25}} = 2.0 \qquad \text{and} \qquad \overline{x} = 103$$

then the test statistic value is

$$z = \frac{\overline{x} - \mu}{\sigma_{\overline{X}}} \Rightarrow z = \frac{\overline{x} - \mu}{\frac{\sigma}{\sqrt{n}}} \Rightarrow z = \frac{103 - 100}{\frac{10}{\sqrt{25}}} = 1.5$$

- That is, the observed value of $\overline{x}$ is 1.5 standard Deviations (of $\overline{X}$) larger than what we expect it to be when H_0 is true.

67

Case I: A Normal Population with Known σ

- Statistic Z is natural measure of distance between $\overline{X}$, estimator of μ, and its expected value when H_0 is true.
- If this distance is too great in a direction consistent with H_a, null hypothesis should be rejected.
- Suppose first that the alternative hypothesis has the form

$$H_a : \mu > \mu_0.$$

- Then an $\overline{x}$ value less than μ_0 certainly does not provide support for H_a.

$$z = \frac{\overline{x} - \mu}{\frac{\sigma}{\sqrt{n}}}$$

- Such an $\overline{x}$ corresponds to a negative value of z (since $\overline{x} - \mu_0$ is negative and the divisor $\sigma / \sqrt{n}$ is positive).

68

Case I: A Normal Population with Known σ

- Similarly, an $\overline{x}$ value that exceeds μ_0 by only a small amount (corresponding to z, which is positive but small) does not suggest that H_0 should be rejected in favor of H_a.
- Rejection of H_0 is appropriate only when $\overline{x}$ considerably exceeds μ_0—that is,
  when the ***z* value** is positive and **large**.
- In summary, appropriate rejection region, based on the test statistic Z rather than $\overline{X}$, has the form $z \geq c$.

69

Case I: A Normal Population with Known σ

- As we have discussed earlier, **cutoff value c** should be chosen to control probability of **type I error** at the **desired level α.**

- The required cutoff c is **z critical value** that captures upper-tail area α under the z curve.

Case I: A Normal Population with Known σ

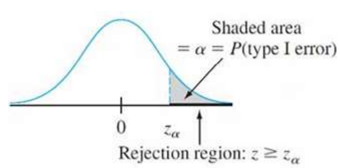
- As an example,
- let $c = 1.645$, value that captures tail area 0.05($z_{0.05} = 1.645$).

Table A.3 Standard Normal Curve Areas *(cont.)* $\Phi(z) = P(Z \le z)$

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545

Then,

$$\begin{aligned} \alpha &= P(\text{type I error}) \\ &= P(H_0 \text{ is rejected when } H_0 \text{ is true}) \\ &= P(Z \ge 1.645 \text{ when } Z \sim N(0,1)) \\ &= 1 - \Phi(1.645) = 1 - 0.95 = 0.05 \end{aligned}$$

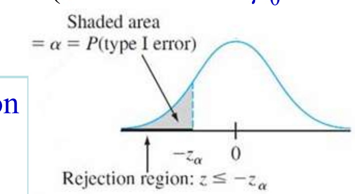


- More generally, rejection region $z \ge z_\alpha$ has type I error probability α.

- The test procedure is *upper-tailed* because rejection region consists only of large values of test statistic.

Case I: A Normal Population with Known σ

- Analogous reasoning for alternative hypothesis $H_a: \mu < \mu_0$ suggests a rejection region of the form $z \le c$, where c is a suitably chosen negative number ($\bar{x}$ is far below μ_0 if and only if z is quite negative).



- Because Z has standard normal distribution when H_0 is true, taking $c = -z_\alpha$ yields P(type I error) $= \alpha$.

This is a ***lower-tailed*** **test.**

- For example, $z_{0.10} = 1.28$ implies that the rejection region $z \le -1.28$ specifies a test with significance level 0.10.

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
−1.2	.1151	.1131	.1112	.1093	.1075	.1056	.1038	.1020	.1003	.0985

74

Case I: A Normal Population with Known σ

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
−1.2	.1151	.1131	.1112	.1093	.1075	.1056	.1038	.1020	.1003	.0985

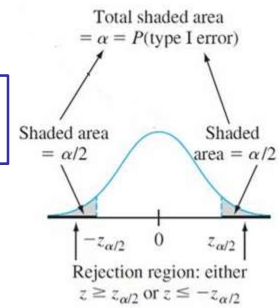
Case I: A Normal Population with Known σ

- Finally, when alternative hypothesis is $H_a: \mu \neq \mu_0$, H_0 should be rejected if $\bar{x}$ is too far to **either side of μ_0**.

- This is equivalent to rejecting H_0 either if $z \geq c$ or if $z \leq -c$.
- Suppose we desire $\alpha = 0.05$. Then,

$0.05 = P(Z \geq c$ or $Z \leq -c$ when Z has a standard normal distribution)

$$= \Phi(-c) + (1 - \Phi(c)) = 2[1 - \Phi(c)]$$



- Thus c is such that $1 - \Phi(c)$, area under the z curve to the right of c, is 0.025 (and not 0.05!).

76

Case I: A Normal Population with Known σ

- From Appendix Table A.3, $c = 1.96$, and the rejection region is $z \geq 1.96$ *or* $z \leq -1.96$.

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
−1.9	.0287	.0281	.0274	.0268	.0262	.0256	.0250	.0244	.0239	.0233
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767

- For any α, the *two-tailed* rejection region $z \geq z_{\alpha/2}$ *or* $z \leq -z_{\alpha/2}$ has type I error probability α (since area $\alpha/2$ is captured under each of two tails of the z curve).

- Again, key reason for using standardized test statistic Z is that because Z has a known distribution when H_0 is true (standard normal), rejection region with desired type I error probability is easily obtained by using appropriate critical value.

77

Case I: A Normal Population with Known σ

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
−1.9	.0287	.0281	.0274	.0268	.0262	.0256	.0250	.0244	.0239	.0233
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767

Case I: A Normal Population with Known σ

- **Test procedure** for **case I** is summarized in accompanying box, and corresponding rejection regions are illustrated in Figure 8.2.

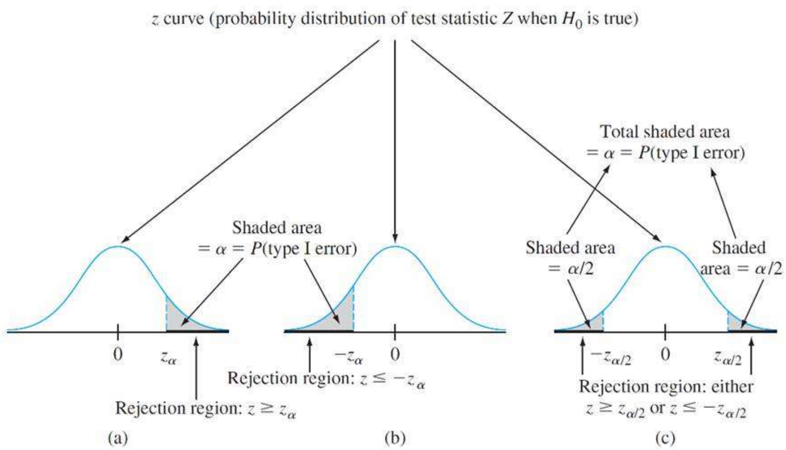


Figure 8.2 Rejection regions for ***z* tests**: (a) **upper-tailed test**; (b) **lower-tailed test**; (c) **two-tailed test**

Case I: A Normal Population with Known σ

Null hypothesis: $H_0: \mu = \mu_0$

Test statistic value : $z = \dfrac{\bar{x} - \mu_0}{\sigma/\sqrt{n}}$

Alternative Hypothesis	**Rejection Region for Level α Test**
$H_a: \mu > \mu_0$	$z \geq z_\alpha$ (upper-tailed test)
$H_a: \mu < \mu_0$	$z \leq -z_\alpha$ (lower-tailed test)
$H_a: \mu \neq \mu_0$	either $z \geq z_{\alpha/2}$ or $z \leq -z_{\alpha/2}$ (two-tailed test)

81

Case I: A Normal Population with Known σ

- Use of the following sequence of steps is recommended when testing hypotheses about a parameter.

1. Identify the parameter of interest and describe it in the context of the problem situation.
2. Determine the null value and state the null hypothesis.
3. State the appropriate alternative hypothesis.
4. Give the formula for the computed value of the test statistic (substituting the null value and the known values of any other parameters, but *not* those of any sample-based quantities).

82

Case I: A Normal Population with Known σ

5. State the rejection region for the selected significance level α.
6. Compute any necessary sample quantities, substitute into the formula for the test statistic value, and compute that value.
7. Decide whether H_0 should be rejected, and state this conclusion in the problem context.

- The formulation of hypotheses (Steps 2 and 3) should be done before examining the data.

83

Example 6

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- A manufacturer of sprinkler systems used for fire protection in office buildings claims that true average system-activation temperature is 130°.
- A sample of $n = 9$ systems, when tested, yields a sample average activation temperature of 131.08°F.
- If the distribution of activation times is normal with standard deviation 1.5°F,
  does the data contradict the manufacturer's claim at significance level $\alpha = 0.01$?

84

Example 6

cont'd

1. Parameter of interest: μ = true average activation temperature.
2. Null hypothesis: H_0: $\mu = 130$ (null value = $\mu_0 = 130$).
3. Alternative hypothesis: H_a: $\mu \neq 130$ (a departure from the claimed value in *either* direction is of concern).
4. Test statistic value:

$\mu_0 = 130$

$\sigma = 1.5$

$n = 9$

$$z = \frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}} = \frac{\bar{x} - 130}{1.5/\sqrt{n}}$$

85

Example 6

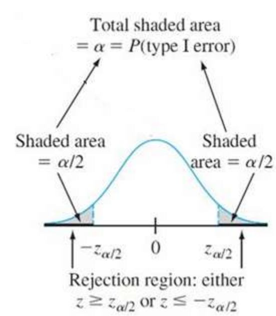
H_a: $\mu \neq 130$ either $z \geq z_{\alpha/2}$ or $z \leq -z_{\alpha/2}$ (two-tailed test)

cont'd

5. Rejection region: The form of H_a implies use of two-tailed test with rejection region *either* $z \geq z_{0.005}$ *or* $z \leq -z_{0.005}$.

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
−2.5	.0062	.0060	.0059	.0057	.0055	.0054	.0052	.0051	.0049	.0038
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952

- From Appendix Table A.3, $z_{0.005} = 2.58$,
  so we reject H_0
  if either $z \geq 2.58$ or $z \leq -2.58$



Example 6 H_a: $\mu \neq 130$ either $z \geq z_{\alpha/2}$ or $z \leq -z_{\alpha/2}$ (two-tailed test)

cont'd

6. Substituting $n = 9$ and $\bar{x} = 131.08$,

$$z = \frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}} = \frac{\bar{x} - 130}{1.5/\sqrt{n}} \Rightarrow z = \frac{131.08 - 130}{1.5/\sqrt{9}} = \frac{1.08}{0.5} = 2.16$$

- That is, the observed sample mean is a bit more than 2 standard deviations above what would have been expected were H_0 true.

7. The computed value $z = 2.16$ does not fall in rejection region ($-2.58 < 2.16 < 2.58$), so H_0 cannot be rejected at significance level 0.01.

Data does not give strong support to the claim that the true average differs from the design value of 130.

End of Section 8.2