1. 前言

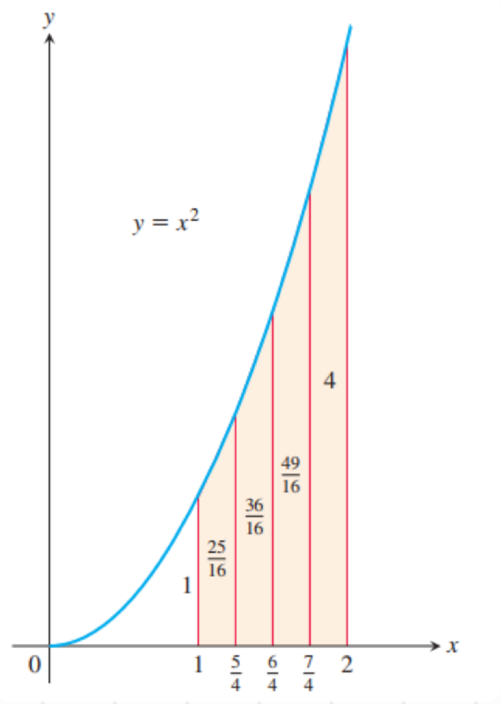
数值积分是数学中的一种技术，用于近似计算函数的定积分。定积分是分析函数在某个区间上的总体变化量，通常表示为该函数图形与x轴之间区域的面积。然而，并不是所有函数的积分都可以用基本的分析方法（如反导数）求解，特别是对于复杂的或无法表达为封闭形式的函数。

在这种情况下，数值积分就成为了一种有用的工具。它通过将积分区间分割成小段，然后近似计算每一小段的面积，最后将这些近似值加总来得到整体积分的估计值。常见的数值积分方法包括：

1. **矩形法则**：将积分区间分为若干小矩形，计算每个矩形的面积，然后将它们相加。这是最简单的方法，但精度较低。
2. **梯形法则**：在每个小区间上用梯形而不是矩形来近似。这通常比矩形法则更精确。
3. **辛普森法则**：使用二次多项式（抛物线）来近似每个小区间上的函数形状，然后计算这些抛物线形状区域的面积。这种方法在很多情况下能提供更好的精度。

2. 近似方法

(i) 矩形 → 黎曼



(ii)

The Trapezoidal Rule

To approximate $\int_a^b f(x)\,dx$, use

$$T = \frac{\Delta x}{2}\left(y_0 + 2y_1 + 2y_2 + \cdots + 2y_{n-1} + y_n\right).$$

The y's are the values of f at the partition points

$$x_0 = a, x_1 = a + \Delta x, x_2 = a + 2\Delta x, \ldots, x_{n-1} = a + (n-1)\Delta x, x_n = b,$$

where $\Delta x = (b-a)/n$.

$\frac{1}{2} \cdot \Delta x \cdot \left[(y_0+y_1)+(y_1+y_2)+\cdots+(y_{n-1}+y_n)\right]$

梯形面积

(iii)

Simpson's Rule

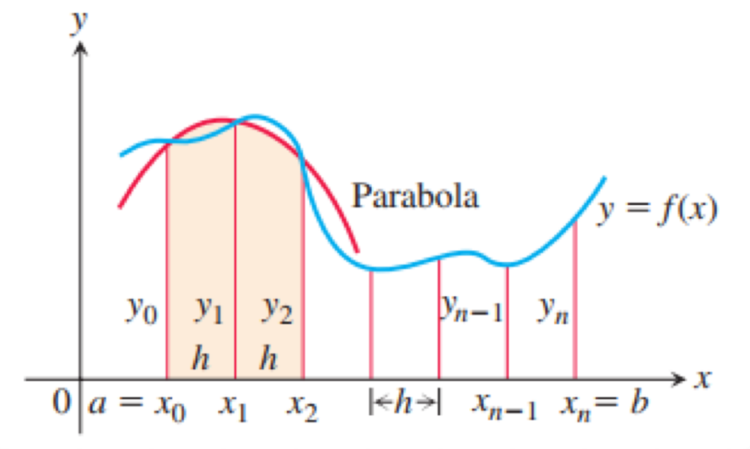
To approximate $\int_a^b f(x)\,dx$, use

$$S = \frac{\Delta x}{3}\left(y_0 + 4y_1 + 2y_2 + 4y_3 + \cdots + 2y_{n-2} + 4y_{n-1} + y_n\right).$$

The y's are the values of f at the partition points

$$x_0 = a, x_1 = a + \Delta x, x_2 = a + 2\Delta x, \ldots, x_{n-1} = a + (n-1)\Delta x, x_n = b.$$

The number n is even, and $\Delta x = (b-a)/n$.



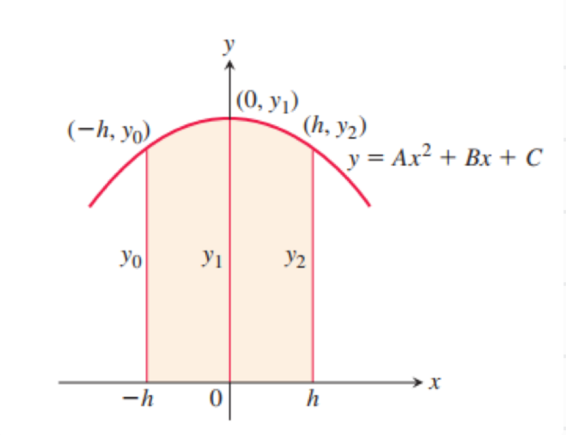
$\boxed{\Delta x \cdot n} \; y$

$(y_0+4y_1+y_2)+(y_2+4y_3+y_4) \cdots$

$\Delta x \cdot \frac{1}{3} \cdot \frac{n}{2}$ 组 $\cdot 6$

Another rule for approximating the definite integral of a continuous function results from using parabolas instead of the straight-line segments that produced trapezoids. As before, we partition the interval $[a, b]$ into n subintervals of equal length $h = \Delta x = (b-a)/n$, but this time we require that n be an *even* number. On each consecutive pair of intervals we approximate the curve $y = f(x) \geq 0$ by a parabola, as shown in Figure 8.9. A typical parabola passes through three consecutive points (x_{i-1}, y_{i-1}), (x_i, y_i), and (x_{i+1}, y_{i+1}) on the curve.

Let's calculate the shaded area beneath a parabola passing through three consecutive points. To simplify our calculations, we first take the case where $x_0 = -h$, $x_1 = 0$, and



$x_2 = h$ (Figure 8.10), where $h = \Delta x = (b-a)/n$. The area under the parabola will be the same if we shift the y-axis to the left or right. The parabola has an equation of the form

$$y = Ax^2 + Bx + C,$$

近似函数

so the area under it from $x = -h$ to $x = h$ is

$$A_p = \int_{-h}^{h} (Ax^2 + Bx + C)\,dx$$

$$= \left[\frac{Ax^3}{3} + \frac{Bx^2}{2} + Cx\right]_{-h}^{h}$$

$$= \frac{2Ah^3}{3} + 2Ch = \frac{h}{3}(2Ah^2 + 6C).$$

积分

对应这个图

Since the curve passes through the three points $(-h, y_0)$, $(0, y_1)$, and (h, y_2), we also have

$$y_0 = Ah^2 - Bh + C, \qquad y_1 = C, \qquad y_2 = Ah^2 + Bh + C,$$

from which we obtain

$$\begin{aligned} C &= y_1, \\ Ah^2 - Bh &= y_0 - y_1, \\ Ah^2 + Bh &= y_2 - y_1, \\ 2Ah^2 &= y_0 + y_2 - 2y_1. \end{aligned}$$

Hence, expressing the area A_p in terms of the ordinates y_0, y_1, and y_2, we have

配对

$$A_p = \frac{h}{3}(2Ah^2 + 6C) = \frac{h}{3}((y_0 + y_2 - 2y_1) + 6y_1) = \frac{h}{3}(y_0 + 4y_1 + y_2).$$

Now shifting the parabola horizontally to its shaded position in Figure 8.9 does not change the area under it. Thus the area under the parabola through (x_0, y_0), (x_1, y_1), and (x_2, y_2) in Figure 8.9 is still

$$\frac{h}{3}(y_0 + 4y_1 + y_2).$$

Similarly, the area under the parabola through the points (x_2, y_2), (x_3, y_3), and (x_4, y_4) is

$$\frac{h}{3}(y_2 + 4y_3 + y_4).$$

Computing the areas under all the parabolas and adding the results gives the approximation

$$\begin{aligned} \int_a^b f(x)\,dx &\approx \frac{h}{3}(y_0 + 4y_1 + y_2) + \frac{h}{3}(y_2 + 4y_3 + y_4) + \cdots \\ &\quad + \frac{h}{3}(y_{n-2} + 4y_{n-1} + y_n) \\ &= \frac{h}{3}(y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + \cdots + 2y_{n-2} + 4y_{n-1} + y_n). \end{aligned}$$

3. 误差分析

THEOREM 1—Error Estimates in the Trapezoidal and Simpson's Rules

If f'' is continuous and M is any upper bound for the values of $|f''|$ on $[a, b]$, then the error E_T in the trapezoidal approximation of the integral of f from a to b for n steps satisfies the inequality

$$|E_T| \leq \frac{M(b-a)^3}{12n^2}. \qquad \text{Trapezoidal Rule}$$

If $f^{(4)}$ is continuous and M is any upper bound for the values of $|f^{(4)}|$ on $[a, b]$, then the error E_S in the Simpson's Rule approximation of the integral of f from a to b for n steps satisfies the inequality

$$|E_S| \leq \frac{M(b-a)^5}{180n^4}. \qquad \text{Simpson's Rule}$$

其实~~类似~~类似泰勒展开

系数推导详见网页…

1. 矩形法则的误差

矩形法则使用函数在小区间上的一个点的值来估计整个区间上的积分。如果函数在区间 $[a, b]$ 上连续，并且在这个区间上有一阶导数，误差可以用泰勒级数展开来估计。

假设我们将积分区间 $[a, b]$ 分成 n 个等长的小区间，每个区间的长度为 $\Delta x = \frac{b-a}{n}$。考虑函数在某个小区间 $[x_i, x_{i+1}]$ 上的误差，其中 $x_i = a + i\Delta x$。根据泰勒公式，函数 $f(x)$ 在点 x_i 处的一阶泰勒展开是：

$$f(x) \approx f(x_i) + f'(c)(x - x_i)$$

其中 c 是 x_i 和 x 之间的某个点。在整个小区间 $[x_i, x_{i+1}]$ 上积分，误差大致为：

$$\int_{x_i}^{x_{i+1}} f(x)\,dx - \Delta x \cdot f(x_i) = \int_{x_i}^{x_{i+1}} f'(c)(x - x_i)\,dx$$

由于 $|x - x_i| \leq \Delta x$，误差大致为 $O(\Delta x^2)$。因此，对于整个区间的总误差，由于有 n 个小区间，误差大致是 $O(n\Delta x^2) = O\left(\frac{(b-a)^2}{n}\right)$。

2. 梯形法则的误差

梯形法则使用线性插值（即连接区间两端点的直线）来近似每个小区间上的函数。误差分析类似于矩形法则，但使用二阶泰勒展开。对于具有二阶连续导数的函数 $f(x)$，每个小区间的误差大致是 $O(\Delta x^3)$，因此整个区间的总误差是 $O(n\Delta x^3) = O\left(\frac{(b-a)^3}{n^2}\right)$。

3. 辛普森法则的误差

辛普森法则使用二次多项式来近似函数在每个小区间上的行为。误差分析涉及到函数的四阶泰勒展开。对于具有四阶连续导数的函数 $f(x)$，每个小区间的误差大致是 $O(\Delta x^5)$，因此整个区间的总误差是 $O(n\Delta x^5) = O\left(\frac{(b-a)^5}{n^4}\right)$。