Proof of Cauchy-Schwarz by Brute Force

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The Cauchy Schwarz inequality states that for any real numbers $\mathbf{a} = (a_1, a_2, \ldots a_n)$ and $\mathbf{b} = (b_1, b_2, \ldots, b_n)$,

$$(a_1b_1 + a_2b_2 + \cdots + a_nb_n) \le (a_1^2 + a_2^2 + \cdots a_n^2)^{\frac{1}{2}}(b_1^2 + b_2^2 + b_n^2)^{\frac{1}{2}}$$

or more concisely,

$$\sum a_ib_i \le \left(\sum a_i^2\right)^{\frac{1}{2}} \left(\sum b_i^2\right)^{\frac{1}{2}},$$

with equality if $\mathbf{a} = \lambda\mathbf{b}$. (Any unmarked sum $\sum$ means $\sum_{i=1}^{n}$.)

The naive and immediate approach to try proving this is by just squaring and expanding. And that is what we will try to do.

Proof

$$\begin{aligned}
\left(\sum_i a_ib_i\right)^2 &= \left(\sum_i a_i^2\right)\left(\sum_i b_i^2\right) \\
\left(\sum_i a_ib_i\right)^2 &= \sum_i (a_ib_i)^2 + \sum_{i\neq j} a_ib_ia_jb_j \\
\left(\sum_i a_i^2\right)\left(\sum_i b_i^2\right) &= \sum_i (a_ib_i)^2 + \sum_{i\neq j} a_i^2b_j^2 \\
\left(\sum_i a_i^2\right)\left(\sum_i b_i^2\right) - \left(\sum_i a_ib_i\right)^2 &= \sum_{1\le i<j\le n} a_i^2b_j^2 + b_i^2a_j^2 - 2a_ia_jb_ib_j \\
&= \sum_{1\le i<j\le n} (a_ib_j - a_jb_i)^2 \ge 0 \quad \square
\end{aligned}$$

Some motivation from Tim Gowers. Note the equality case, $a_i = \lambda b_i$. How would we represent this, in a terse mathematical way? Feynman famously claims in his lectures (II-25-6) (paraphrased)

> *All of the laws of physics can be contained in one equation.* That equation is
> $$U = 0,$$
> where $U = U_1 + U_2 + U_3 + \cdots$, where $U_1 = (F - ma)^2$ and $U_2 = (E - mc^2)^2$ and so on.

Following Feynman's lead, we find that

$$\sum \left(a_i - \lambda b_i\right)^2 = 0.$$

Instead of dealing with λ, we will say that $\frac{a_i}{b_i} = \frac{a_j}{b_j} \iff a_i b_j - a_j b_i$ for a more symmetric condition. Now using our trick,

$$\sum_{(i,j)} (a_i b_j - a_j b_i)^2 \geq 0.$$

Expanding gives

$$\sum (a_i^2 b_j^2 + a_j^2 b_i^2 - 2a_i b_j a_j b_i) = 2\left(\sum a_i^2\right)\left(\sum b_j^2\right) - 2\left(\sum a_i b_i\right)^2 \geq 0. \quad \square$$