3.3 Expected Values

The Expected Value of *X*

Expected Values

- Consider university having 15,000 students
- Let X = number of courses for which randomly selected student is registered
- pmf of X follows

x	1	2	3	4	5	6	7
$p(x)$	.01	.03	.13	.25	.39	.17	.02
Number registered	150	450	1950	3750	5850	2550	300

- since p(1)=0.01, we know that (0.01) (15,000) = 150 of student are registered for one course

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Expected Values

Cont.

$$\text{average no. of courses per student} = \frac{\text{total no. of courses taken by all students}}{\text{total no. of students}}$$

(*average value of X in Population*)

x	1	2	3	4	5	6	7
$p(x)$	.01	.03	.13	.25	.39	.17	.02
Number registered	150	450	1950	3750	5850	2550	300

Each of 150 students is taking one course

150 contribute 1(150) courses to total

Each of 450 students is taking two courses

450 contribute 2(450) courses to total

$$\text{Population averge value of } X = \frac{1(150)+2(450)+3(1950)+\cdots+7(300)}{15{,}000} = 4.57$$

$$1\cdot p(1)+2\cdot p(2)+3\cdot p(3)+4\cdot p(4)+5\cdot p(5)+6\cdot p(6)+7\cdot p(7)$$

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Expected Values

Cont.

$$\textit{Population averge value of } X = 1\cdot p(1) + 2\cdot p(2) + 3\cdot p(3) + 4\cdot p(4) + 5\cdot p(5) + 6\cdot p(6) + 7\cdot p(7)$$

To compute population average value of X ,
only possible values of X along with their probabilities are needed

⇩

Population size is irrelevant as long as pmf is given

⇩

Average or mean value of X is then weighted average of possible values 1, 2, …, 7
Where weights are probabilities of those values

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Expected Value of X

Definition

Let X be discrete random variable with set of possible values D and pmf $p(x)$

Expected Value or **Mean value** of X, denoted by

- $E(X)$ or
- μ_X

is

$$E(X) = \mu_X = \sum_{x \in D} x \cdot p(x)$$

When it is clear to which X the expected value refers, μ rather than μ_X is often used.

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Example 3.16

Consider a university having 15,000 students and
let X = of courses for which randomly selected student is registered.
The pmf of X follows.

x	1	2	3	4	5	6	7
$p(x)$	.01	.03	.13	.25	.39	.17	.02
Number registered	150	450	1950	3750	5850	2550	300

$$\boxed{E(X) = \mu_X = \sum_{x \in D} x \cdot p(x)}$$

$$\begin{aligned} E(X) = \mu_X = \mu &= [1 \cdot p(1) + 2 \cdot p(2) + 3 \cdot p(3) + 4 \cdot p(4) + 5 \cdot p(5) + 6 \cdot p(6) + 7 \cdot p(7)] \\ &= [(1)(0.01) + (2)(0.03) + (3)(0.13) + (4)(0.25) + (5)(0.39) + (6)(0.17) + (7)(0.02)] \\ &= 0.01 + 0.06 + 0.39 + 1.00 + 1.95 + 1.02 + 0.14 \\ &= 4.57 \end{aligned}$$

If we think of population as consisting of the X values 1, 2, . . . , 7, then $\mu = 4.57$ is **Population Mean**.

In the sequel, we will often refer to μ as the ***Population Mean*** rather than the **Mean of X in the population.**

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Example 3.16

cont'd

Notice that
μ here is not 4, the ordinary average of 1, . . . , 7,
because distribution puts more weight on 4, 5, and 6 than on other X values.

x	1	2	3	4	5	6	7
$p(x)$	.01	.03	.13	.25	.39	.17	.02
Number registered	150	450	1950	3750	5850	2550	300

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Example 3.17

APGAR เป็นการตรวจที่ใช้ไปทั่วโลกในห้องคลอดเพื่อ ที่จะประเมินสุขภาพทั่วไป และความสมบูรณ์ของทารกแรกเกิด

○ Just after birth, each newborn child is rated on a scale called Apgar Scale. Possible ranges are 0, 1, …, 10, with child's rating determined by color (สีผิว), muscle tone (ความตึงตัวของกล้ามเนื้อ), respiratory effort (ความสามารถในการหายใจ), heartbeat (ภาวะการเต้นของหัวใจ), and reflex irritability (การตอบสนองต่อการกระตุ้น)

(the best possible score is 10).

○ Let X be Apgar score of a randomly selected child born at a certain hospital during next year, and suppose that pmf of X is

x	0	1	2	3	4	5	6	7	8	9	10
$p(x)$	.002	.001	.002	.005	.02	.04	.18	.37	.25	.12	.01

Then mean of value X is

$$E(X) = \mu = 0(.002) + 1(.001) + 2(.002) + \cdots + 8(.25) + 9(.12) + 10(.01) = 7.15$$

○ Again μ is not a possible value of variable X.

○ Also, because variable refers to future child, there is no concrete existing population to which μ refers.

○ Instead, we think of pmf as model for conceptual population consisting of value 0, 1, 2, … ,10.

○ Mean value of this conceptual population is $\mu=7.15$.

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Example 3.18

Let X=1, if a randomly selected component needs warranty service and = 0 otherwise.

Then X is Bernoulli random variable with pmf

$$p(x) = \begin{cases} 1-p & x = 0 \\ p & x = 1 \\ 0 & x \neq 0, 1 \end{cases}$$

from which

$$E(X) = 0 \cdot p(0) + 1 \cdot p(1) = 0(1-p) + 1(p) = p.$$

○ That is, expected value of X is just probability that X takes on value 1.

○ If we conceptualize a population consisting of *0*s in proportion *1-p* and *1*s in proportion *p*, then

○ Population average is $\mu=p$

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$p(1) = P(X = 1) = P(B) = p$

Ex. 3.12

Example 3.19

$p(2) = P(X = 2) = P(GB) = P(G) \cdot P(B) = (1-p) \cdot p$

$p(3) = P(X = 3) = P(GGB) = P(G) \cdot P(G) \cdot P(B) = (1-p)^2 \cdot p$

$\vdots$

The general form for pmf of X =number of children born up to and including the first boy is

$$p(x) = \begin{cases} p(1-p)^{x-1} & x = 1, 2, 3, \ldots \\ 0 & \text{otherwise} \end{cases}$$

From the definition

$$E(X) = \sum_{D} x \cdot p(x) = \sum_{x=1}^{\infty} xp(1-p)^{x-1} = p \sum_{x=1}^{\infty} \left[-\frac{d}{dp} (1-p)^x \right] \quad (3.9)$$

If we interchange the order of taking derivative and summation, the sum is that geometric series.

After sum is computed, derivative is taken, and final results is

$$E(X) = \frac{1}{p}$$

If *p is near 1*, we expect to see a boy very soon, whereas if *p is near 0*, we expect many births before the first boy.

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Cont.

Example 3.19

For $p = 0.5$, $E(X) = 1/p = 1/0.5 = 2$

There is another frequently used interpretation of μ . Consider the pmf

$$p(x) = \begin{cases} (.5) \cdot (.5)^{x-1} & \text{if } x = 1, 2, 3, \ldots \\ 0 & \text{otherwise} \end{cases}$$

○ This is pmf of X = the number of tosses of a fair coin necessary to obtain the first H (a special case of Example 3.19).

○ Suppose we observe value x from this pmf (toss a coin until H appears), then observe independently another value (keep tossing), then another, and so on.

○ If after observing a very large number of x values, we average them, resulting sample average will be very near to $\mu=2$.

○ This is , μ can be interpreted as the long-run average observed value of X when experiment is performed repeatedly

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The Variance of X

The Variance of X

- Expected value of X describes where probability distribution is centered
- Using physical analogy of placing point mass p(x) at value x on one-dimensional axis, if axis were then supported by a fulcrum placed at μ, there would be no tendency for axis to tilt.

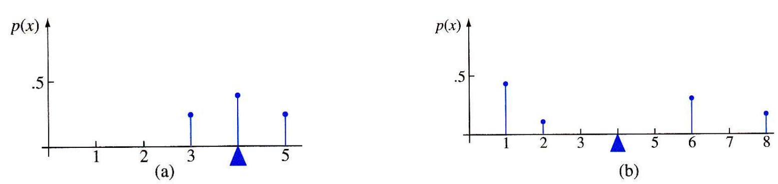


Figure 3.7 Two different probability distributions with $\mu = 4$

- Although both distributions have the same center μ,
  - distribution of Figure 3.7(b) has greater spread or variability or dispersion than does that of Figure 3.7(a).
- We will use **Variance of X** to assess the amount of variability in (distribution of) X

The Variance of X

Definition

Let X have pmf $p(x)$ and expected value μ.

Then **Variance** of X, denoted by $V(X)$ or σ^2_X, or just σ^2, is

$$V(X) = \sum_D (x-\mu)^2 \cdot p(x) = E\left[(X-\mu)^2\right]$$

Standard Deviation (SD) of X is

$$\sigma_X = \sqrt{\sigma_X^2}$$

○ If most of probability distribution is close to μ, then σ^2 will be relatively small.

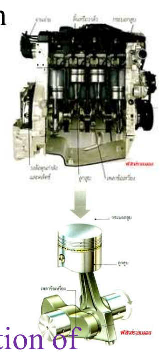
○ However, if there are x values far from μ that have large $p(x)$, then σ^2 will be quite large.

Example 3.24

x	4	6	8
$p(x)$	.5	.3	.2

If X is the number of cylinders on the next car to be tuned at a service facility

[p(4)=0.5, p(6) = 0.3, p(8) = 0.2, from which μ = 5.4], then



$$V(X) = \sum_D (x-\mu)^2 \cdot p(x) = E\left[(X-\mu)^2\right]$$

$$V(X) = \sigma^2 = \sum_{x=4}^{8} (x-5.4)^2 \cdot p(x)$$

$$= (4-5.4)^2(.5) + (6-5.4)^2(.3) + (8-5.4)^2(.2) = 2.44$$

The standard deviation of X is $\sigma = \sqrt{2.44} = 1.562$.

When pmf $p(x)$ specifies mathematical model for distribution of population values, both σ^2 and σ measure the spread of values in population; σ^2 is population variance, and σ is population standard deviation.

A Shortcut Formula for σ^2

A Shortcut Formula for σ^2

$V(X) = \sum_D (x-\mu)^2 \cdot p(x) = E\left[(X-\mu)^2\right]$

The number of arithmetic operations necessary to compute σ^2 can be reduced by using an alternative formula.

Proposition

$$V(X) = \sigma^2 = \left[\sum_D x^2 \cdot p(x)\right] - \mu^2 = E(X^2) - [E(X)]^2$$

In using this formula, $E(X^2)$ is computed first without any subtraction; then $E(X)$ is computed, squared, and subtracted (once) from $E(X^2)$.

A Shortcut Formula for σ^2

Proof of the Shortcut Formula

Expand $(x-\mu)^2$ in the definition of σ^2 to obtain $x^2-2\mu x+\mu^2$,and then carry $\sum$ through to each of the three terms:

$$\sigma^2 = \sum_D x^2 \cdot p(x) - 2\mu \cdot \sum_D x \cdot p(x) + \mu^2 \sum_D p(x)$$

$$= E(X^2) - 2\mu \cdot \mu^2 = E(X^2) - \mu^2$$

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Example 3.25

The pmf of the number of cylinders X on next car to be tuned at a certain facility was given in Example 3.24 as

p(4) = 0.5, p(6) = 0.3, and p(8) = 0.2, from which $\mu = 5.4$, and

$$V(X) = \sigma^2 = \left[\sum_D x^2 \cdot p(x)\right] - \mu^2 = E(X^2) - [E(X)]^2$$

x	4	6	8
$p(x)$	.5	.3	.2

Thus

$$E(X^2) = (4^2)(0.5) + (6^2)(0.3) + (8^2)(0.2) = 31.6$$

$$V(X) = \sigma^2 = E(X^2) - [E(X)]^2$$

$$= 31.6 - (5.4)^2 = 2.44$$

As in Example 3.24.

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3.4 Binomial Probability Distribution

The Binomial Probability Distribution

There are many experiments that conform either exactly or approximately to the following list of requirements:

1. Experiment consists of sequence of n smaller experiments called *trials,* where n is fixed in advance of experiment.

2. Each trial can result in one of the same two possible outcomes (dichotomous trials), which we generically denote by success (S) and failure (F).

3. Trials are independent, so that outcome on any particular trial does not influence outcome on any other trial.

4. Probability of success $P(S)$ is <u>constant</u> from trial to trial; we denote this probability by p.

The Binomial Probability Distribution

Definition

An experiment for which Conditions 1–4 are satisfied is called a **Binomial Experiment.**

Example 3.27

- The same coin is tossed successively and independently n times.
- We arbitrarily use
  - S to denote outcome H (heads) and
  - F to denote the outcome T (tails).
- Then this experiment satisfies Conditions 1–4.

- Tossing a thumbtack n times, with
  - S = point up and
  - F = point down, also results in a binomial experiment.

Many experiments involve a sequence of independent trials for which there are more than two possible outcomes on any one trial.

Binomial experiment can then be created by dividing the possible outcomes into two groups.

Example 3.28

- Color of pea seeds is determined by a single genetic locus.
- If two alleles at this locus are *AA* or *Aa* (genotype), then pea will be yellow (phenotype), and
- if allele is *aa*, pea will be green.
- Suppose we pair of *20 Aa* seeds and cross the two seeds in each of ten pairs to obtain ten new genotypes
- Call each new genotype a *success S* if it is *aa* and a failure otherwise
- Then with this identification of *S* and *F*, the experiment is binomial with $n=10$ and $p=P$ *(aa genotype)*
- If each member of pair is equally likely to contribute *a or A*, then $p=P(a).P(a)$

$$p = P(a).P(a) = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{1}{4}$$

Example 3.29

- Suppose a certain city has 50 licensed restaurants, of which 15 currently have at least one serious health code violation and the other 35 have no serious violation.
- There are five inspectors, each of whom will inspect one restaurant during the coming week.
- Name of each restaurant is written on a different slip of paper, and after slips are thoroughly mixed, each inspector in turn draws one of the slips *without replacement*.
- Label the *i*th trial as a success if the *i*th restaurant selected (i=1,2,…,5) has no serious violations. Then

Cont.

Example 3.29

$$P(S\ on\ 1^{st}\ trial=\frac{35}{50}=0.7$$
$$P(S\ on\ 2^{nd}\ trial \mid S\ on\ 1^{st}\ trial)=\frac{34}{49}=0.6938$$
$$P(S\ on\ 3^{rd}\ trial \mid S\ on\ 2^{nd}\ trial)=\frac{33}{48}=0.6875$$
$$P(S\ on\ 4^{th}\ trial \mid S\ on\ 3^{rd}\ trial)=\frac{32}{47}=0.6800$$
$$P(S\ on\ 5^{th}\ trial \mid S\ on\ 4^{th}\ trial)=\frac{31}{46}=0.6739$$

$$P(S \text{ on first trial}) = \frac{35}{50} = .70$$

$$\begin{aligned} P(S \text{ on second trial}) &= P(SS) + P(FS) \\ &= P(\text{second } S \mid \text{first } S)\, P(\text{first } S) \\ &\quad + P(\text{second } S \mid \text{first } F)\, P(\text{first } F) \\ &= \frac{34}{49}\cdot\frac{35}{50} + \frac{35}{49}\cdot\frac{15}{50} = \frac{35}{50}\left(\frac{34}{49}+\frac{15}{49}\right) = \frac{35}{50} = .70 \end{aligned}$$

Similarly, it can be shown that $P(S \text{ on } i\text{th trial}) = .70$ for $i = 3, 4, 5$. However,

$$P(S \text{ on fifth trial} \mid SSSS) = \frac{31}{46} = .67$$

whereas

$$P(S \text{ on fifth trial} \mid FFFF) = \frac{35}{46} = .76$$

The experiment is not binomial because the trials are not independent. In general, if sampling is without replacement, the experiment will not yield independent trials. If each slip had been replaced after being drawn, then trials would have been independent, but this might have resulted in the same restaurant being inspected by more than one inspector. ■

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Example 3.30

A certain state has 500,000 licensed drivers, of whom 400,000 are insured. A sample of 10 drivers is chosen without replacement. The ith trial is labeled S if the ith driver chosen is insured. Although this situation would seem identical to that of Example 3.29, the important difference is that the size of the population being sampled is very large relative to the sample size. In this case

$$P(S \text{ on } 2 \mid S \text{ on } 1) = \frac{399{,}999}{499{,}999} = .80000$$

and

$$P(S \text{ on } 10 \mid S \text{ on first } 9) = \frac{399{,}991}{499{,}991} = .799996 \approx .80000$$

These calculations suggest that although the trials are not exactly independent, the conditional probabilities differ so slightly from one another that for practical purposes the trials can be regarded as independent with constant $P(S) = .8$. Thus, to a very good approximation, the experiment is binomial with $n = 10$ and $p = .8$. ■

We will use the following rule of thumb in deciding whether a "without-replacement" experiment can be treated as a binomial experiment.

.20

The Binomial Probability Distribution

Rule

Consider sampling without replacement from a dichotomous population of size *N*.

If sample size (number of trials) *n* is at most 5% of population size, experiment can be analyzed as though it were exactly a **Binomial Experiment**.

Ex.

$N = 50$, $n = 5$ ➔ $n/N = 5/50 = 0.1 > 0.05$: binomial experiment is not a good approximation

$N = 500{,}000$, $n = 10$ ➔ $n/N = 10/500{,}000 = 0.00002 < 0.05$: binomial experiment is a good approximation

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Binomial Random Variable and Distribution

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The Binomial Random Variable and Distribution

In most binomial experiments, it is the total number of *S*'s, rather than knowledge of exactly which trials yielded *S*'s, that is of interest.

Definition

Binomial Random Variable *X* associated with a binomial experiment consisting of *n* trials is defined as

X = the **number** of ***S***'s among the ***n*** **trials**

ตัวแปรสุ่มไบโนเมียล X คือ จำนวนครั้งของความสำเร็จในการทดลอง n ครั้ง

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The Binomial Random Variable and Distribution

Suppose, for example, that $n = 3$.

Then there are eight possible outcomes for the experiment:

SSS SSF SFS SFF FSS FSF FFS FFF

From definition of *X*,

$X(SSS)=3$, $X(SSF)=2$, $X(SFS)=2$, $X(SFF)=1$, $X(FSS)=2$, $X(FSF)=1$, $X(FFS)=1$, and $X(FFF)=0$

Possible values for X in ***n*-trial** experiment are

$$x = 0, 1, 2, 3, \ldots, n$$

We will often write **X~Bin (*n*, *p*)** to indicate that X is binomial rv based on **n trials** with **success probability *p***

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Binomial Random Variable and Distribution

Notation

Because **pmf** of binomial rv X depends on two parameters $\boldsymbol{n}$ and $\boldsymbol{p}$, we denote the pmf by $\boldsymbol{b(x; n, p)}$.

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Binomial Random Variable and Distribution

- Consider first the case $n = 4$ for which each outcome, its probability, and corresponding x value are listed in Table 3.1.

Outcome	x	Probability	Outcome	x	Probability
$SSSS$	4	p^4	$FSSS$	3	$p^3(1-p)$
$SSSF$	3	$p^3(1-p)$	$FSSF$	2	$p^2(1-p)^2$
$SSFS$	3	$p^3(1-p)$	$FSFS$	2	$p^2(1-p)^2$
$SSFF$	2	$p^2(1-p)^2$	$FSFF$	1	$p(1-p)^3$
$SFSS$	3	$p^3(1-p)$	$FFSS$	2	$p^2(1-p)^2$
$SFSF$	2	$p^2(1-p)^2$	$FFSF$	1	$p(1-p)^3$
$SFFS$	2	$p^2(1-p)^2$	$FFFS$	1	$p(1-p)^3$
$SFFF$	1	$p(1-p)^3$	$FFFF$	0	$(1-p)^4$

Table 3.1 Outcomes and Probabilities for a Binomial Experiment with four Trials

- For example,

$$\begin{aligned} P(SSFS) &= P(S)\cdot P(S)\cdot P(F)\cdot P(S) \qquad (\textit{indenpendent trials}) \\ &= p\cdot p\cdot(1-p)\cdot p \qquad (\textit{cons}\tan t\ P(S)) \\ &= p^3\cdot(1-p) \end{aligned}$$

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Binomial Random Variable and Distribution

- In this special case, we wish $b(x;\ 4,\ p)$ for $x = 0,1,2,3$, and 4
- For $b(3;\ 4,\ p)$, and $x = 3$
  - Sum probabilities associated with each such outcome

$$b(3;4,p) = P(FSSS) + P(SFSS) + P(SSFS) + P(SSSF) = 4p^3 \cdot (1-p)$$

- There are four outcomes with $x=3$ and each has probability $p^3(1-p)$
  - *Order of S*'s and *F*'s is not important, but only the number of *S*'s, so

$$b(3;4,p) = \begin{Bmatrix} \text{number of outcomes} \\ \text{with } X = 3 \end{Bmatrix} \cdot \begin{Bmatrix} \text{probability of any particular outcome} \\ \text{with } X = 3 \end{Bmatrix}$$

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Binomial Random Variable and Distribution

- Similarly, $b(2;\ 4,\ p) = 6p^2(1-p)^2$, which is also product of the number of outcomes with $X=2$ and probability of any such outcome
  - In general,

$$b(x;4,p) = \begin{Bmatrix} \text{number of sequences of} \\ \text{length } n \text{ consisting of } x\ S's \end{Bmatrix} \cdot \begin{Bmatrix} \text{probability of any} \\ \text{particular such sequence} \end{Bmatrix}$$

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Binomial Random Variable and Distribution

○ Since ordering of S's and F's is not important, the second factor in the previous equation is

$$p^x(1-p)^{n-x}$$

(e.g., the first x trials resulting in S and the last n-x resulting in F.)

○ First factor is the number of ways of choosing x of n *trials* to be S's

⇩

Number of combinations of size x that can be constructed from n distinct objects (trial here).

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The Binomial Random Variable and Distribution

THEOREM

$$b(x;n,p)=\begin{cases}\binom{n}{x}p^x(1-p)^{n-x} & x=0,1,2,\ldots,n\\ 0 & otherwise\end{cases}$$

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Example 3.31

- Each of six randomly selected cola drinkers is given glass containing cola S and one containing cola F.
- Glasses are identical in appearance except for a code on the bottom to identify cola.
- Suppose there is actually no tendency among cola drinkers to prefer one cola to the other.

Then $p = P$(a selected individual prefers S) = 0.5, so with
X = the number among the six who prefer S,
$X \sim \text{Bin}(6,0.5)$.

Thus

$$P(X = 3) = b(3; 6, 0.5) = \binom{6}{3}(0.5)^3(0.5)^3 = 20(0.5)^6 = 0.313$$

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Example 3.31

cont'd

Probability that at least three prefer S is

$$P(X \geq 3) = \sum_{x=3}^{6} b(x; 6, 0.5)$$

$$= \sum_{x=3}^{6} \binom{6}{x} (0.5)^x(0.5)^{6-x}$$

$$= 0.656$$

and Probability that at most one prefers S is

$$P(X \leq 1) = \sum_{x=0}^{1} b(x; 6, 0.5)$$

$$= 0.109$$

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Using Binomial Tables

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Using Binomial Tables

Even for a relatively small value of n, the computation of binomial probabilities can be tedious.

Appendix Table A.1 tabulates the cdf $F(x) = P(X \leq x)$ for $n = 5, 10, 15, 20, 25$ in combination with selected values of p.

Notation

For $X \sim \text{Bin}(n,p)$, *the cdf will be denoted by*

$$P(X \leq x) = B(x; n, p) = \sum_{y=0}^{x} b(y; n, p) \qquad x = 0,1,2,\ldots,n$$

Various other probabilities can then be calculated using the proposition on cdf's. (from Section 3.2 : Probability Distributions for Discrete Random Variables)

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Using Binomial Tables

Table A.1 Cumulative Binomial Probabilities

$$B(x; n, p) = \sum_{y=0}^{x} b(y; n, p)$$

$$P(X \le x) = B(x; n, p) = \sum_{y=0}^{x} b(y; n, p) \qquad x = 0,1,2,\ldots,n$$

a. $n = 5$

		p														
		0.01	**0.05**	**0.10**	**0.20**	**0.25**	**0.30**	**0.40**	**0.50**	**0.60**	**0.70**	**0.75**	**0.80**	**0.90**	**0.95**	**0.99**
	0	.951	.774	.590	.328	.237	.168	.078	.031	.010	.002	.001	.000	.000	.000	.000
	1	.999	.977	.919	.737	.633	.528	.337	.188	.087	.031	.016	.007	.000	.000	.000
x	2	1.000	.999	.991	.942	.896	.837	.683	.500	.317	.163	.104	.058	.009	.001	.000
	3	1.000	1.000	1.000	.993	.984	.969	.913	.812	.663	.472	.367	.263	.081	.023	.001
	4	1.000	1.000	1.000	1.000	.999	.998	.990	.969	.922	.832	.763	.672	.410	.226	.049

b. $n = 10$

		p														
		0.01	**0.05**	**0.10**	**0.20**	**0.25**	**0.30**	**0.40**	**0.50**	**0.60**	**0.70**	**0.75**	**0.80**	**0.90**	**0.95**	**0.99**
	0	.904	.599	.349	.107	.056	.028	.006	.001	.000	.000	.000	.000	.000	.000	.000
	1	.996	.914	.736	.376	.244	.149	.046	.011	.002	.000	.000	.000	.000	.000	.000
	2	1.000	.988	.930	.678	.526	.383	.167	.055	.012	.002	.000	.000	.000	.000	.000
	3	1.000	.999	.987	.879	.776	.650	.382	.172	.055	.011	.004	.001	.000	.000	.000
	4	1.000	1.000	.998	.967	.922	.850	.633	.377	.166	.047	.020	.006	.000	.000	.000
x	5	1.000	1.000	1.000	.994	.980	.953	.834	.623	.367	.150	.078	.033	.002	.000	.000
	6	1.000	1.000	1.000	.999	.996	.989	.945	.828	.618	.350	.224	.121	.013	.001	.000
	7	1.000	1.000	1.000	1.000	1.000	.998	.988	.945	.833	.617	.474	.322	.070	.012	.000
	8	1.000	1.000	1.000	1.000	1.000	1.000	.998	.989	.954	.851	.756	.624	.264	.086	.004
	9	1.000	1.000	1.000	1.000	1.000	1.000	1.000	.999	.994	.972	.944	.893	.651	.401	.096

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Using Binomial Tables

$$P(X \le x) = B(x; n, p) = \sum_{y=0}^{x} b(y; n, p) \qquad x = 0,1,2,\ldots,n$$

c. $n = 15$

		p														
		0.01	**0.05**	**0.10**	**0.20**	**0.25**	**0.30**	**0.40**	**0.50**	**0.60**	**0.70**	**0.75**	**0.80**	**0.90**	**0.95**	**0.99**
	0	.860	.463	.206	.035	.013	.005	.000	.000	.000	.000	.000	.000	.000	.000	.000
	1	.990	.829	.549	.167	.080	.035	.005	.000	.000	.000	.000	.000	.000	.000	.000
	2	1.000	.964	.816	.398	.236	.127	.027	.004	.000	.000	.000	.000	.000	.000	.000
	3	1.000	.995	.944	.648	.461	.297	.091	.018	.002	.000	.000	.000	.000	.000	.000
	4	1.000	.999	.987	.836	.686	.515	.217	.059	.009	.001	.000	.000	.000	.000	.000
	5	1.000	1.000	.998	.939	.852	.722	.402	.151	.034	.004	.001	.000	.000	.000	.000
	6	1.000	1.000	1.000	.982	.943	.869	.610	.304	.095	.015	.004	.001	.000	.000	.000
x	7	1.000	1.000	1.000	.996	.983	.950	.787	.500	.213	.050	.017	.004	.000	.000	.000
	8	1.000	1.000	1.000	.999	.996	.985	.905	.696	.390	.131	.057	.018	.000	.000	.000
	9	1.000	1.000	1.000	1.000	.999	.996	.966	.849	.597	.278	.148	.061	.002	.000	.000
	10	1.000	1.000	1.000	1.000	1.000	.999	.991	.941	.783	.485	.314	.164	.013	.001	.000
	11	1.000	1.000	1.000	1.000	1.000	1.000	.998	.982	.909	.703	.539	.352	.056	.005	.000
	12	1.000	1.000	1.000	1.000	1.000	1.000	1.000	.996	.973	.873	.764	.602	.184	.036	.000
	13	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	.995	.965	.920	.833	.451	.171	.010
	14	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	.995	.987	.965	.794	.537	.140

(continued)

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Using Binomial Tables

Table A.1 Cumulative Binomial Probabilities *(cont.)*

$B(x; n, p) = \sum_{y=0}^{x} b(y; n, p)$

d. $n = 20$

$$P(X \le x) = B(x;n,p) = \sum_{y=0}^{x} b(y;n,p) \qquad x = 0,1,2,\ldots,n$$

x \ p	0.01	0.05	0.10	0.20	0.25	0.30	0.40	0.50	0.60	0.70	0.75	0.80	0.90	0.95	0.99
0	.818	.358	.122	.012	.003	.001	.000	.000	.000	.000	.000	.000	.000	.000	.000
1	.983	.736	.392	.069	.024	.008	.001	.000	.000	.000	.000	.000	.000	.000	.000
2	.999	.925	.677	.206	.091	.035	.004	.000	.000	.000	.000	.000	.000	.000	.000
3	1.000	.984	.867	.411	.225	.107	.016	.001	.000	.000	.000	.000	.000	.000	.000
4	1.000	.997	.957	.630	.415	.238	.051	.006	.000	.000	.000	.000	.000	.000	.000
5	1.000	1.000	.989	.804	.617	.416	.126	.021	.002	.000	.000	.000	.000	.000	.000
6	1.000	1.000	.998	.913	.786	.608	.250	.058	.006	.000	.000	.000	.000	.000	.000
7	1.000	1.000	1.000	.968	.898	.772	.416	.132	.021	.001	.000	.000	.000	.000	.000
8	1.000	1.000	1.000	.990	.959	.887	.596	.252	.057	.005	.001	.000	.000	.000	.000
9	1.000	1.000	1.000	.997	.986	.952	.755	.412	.128	.017	.004	.001	.000	.000	.000
10	1.000	1.000	1.000	.999	.996	.983	.872	.588	.245	.048	.014	.003	.000	.000	.000
11	1.000	1.000	1.000	1.000	.999	.995	.943	.748	.404	.113	.041	.010	.000	.000	.000
12	1.000	1.000	1.000	1.000	1.000	.999	.979	.868	.584	.228	.102	.032	.000	.000	.000
13	1.000	1.000	1.000	1.000	1.000	1.000	.994	.942	.750	.392	.214	.087	.002	.000	.000
14	1.000	1.000	1.000	1.000	1.000	1.000	.998	.979	.874	.584	.383	.196	.011	.000	.000
15	1.000	1.000	1.000	1.000	1.000	1.000	1.000	.994	.949	.762	.585	.370	.043	.003	.000
16	1.000	1.000	1.000	1.000	1.000	1.000	1.000	.999	.984	.893	.775	.589	.133	.016	.000
17	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	.996	.965	.909	.794	.323	.075	.001
18	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	.999	.992	.976	.931	.608	.264	.017
19	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	.999	.997	.988	.878	.642	.182

(continued)

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Using Binomial Tables

Table A.1 Cumulative Binomial Probabilities *(cont.)*

$B(x; n, p) = \sum_{y=0}^{x} b(y; n, p)$

e. $n = 25$

$$P(X \le x) = B(x;n,p) = \sum_{y=0}^{x} b(y;n,p) \qquad x = 0,1,2,\ldots,n$$

x \ p	0.01	0.05	0.10	0.20	0.25	0.30	0.40	0.50	0.60	0.70	0.75	0.80	0.90	0.95	0.99
0	.778	.277	.072	.004	.001	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000
1	.974	.642	.271	.027	.007	.002	.000	.000	.000	.000	.000	.000	.000	.000	.000
2	.998	.873	.537	.098	.032	.009	.000	.000	.000	.000	.000	.000	.000	.000	.000
3	1.000	.966	.764	.234	.096	.033	.002	.000	.000	.000	.000	.000	.000	.000	.000
4	1.000	.993	.902	.421	.214	.090	.009	.000	.000	.000	.000	.000	.000	.000	.000
5	1.000	.999	.967	.617	.378	.193	.029	.002	.000	.000	.000	.000	.000	.000	.000
6	1.000	1.000	.991	.780	.561	.341	.074	.007	.000	.000	.000	.000	.000	.000	.000
7	1.000	1.000	.998	.891	.727	.512	.154	.022	.001	.000	.000	.000	.000	.000	.000
8	1.000	1.000	1.000	.953	.851	.677	.274	.054	.004	.000	.000	.000	.000	.000	.000
9	1.000	1.000	1.000	.983	.929	.811	.425	.115	.013	.000	.000	.000	.000	.000	.000
10	1.000	1.000	1.000	.994	.970	.902	.586	.212	.034	.002	.000	.000	.000	.000	.000
11	1.000	1.000	1.000	.998	.980	.956	.732	.345	.078	.006	.001	.000	.000	.000	.000
12	1.000	1.000	1.000	1.000	.997	.983	.846	.500	.154	.017	.003	.000	.000	.000	.000
13	1.000	1.000	1.000	1.000	.999	.994	.922	.655	.268	.044	.020	.002	.000	.000	.000
14	1.000	1.000	1.000	1.000	1.000	.998	.966	.788	.414	.098	.030	.006	.000	.000	.000
15	1.000	1.000	1.000	1.000	1.000	1.000	.987	.885	.575	.189	.071	.017	.000	.000	.000
16	1.000	1.000	1.000	1.000	1.000	1.000	.996	.946	.726	.323	.149	.047	.000	.000	.000
17	1.000	1.000	1.000	1.000	1.000	1.000	.999	.978	.846	.488	.273	.109	.002	.000	.000
18	1.000	1.000	1.000	1.000	1.000	1.000	1.000	.993	.926	.659	.439	.220	.009	.000	.000
19	1.000	1.000	1.000	1.000	1.000	1.000	1.000	.998	.971	.807	.622	.383	.033	.001	.000
20	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	.991	.910	.786	.579	.098	.007	.000
21	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	.998	.967	.904	.766	.236	.034	.000
22	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	.991	.968	.902	.463	.127	.002
23	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	.998	.993	.973	.729	.358	.026
24	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	.999	.996	.928	.723	.222

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Example 3.32

- Suppose that *20%* of all copies of a particular textbook fail a certain binding strength test.

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- Let X denote the number among 15 randomly selected copies that fail the test.
- Then X has a binomial distribution with $n = 15$ and $p = 0.2$

1. Probability that <u>at most 8 fail</u> the test is

$$P(X \le 8) = \sum_{y=0}^{8} b(y;15,0.2) = B(8;15,0.2)$$

0.999

c. $n = 15$

		p														
		0.01	0.05	0.10	0.20	0.25	0.30	0.40	0.50	0.60	0.70	0.75	0.80	0.90	0.95	0.99
	0	.860	.463	.206	.035	.013	.005	.000	.000	.000	.000	.000	.000	.000	.000	.000
	1	.990	.829	.549	.167	.080	.035	.005	.000	.000	.000	.000	.000	.000	.000	.000
	2	1.000	.964	.816	.398	.236	.127	.027	.004	.000	.000	.000	.000	.000	.000	.000
	3	1.000	.995	.944	.648	.461	.297	.091	.018	.002	.000	.000	.000	.000	.000	.000
	4	1.000	.999	.987	.836	.686	.515	.217	.059	.009	.001	.000	.000	.000	.000	.000
	5	1.000	1.000	.998	.939	.852	.722	.402	.151	.034	.004	.001	.000	.000	.000	.000
	6	1.000	1.000	1.000	.982	.943	.869	.610	.304	.095	.015	.004	.001	.000	.000	.000
x	7	1.000	1.000	1.000	.996	.983	.950	.787	.500	.213	.050	.017	.004	.000	.000	.000
	8	1.000	1.000	1.000	.999	.996	.985	.905	.696	.390	.131	.057	.018	.000	.000	.000
	9	1.000	1.000	1.000	1.000	.999	.996	.966	.849	.597	.278	.148	.061	.002	.000	.000

Cont' d

Example 3.32

2. Probability that <u>exactly 8 fail</u> is

$$P(X = 8) = P(X \le 8) - P(X \le 7) = B(8; 15,0.2) - B(7; 15, 0.2)$$

c. $n = 15$

		p														
		0.01	0.05	0.10	0.20	0.25	0.30	0.40	0.50	0.60	0.70	0.75	0.80	0.90	0.95	0.99
	0	.860	.463	.206	.035	.013	.005	.000	.000	.000	.000	.000	.000	.000	.000	.000
	1	.990	.829	.549	.167	.080	.035	.005	.000	.000	.000	.000	.000	.000	.000	.000
	2	1.000	.964	.816	.398	.236	.127	.027	.004	.000	.000	.000	.000	.000	.000	.000
	3	1.000	.995	.944	.648	.461	.297	.091	.018	.002	.000	.000	.000	.000	.000	.000
	4	1.000	.999	.987	.836	.686	.515	.217	.059	.009	.001	.000	.000	.000	.000	.000
	5	1.000	1.000	.998	.939	.852	.722	.402	.151	.034	.004	.001	.000	.000	.000	.000
	6	1.000	1.000	1.000	.982	.943	.869	.610	.304	.095	.015	.004	.001	.000	.000	.000
x	7	1.000	1.000	1.000	.996	.983	.950	.787	.500	.213	.050	.017	.004	.000	.000	.000
	8	1.000	1.000	1.000	.999	.996	.985	.905	.696	.390	.131	.057	.018	.000	.000	.000
	9	1.000	1.000	1.000	1.000	.999	.996	.966	.849	.597	.278	.148	.061	.002	.000	.000

$$B(8; 15,0.2) = 0.999$$

$$B(7; 15,0.2) = 0.996$$

$$P(X = 8) = 0.999 - 0.996 = 0.003$$

Cont' d

Example 3.32

3. Finally, Probability that **between** 4 and 7, **inclusive**, fail is

$$\begin{aligned} P(4 \le X \le 7) &= P(X = 4, 5, 6, or\ 7) \\ &= P(X \le 7) - P(X \le 3) \\ &= B(7; 15, 0.2) - B(3; 15, 0.2) \\ &= 0.996 - 0.648 \\ &= 0.348 \end{aligned}$$

c. $n = 15$

		p														
		0.01	**0.05**	**0.10**	**0.20**	**0.25**	**0.30**	**0.40**	**0.50**	**0.60**	**0.70**	**0.75**	**0.80**	**0.90**	**0.95**	**0.99**
	0	.860	.463	.206	.035	.013	.005	.000	.000	.000	.000	.000	.000	.000	.000	.000
	1	.990	.829	.549	.167	.080	.035	.005	.000	.000	.000	.000	.000	.000	.000	.000
	2	1.000	.964	.816	.398	.236	.127	.027	.004	.000	.000	.000	.000	.000	.000	.000
	3	1.000	.995	.944	.648	.461	.297	.091	.018	.002	.000	.000	.000	.000	.000	.000
	4	1.000	.999	.987	.836	.686	.515	.217	.059	.009	.001	.000	.000	.000	.000	.000
	5	1.000	1.000	.998	.939	.852	.722	.402	.151	.034	.004	.001	.000	.000	.000	.000
	6	1.000	1.000	1.000	.982	.943	.869	.610	.304	.095	.015	.004	.001	.000	.000	.000
x	7	1.000	1.000	1.000	.996	.983	.950	.787	.500	.213	.050	.017	.004	.000	.000	.000
	8	1.000	1.000	1.000	.999	.996	.985	.905	.696	.390	.131	.057	.018	.000	.000	.000
	9	1.000	1.000	1.000	1.000	.999	.996	.966	.849	.597	.278	.148	.061	.002	.000	.000

Notice that this latter probability is the difference between entries in the $x = 7$ and $x = 3$ rows, *not* the $x = 7$ and $x = 4$ rows. ■

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Mean and Variance of *X*

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The Mean and Variance of X

For $n = 1$, Binomial distribution becomes Bernoulli distribution.

Example 3.18 Let $X = 1$ if a randomly selected component needs warranty service and $= 0$ otherwise. Then X is a Bernoulli rv with pmf

$$p(x) = \begin{cases} 1-p & x = 0 \\ p & x = 1 \\ 0 & x \neq 0, 1 \end{cases}$$

- Mean value of a Bernoulli variable is $\mathbf{E(X)} = \mu = p$,
  so the expected number of S's on any single trial is p.
- Since a binomial experiment consists of n **trials**, intuition suggests that for $X \sim \text{Bin}(n, p)$, $E(X) = np$,

  product of number of trials and probability of success on single trial.

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Mean and Variance of X

Proposition

$$\begin{aligned} &\text{if } X \sim Bin(n, p), \text{then} \\ &\quad E(X) = np, \\ &\quad V(X) = np(1-p), \text{ and} \\ &\quad \sigma_X = \sqrt{npq} \qquad (\text{where } q = 1-p) \end{aligned}$$

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Proof of E(X)

Example 3.34

- If 75% of all purchases at a certain store are made with a credit card
- X is the number among ten randomly selected purchases made with a credit card, then

$$X \sim Bin(10, 0.75)$$

Thus,

$$\begin{aligned} E(X) &= np \\ &= (10)(0.75) \\ &= 7.5 \end{aligned}$$

$$\begin{aligned} V(X) &= npq \\ &= np(1-p) \\ &= (10)(0.75)(0.25) \\ &= 1.875 \end{aligned}$$

$$\begin{aligned} \sigma &= \sqrt{V(X)} \\ &= \sqrt{1.875} \end{aligned}$$

Again, even though X can take on only integer values, $E(X)$ need not be integer.

If we perform a large number of independent binomial experiments, each with $n = 10$ trials and $p = 0.75$, then the average number of S's per experiment will be close to 7.5.

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