Some Properties of the Natural Logarithm

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I just wanted to re-derive all the properties of $\log x$ (to base e), to convince myself nothing here was circular. I don't like using ln so log will have to do.

$$\log x \stackrel{\text{def}}{=} \int_1^x \frac{1}{t}\,dt \quad (x > 0)$$

1. $\log 1 = 0$.
2. $\log(xy) = \log x + \log y$

$$\log(xy) = \int_1^{xy} \frac{1}{t}\,dt = \int_1^x \frac{1}{t}\,dt + \int_x^{xy} \frac{1}{t}\,dt$$

For the second integral, $t \mapsto \frac{t}{x}$

$$\int_x^{xy} \frac{1}{t}\,dt = \int_1^y \frac{1}{t}\,dt$$

Therefore $\log xy = \log x + \log y$.
3. $\log(x/y) = \log x - \log y$

$$\log(x/y) = \int_1^{x/y} \frac{1}{t}\,dt = \int_1^x \frac{1}{t}\,dt + \int_x^{x/y} \frac{1}{t}\,dt$$

For the second integral $t \mapsto \frac{y}{x}t$

$$\int_x^{x/y} \frac{1}{t}\,dt = \int_y^1 \frac{1}{t}\,dt = -\int_1^y \frac{1}{t}\,dt = -\log y$$

$$\log(x/y) = \log x - \log y$$

4. $\log(x^y) = y \log x$
Let $u = t^{1/y}$, $dt = yu^{y-1}du$

$$\log(x^y) = \int_1^{x^y} \frac{1}{t}\,dt = \int_1^x \frac{1}{u^y} yu^{y-1}\,du = y\int_1^x \frac{1}{u}\,du = y\log x$$

5. $\log x$ is continuous and differentiable on $\mathbb{R}$ (this follows from Fundamental theorem of calculus)
6. $\frac{d}{dx} \log x = \frac{1}{x}$ (follows from leibnitz rule). Also, $\log x$ is monotonically increasing.
7. As $\log x$ is monotonically increasing and continuous, it is invertible.
8. $\log e = 1$
Define $e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n$. It may be shown that this limit exists, converges etc. Take log on both sides (which is justified as log is continuous, and the limit is positive)

$$\log e = \lim_{n \to \infty} \log \left(1 + \frac{1}{n}\right)^n = \lim_{n \to \infty} n \log \left(1 + \frac{1}{n}\right) = \lim_{x \to 0} \frac{\log(1+x)}{x}$$

Now for the last limit use L'Hopital, to find $\log e = 1$.
9. $\exp(\log x) = x$
Here we will use the chain rule, and the fact that $(e^x)' = e^x$ (which can be shown equivalent to the previous definition of e) Let $f(x) = e^{\log x}$

$$f'(x) = e^{\log x} \frac{1}{x}$$

$$f''(x) = e^{\log x} \frac{1}{x^2} - e^{\log x} \frac{1}{x^2} = 0$$

$$f'' = 0 \implies f' = c \implies f = cx + d$$

$$f(1) = 1 = c + d, \quad f(e) = e = ce + d \implies c = 1, d = 0$$

$$f(x) = x$$

$$e^{\log x} = x$$

10. $\log(e^x) = x$

$$e^x = \lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^n$$

$$\log e = \lim_{n \to \infty} \log \left(1 + \frac{x}{n}\right)^n = \lim_{n \to \infty} n \log \left(1 + \frac{x}{n}\right) = \lim_{u \to 0} \frac{\log(1+xu)}{u}$$

Use L'hopital and the result follows.
11. e^x and $\log x$ are inverses.