8 Tests of Hypotheses Based on a Single Sample

8.2 Tests About a Population Mean

Tests About a Population Mean

Confidence intervals for a population mean μ focused on three different cases.

We now develop test procedures for these cases.

Case I: A Normal Population with Known σ

Case I: A Normal Population with Known σ

Although the assumption that the value of σ is known is rarely met in practice, this case provides a good starting point because of the ease with which general procedures and their properties can be developed.

The null hypothesis in all three cases will state that μ has a particular numerical value, the *null value*, which we will denote by μ_0. Let $X_1, \ldots, X_n$ represent a random sample of size *n* from the normal population.

Case I: A Normal Population with Known σ

Then the sample mean $\overline{X}$ has a normal distribution with expected value $\mu_{\overline{X}} = \mu$ and standard deviation $\sigma_{\overline{X}} = \sigma/\sqrt{n}$.

When H_0 is true, $\mu_{\overline{X}} = \mu_0$. Consider now the statistic Z obtained by standardizing $\overline{X}$ under the assumption that H_0 is true:

$$Z = \frac{\overline{X} - \mu_0}{\sigma/\sqrt{n}}$$

[illegible]

Case I: A Normal Population with Known σ

Substitution of the computed sample mean $\bar{x}$ gives z, the distance between $\bar{x}$ and μ_0 expressed in "standard deviation units."

For example, if the null hypothesis is

H_0: $\mu = 100$, $\sigma_{\bar{X}} = \sigma/\sqrt{n} = 10/\sqrt{25} = 2.0$, and $\bar{x} = 103$, then the test statistic value is $z = (103 - 100)/2.0 = 1.5$.

$$\frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}}$$

That is, the observed value of $\bar{x}$ is 1.5 standard Deviations (of $\bar{X}$) larger than what we expect it to be when H_0 is true.

Case I: A Normal Population with Known σ

The statistic Z is a natural measure of the distance between $\overline{X}$, the estimator of μ, and its expected value when H_0 is true. If this distance is too great in a direction consistent with H_a, the null hypothesis should be rejected.

Suppose first that the alternative hypothesis has the form $H_a : \mu > \mu_0$. Then an $\bar{x}$ value less than μ_0 certainly does not provide support for H_a.

Such an $\bar{x}$ corresponds to a negative value of z (since $\bar{x} - \mu_0$ is negative and the divisor $\sigma/\sqrt{n}$ is positive).

Case I: A Normal Population with Known σ

Similarly, an $\bar{x}$ value that exceeds μ_0 by only a small amount (corresponding to *z*, which is positive but small) does not suggest that H_0 should be rejected in favor of H_a.

The rejection of H_0 is appropriate only when $\bar{x}$ considerably exceeds μ_0—that is, when the *z* value is positive and large. In summary, the appropriate rejection region, based on the test statistic *Z* rather than $\bar{X}$, has the form $z \geq c$.

Case I: A Normal Population with Known σ

As we have discussed earlier, the cutoff value *c* should be chosen to control the probability of a type I error at the desired level α.

This is easily accomplished because the distribution of the test statistic *Z* when H_0 is true is the standard normal distribution (that's why μ_0 was subtracted in standardizing).

The required cutoff *c* is the *z* critical value that captures upper-tail area α under the *z* curve.

Case I: A Normal Population with Known σ

As an example, let c = 1.645, the value that captures tail area .05($z_{.05}$ = 1.645). Then,

α = P(type I error)

= P(H_0 is rejected when H_0 is true)

=P($Z \geq$ 1.645 when $Z \sim N$(0,1))

= 1 – Φ(1.645) =.05

More generally, the rejection region $z \geq z_\alpha$ has type I error probability α.

The test procedure is *upper-tailed* because the rejection region consists only of large values of the test statistic.

Case I: A Normal Population with Known σ

Analogous reasoning for the alternative hypothesis H_a: $\mu < \mu_0$ suggests a rejection region of the form $z \leq c$, where c is a suitably chosen negative number ($\bar{x}$ is far below μ_0 if and only if z is quite negative).

Because Z has a standard normal distribution when H_0 is true, taking $c = -z_\alpha$ yields P(type I error) = α.

This is a *lower-tailed* test. For example, $z_{.10} = 1.28$ implies that the rejection region $z \leq -1.28$ specifies a test with significance level .10.

Case I: A Normal Population with Known σ

Finally, when the alternative hypothesis is H_a: $\mu \neq \mu_0$, H_0 should be rejected if $\bar{x}$ is too far to either side of μ_0. This is equivalent to rejecting H_0 either if $z \geq c$ or if $z \leq -c$. Suppose we desire $\alpha = .05$. Then,

$$.05 = P(Z \geq c \text{ or } Z \leq -c \quad \text{when } Z \text{ has a standard normal distribution})$$

$$= \Phi(-c) + 1 - \Phi(c) = 2[1 - \Phi(c)]$$

Case I: A Normal Population with Known σ

Thus *c* is such that $1 - \Phi(c)$, the area under the *z* curve to the right of *c*, is .025 (and not .05!).

From Appendix Table A.3, *c* = 1.96 , and the rejection region is $z \geq 1.96$ *or* $z \leq -1.96$.

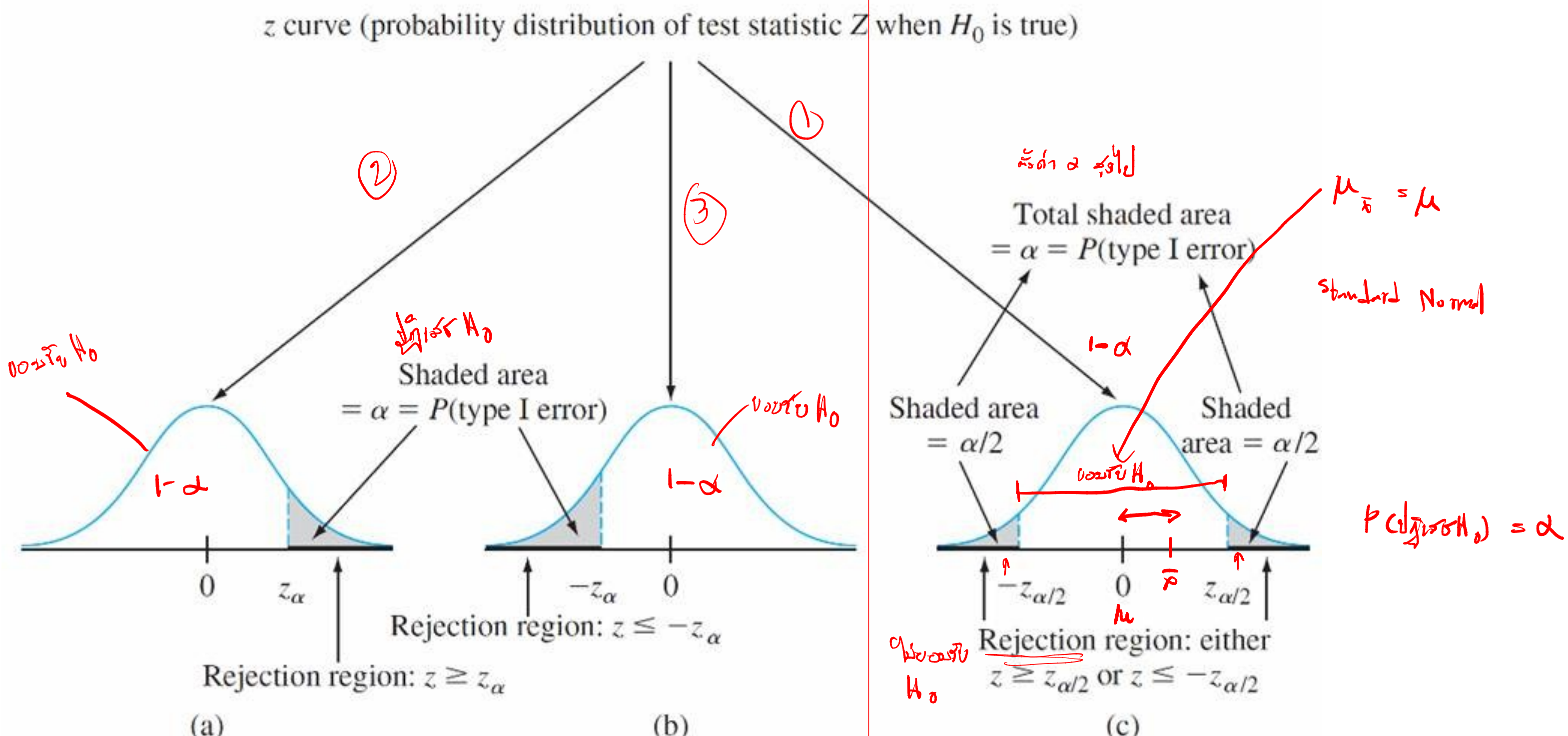
For any α, the *two-tailed* rejection region $z \geq z_{\alpha/2}$ *or* $z \leq -z_{\alpha/2}$ has type I error probability α (since area $\alpha/2$ is captured under each of the two tails of the *z* curve).

Case I: A Normal Population with Known σ

Again, the key reason for using the standardized test statistic Z is that because Z has a known distribution when H_0 is true (standard normal), a rejection region with desired type I error probability is easily obtained by using an appropriate critical value.

Case I: A Normal Population with Known σ

The test procedure for case I is summarized in the accompanying box, and the corresponding rejection regions are illustrated in Figure 8.2.



Rejection regions for *z* tests: (a) upper-tailed test; (b) lower-tailed test; (c) two-tailed test

Figure 8.2

Case I: A Normal Population with Known σ

Null hypothesis: $H_0 : \mu = \mu_0$

Test statistic value : $z = \dfrac{\bar{x} - \mu_0}{\sigma/\sqrt{n}}$

Alternative Hypothesis	**Rejection Region for Level α Test**
$H_a\colon \mu > \mu_0$	$z \geq z_\alpha$ (upper-tailed test)
$H_a\colon \mu < \mu_0$	$z \leq -z_\alpha$ (lower-tailed test)
$H_a\colon \mu \neq \mu_0$ either $z \geq z_{\alpha/2}$ or $z \leq -z_{\alpha/2}$ (two-tailed test)	either $z \geq z_{\alpha/2}$ or $z \leq -z_{\alpha/2}$ (two-tailed test)

Case I: A Normal Population with Known σ

Use of the following sequence of steps is recommended when testing hypotheses about a parameter.

1. Identify the parameter of interest and describe it in the context of the problem situation.

2. Determine the null value and state the null hypothesis.

3. State the appropriate alternative hypothesis.

Case I: A Normal Population with Known σ

4. Give the formula for the computed value of the test statistic (substituting the null value and the known values

of any other parameters, but *not* those of any samplebased quantities).

5. State the rejection region for the selected significance level α.

6. Compute any necessary sample quantities, substitute into the formula for the test statistic value, and compute that value.

Case I: A Normal Population with Known σ

7. Decide whether H_0 should be rejected, and state this conclusion in the problem context.

The formulation of hypotheses (Steps 2 and 3) should be done before examining the data.

Example 6

A manufacturer of sprinkler systems used for fire protection in office buildings claims that the true average system-activation temperature is 130°.

μ

A sample of n = 9 systems, when tested, yields a sample average activation temperature of 131.08°F.

$\bar{x}$

σ

If the distribution of activation times is normal with standard deviation 1.5°F, does the data contradict the manufacturer's claim at significance level α = .01?

Example 6

cont'd

1. Parameter of interest: μ = true average activation temperature.

2. Null hypothesis: H_0: $\mu = 130$ (null value = $\mu_0 = 130$).

3. Alternative hypothesis: H_a: $\mu \neq 130$ (a departure from the claimed value in *either* direction is of concern).

4. Test statistic value:

$$z = \frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}} = \frac{\bar{x} - 130}{1.5/\sqrt{n}}$$

Example 6

cont'd

5. Rejection region: The form of H_a implies use of a two-tailed test with rejection region *either* $z \geq z_{.005}$ *or* $z \leq -z_{.005}$. From Appendix Table A. $z_{.005} = 2.58$, so we reject H_0 if either $z \geq 2.58$ or $z \leq -2.58.$

6. Substituting n = 9 and $\bar{x}$ = 131.08,

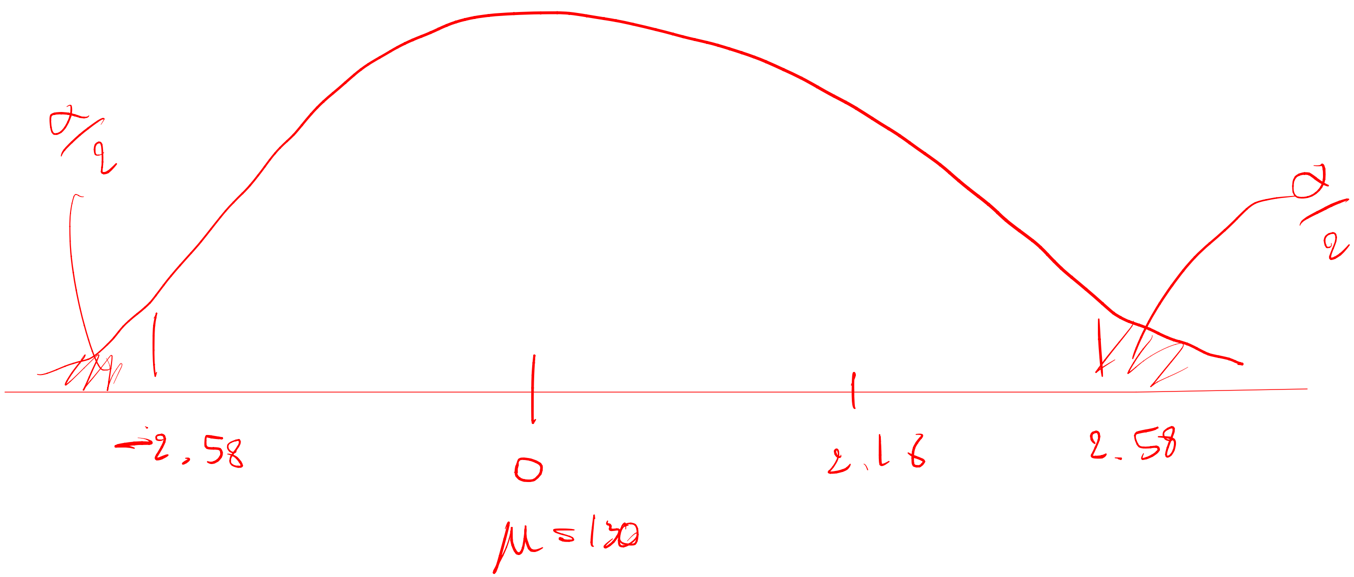
$$z = \frac{131.08 - 130}{1.5/\sqrt{9}} = \frac{1.08}{.5} = 2.16$$

That is, the observed sample mean is a bit more than 2 standard deviations above what would have been expected were H_0 true.

Example 6

cont'd

7. The computed value $z = 2.16$ does not fall in the rejection region ($-2.58 < 2.16 < 2.58$), so H_0 cannot be rejected at significance level .01. The data does not give strong support to the claim that the true average differs from the design value of 130.



Case I: A Normal Population with Known σ

***β* and Sample Size Determination** The *z* tests for case I are among the few in statistics for which there are simple formulas available for β, the probability of a type II error.

Consider first the upper-tailed test with rejection region $z \geq z_\alpha$.

This is equivalent to $\bar{x} \geq \mu_0 + z_\alpha \cdot \sigma/\sqrt{n}$, so H_0 will not be rejected if $\bar{x} < \mu_0 + z_\alpha \cdot \sigma/\sqrt{n}$.

Case I: A Normal Population with Known σ

Now let μ' denote a particular value of μ that exceeds the null value μ_0. Then,

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$$\beta(\mu') = P(H_0 \text{ is not rejected when } \mu = \mu')$$

$$= P(\overline{X} < \mu_0 + z_\alpha \cdot \sigma/\sqrt{n} \text{ when } \mu = \mu')$$

$$= P\left(\frac{\overline{X} - \mu'}{\sigma/\sqrt{n}} < z_\alpha + \frac{\mu_0 - \mu'}{\sigma/\sqrt{n}} \text{ when } \mu = \mu'\right)$$

$$= \Phi\left(z_\alpha + \frac{\mu_0 - \mu'}{\sigma/\sqrt{n}}\right)$$

Case I: A Normal Population with Known σ

As μ' increases, $\mu_0 - \mu'$ becomes more negative, so $\beta(\mu')$ will be small when μ' greatly exceeds μ_0 (because the value at which Φ is evaluated will then be quite negative).

Error probabilities for the lower-tailed and two-tailed tests are derived in an analogous manner.

If σ is large, the probability of a type II error can be large at an alternative value μ' that is of particular concern to an investigator.

Case I: A Normal Population with Known σ

Suppose we fix α and also specify β for such an alternative value. In the sprinkler example, company officials might view $\mu' = 132$ as a very substantial departure from H_0: $\mu = 130$ and therefore wish $\beta(132) = .10$

in addition to $\alpha = .01$.

More generally, consider the two restrictions
P(type I error) $= \alpha$ and $\beta(\mu') = \beta$ for specified α, μ' and β.

Case I: A Normal Population with Known σ

Then for an upper-tailed test, the sample size *n* should be chosen to satisfy

$$\Phi\left(z_\alpha + \frac{\mu_0 - \mu'}{\sigma/\sqrt{n}}\right) = \beta$$

This implies that

$$-z_\beta = \begin{array}{l} z \text{ critical value that} \\ \text{captures lower-tail area } \beta \end{array} = z_\alpha + \frac{\mu_0 - \mu'}{\sigma/\sqrt{n}}$$

Case I: A Normal Population with Known σ

It is easy to solve this equation for the desired *n*. A parallel argument yields the necessary sample size for lower- and two-tailed tests as summarized in the next box.

Alternative Hypothesis	**Type II Error Probability for a Level a Test**
H_a: $\mu > \mu_0$	$\Phi\left(z_\alpha + \frac{\mu_0 - \mu'}{\sigma/\sqrt{n}}\right)$
H_a: $\mu < \mu_0$	$1 - \Phi\left(-z_\alpha + \frac{\mu_0 - \mu'}{\sigma/\sqrt{n}}\right)$
H_a: $\mu \neq \mu_0$	$\Phi\left(z_{\alpha/2} + \frac{\mu_0 - \mu'}{\sigma/\sqrt{n}}\right) - \Phi\left(-z_{\alpha/2} + \frac{\mu_0 - \mu'}{\sigma/\sqrt{n}}\right)$

Case I: A Normal Population with Known σ

where $\Phi(z)$ = the standard normal cdf.

The sample size n for which a level α test also has $\beta(\mu') = \beta$ at the alternative value μ' is

$$n = \begin{cases} \left[\dfrac{\sigma(z_\alpha + z_\beta)}{\mu_0 - \mu'}\right]^2 & \text{for a one-tailed (upper or lower) test} \\ \left[\dfrac{\sigma(z_{\alpha/2} + z_\beta)}{\mu_0 - \mu'}\right]^2 & \text{for a two-tailed test (an approximate solution)} \end{cases}$$

Example 7

Let μ denote the true average tread life of a certain type of tire.

Consider testing H_0: $\mu = 30{,}000$ versus H_a: $\mu > 30{,}000$ based on a sample of size $n = 16$ from a normal population distribution with $\sigma = 1500$.

A test with $\alpha = .01$ requires $z_\alpha = z_{.01} = 2.33$.

Example 7

cont'd

The probability of making a type II error when μ = 31,000 is

$$\beta(31{,}000) = \Phi\left(2.33 + \frac{30{,}000 - 31{,}000}{1500/\sqrt{16}}\right)$$

$$\beta(31{,}000) = \Phi\left(2.33 + \frac{30{,}000 - 31{,}000}{1500/\sqrt{16}}\right) = \Phi(-.34)$$

$$\beta(31{,}000) = \Phi\left(2.33 + \frac{30{,}000 - 31{,}000}{1500/\sqrt{16}}\right) = \Phi(-.34) = .3669$$

Example 7

cont'd

Since $z_{.1} = 1.28$, the requirement that the level .01 test also have $\beta(31{,}000) = .1$ necessitates

$$n = \left[\frac{1500(2.33 + 1.28)}{30{,}000 - 31{,}000}\right]^2$$

$$= (-5.42)^2$$

$$= 29.32$$

The sample size must be an integer, so $n = 30$ tires should be used.