2 Probability

Introduction

- Probability refers to study of randomness and uncertainty
- In any situation, one of a number of possible outcomes may occur.
- Probability provides methods for quantifying the chances, or likelihoods, associated with various outcomes
- Language of probability is constantly used in informal manner
  - It is likely that Dow-Jones average will increase by the end of this year
  - It is expected that at least 20,000 concert tickets will be sold
  - There is a 50-50 chance that I will pass probability and statistics subject

2.1 Sample Spaces and Events

Sample Spaces and Events

- **Experiment** is any activity or process whose outcome is subject to uncertainty.

- Although the word *experiment* generally suggests a planned or carefully controlled laboratory testing situation, we use it here in a much wider sense.

- Thus experiments that may be of interest include
  - tossing a coin once or several times,
  - selecting a card or cards from a deck,
  - ascertaining the commuting time from home to work on a particular morning,
  - obtaining blood types from a group of individuals

The Sample Space of an Experiment

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The Sample Space of an Experiment

- **Definition**
  - **Sample Space** of experiment is set of all possible outcomes of experiment ➡ denoted by $\mathcal{S}$

- ***Example 1 :***
  - One experiment consists of examining a single fuse to see whether it is defective.

- Sample space for this experiment can be shown as set of

$$\mathcal{S} = \{N, D\},$$

- where N represents not defective,
  D represents defective

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Example 1

cont'd

- **Example 2** : experiment involves tossing a thumbtack
- Sample space is the set of $\mathcal{S}$ = {U, D}
  where U : it landed point up
  D : it landed point down

- **Example 3** : experiment consist of observing gender of the next child born at the local hospital
- Sample space will be $\mathcal{S}$ = {M, F}
- Where M : Male
  F : Female

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Events

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Events

- In our study of probability, we will be interested not only in individual outcomes of $\mathcal{S}$ but also in various collections of outcomes from $\mathcal{S}$.

Definition

- **Event** is any collection (subset) of outcomes contained in sample space $\mathcal{S}$.

- **Event** is **simple** if it consists of exactly one outcome and **compound** if it consists of more than one outcome

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Events

- When an experiment is performed, a particular **event *A*** is said to occur if resulting experimental outcome is contained in *A*

- In general, exactly **one simple event** will occur, but many **compound events** will occur simultaneously

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Example 5

- Consider an experiment in which each of three vehicles taking a particular freeway exit turns left (*L*) or right (*R*) at the end of the exit ramp.

Eight possible outcomes

⇓

sample space

L L L	⇒ E_1
R L L	⇒ E_2
L R L	⇒ E_3
L L R	⇒ E_4
L R R	⇒ E_5
R L R	⇒ E_6
R R L	⇒ E_7
R R R	⇒ E_8

Eight simple events

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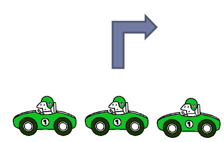
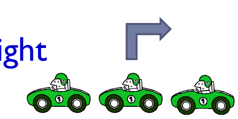
Example 5 cont'd

- Some **compound events** include

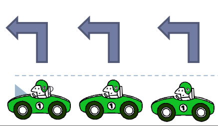
A = {*RLL*, *LRL*, *LLR*} = Event that exactly one of three vehicles turns right

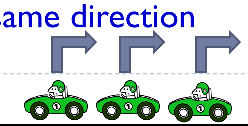


B = {*LLL*, *RLL*, *LRL*, *LLR*} = Event that at most one of vehicles turns right



C = {*LLL*, *RRR*} = Event that all three vehicles turn in same direction





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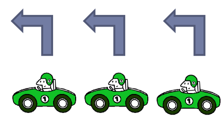
Example 5

cont'd

- Suppose that when experiment is performed, outcome is ***LLL***

- Then simple event E_1 has occurred and so also have events B and C (but not A).

A = {*RLL*, *LRL*, *LLR*}
B = {*LLL*, *RLL*, *LRL*, *LLR*}
C = {*LLL*, *RRR*}



L L L → E_1
R L L → E_2
L R L → E_3
L L R → E_4
L R R → E_5
R L R → E_6
R R L → E_7
R R R → E_8

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Some Relations from Set Theory

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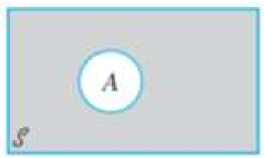
Some Relations from Set Theory

- **Event** is just a **Set**, so relationships and results from elementary **set theory** can be used to study events.
- The following operations will be used to create new events from given events.

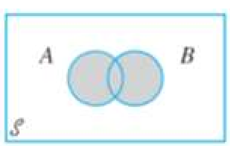
Definition

1. **Complement of an event *A***, denoted by ***A′***, is
   - set of all outcomes in $\mathcal{S}$ that are not contained in *A*.



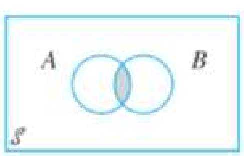
Some Relations from Set Theory

2. **Union** of two events *A* and *B*, denoted by ***A*** ∪ ***B*** and read "*A or B*,"



All outcomes are ***either in A or in B or in both events***

3. **Intersection** of two events *A* and *B*, denoted by ***A*** ∩ ***B*** and read "*A and B*,"

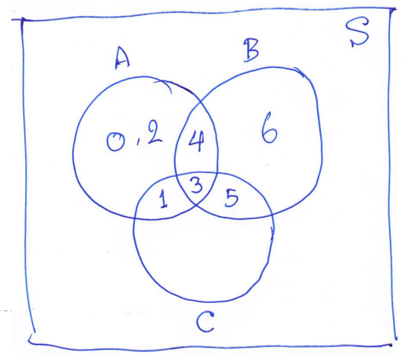


All outcomes that are **in *both A and B***.

Example 8

- For experiment in which the number of pumps in use at a single six-pump gas station is observed,
  let $A = \{0, 1, 2, 3, 4\}$,
  $B = \{3, 4, 5, 6\}$, and
  $C = \{1, 3, 5\}$.

Sample space = {0, 1, 2, 3, 4, 5, 6}



Then
$A' = \{5, 6\}$,
$A \cup B = \{0, 1, 2, 3, 4, 5, 6\} = \mathcal{S}$,
$A \cup C = \{0, 1, 2, 3, 4, 5\}$,
$A \cap B = \{3, 4\}$,
$A \cap C = \{1, 3\}$,
$(A \cap C)' = \{0, 2, 4, 5, 6\}$

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Some Relations from Set Theory

- Sometimes A and B have **no outcomes** in common, so that intersection of A and B contains no outcomes.

- **Definition**
- Let $\varnothing$ denote *null event* (the event consisting of no outcomes whatsoever).

- When $A \cap B = \varnothing$, A and B are said to be **mutually exclusive** or **disjoint** events.

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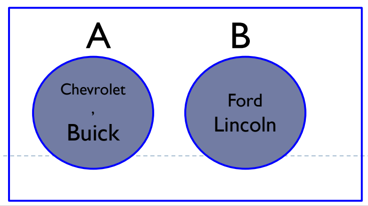
Example 10

- Small city has three automobile dealerships:
  - GM dealer selling Chevrolets and Buicks;
  - Ford dealer selling Fords and Lincolns; and
  - Toyota dealer.
- If experiment consists of observing brand of next car sold, then events

  A = {Chevrolet, Buick} and
  B = {Ford, Lincoln} are mutually exclusive
- because next car sold cannot be both GM product and Ford product (at least until the two companies merge!).



Some Relations from Set Theory

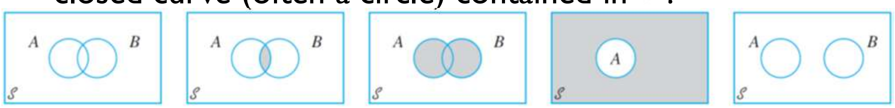
- Operations of union and intersection can be extended to more than two events.
- For any three events A, B, and C, the event

  $A \cup B \cup C$ is set of outcomes contained in at least one of three events

  $A \cap B \cap C$ is set of outcomes contained in all three events.
- Given events $A_1, A_2, A_3, \ldots,$ these events are said to be mutually exclusive (or pairwise disjoint) if no two events have any outcomes in common.

Some Relations from Set Theory

- Pictorial representation of events and manipulations with events is obtained by using **Venn Diagrams**.
- To construct a Venn diagram, draw a rectangle whose interior will represent the sample space $\mathcal{S}$.
- Then any event A is represented as the interior of a closed curve (often a circle) contained in $\mathcal{S}$.



(a) Venn diagram of events A and B (b) Shaded region is $A \cap B$ (c) Shaded region is $A \cup B$ (d) Shaded region is A' (e) Mutually exclusive events

Figure 2.1 shows examples of Venn diagrams.

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2.2 Axioms, Interpretations, and Properties of Probability

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Axioms, Interpretation, and Properties of Probability

- Given experiment and sample space $\mathcal{S}$,

  *objective of probability is to assign to each event A denoted by **P(A)**, called Pobability of Event A,*

  - *which will give a precise measure of the **chance** that **A** will **occur**.*

- To ensure that probability assignments will be consistent with our intuitive notions of probability, all assignments should satisfy the following axioms (basic properties) of probability.

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Axioms, Interpretation, and Properties of Probability

Axiom 1 (สัจพจน์ : คำกล่าวที่ถือว่าเป็นความจริงโดยไม่ต้องพิสูจน์)

For any event A, $P(A) \geq 0$.

chance of A occurring should be nonnegative.

Axiom 2

$P(\mathcal{S}) = 1$ ความน่าจะเป็นของ Sample Space เท่ากับ 1

sample space is by definition the event that must occur when the experiment is performed ($\mathcal{S}$ contains all possible outcomes), so Axiom 2 says that **maximum possible probability of 1** is assigned $\mathcal{S}$

Axiom 3

If $A_1, A_2, A_3, \ldots$ is an infinite collection of disjoint events, then

$$P(A_1 \cup A_2 \cup A_3 \cup \ldots) = \sum_{i=1}^{\infty} P(A_i)$$

Third axiom formalizes the idea that if we wish probability that at least one of a number of events will occur and no two of events can occur simultaneously, then chance of at least one occurring is sum of the chances of individual events.

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Axioms , Interpretation, and Properties of Probability

- **Proposition** (ประพจน์ : ข้อความที่จะต้องเป็นจริงหรือเท็จอย่างใดอย่างหนึ่ง)

$$P(\varnothing) = 0$$

- where $\varnothing$ is the null event (the event containing no outcomes whatsoever).

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Example 11

- Consider tossing a thumbtack in the air.
- When it comes to rest on the ground, either its point will be up (outcome U) or down (outcome D).

Sample space for this event is $\mathscr{S} = \{U, D\}$.

- Axioms specify $P(\mathscr{S}) = 1$, so probability assignment will be completed by determining $P(U)$ and $P(D)$.
- Since U and D are disjoint and their union is $\mathscr{S}$, the foregoing proposition implies that

$$1 = P(\mathscr{S}) = P(U) + P(D)$$

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Example 11

cont'd

- It follows that $P(D) = 1 - P(U)$.
- One possible assignment of probabilities is

$$P(U) = 0.5, P(D) = 0.5,$$

whereas another possible assignment is

$$P(U) = 0.75, P(D) = 0.25.$$

- In fact, letting p represent any fixed number between 0 and 1,
- $P(U) = p$,
- $P(D) = 1 - p$ is an assignment consistent with the axioms.

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Example 12

- Consider testing batteries coming off an assembly line one by one until one having a voltage within prescribed limits is found.

The simple events are

$E_1 = \{S\}$, $E_2 = \{FS\}$,

$E_3 = \{FFS\}$, $E_4 = \{FFFS\}, \ldots$

- Suppose the probability of any particular battery being satisfactory is 0.99.

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Example 12

cont'd

Then it can be shown that

$P(E_1) = 0.99$, $E_1 = \{S\}$,
$P(E_2) = (0.01)(0.99)$, $E_2 = \{FS\}$,
$P(E_3) = (0.01)^2(0.99)$, . . . is an assignment of probabilities to the simple events that satisfies the axioms.

- In particular, because the E_is are disjoint and
  $$\mathcal{S} = E_1 \cup E_2 \cup E_3 \cup \ldots,$$
  it must be the case that
  $$1 = P(S) = P(E_1) + P(E_2) + P(E_3) + \cdots$$
  $$= 0.99[1 + 0.01 + (0.01)^2 + (0.01)^3 + \cdots]$$

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$1 = 0.99[1 + 0.01 + (0.01)^2 + (0.01)^3 + \cdots]$

Example 12

cont'd

- Here we have used formula for sum of a geometric series:

$$a + ar + ar^2 + ar^3 + \cdots = \frac{a}{1 - r}$$

- However, another legitimate (according to the axioms) probability assignment of the same "geometric" type is obtained by replacing 0.99 by any other number p between 0 and 1 (and 0.01 by $1 - p$).

$$p = 0.99 \qquad a = p = 0.99$$
$$1 - p = 1 - 0.99 = 0.01 \qquad r = (1 - p) = 0.01$$

$$p + p(1-p) + p(1-p)^2 + p(1-p)^3 + \ldots = \frac{p}{(1-(1-p))} = 1$$

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Interpreting Probability

Interpreting Probability

- Examples 11 and 12 show that **axioms** do not completely **determine assignment of probabilities** to events.
- **Axioms** serve only to **rule** out assignments inconsistent with our intuitive notions of probability.
- In the tack-tossing experiment of Example 11, two particular assignments were suggested.

$$P(U) = 0.5, P(D) = 0.5$$
$$P(U) = 0.75, P(D) = 0.25$$

- Appropriate or correct assignment depends on the nature of the thumbtack and also on one's interpretation of probability.

Interpreting Probability

- Interpretation that is most frequently used and most easily understood is based on notion of **Relative Frequencies** *(ความถี่สัมพัทธ์)*.

- Consider experiment that can be repeatedly performed in an identical and independent fashion, and
- let ***A*** be an event consisting of fixed set of outcomes of the experiment.

- Simple examples of such repeatable experiments include the tack-tossing experiments previously discussed.

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Interpreting Probability

- If experiment is performed ***n*** times, on some of replications
  - event *A* will occur (outcome will be in set *A*), and on others,
  - *A* will not occur.

- Let ***n*(*A*)** denote number of replications on which *A* does **occur**.

- ***Relative Frequency*** *(ความถี่สัมพัทธ์)* of occurrence of event *A* in the sequence of *n* replications can be expressed as :

จำนวนครั้งที่เกิดเหตุการณ์ A ➡ $n(A)$

จำนวนครั้งของการทดลองทั้งหมด ➡ n

$$\frac{n(A)}{n}$$

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Interpreting Probability

- For example, let *A* be event that package sent within the state of California for 2nd day delivery actually arrives within one day.

- Results from sending 10 such packages (the first 10 replications) are as follows:

Package #	1	2	3	4	5	6	7	8	9	10
Did *A* occur?	N	Y	Y	Y	N	N	Y	Y	N	N
Relative frequency of *A*	0	.5	.667	.75	.6	.5	.571	.625	.556	.5
$\frac{n(A)}{n}$ →	$\frac{0}{1}$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{3}{4}$	$\frac{3}{5}$	$\frac{3}{6}$	$\frac{4}{7}$	$\frac{5}{8}$	$\frac{5}{9}$	$\frac{5}{10}$

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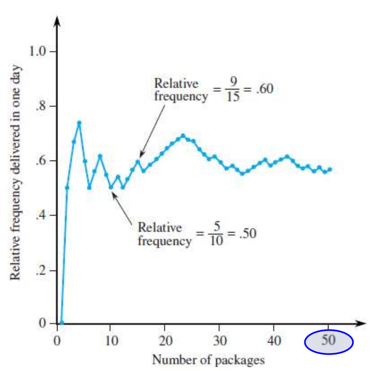
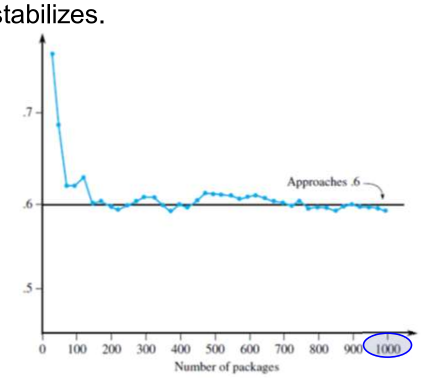
Interpreting Probability

- Figure 2.2(a) shows how the relative frequency *n*(*A*)/*n* fluctuates rather substantially over the course of first 50

But as number of replications continues to increase, Figure 2.2(b) illustrates how relative frequency stabilizes.



Behavior of relative frequency (a) Initial fluctuation



Behavior of relative frequency (b) Long-run stabilization

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Figure 2.2

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Interpreting Probability

- More generally, empirical evidence, based on results of many such repeatable experiments, indicates that any relative frequency of this sort will stabilize as number of replications *n* increases.

- That is, as *n* gets arbitrarily large,
  - ***n*(*A*)/*n*** approaches **limiting value** referred to as

***Limiting (or long-run) relative frequency* of event *A*.**
(ขอบเขตความถี่สัมพัทธ์ของเหตุการณ์ A)

- ***Objective interpretation of probability*** identifies this limiting relative frequency with ***P*(*A*).**

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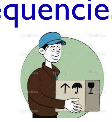
Interpreting Probability

- Suppose that probabilities are assigned to events in accordance with their limiting relative frequencies.

- Then a statement such as



"Probability of package being delivered within one day of mailing is **0.6**"

means

"a large number of mailed packages, roughly 60% will arrive within one day"

- Similarly, if *B* is event that appliance of particular type will need service while under warranty, then $P(B) = 0.1$ is interpreted to mean that in long run 10% of such appliances will need warranty service.

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Interpreting Probability

- When we speak of a fair coin, we shall mean

$$P(H) = P(T) = 0.5,$$

- and a fair die is one for which limiting relative frequencies of six outcomes are all $\frac{1}{6}$, suggesting probability assignments

$$P(\{1\}) = \cdot\cdot\cdot = P(\{6\}) = \frac{1}{6}.$$

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More Probability Properties

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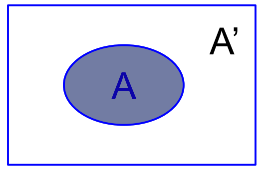
More Probability Properties

Proposition

- For any event A,

$$P(A) + P(A') = 1$$

$$P(A) = 1 - P(A')$$



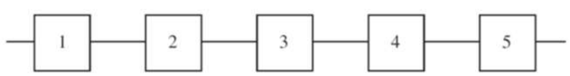
○ In general, this proposition is useful when event of interest can be expressed as "at least . . . ," since then the complement "less than . . ." may be easier to work with

(in some problems, "more than . . ." is easier to deal with than "at most . . .").

○ When you are having difficulty calculating $P(A)$ directly, think of determining $P(A')$

Example 13

- Consider a system of five identical components connected in series, as illustrated in Figure 2.3.



A system of five components connected in a series

Figure 2.3

- Denote a component that fails by F and one that doesn't fail by S (for success).

- Let A be event that *system* fails.
- For A to occur, at least one of the individual components must fail.

Example 13

cont'd

- Outcomes in *A* include *SSFSS* (1, 2, 4, and 5 all work, but 3 does not), *FFSSS*, and so on.
- There are in fact 31 different outcomes in *A*.
- However, A', event that system works, consists of single outcome *SSSSS*.
- We will see in Section 2.5 that if 90% of all components do not fail and different components fail independently of one another, then

$$P(A') = P(SSSSS) = 0.9^5 = 0.59.$$

- Thus $P(A) = 1 - 0.59 = 0.41$; so among a large number of such systems, roughly 41% will fail.

FFFFF
FFFFS
FFFSF
FFFSS
FFSFF
FFSFS
FFSSF
FFSSS
FSFFF
FSFFS
FSFSF
FSFSS
FSSFF
FSSFS
FSSSF
FSSSS
SFFFF
SFFFS
SFFSF
SFFSS
SFSFF
SFSFS
SFSSF
SFSSS
SSFFF
SSFFS
SSFSF
SSFSS
SSSFF
SSSFS
SSSSF
A

SSSSS A'

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More Probability Properties

Proposition

- For any event *A*, $P(A) \leq 1$.
- This is because

$$1 = P(A) + P(A') \geq P(A); \quad \text{since } P(A') \geq 0$$

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More Probability Properties

- When events A and B are mutually exclusive,

$$P(A \cup B) = P(A) + P(B).$$

- For events that are not mutually exclusive, adding $P(A)$ and $P(B)$ results in "doublecounting" outcomes in intersection. The next result shows how to correct for this.

Proposition

- For any two events A and B,

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

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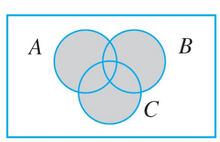
More Probability Properties

- Probability of a union of more than two events can be computed analogously.

For any three events A, B, and C,

$$\begin{aligned} P(A \cup B \cup C) = P(A) + P(B) + P(C) \\ - P(A \cap B) - P(A \cap C) - P(B \cap C) \\ + P(A \cap B \cap C) \end{aligned}$$

This can be verified by examining a Venn diagram of $A \cup B \cup C$, which is shown in Figure 2.6.



$A \cup B \cup C$

Figure 2.6

When $P(A)$, $P(B)$, and $P(C)$ are added, certain intersections are counted twice, so they must be subtracted out, but this results in $P(A \cap B \cap C)$ being subtracted once too often.

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Determining Probabilities Systematically

(การพิจารณาความน่าจะเป็นอย่างเป็นระบบ)

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Determining Probabilities Systematically

- Consider a **sample space** that is either
  - **finite** or
  - **“countably infinite”** (the latter means that outcomes can be listed in an infinite sequence, so there is
    - a first outcome, a second outcome, a third outcome, and so on
- For example, the battery testing scenario of Example 12)
- Let E_1, E_2, E_3, … denote the corresponding simple events, each consisting of a single outcome.

$E_1 = \{S\}$,
$E_2 = \{FS\}$,
$E_3 = \{FFS\}$,
$E_4 = \{FFFS\}$,
. . . .

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Determining Probabilities Systematically

- Sensible strategy for probability computation is to first determine each simple event probability, with the requirement that

$$\sum P(E_i) = 1$$

- Then probability of any compound event A is computed by adding together the $P(E_i)$'s for all E_i's in A:

$$P(A) = \sum_{all\ E_i\text{'s in } A} P(E_i)$$

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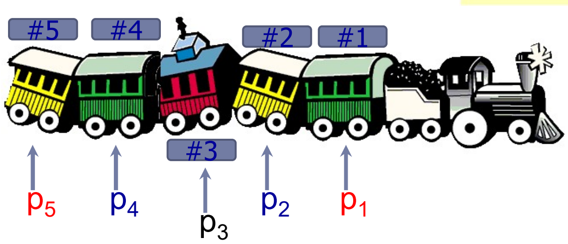
Example 15

- During off-peak hours a commuter train has five cars.
- Suppose a commuter is

twice as likely to select the middle car (#3) as to select either adjacent car (#2 or #4), and is $p_3 = 2p_2 = 2p_4$

twice as likely to select either adjacent car as to select either end car (#1 or #5). $p_2 = p_4 = 2p_1 = 2p_5$



- Let $p_i = P(\text{car } i \text{ is selected}) = P(E_i)$.

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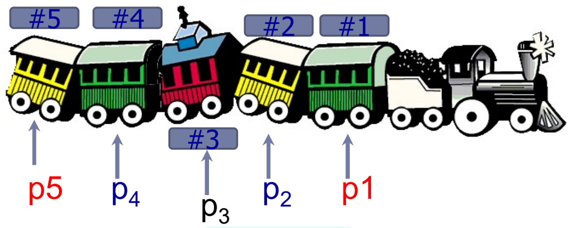
Example 15 (cont.)



Then we have

$$p_3 = 2p_2 = 2p_4$$

$$p_2 = p_4 = 2p_1 = 2p_5$$

$$p_1 = p_5$$

- This gives $\sum_{i=1}^{5} P(E_i) = 1$ ⇨ $P(E_1) + P(E_2) + P(E_3) + P(E_4) + P(E_5) = 1$

$\sum P(E_i) = 1$

$$p_1 + p_2 + p_3 + p_4 + p_5 = 1$$

แทนค่าสมการให้อยู่ในเทอมของ p_1 $p_1 + 2p_1 + 4p_1 + 2p_1 + p_1 = 1$ ⇨ $10p_1 = 1$

$$p_1 = 0.1$$

- implying $p_1 = p_5 = 0.1$, $p_2 = p_4 = 0.2$, $p_3 = 0.4$.
- Probability that one of the three middle cars is selected (a compound event) is then $p_2 + p_3 + p_4 = 0.8$. $P(A) = \sum_{\text{all } E_i\text{'s in } A} P(E_i)$

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Equally Likely Outcomes

(ผลลัพธ์ที่มีโอกาสเกิดอย่างความเท่าเทียมกัน)

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Equally Likely Outcomes

- In many experiments consisting of N outcomes, it is reasonable to assign equal probabilities to **all N** simple events.
- Example :    
  - Tossing a fair coin or fair die once or twice (or any fixed number of times), or
  - Selecting one or several cards from a well-shuffled deck of 52.
  - With $p = P(E_i)$ for every i,

$$\sum_{i=1}^{N} P(E_i) = 1 \Rightarrow \sum_{i=1}^{N} p = N \times p = 1 \Rightarrow N \times p = 1 \Rightarrow p = \frac{1}{N}$$

- That is, if there are N equally likely outcomes, the probability for each is $1/N$.

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Equally Likely Outcomes

- Now consider event A, with $N(A)$ denoting the number of outcomes contained in A. Then

Probability ของแต่ละ Event

จำนวน outcome ใน Event A

$$P(A) = \sum_{E_i \text{ in } A} P(E_i) = \sum_{E_i \text{ in } A} \frac{1}{N} = \frac{N(A)}{N}$$

จำนวน outcome ใน Sample Space

- Thus when outcomes are equally likely, computing probabilities reduces to counting: determine both
  - the number of outcomes $N(A)$ in A and
  - the number of outcomes N in $\mathcal{S}$,
- and form their ratio.

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Example 16

- You have **six** unread **mysteries** on your bookshelf and **six** unread **science fiction** books.
  - **First three** of each type are **hardcover**, and
  - **Last three** are **paperback**.
- Consider
  - randomly selecting **one** of the six mysteries and then
  - randomly selecting **one** of the six science fiction books
- Number the mysteries 1, 2, 3, 4, 5, and , 6, and do
- the same for the science fiction books.

Example 16

cont'd

- Then each outcome is a pair of numbers
- There are $N = 36$ possible outcomes (Sample Space)

Science fiction books

Mysteries

	1	2	3	4	5	6
1	(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)
2	(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
3	(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)
4	(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)
5	(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(5,6)
6	(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)

- With random selection as described, the 36 outcomes are equally likely

$$P(E_{ij}) = \frac{1}{36}$$

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Example 16

cont'd

- Suppose : Event A are **both selected books** are **paperbacks**
- We have nine outcomes

Science fiction books

Mysteries

	1	2	3	4	5	6
1	(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)
2	(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
3	(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)
4	(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)
5	(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(5,6)
6	(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)

- So probability of the event *A* that both selected books are paperbacks is

$$P(A) = \frac{N(A)}{N} = \frac{9}{36} = 0.25$$

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