4

Continuous Random Variables and Probability Distributions

4.4 The Exponential and Gamma Distributions

The Exponential Distribution

The Exponential Distributions

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ต่าง

The family of exponential distributions provides probability models that are very widely used in engineering and science disciplines.

Definition

X is said to have an **exponential distribution** with parameter λ (λ > 0) if the pdf of *X* is

$$f(x; \lambda) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & \text{otherwise} \end{cases} \qquad \textbf{(4.5)}$$

The Exponential Distributions

Some sources write the exponential pdf in the form $(1/\beta)e^{-x/\beta}$, so that $\beta = 1/\lambda$. The expected value of an exponentially distributed random variable X is

$$E(X) = \int_0^\infty x\lambda e^{-\lambda x}\,dx$$

Obtaining this expected value necessitates doing an integration by parts. The variance of X can be computed using the fact that $V(X) = E(X^2) - [E(X)]^2$.

The determination of $E(X^2)$ requires integrating by parts twice in succession.

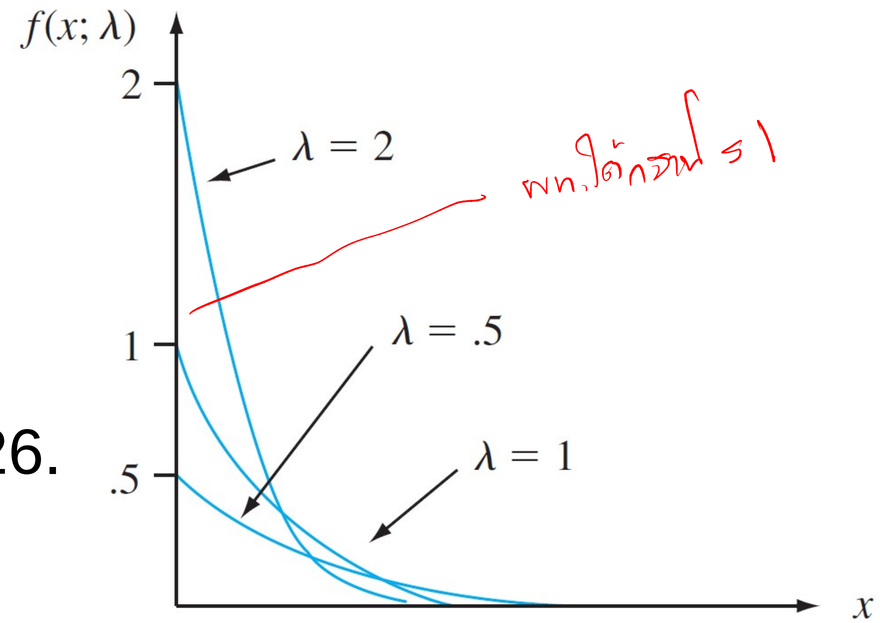
The Exponential Distributions

The results of these integrations are as follows:

$$\mu = \frac{1}{\lambda} \qquad \sigma^2 = \frac{1}{\lambda^2}$$

Both the mean and standard deviation of the exponential distribution equal $1/\lambda$.

Graphs of several exponential pdf's are illustrated in Figure 4.26.



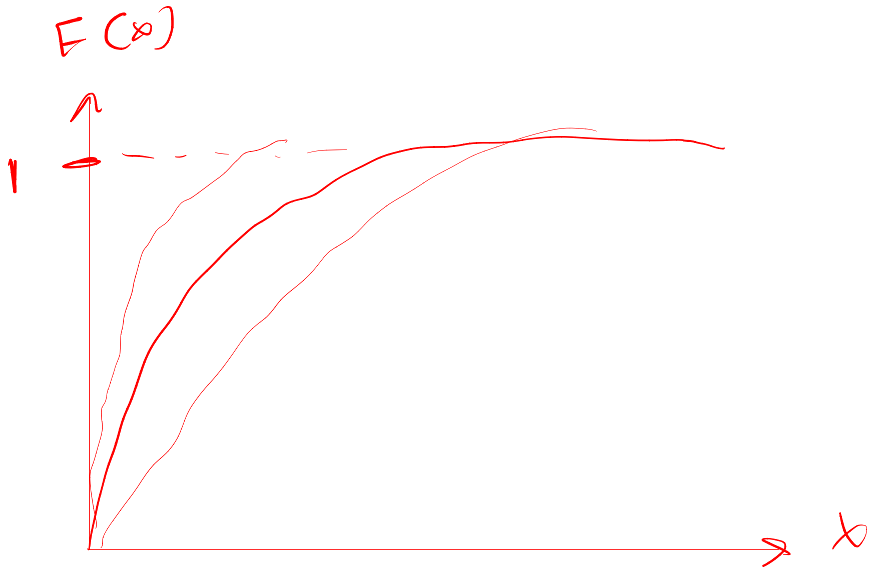
Exponential density curves

Figure 4.26

The Exponential Distributions

The exponential pdf is easily integrated to obtain the cdf.

$$F(x; \lambda) = \begin{cases} 0 & x < 0 \\ 1 - e^{-\lambda x} & x \geq 0 \end{cases}$$



Example 21

The article “Probabilistic Fatigue Evaluation of Riveted Railway Bridges” (*J. of Bridge Engr.,* 2008: 237–244) suggested the exponential distribution with mean value 6 MPa as a model for the distribution of stress range in certain bridge connections.

Let’s assume that this is in fact the true model. Then $E(X) = 1/\lambda = 6$ implies that $\lambda = .1667$.

Example 21

cont'd

The probability that stress range is at most 10 MPa is

$$P(X \leq 10) = F(10\ ;\ .1667)$$

$$= 1 - e^{-(.1667)(10)}$$

$$= 1 - .189$$

$$= .811$$

Example 21

cont'd

The probability that stress range is between 5 and 10 MPa is

$$P(5 \le X \le 10) = F(10; .1667) - F(5; .1667)$$

$$= (1 - e^{-1.667}) - (1 - e^{-.8335})$$

$$= .246$$

The Exponential Distributions

The exponential distribution is frequently used as a model for the distribution of times between the occurrence of successive events, such as customers arriving at a service facility or calls coming in to a switchboard.

The Exponential Distributions

Proposition

Suppose that the number of events occurring in any time interval of length t has a Poisson distribution with parameter αt (where α, the rate of the event process, is the expected number of events occurring in 1 unit of time) and that numbers of occurrences in nonoverlapping intervals are independent of one another. Then the distribution of elapsed time between the occurrence of two successive events is exponential with parameter $\lambda = \alpha$.

The Exponential Distributions

Although a complete proof is beyond the scope of the text, the result is easily verified for the time X_1 until the first event occurs:

$$P(X_1 \leq t) = 1 - P(X_1 > t) = 1 - P\,[\text{no events in } (0, t)]$$

$$= 1 - \frac{e^{-\alpha t} \cdot (\alpha t)^0}{0!} = 1 - e^{-\alpha t}$$

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which is exactly the cdf of the exponential distribution.

Example 22

CDF

ครั้ง/วัน

Suppose that calls are received at a 24-hour “suicide hotline” according to a Poisson process with rate $\alpha = .5$ call per day.

Then the number of days X between successive calls has an exponential distribution with parameter value .5, so the probability that more than 2 days elapse between calls is

$$P(X > 2) = 1 - P(X \leq 2)$$

$(\cancel{1} + e^{-\alpha t})$

$$= \cancel{1} - F(2; .5)$$

$$= e^{-(.5)(2)}$$

Example 22

$p(x > 2\text{ days})$ = .368

$\mu = \frac{1}{\lambda} = \frac{1}{\alpha}$

The expected time between successive calls is 1/.5 = 2 days.

The Exponential Distributions

Another important application of the exponential distribution is to model the distribution of component lifetime.

A partial reason for the popularity of such applications is the **“memoryless” property** of the exponential distribution.

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Suppose component lifetime is exponentially distributed with parameter λ.

The Exponential Distributions

After putting the component into service, we leave for a period of t_0 hours and then return to find the component still working; what now is the probability that it lasts at least an additional t hours?

In symbols, we wish $P(X \geq t + t_0 \mid X \geq t_0)$.

By the definition of conditional probability,

$P(A \mid B)$

$$P(X \geq t + t_0 \mid X \geq t_0) = \frac{P[(X \geq t + t_0) \cap (X \geq t_0)]}{P(X \geq t_0)} \qquad \frac{P(A \cap B)}{P(B)}$$

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The Exponential Distributions

But the event $X \geq t_0$ in the numerator is redundant, since both events can occur if $X \geq t + t_0$ and only if. Therefore,

$\underbrace{1-(1-e^{-\lambda(t+t_0)})}_{P(X \leq t+t_0)}$

$$P(X \geq t + t_0 \mid X \geq t_0) = \frac{P(X \geq t + t_0)}{P(X \geq t_0)} = \frac{1 - F(t + t_0;\lambda)}{1 - F(t_0;\lambda)} = e^{-\lambda t}$$

$= \dfrac{e^{-\lambda t - \lambda t_0}}{\cancel{e^{-\lambda t_0}}}$

$1-(1-e^{-\lambda(t_0)})$

$1-(1-e^{-\lambda t})$

$P(X \geq t)$

This conditional probability is identical to the original probability $P(X \geq t)$ that the component lasted t hours.

The Exponential Distributions

Thus *the distribution of additional lifetime is exactly the same as the original distribution of lifetime,* so at each point in time the component shows no effect of wear.

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In other words, the distribution of remaining lifetime is independent of current age.