2.3 Counting Techniques

(เทคนิคการนับ)

Counting Techniques

- When various outcomes of an experiment are equally

Same probability is assigned to each simple event

Task of computing probabilities reduces to counting

Letting

- N denote number of outcomes in a sample space and (2.1)
- $N(A)$ represent number of outcomes contained in event A

$$P(A) = \frac{N(A)}{N}$$

Product Rule for Ordered Pairs

(กฎการคูณสำหรับคู่ที่มีการเรียงลำดับ)

Product Rule for Ordered Pairs

- Our **first counting rule** applies to any situation in which a set (event) consists of ordered pairs of objects and we wish to count the number of such pairs
- By an ordered pair, we mean that,
  if O_1 and O_2 are objects, then
  the pair (O_1, O_2) is different from the pair (O_2, O_1).

Product Rule for Ordered Pairs

- For example

Los Angeles → Chicago → New York

(American, United)
(United , American)
(United, United)

- if an individual selects one airline for a trip from Los Angeles to Chicago and (after transacting business in Chicago)
- Second one for continuing on to New York,
- One possibility is
  (American, United), another is
  (United, American), and still another is
  (United, United).

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Product Rule for Ordered Pairs

Proposition

- If the **first element or object** of an ordered pair can be selected in n_1 ways, and for each of these n_1 ways the **second element** of the pair can be selected in n_2 ways, then the number of pairs is $n_1 n_2$.
- An alternative interpretation involves carrying out an operation that consists of two stages.
- If the **first stage** can be performed in any one of n_1 ways, and for each such way there are n_2 ways to perform the **second stage**, then $n_1 n_2$ is the number of ways of carrying out the two stages in sequence.

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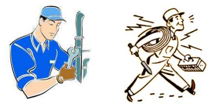
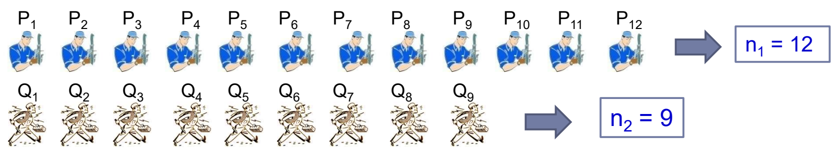
Example 17



- Homeowner doing some remodeling requires services of both plumbing contractor (ผู้รับเหมางานประปา) and electrical contractor (ผูรับเหมางานไฟฟ้า).
- If there are 12 plumbing contractors and 9 electrical contractors available in the area, in how many ways can the contractors be chosen?



- If we denote plumbers by $P_1, P_2, P_3, \ldots, P_{12}$ and electricians by $Q_1, Q_2, Q_3, \ldots, Q_9$
- We wish the number of pairs of the form (P_i, Q_j) ➡ 2-tuple
- **Product rule** yields $N = (12)\text{x}(9) = 108$ possible ways of choosing the two types of contractors

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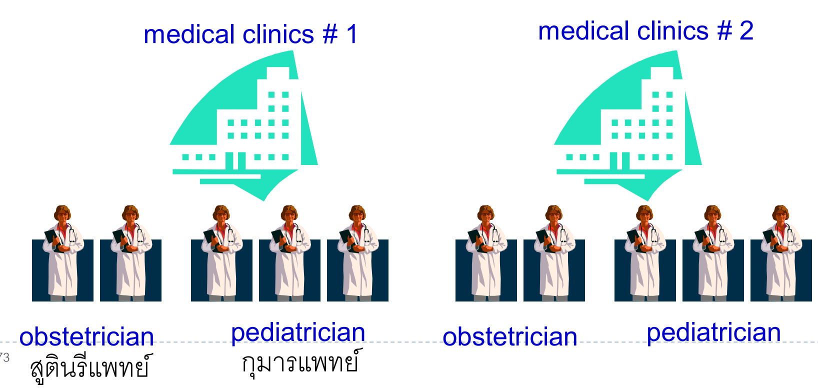
Example 18

- Family has just moved to a new city and requires the services of both an obstetrician and a pediatrician. There are two easily accessible medical clinics, each having two obstetricians and three pediatricians.



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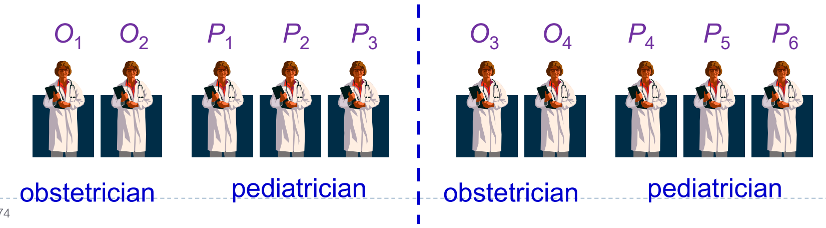
Example 18

- Family will obtain maximum health insurance benefits by joining a clinic and selecting both doctors from that clinic.
- In how many ways can this be done?
- Denote the obstetricians by O_1, O_2, O_3, and O_4 and the pediatricians by P_1, P_2, P_3, P_4, P_5, P_6.



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Example 18

cont'd

- Then we wish the number of pairs (O_i, P_j) for which O_i and P_j are associated with the same clinic.
- Because there are four obstetricians, $n_1 = 4$, and for each there are three choices of pediatrician, so $n_2 = 3$.
- Applying the product rule gives $N = n_1 n_2 = 12$ possible choices.

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Product Rule for Ordered Pairs

- In many counting and probability problems, configuration called a **tree diagram** can be used to represent pictorially all the possibilities.
- Tree diagram associated with Example 18 appears in Figure 2.7.

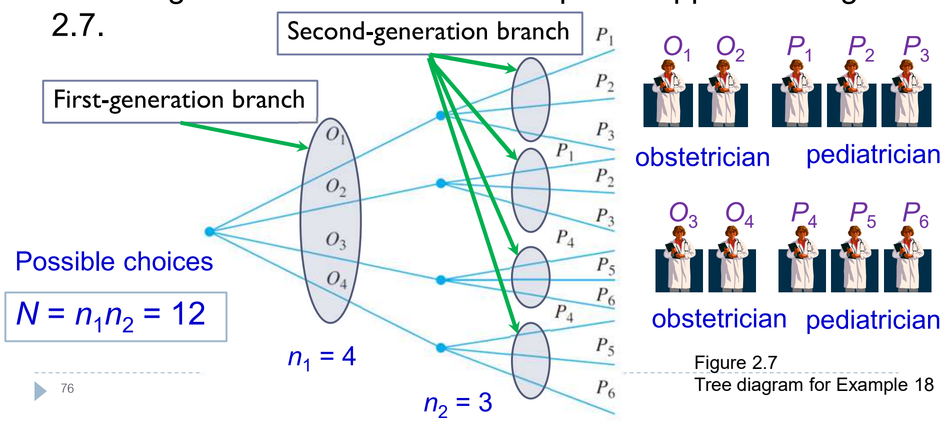

Figure 2.7
Tree diagram for Example 18

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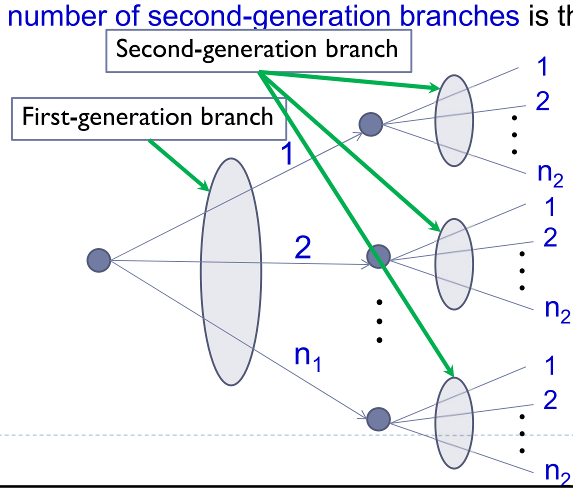
Product Rule for Ordered Pairs

- Generalizing, suppose there are n_1 first-generation branches, and for each first generation branch there are n_2 second-generation branches.
- Total number of second-generation branches is then n_1n_2.



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A More General Product Rule

(กฎการคูณทั่วไปอื่นๆ)

A More General Product Rule

- If six-sided die is tossed **five times in succession** rather than just twice, then each possible outcome is an ordered collection of five numbers

  such as (1, 3, 1, 2, 4) or (6, 5, 2, 2, 2).

  5 objects　　　5 objects

  Each outcome of the die-tossing experiment is **5-tuple**

- We will call an ordered collection of ***k* objects** a ***k-tuple*** (so a pair is a 2-tuple and a triple is a 3-tuple)

A More General Product Rule

Product Rule for *k*-Tuples

- Suppose a set consists of ordered collections of k elements (k-tuples) and that
- there are n_1 possible choices for the first element;
- for each choice of the first element, there are n_2 possible choices of the second element; . . . ;
- for each possible choice of the first $k - 1$ elements, there are n_k choices of the kth element.
- Then there are $n_1 n_2 \cdot \cdot \cdot \cdot n_k$ possible k-tuples.

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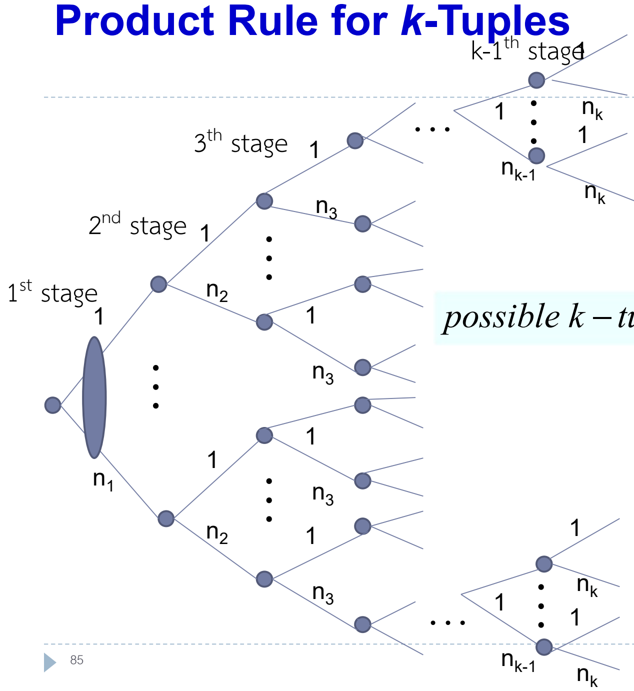
Product Rule for *k*-Tuples



$$possible\ k-tuples = n_1 \times n_2 \times \ldots \times n_k$$

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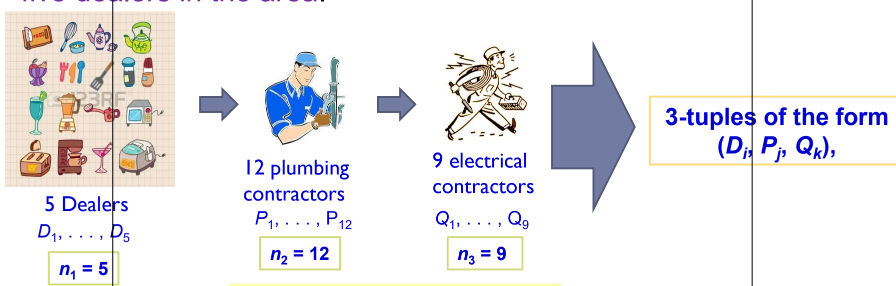
Example 19

Example 17 continued…

- Suppose the home remodeling job involves first purchasing several kitchen appliances.
- They will all be purchased from the same dealer, and there are five dealers in the area.



$$N = n_1 \times n_2 \times n_3 = 5 \times 12 \times 9 = 540$$

- so there are 540 ways to choose first an appliance dealer, then a plumbing contractor, and finally an electrical contractor.

Permutations and Combinations

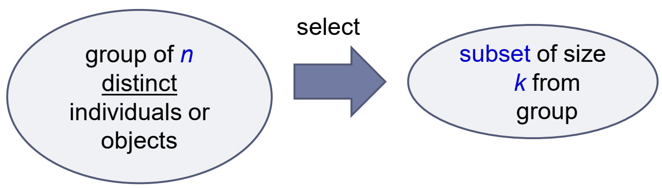
(วิธีเรียงสับเปลี่ยน) (วิธีการจัดหมู่)

Permutations and Combinations

- Consider a group of *n* distinct individuals or objects

  there is some characteristic that differentiates any particular individual or object from any other

- How many ways are there to select a subset of size *k* from the group?



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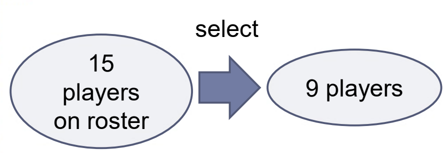
Permutations and Combinations

Example

- if a Little League team has 15 players on its roster,
- how many ways are there to select 9 players to form a starting lineup?

2013 OKLAHOMA CITY THUNDER SUMMER LEAGUE ROSTER

NO.	PLAYER	POS.	HT.	WT.	DOB	FROM	NBA YRS.
12	Steven Adams	C	7-0	250	07/28/93	Pittsburgh/New Zealand	R
42	Ron Anderson	F	6-9	255	09/12/89	South Florida/USA	R
13	Dwight Buycks	G	6-3	190	03/06/89	Marquette/USA	R
15	Reggie Jackson	G	6-3	208	04/16/90	Boston College/USA	2
47	Grant Jerrett	F	6-10	220	07/08/93	Arizona/USA	R
3	Perry Jones	F	6-11	235	09/24/91	Baylor/USA	1
8	Kyle Kuric	G	6-4	195	08/25/89	Louisville/USA	R
11	Jeremy Lamb	G	6-5	180	05/30/92	Connecticut/USA	1
14	DeAndre Liggins	G	6-6	209	03/31/88	Kentucky/USA	2
33	Daniel Orton	C	6-10	255	08/06/90	Kentucky/USA	3
21	Andre Roberson	F	6-7	190	12/04/91	Colorado/USA	R
44	Andrew Smith	C	6-11	240	09/09/90	Butler/USA	R
2	Michael Snaer	G	6-4	180	06/21/90	Florida State/USA	R
7	Tony Taylor	G	6-0	191	09/09/90	George Washington/USA	R



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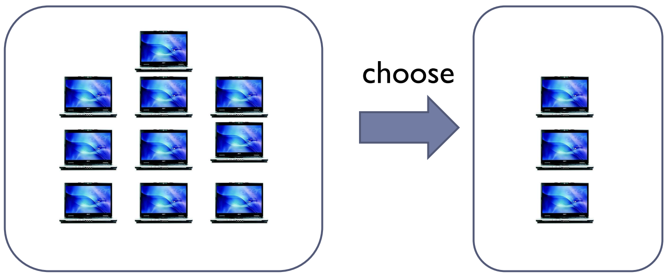
Permutations and Combinations

Example

- Or if a university bookstore sells ten different laptop computers but has room to display only three of them, in how many ways can three be chosen?



Permutations and Combinations

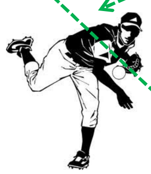
- An answer to the general question just posed requires that we distinguish between two cases.

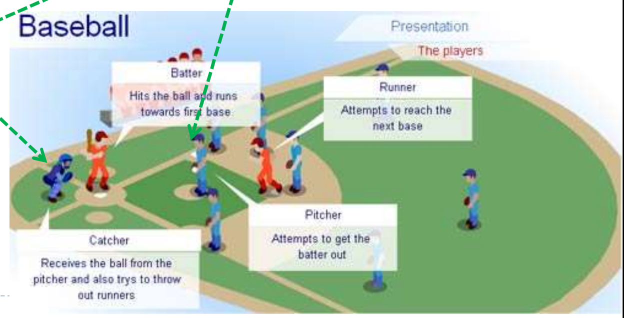
1) The order of selection is important

2) The order of selection is not important

Permutations and Combinations

- In some situations, such as the baseball scenario, the order of selection is important.
- For example, Angela being the pitcher and Ben the catcher gives a different lineup from the one in which Angela is catcher and Ben is pitcher.





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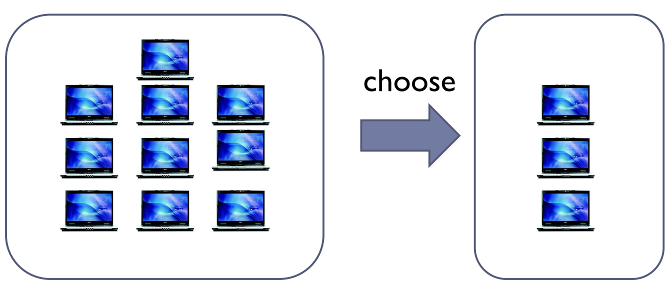
Permutations and Combinations

- Often, though, order is not important and one is interested only in which individuals or objects are selected, as would be the case in the laptop display scenario.



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Permutations and Combinations

- **Definition**
- An **ordered subset** is called a **Permutation**.(วิธีเรียงสับเปลี่ยน)
  - Number of permutations of size ***k*** that can be formed from the ***n*** individuals or objects in a group

denoted by ➡ $P_{k,n}$

- An **unordered subset** is called a **Combination**. (วิธีการจัดหมู่)
  - Number of combination of size ***k*** that can be formed from the ***n*** individuals or objects in a group

denoted by ➡ $C_{k,n}$ or $\binom{n}{k}$ ➡ read "*n* choose *k*"

This notation is quite common use in probability books.

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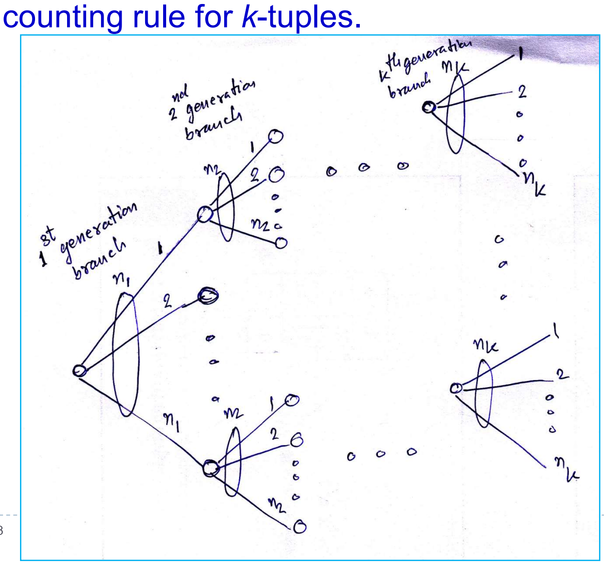
Permutations and Combinations

- Number of permutations can be determined by using counting rule for *k*-tuples.



Tree diagram

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Permutations and Combinations

- Suppose, for example, that a college of engineering has seven departments, which we denote by *a*, *b*, *c*, *d*, *e*, *f*, and *g*.
- Each department has one representative on the college's student council.
- From these **seven representatives**,
  - one is to be chosen **chair**,
  - another is to be selected **vice-chair**, and
  - a third will be **secretary**.

- How many ways are there to select the three officers?

That is, how many permutations of size 3 can be formed from 7 representatives?

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Permutations and Combinations

- To answer this question, think of forming a **triple (3-tuple)**
  - The first element is the chair,
  - The second is the vice-chair, and
  - The third is the secretary.
- One such triple is (*a*, *g*, *b*), another is (*b*, *g*, *a*), and yet another is (*d*, *f*, *b*).
  - Now **the chair** can be selected in any of n_1 **= 7 ways**.
- For each way of selecting the chair, there are n_2 **= 6 ways** to select **the vice-chair**, and hence $7 \times 6 = 42$ (chair, vice-chair) pairs.

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Permutations and Combinations

- Finally, for each way of selecting a chair and vice-chair, there are n_3 **= 5 ways** of choosing the secretary.
- This gives

$$P_{3,7} = 7 \times 6 \times 5 = 210$$

Number of permutations of size 3 that can be formed from 7 distinct individuals

- Tree diagram representation would show three generations of branches.
- Expression for $P_{3,7}$ can be rewritten with the aid of *factorial notation*.

Recall that $7! = 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$

"7 factorial" is compact notation for descending product of integers

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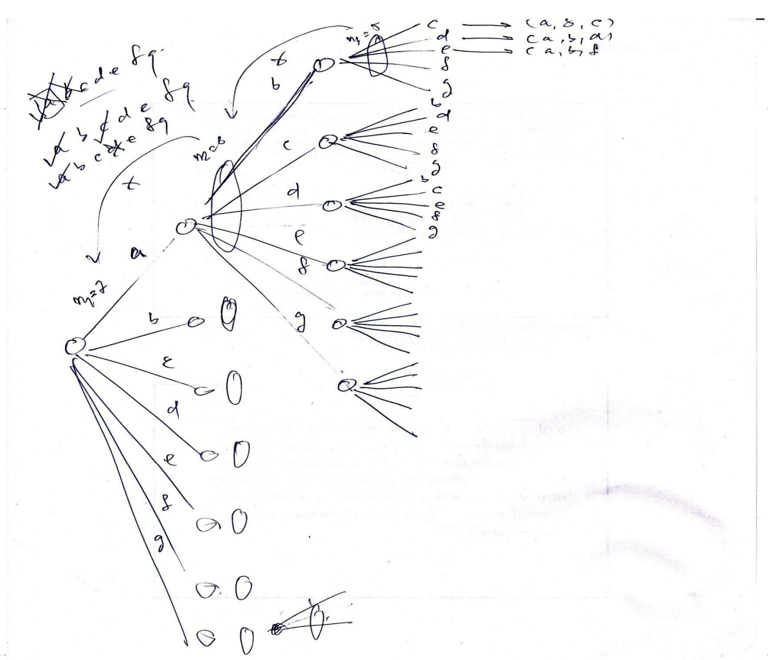

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Permutations and Combinations

- More generally, for any positive integer m,

$$m! = m \times (m-1) \times (m-2) \times (m-3) \times \cdots \times 3 \times 2 \times 1$$

This gives **1! = 1**, and we also define **0! = 1**. Then

$$P_{3,7} = 7 \times 6 \times 5 = \boxed{7 \times (7-1)(7-(3-1))}$$

$$= 7 \times 6 \times 5 \times \left(\frac{4!}{4!}\right) = 7 \times 6 \times 5 \times \left(\frac{4!}{(7-3)!}\right) = \frac{7 \times 6 \times 5 \times 4!}{(7-3)!} = \frac{7!}{4!}$$

- More generally,

$$P_{k,n} = n \times (n-1) \times (n-2) \times (n-3) \cdots (n-(k-2)) \times (n-(k-1))$$

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Permutations and Combinations

$$P_{k,n} = n \times (n-1) \times (n-2) \times (n-3) \cdots (n-(k-2)) \times (n-(k-1))$$

- Multiplying and dividing this by ***(n – k)!*** gives a compact expression for the number of permutations.

$$P_{k,n} = n \times (n-1) \times (n-2) \times (n-3) \cdots (n-(k-2)) \times (n-(k-1)) \times \boxed{\frac{(n-k)!}{(n-k)!}}$$

$$= \frac{n \times (n-1) \times (n-2) \times (n-3) \cdots (n-(k-2)) \times (n-(k-1)) \times (n-k)!}{(n-k)!}$$

- **Proposition** $$P_{k,n} = \frac{n!}{(n-k)!}$$

Number of permutations of size k that can be formed from n distinct individuals

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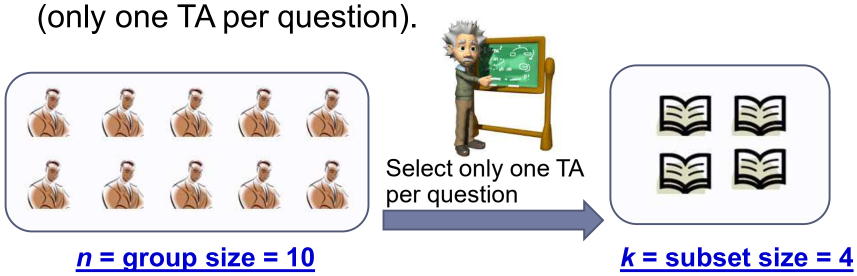
Example 21 (permutations)

- There are ten teaching assistants (10 TAs) available for grading papers in a calculus course at a large university.
- First exam consists of four questions, and the professor wishes to select a different TA to grade each question (only one TA per question).



- How many ways can the TAs be chosen for grading?

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Example 21

cont'd

- The number of permutations is

$$P_{k,n} = \frac{n!}{(n-k)!}$$

$$P_{4,10} = \frac{10!}{(10-4)!} = \frac{10!}{6!} = 10 \times 9 \times 8 \times 7 = 5{,}040$$

- That is, the professor could give 5040 different four-question exams without using the same assignment of graders to questions, by which time all the teaching assistants would hopefully have finished their degree programs!

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Permutations and Combinations

- Now let's move on to **combinations** (i.e., **un**ordered subsets).
- Again refer to the student council scenario, and suppose that three of the seven representatives are to be selected to attend a statewide convention.

order of selection is not important

- The number of combinations of size 3 that can be formed from the 7 individuals is

"*n* choose *k*" $\binom{7}{3}$

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Permutations and Combinations

- Consider for a moment combination *a, c, g*.
- These three individuals can be ordered in 3! = 6 ways to produce permutations:

a, c, g *a, g, c* *c, a, g* *c, g, a* *g, a, c* *g, c, a*

- Similarly, there are 3! = 6 ways to order combination *b, c, e* to produce permutations, and
- in fact 3! ways to order any particular combination of size 3 to produce permutations.

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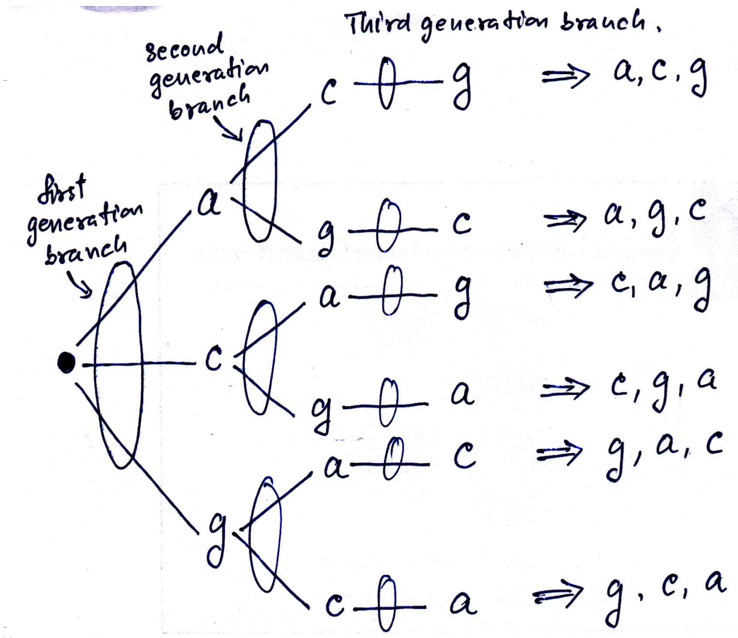

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Permutations and Combinations

- This implies the following relationship between the number of combinations and the number of permutations:

$$P_{3,7} = (3!).\binom{7}{3} \quad \Rightarrow \quad \binom{7}{3} = \frac{P_{3,7}}{3!} = \frac{7!}{(3!)(4!)} = \frac{7\times6\times5}{3\times2\times1} = 35$$

- It would not be too difficult to list the 35 combinations, but there is no need to do so if we are interested only in how many there are.
- Notice that the number of permutations 210 far exceeds the number of combinations; the former is larger than the latter by a factor of 3! since that is how many ways each combination can be ordered.

generalizing $$P_{k,n} = k!\binom{n}{k}$$

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Permutations and Combinations

- Generalizing foregoing line of reasoning gives a simple relationship between the number of permutations and the number of combinations that yields a concise expression for latter quantity.

$$P_{k,n} = k!\binom{n}{k}$$

- **Proposition**

$$\binom{n}{k} = \frac{P_{k,n}}{k!} = \frac{n!}{k! \times (n-k)!}$$

- Notice that

$\binom{n}{n} = 1$ and $\binom{n}{0} = 1$ Since there is only one way to choose a set of (all) *n* elements or of no elements

$\binom{n}{1} = n$ Since there are *n* subsets of size 1

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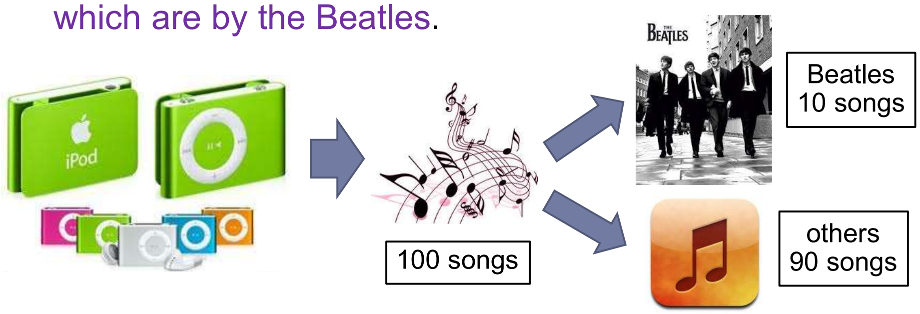
Example 22 : Permutation

- A particular iPod playlist contains 100 songs, 10 of which are by the Beatles.



- Suppose the shuffle feature is used to play the songs in random order (the randomness of the shuffling process is investigated in "Does Your iPod *Really* Play Favorites?"

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Example 22

- What is **probability** that the first Beatles song heard is the fifth song played?

NB NB NB NB B

- In order for this event to occur, it must be the case that
  - First four songs played are not Beatles' songs (NBs) and
  - Fifth song is by the Beatles (B).
- Random shuffle assumption implies that any particular set of 5 songs from amongst the 100 has the same chance of being selected as the first five played as does any other set of five songs; each outcome is equally likely.

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Example 22

cont'd

- Therefore desired probability is

$$\frac{\textit{Number of outcomes for which event of interest occurs}}{\textit{Number of possible outcomes}}$$

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Example 22

cont'd

- Number of ways to select the first five songs is
  100 x 99 x 98 x 97 x 96

- Number of ways to select these five songs so that the first four are NBs and the next is B is
  90 x 89 x 88 x 87 x 10

$$P(1^{st}\ B\ is\ the\ 5^{th}\ song\ played) = \frac{90\times89\times88\times87\times10}{100\times99\times98\times97\times96} = \frac{P_{4,90}\times10}{P_{5,100}} = 0.0679$$

Order of song is important ➡ Permutation

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Example 23 : Combination

- University warehouse has received a shipment of 25 printers, of which
  - 10 are laser printers and
  - 15 are inkjet model

Laser printer

Inkjet printer

- If 6 of these 25 are selected at random to be checked by a particular technician,
- What is the probability that exactly 3 of those selected are inkjets printers (so that the other 3 are lasers?)

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Example 23 : Combination (cont.)

- Let D_3 = {exactly 3 of 6 selected are inkjet printers}
- Assuming that any particular set of 6 printers is as likely to be chosen as is any other set of 6, we have equally likely outcome, so

$$P(D_3) = \frac{N(D_3)}{N}$$

Number of way of choosing 3 laser printers and 3 Inkjet models

No. of way of choosing 3 of 15 Inkjet models $\Rightarrow \binom{15}{3}$

No. of way of choosing 3 of 10 laser printers $\Rightarrow \binom{10}{3}$

Number of way of choosing 6 printers from the 25 $\Rightarrow \binom{25}{6}$

$$P(D_3) = \frac{N(D_3)}{N} = \frac{\binom{15}{3}\binom{10}{3}}{\binom{25}{6}} = \frac{\frac{15!}{3!12!} \cdot \frac{10!}{3!7!}}{\frac{25!}{6!19!}} = 0.3083$$

Example 23 : Combination (cont.)

- What is the probability that at least 3 inkjet printers are selected?
- Let
  - D_3 = {exactly 3 of 6 selected are inkjet printers}
  - D_4 = {exactly 4 of 6 selected are inkjet printers}
  - D_5 = {exactly 5 of 6 selected are inkjet printers}
  - D_6 = {exactly 6 of 6 selected are inkjet printers}

$$P(D_3 \cup D_4 \cup D_5 \cup D_6) = P(D_3) + P(D_4) + P(D_5) + P(D_6)$$

$$= \frac{\binom{15}{3}\binom{10}{3}}{\binom{25}{6}} + \frac{\binom{15}{4}\binom{10}{2}}{\binom{25}{6}} + \frac{\binom{15}{5}\binom{10}{1}}{\binom{25}{6}} + \frac{\binom{15}{6}\binom{10}{0}}{\binom{25}{6}}$$

$$= 0.8530$$

2.4 Conditional Probability

(ความน่าจะเป็นแบบมีเงื่อนไข)

Conditional Probability

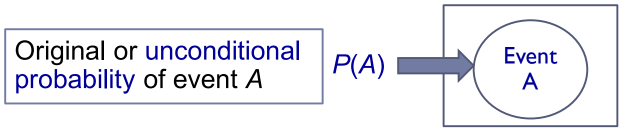
- Probabilities assigned to various events depend on what is known about experimental situation when assignment is made

- Subsequent to initial assignment, partial information relevant to outcome of experiment may become available.

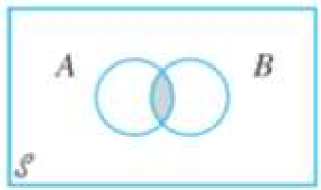
- Such information may cause us to revise some of our probability assignments.

Conditional Probability

- For particular event A, we have used $P(A)$ to represent probability, assigned to A; we now think of $\boldsymbol{P(A)}$ as **original**, or **unconditional probability of the event *A***



- In this section, we examine how the information "an event B has occurred" affects probability assigned to A



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Conditional Probability

For example

Event A → **Individual** having a particular **disease** in the presence of certain **symptoms**

blood test is performed on individual

↓

Event B → result is **negative** (B = 5 negative blood test)

↓

probability of having the **disease** will **change**

↓

(it should decrease, but not usually to zero, since blood tests are not infallible)

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Conditional Probability

P(A | B) → **Conditional probability of A given that Event B has occurred**

B is the "conditioning event"

Example

Event A → Randomly selected student at your university obtained all desired classes during previous term's registration cycle
Presumably P(A) is not very large

Event B → Selected student is athlete who gets special registration priority

P(A|B) should be substantially larger than P(A), although perhaps still not close to 1

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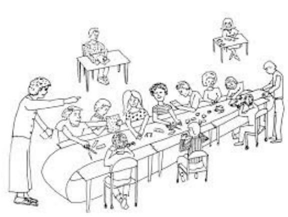
Example 24

- Complex components are assembled in a plant that uses two different assembly lines, *A* and *A′*



Line *A*

Line *A′*

- Line *A* uses older equipment than *A′*
- Line *A* is somewhat slower and less reliable than *A′*

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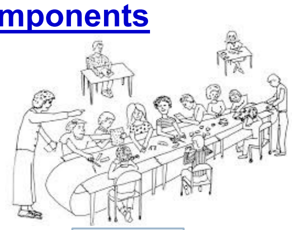
Example 24

- Suppose on a given, line A and A' has **assembled 18 components**



Line A

Line A'

- 8 components
  - 2 defective (B)
  - 6 nondefective (B')
- 10 components
  - 1 defective
  - 9 nondefective

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Example 24

cont'd

- This information is summarized in the accompanying table

defective | nondefective

		Condition		
		B	B'	
Line	A	2	6	8 components
	A'	1	9	10 components

- Unaware of this information, sales manager randomly selects 1 of these 18 components for a demonstration.
- Prior to the demonstration

$$P(line\ A\ component\ selected = P(A) = \frac{N(A)}{N} = \frac{8}{18} = 0.44$$

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Example 24

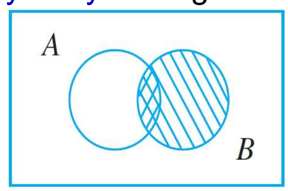
cont'd

However, if chosen component turns out to be defective

Event B has occurred

So component must have been 1 of 3 in B column of table

- Since these 3 components are equally likely among themselves after *B* has occurred,



		Condition	
		B	B'
Line	A	2	6
	A'	1	9

$$P(A \mid B) = \frac{2}{3} = \frac{2/18}{3/18} = \frac{P(A \cap B)}{P(B)} = 0.667 \quad (2.2)$$

Conditional Probability

- In Equation (2.2), conditional probability is expressed as a ratio of unconditional probabilities:

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

Numerator is probability of intersection of two events

Denominator is the probability of conditioning event *B*

- A Venn diagram illuminates this relationship

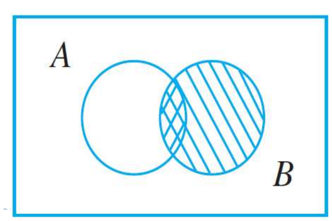
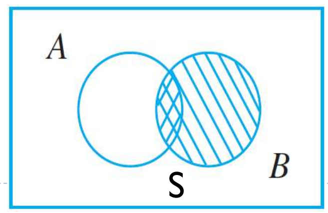


Figure 2.8 Motivating the definition of conditional probability

Conditional Probability

- Given that ***B*** has occurred, relevant sample space is no longer ***S*** but consists of outcomes in *B*;
- ***A*** has occurred if and only if one of outcomes in intersection occurred, so conditional probability of *A* given *B* is proportional to $P(A \cap B)$.
- Proportionality constant $1/P(B)$ is used to ensure that the probability $P(B \mid B)$ of the new sample space *B* equals 1.



$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

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Definition of Conditional Probability

(นิยามของความน่าจะเป็นอย่างมีเงื่อนไข)

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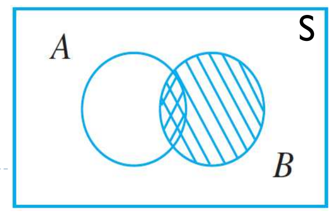
Definition of Conditional Probability

Definition

For any two events *A* and *B* with *P*(*B*) > 0, the **conditional probability of *A* given that *B* has occurred** is defined by

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} \qquad (2.3)$$



Example 25

- Suppose that of all individuals buying a certain digital camera,
  - 60% include an optional memory card in their purchase,
  - 40% include an extra battery, and
  - 30% include both a card and battery

- Consider randomly selecting a buyer and let

  A = {memory card purchased} and
  B = {battery purchased}

  Then

  $P(A) = 0.60$,
  $P(B) = 0.40$,
  $P(\text{both purchased}) = P(A \cap B) = 0.30$

Example 25

cont'd

- Given that selected individual purchased **extra battery (event B)**, probability that **optional memory card (event A)** was also purchased is

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{0.30}{0.40} = 0.75$$

- That is, of all those purchasing an extra battery, 75% purchased an optional memory card.
- Similarly,

$$P(battery \mid memory\ card) = P(B \mid A) = \frac{P(A \cap B)}{P(A)} = \frac{0.30}{0.60} = 0.50$$

- Notice that

$$P(A \mid B) \neq P(A) \ and \ P(B \mid A) \neq P(B)$$

Event whose probability is desired might be union or intersection of other events, and the same could be true of the conditioning event

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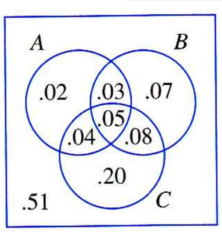
Example

- News magazine publishes three columns entitled
  - "Art" (A),
  - "Books" (B), and
  - "Cinema" (C)
- Reading habits of randomly selected reader with respect to these columns are

Read regularly	A	B	C	A∩B	A∩C	B∩C	A∩B∩C
Probability	**0.14**	**0.23**	**0.37**	**0.08**	**0.09**	**0.13**	**0.05**



Venn diagram

We thus have $P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{0.08}{0.23} = 0.348$

$$P(A \mid B \cup C) = \frac{P(A \cap (B \cup C))}{P(B \cup C)} = \frac{0.04 + 0.05 + 0.03}{0.03 + 0.05 + 0.07 + 0.08 + 0.04 + 0.2} = \frac{0.12}{0.47} = 0.255$$

$$P(A \mid reads\ at\ least\ one) = P(A \mid (A \cup B \cup C)) = \frac{P(A \cap (A \cup B \cup C))}{P(A \cup B \cup C)} = \frac{P(A)}{P(A \cup B \cup C)} = \frac{0.14}{0.49} = 0.286$$

and

$$P(A \cup B \mid C) = \frac{P((A \cup B) \cap C)}{P(C)} = \frac{0.04 + 0.05 + 0.08}{0.37} = 0.459$$

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Multiplication Rule for $P(A \cap B)$

Multiplication Rule for $P(A \cap B)$

- Definition of conditional probability yields the following result, obtained by multiplying both sides of Equation (2.3) by $P(B)$. $P(A|B) = \frac{P(A \cap B)}{P(B)}$ ⟸ conditional probability of A given that B has occurred

  Multiplication Rule

$$P(A \cap B) = P(A|B) \cdot P(B)$$

  This rule is important because it is often the case that $P(A \cap B)$ is desired, whereas both $P(B)$ and $P(A|B)$ can be specified from problem description.

  Consideration of $P(B|A)$ gives $P(A \cap B) = P(B|A) \cdot P(A)$

Example 27

- Four individuals have responded to a request by a blood bank for blood donations.
- None of them has donated before, so their blood types are unknown

มีเพียงคนเดียวที่มีเลือดกรุ๊ปนี้

- Suppose only type O+ is desired and only one of four actually has this type.
- If potential donors are selected in random order for typing,
- What is probability that **at least three individuals** must be typed to obtain the desired type?

ความน่าจะเป็นที่อย่างน้อย 3 คน ต้องถูกเจาะเลือดเพื่อหากลุ่มเลือดที่ต้องการมีค่าเท่าไร?

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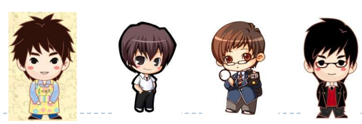
Example 27 cont'd



not O+ not O+ not O+ O+

- Making the identification

B = {first type not O+} and
A = {second type not O+},

$$P(B)=\frac{3}{4}$$

- Given that first type is not O+, two of the three individuals left are not O+, so

$$P(A\mid B)=\frac{2}{3}$$

- Multiplication rule now gives

$$\begin{aligned} P(\text{at least three individuals are typed}) &= P(A\cap B)\\ &= P(A\mid B)\cdot P(B)\\ &= \frac{2}{3}\cdot\frac{3}{4}=\frac{6}{12}\\ &= 0.5 \end{aligned}$$

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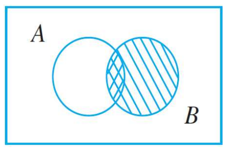
Example 27

cont'd

- Multiplication rule is most useful when experiment consists of several stages in succession.
- Conditioning event B then describes outcome of first stage and A outcome of the second, so that
  $P(A \mid B)$ —conditioning on what occurs first—will often be known

conditional probability of A given that B has occurred $\Rightarrow$ $P(A \mid B) = \dfrac{P(A \cap B)}{P(B)}$ $\Rightarrow$ $P(A \cap B) = P(A \mid B) \cdot P(B)$

Multiplication Rule for $P(A \cap B)$



- Rule is easily extended to experiments involving more than two stages.

Example 27

$P(A \mid B) = \dfrac{P(A \cap B)}{P(B)}$ $\Rightarrow$ $P(A \cap B) = P(A \mid B) \cdot P(B)$ cont'd

- For example,

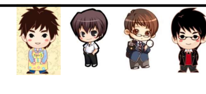
$$P(A_1 \cap A_2 \cap A_3) = P(A_3 \mid A_1 \cap A_2) \cdot P(A_1 \cap A_2)$$

$$= P(A_3 \mid A_1 \cap A_2) \cdot P(A_2 \mid A_1) \cdot P(A_1) \qquad (2.4)$$

where A_1 occurs **first**, followed by A_2, and finally A_3.

Example 2.28

B = {first type not O+} and
A = {second type not O+},
C = {third type is O+}



- For the blood typing experiment of Example 2.27

$$P(A \cap B \cap C) = P(C \mid B \cap A) \cdot P(A \mid B) \cdot P(B)$$

$$P(\text{third type is O+}) = P(\text{third is} \mid \text{first isn't} \cap \text{second isn't}) \cdot P(\text{second isn't} \mid \text{first isn't}) \cdot P(\text{first isn't}) = \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{3}{4} = \frac{1}{4} = 0.25$$

- When experiment of interest consists of a sequence of several stages, it is convenient to represent these with a tree diagram.
- Once we have an appropriate tree diagram, probabilities and conditional probabilities can be entered on the various branches; this will make repeated use of the multiplication rule quite straightforward

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Example 2.29

brand 1 (50%)

brand 2 (30%)

brand 3 (20%)

- A chain of video stores sells three different brands of DVD players.
- Of its DVD player sales,
  - 50% are brand 1 (the least expensive),
  - 30% are brand 2, and
  - 20% are brand 3.
- Each manufacturer offers a 1-year warranty on parts and labor.
- It is known that
  - 25% of brand 1's DVD players require warranty repair work,
  - whereas the corresponding percentages for brands 2 and 3 are 20% and 10%, respectively.

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Example 2.29

brand 1 (50%)

brand 2 (30%)

brand 3 (20%)

1. What is probability that a randomly selected purchaser has bought a brand 1 DVD player that will need repair while under warranty?

2. What is probability that a randomly selected purchaser has a DVD player that will need repair while under warranty?

3. If a customer returns to store with DVD player that needs warranty repair work,

   what is probability that it is

   a brand 1 DVD player

   a brand 2 DVD player"

   a brand 3 DVD player?

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Example 2.29 (cont.)

- The first stage of the problem involves a customer selecting one of the three brands of DVD player
- Let A_i={brand i is purchased}, for i=1, 2, and 3.
- Then $P(A_1)=0.50$, $P(A_2)=0.30$, and $P(A_3)=0.20$.
- Once a brand of DVD player is selected, the second stage involves observing whether the selected DVD player needs warranty repair.
- With B = {needs repair} and B' = {doesn't need repair}, the given information implies that

  $P(B|A_1)=0.25$, $P(B|A_2)=0.20$, and $P(B|A_3)=0.10$

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Example 2.29 (cont.)

- Tree diagram representing this experimental situation is shown in Figure 2.10.

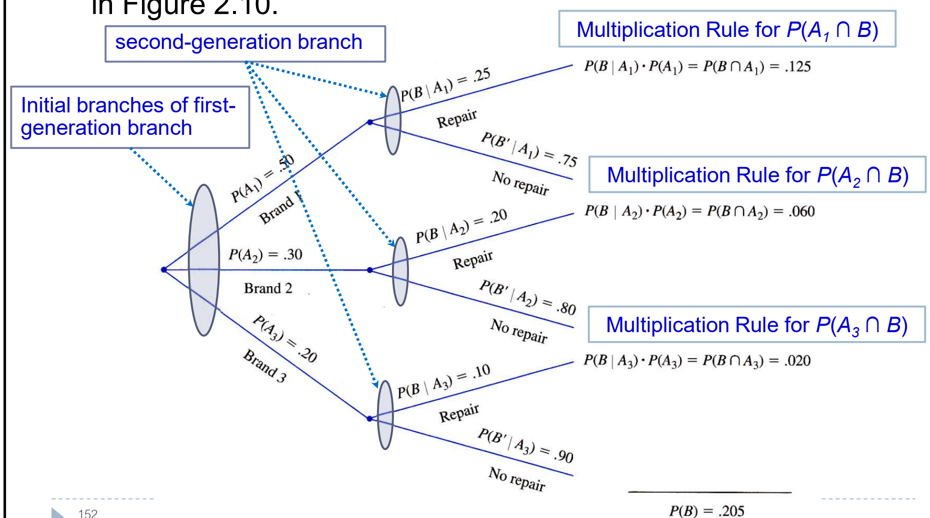

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Figure 2.10 Tree diagram for example 2.29

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Example 2.29 (cont.)

- Initial branches correspond to different brands of DVD players;
- there are two second-generation branches emanating from the tip of each initial branch,
  - one for "needs repair" and
  - the other for "doesn't need repair."
- Probability $P(A_i)$ appears on the i^{th} initial branch, whereas conditional probabilities $P(B|A_i)$ and $P(B'|A_i)$ appear on second generation branches.
- To the right of each second-generation branch corresponding to the occurrence of B, we display the product of probabilities on branches leading out to that point.
- This is simply multiplication rule in action

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Example 2.29 (cont.)

What is probability that a randomly selected purchaser has bought a brand 1 DVD player that will need repair while under warranty?

Answer to question posed in 1 is thus

$$P(A_1 \cap B) = P(B \mid A_1) \cdot P(A_1) = 0.125$$

What is probability that a randomly selected purchaser has a DVD player that will need repair while under warranty?

Answer to question 2 is

$$\begin{aligned} P(B) &= P[(brand\ 1\ and\ repair)\ or\ (brand\ 2\ and\ repair)\ or\ (brand\ 3\ and\ repair)] \\ &= P(A_1 \cap B) + P(A_2 \cap B) + P(A_3 \cap B) \\ &= 0.125 + 0.060 + 0.020 = 0.205 \end{aligned}$$

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Example 2.29 (cont.)

If a customer returns to store with DVD player that needs warranty repair work,

what is probability that it is

a brand 1 DVD player

a brand 2 DVD player"

a brand 3 DVD player?

- Finally,

$$P(A_1 \mid B) = \frac{P(A_1 \cap B)}{P(B)} = \frac{0.125}{0.205} = 0.61$$

$$P(A_2 \mid B) = \frac{P(A_2 \cap B)}{P(B)} = \frac{0.060}{0.205} = 0.29$$

- and

$$P(A_3 \mid B) = 1 - P(A_1 \mid B) - P(A_2 \mid B) = 0.10$$

- Initial or prior probability of brand 1 is 0.50.
  - Once it is known that the selected DVD player needed repair, the posterior probability of brand 1 increases to 0.61.
- This is because brand 1 DVD players are more likely to need warranty repair than are the other brands.
- The posterior probability of brand 3 is $P(A_3|B)=0.10$, which is much less than the prior probability $P(A_3)=0.20$

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Bayes' Theorem

(ทฤษฎีของเบย์)

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Bayes' Theorem

- Computation of posterior probability $P(A_j|B)$ from
  - given prior probabilities $P(A_i)$ and
  - conditional probabilities $P(B|A_i)$ occupies a central position in elementary probability.

- General rule for such computations, which is really just a simple application of multiplication rule, goes back to Reverend Thomas Bayes, who lived in the eighteenth century.
- To state it we first need another result.
  - Recall that events $A_1, \ldots, A_k$ are mutually exclusive if no two have any common outcomes.
  - Events are *exhaustive* if one A_i must occur, so that

$$A_1 \cup A_2 \cup \cdots \cup A_k = S$$

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Bayes' Theorem

The Law of Total Probability

- Let $A_1, \ldots, A_k$ be mutually exclusive and exhaustive events. Then for any other event B,

$$\begin{aligned} P(B) &= P(B \mid A_1)P(A_1) + P(B \mid A_2)P(A_2) + \cdots + P(B \mid A_k)P(A_k) \\ &= \sum_{i=1}^{k} P(B \mid A_i)P(A_i) \end{aligned} \quad (2.5)$$

Bayes' Theorem

Proof

- Because the A_i s are mutually exclusive and exhaustive, if B occurs it must be in conjunction with exactly one of the A_j's
- That is, $B = (A_1 \cap B) \cup \cdots \cup (A_k \cap B)$
- where events $A_i \cap B)$ are mutually exclusive
- This "partitioning of B" is illustrated in Figure 2.11. Thus

$$P(B) = \sum_{i=1}^{k} P(A_i \cap B) = \sum_{i=1}^{k} P(B \mid A_i)P(A_i)$$

as desired.

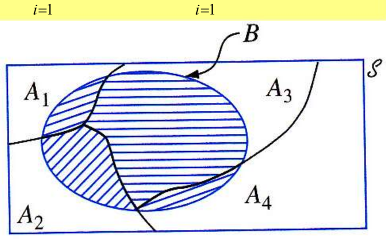

Figure 2.11 Partition of B by mutually exclusive and exhaustive A_i's

Bayes' Theorem

- An example of the use of Equation (2.5) appeared in answering question 2 of Example 2.29,
- where

  A_1 = {brand 1},

  A_2 = {brand 2},

  A_3 = {brand 3}, and B = {repair}

Law of Total Probability

$$P(B) = P(B \mid A_1)P(A_1) + P(B \mid A_2)P(A_2) + \cdots + P(B \mid A_k)P(A_k)$$
$$= \sum_{i=1}^{k} P(B \mid A_i)P(A_i)$$

Equation (2.5)

answering question 2 of Example 2.29

$$P(B) = P[(brand\ 1\ and\ repair)\ or\ (brand\ 2\ and\ repair)\ or\ (brand\ 3\ and\ repair)]$$
$$= P(A_1 \cap B) + P(A_2 \cap B) + P(A_3 \cap B)$$
$$= 0.125 + 0.060 + 0.020 = 0.205$$

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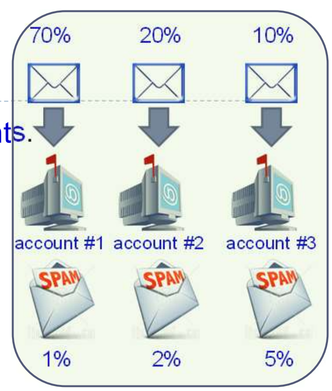
Example 30



- An individual has 3 different email accounts.
- Most of her messages, in fact
  - 70%, come into account #1, whereas
  - 20% come into account #2 and
  - the remaining 10% into account #3
- Of messages into account #1, only 1% are spam, whereas the corresponding percentages for accounts #2 and #3 are 2% and 5%, respectively.
- What is probability that a randomly selected message is spam?

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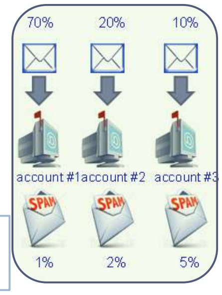
Example 30

cont'd

- To answer this question, let's first establish some notation:

  A_i = {message is from account # i} for i = 1, 2, 3,

  B = {message is spam}



- Then the given percentages imply that

  $P(A_1) = 0.70$,

  $P(A_2) = 0.20$,

  $P(A_3) = 0.10$

messages into account #2, 2% are spam

$P(B|A_1) = .01$, $P(B|A_2) = .02$, $P(B|A_3) = .05$

messages into account #1, 1% are spam

messages into account #3, 5% are spam

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Example 30

cont'd

- Now it is simply a matter of substituting into the equation for the **law of total probability**:

$$P(B) = P(B \mid A_1)P(A_1) + P(B \mid A_2)P(A_2) + \cdots + P(B \mid A_k)P(A_k)$$
$$= \sum_{i=1}^{k} P(B \mid A_i)P(A_i)$$

Equation (2.5)

$P(B) = (0.01)(0.70) + (0.02)(0.20) + (0.05)(0.10) = 0.016$

In the long run, 1.6% of this individual's messages will be spam.

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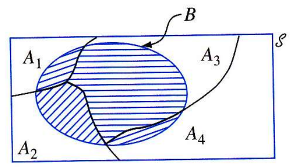
Bayes' Theorem



Bayes' Theorem

Let $A_1, A_2, \ldots, A_k$ be a collection of *k* mutually exclusive and exhaustive events with *prior* probabilities $P(A_i)$ $(i=1,2,\ldots,k)$.

Then for any other event B for which $P(B)>0$, the *posterior* probability of A_j given that B has occurred is

Multiplication Rule for $P(A \cap B)$

$$P(A_j \mid B) = \frac{P(A_j \cap B)}{P(B)} = \frac{P(B \mid A_j)P(A_j)}{\boxed{\sum_{i=1}^{k} P(B \mid A_i).P(A_i)}} \qquad j = 1,2,\ldots,k \qquad (2.6)$$

law of total probability

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Example

- Incidence of a rare disease.
- Only 1 in 1,000 adults in afflicted with a rare disease for which a diagnostic test has been developed.
- The test is such that when

  an individual actually has the disease, a positive result will occur **99%** of the time,

  whereas

  an individual without the disease will show a positive test result only **2%** of the time.
- If a randomly selected individual is tested and the **result is positive**, what is **probability** that **individual has the disease?**

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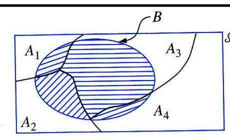
Example (cont.)



- To use Bayes' theorem,

$$P(A_j \mid B) = \frac{P(A_j \cap B)}{P(B)} = \frac{P(B \mid A_j)P(A_j)}{\sum_{i=1}^{k} P(B \mid A_i).P(A_i)} \qquad j = 1,2,\ldots,k$$

- Let
  - A_1 = {individual has the disease},
  - A_2 = {individual does not have the disease}, and
  - B = {positive test result}
- Then
  - $P(A_1) = 0.001$, Only 1 in 1,000 adults in afflicted with rare disease
  - $P(A_2) = 0.999$,
  - $P(B|A_1) = 0.99$, and individual has the disease, a positive result will occur **99%**
  - $P(B|A_2) = 0.02$ individual without disease will show positive test result only **2%**

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Example (cont.)

- Then tree diagram for this problem is in Figure 2.12

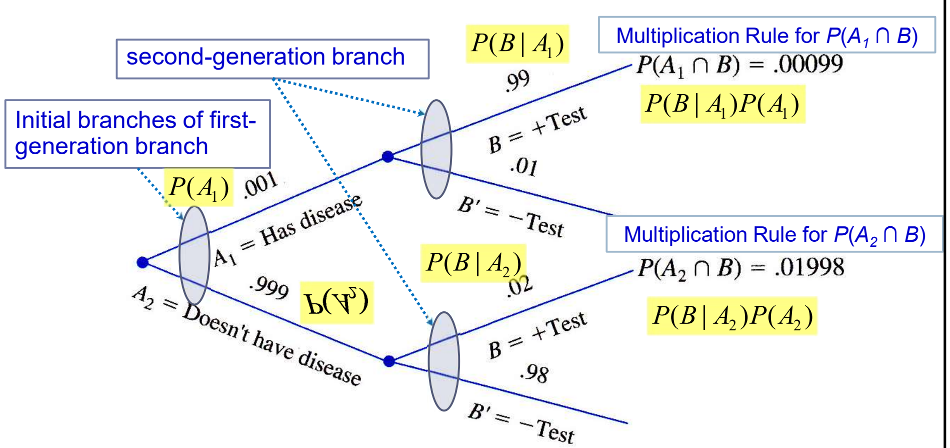

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Figure 2.12 Tree diagram for the rare-disease problem

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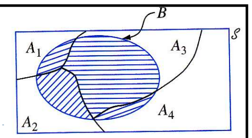
Example (cont.)



- Next to each branch corresponding to a positive test result, multiplication rule yields the recorded probabilities
- Therefore,

law of total probability

$$P(B) = P(B \mid A_1)P(A_1) + P(B \mid A_2)P(A_2) = \sum_{i=1}^{2} P(B \mid A_i)P(A_i)$$ Equation (2.5)

$$P(B) = (0.99 \times 0.001) + (0.02 \times 0.999)$$

$$P(B) = 0.00099 + 0.01998 = 0.02097$$

- From which we have

$$P(A_1 \mid B) = \frac{P(A_1 \cap B)}{P(B)} = \frac{0.00099}{0.02097} = 0.047$$

If a randomly selected individual is tested and the **result is positive**, what is **probability** that **individual has the disease?**

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