Algorithms Assignment 1

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1 Introduction

This assignment is shared between algorithms sections.
Credit: Assit Prof. Brunelle & Assit Prof. Hott

PROBLEM 1 *Asymptotic*

Prove or disprove each of the following conjectures.

1. $2^{n+1} = O(2^n)$.
   $2^n \cdot 2 \leq c2^n$
   $2 \leq c$
   For any n_0 and $c \geq 2$, $2^{n+1} = O(2^n)$

2. $2^{2n} = O(2^n)$.
   $2^{2n} \leq c2^n$
   $(2^n)^2 \leq c2^n$
   $2^n \leq c$
   We can't choose a c that will bound 2^n, so $2^{n+1} \neq O(2^n)$

3. Given that: $\forall \varepsilon > 0,\ \log(n) = o(n^\varepsilon)$,

   show:

   $\forall \varepsilon, k > 0,\ \log^k(n) = o(n^\varepsilon) - > \log^k(n) < cn^\epsilon$

   $\log n = o(n^\epsilon)$
   $\log^k n < c^k n^{\epsilon k}$, which is just the same as saying $\log^k(n) < cn^\epsilon$

PROBLEM 2 *Solving Recurrences*

Prove a (as tight as possible) O (big-Oh) asymptotic bound on the following recurrences. You may use any base cases you'd like.

1. $T(n) = 4T(\frac{n}{3}) + n\log n$
   $f(n) = n\log_2(n)$
   Is $f(n) = O(n^{\log_3 4-\varepsilon})$?
   Choose ε such that $\log_3 4 - \varepsilon = 1.25$.
   $n\log(n) = O(n^{1.25})$
   $n\log(n) \leq cn^{1.25}$
   $\log_2(n)/n^{1/4} \leq c$
   Take lim $n \to +\infty \log_2(n)/n^{1/4} = 0$
   For an arbitrary $n_0 = 2, c = 2, f(n) \in)(n^{log_3 4-\epsilon})$ implies $T(n) \in \Theta(n^{\log_3 4})$

2. $T(n) = 3T(\frac{n}{3} - 2) + \frac{n}{2}$
   case 2:
   $a = 3, b = 3, f(n) = n/2\ f(n) \in \Theta(nlogn)$, for k = 0
   $n/2 = cn\log n$
   $1/(2\log n) = c$
   $For\ n_0 = 5, c = 1, f(n) = \Theta(n\log n)$
   This implies $T(n) = \Theta(n\log n)$

3. $T(n) = 2T(\sqrt{n}) + n$
   By substitution, $T(2^m) = 2T(2^{m/2}) + 2^m, n = 2^m, m = log_2(n)\log_2(n)$
   $S(m) = T(2^m)$
   $S(m/2) = T(2^{m/2})$
   $S(m) = 2S(m/2) + 2^m$
   Case 3: $a = 2, b = 2, f(m) = 2^m, \log_b a$
   First, show $f(m) \in \Omega(m^{1+\epsilon})$
   $2^m \geq cm^{1.25}$
   For $c = 1, m_0 = 0$, $2^m = \Omega(m^{1.25})$
   Also, $2f(m/2) \leq kf(m)$
   $2 \cdot 2^{m/2} \leq k2^m$ for $k = 2$ and $m_0 = 0$, this is true.
   So then $S(m) = \Theta(2^m)$
   $T(2^m) = \Theta(2^m)$
   $T(n) = \Theta(2^{\log_2 n})$
   $T(n) = \Theta(n)$

PROBLEM 3 *Where is Batman when you need him?*

As the newly-hired commissioner of the Gotham City Police Department, James Gordon's first act is to immediately fire all of the dirty cops, stamping out Gotham's widespread police corruption. To do this, Commissioner Gordon must first figure out which officers are honest and which are dirty. There are n officers in the department. The majority ($> n/2$) of the officers are honest, and every officer knows whether or not each other officer is dirty. He will identify the dirty cops by asking the officers, in pairs, to indicate whether the other is dirty. Honest officers will always answer truthfully, dirty cops may answer arbitrarily. Thus the following responses are possible:

Officer A	Officer B	Implication
"B is honest"	"A is honest"	Either both are honest or both are dirty
"B is honest"	"A is dirty"	At least one is dirty
"B is dirty"	"A is honest"	At least one is dirty
"B is dirty"	"A is dirty"	At least one is dirty

1. A group of n officers is uncorrupted if more than half are honest. Suppose we start with an uncorrupted group of n officers. Describe a method that uses only $\lfloor n/2 \rfloor$ pair-wise tests between officers to find a smaller uncorrupted group of at most $\lceil n/2 \rceil$ officers. Prove that your method satisfies each of the three requirements.

   Take pairs of two officers until you either have 0 or 1 officers left. Do a pairwise test on each of those pairs. If the result of this test is the first case, or both responses are honest, then pick one of the officers to keep (put in the smaller n/2 group). Otherwise, discard the pair. If there were 0 leftover officers when you made the pairs, then you are done, if there is 1 leftover officer, look at the number of people kept. If this number is odd, discard the 1 officer left out, if the number of people kept is even, then add this last officer to the smaller n/2 group. Since we are creating pairs of officers to test on, we will always be doing at most n/2 tests. Since we are keeping track of how many advancements are made and deciding how to handle the non even n case, we will only ever get $floor(n/2)$ officers in the new group. And because we only discard when we know there is at least one bad officer, we maintain a majority of uncorrupted officers, at least $floor(n/2) + 1$.

2. Using this approach, devise an algorithm that identifies which officers are honest and which are dirty using only $\Theta(n)$ pairwise tests. Prove the correctness of your algorithm, and prove that only $\Theta(n)$ tests are used.

   Follow the above approach with any n group of uncorrupted officers on smaller and smaller $(n/2)$ groups until you get 1 known good officer. With this 1 known good officer, you can compare with all the other officers to determine who is honest and who is dirty. Since we cover both the even and odd cases of n, and ensure at each level of splitting that we maintain a majority of honest officers, we end up with either a good-good pair or a good-good pair plus a bad officer. In either case, we can surely find one good officer, which will always allow us to determine the aligment of all other officers. This algorithm is described by the recurrence $T(n) = T(n/2) + n/2$. We are splitting the problem size by half each recursion and do at most n/2

compares per level. By the master method (case 3):
$a = 1, b = 2, \log_2 1 = 0, f(n) = n/2$
$f(n) \in \Omega(n^{0+\epsilon})$, choose ϵ to be 1
$f(n) \in \Omega(n)$
$n/2 = cn$
For $c = 1/2,\ f(n) \in \Omega(n)$ This implies $T(n) \in \Theta(n/2)$, which is $\Theta(n)$

3. Prove that a conspiracy of $\lfloor n/2 \rfloor + 1$ dirty officers (who may share a plan) can foil *any* attempt to find a honest officer. I.e., not only will method above not work, but that there is no way *at all* for Commissioner Gordon to identify even one honest officer if there is not an honest majority.

   Consider that the number of honest officers $< n/2$ in this case. Since dirty cops hold a majority, The dirty cops can act as if they were honest officers, in that they identify themselves all as honest, and all honest cops as dirty cops. Because of this, the two groups are symmetric when comparing between pairs and therefore indistinguishable, since the only test we can use is the one described above.

PROBLEM 4 *Karatsuba Example*

Illustrate the Karatsuba algorithm on 20194102×37591056. Use 2-digit multiplication as your base case.

Problem 4: Karatsuba

20194102×37591056

a: 2019 b: 4102 c: 3759 d: 1056

$10^8(ac) + 10^4((a+b)(c+d) - ac - bd) + bd$

2019×3759 6121×4815 4102×1056

a: 20 b: 19 c: 37 d: 59 61 21 48 15 a: 41 b: 02 c: 10 d: 56

ac $10^4(20 \times 37) + 10^2((39 \times 96) - 20 \times 37 - 19 \times 59) + 19 \times 59 = 7589421$

$10^4(61 \times 48) + 10^2((82 \times 63) - 61 \times 48 - 21 \times 15) + 21 \times 15$
$= 29472615$

bd $10^4(41 \times 10) + 10^2((43 \times 66) - 41 \times 10 - 56 \times 02) + 56 \times 02$
$= 4331712$

$7.591176192 \times 10^{14}$