Curvature in Control: Exploring the Tait–Kneser Theorem

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1 Introduction

The study of curvature is a fundamental concept in differential geometry, which has been studied by mathematicians since the early 20th century. Peter Tait and Adolf Kneser were two of the pioneering mathematicians in this field. The Tait–Kneser theorem states that for curves with monotonic curvature—that is, curvature that changes in only one direction—osculating circles, which touch the curve at a single point with matching curvature, are both nested and disjoint. In other words, they are arranged like Russian dolls and do not intersect. It was first published by Peter Tait in 1896, and then re-discovered and published by Adolf Kneser in 1912. This theorem is of high significance in differential geometry, because it highlights the relationship between osculating circles and curvature. Moreover, the Tait-Kneser theorem has an abundant of real-world applications in computer-aided design, architectural modeling, and improving algorithmic efficiency.

In this paper, we explain the geometric intuition behind the Tait-Kneser theorem. More specifically, we explain how to calculate osculating circles at different points on a curve, and then relate them to the theorem. Following that, we analyze a counterexample to understand what happens when we violate the monotonic curvature condition. Finally, we provide a computationally efficient method to generalize centers of osculating circles, so that the equations of these circles can be easily calculated.

2 Definitions and Theorem

Definition 2.1 (Osculating Circle)**.** An osculating circle is the circle that best approximates a plane curve at a given point. It has the same position, tangent direction, and curvature as the curve at that point.

Definition 2.2 (Monotonic Curvature)**.** Monotonic curvature is a property of a curve that is present if κ is either strictly increasing or strictly decreasing over an interval.

Theorem 2.1 (Tait-Kneser)**.** *Let γ be a curve that has monotonic curvature along some interval. Then the osculating circles at any point inside that interval are nested and disjoint.*

3 Circles of Curvature

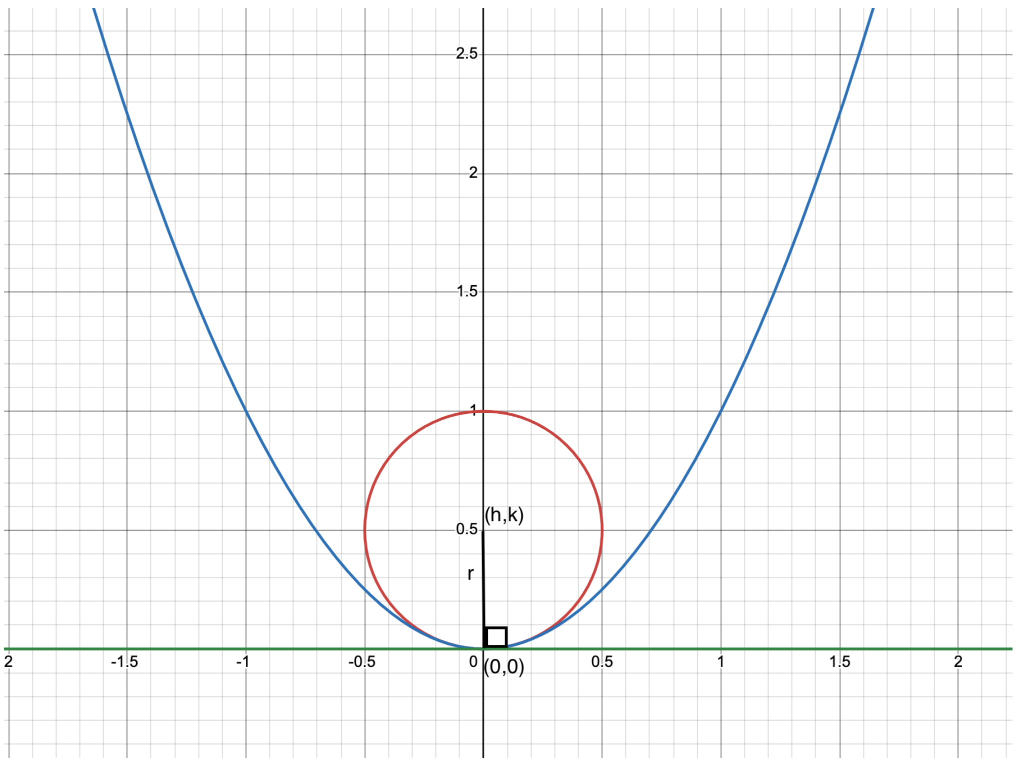

Figure 1: Visualization of the concept of an osculating circle at a single point on a curve

In order to understand the Tait-Kneser Theorem, we need to be comfortable with the concept of osculating circles. We will start off by exploring the computations of these circles of curvature.

Let $\gamma(t) = \langle x(t), y(t) \rangle$
For a parabola, we define $\gamma(t) = \langle t, t^2 \rangle$.

In order to calculate the circle of curvature, we must first find the curvature of a curve (in our case, a parabola) at a specific point. Let that point be $(0, 0)$.

$$\dot{\gamma}(t) = \langle 1, 2t \rangle$$
$$\ddot{\gamma}(t) = \langle 0, 2 \rangle$$

By the definition of curvature, we have

$$\begin{aligned}
\kappa &= \frac{|\ddot{\gamma} \times \dot{\gamma}|}{|\dot{\gamma}|^3} \\
&= \frac{|\langle 0, 2 \rangle \times \langle 1, 2t \rangle|}{|\langle 1, 2t \rangle|^3} \\
&= \frac{2}{(1 + 4t^2)^{\frac{3}{2}}}
\end{aligned}$$

Let r be the radius. For $t = 0$, $\kappa = \frac{2}{(1+4(0)^2)^{\frac{3}{2}}} = 2$, and so r equals $1/2$.

Following this, we calculate the derivative at the point $(0, 0)$.
$\dot{\gamma}(0) = \langle 1, 2(0) \rangle = \dot{\gamma}(0) = \langle 1, 0 \rangle$
Hence, the slope of the tangent line at $(0, 0)$ is 0.

We then proceed to calculate the equation of the line which contains the radius. Since the radius is perpendicular to the tangent line, the slope of the line in question is undefined (meaning we have a vertical line). So, the equation of the perpendicular line that contains the radius is $x = 0$.

Let our circle be of the form $(x - h)^2 + (y - k)^2 = r^2$ with center (h, k). Since the center lies on the line $x = 0$, we can plug it in to obtain $h = 0$.

We can plug in our new expression for h into the equation of the circle, using the point $(0, 0)$.

$$(0 - 0)^2 + (0 - k)^2 = (\frac{1}{2})^2$$
$$k^2 = \frac{1}{4}$$
$$k = \frac{1}{2} \quad \text{or} \quad k = -\frac{1}{2}$$

We choose $k = 1/2$, because the center of the circle clearly lies above the parabola, and so we have $(h, k) = (0, 1/2)$.
The equation of our circle of curvature at the point $(0, 0)$ on γ is $x^2 + (y - \frac{1}{2})^2 = \frac{1}{4}$.

Employing the same technique, we compute the corresponding circles of curvature for $t = 1$ and $t = 2$.

$$\to t = 1 : (x + 4)^2 + (y - \frac{7}{2})^2 = \frac{125}{4}$$

$$\rightarrow t = 2 : (x+32)^2 + (y - \frac{25}{2})^2 = \frac{4913}{4}$$

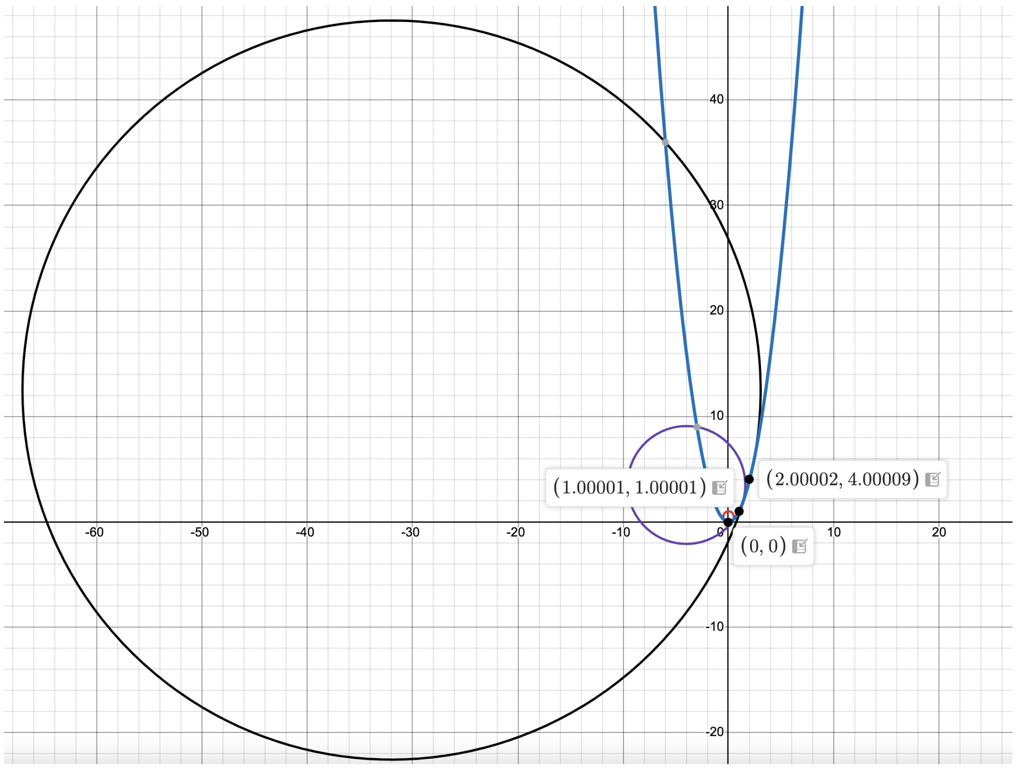

Figure 2: Osculating circles capturing the geometry of $\gamma(t)$ at $t = 0, 1, 2$

After graphing the osculating circles at $t = 0, 1, 2$, we observe that the radius of each circle increases with t, reflecting the fact that the curvature decreases monotonically. Because the curvature is strictly monotonic, the osculating circles are nested and have disjoint interiors, as highlighted in Figure 2, which agrees with the Tait-Kneser Theorem.

To better understand the role of the monotonic curvature condition, we will investigate an example where the theorem fails.

4 Contradiction

To see what happens when the curvature is not monotonic, let us consider the curve $\gamma(t) = \langle t, sin(t) \rangle$. The curve is smooth and differentiable everywhere, but its curvature does not change monotonically, and instead, oscillates as t increases.

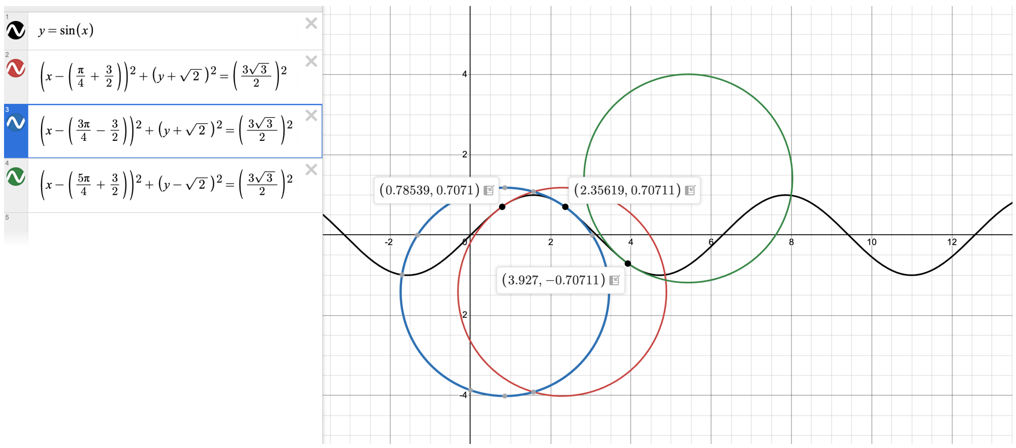

Figure 3: Osculating circles intersecting on a non-monotonic curvature curve.

Since $sin(t)$ has inflection points at $t = \pi n$ where $\kappa = 0$, the curvature alternates between increasing and decreasing values. Thus, the osculating circles intersect each other and fail to nest as shown in Figure 3.

This violates the Tait-Kneser Theorem, as it demonstrates that monotonic curvature is the essential criterion for this theorem. Without it, there is no guarantee that the osculating circles will remain nested and disjoint.

5 Generalizing Centers of Osculating Circles

On a technical note, for the purpose of computational feasibility, we can actually derive the center of an osculating circle using a fully vector-calculus based approach.

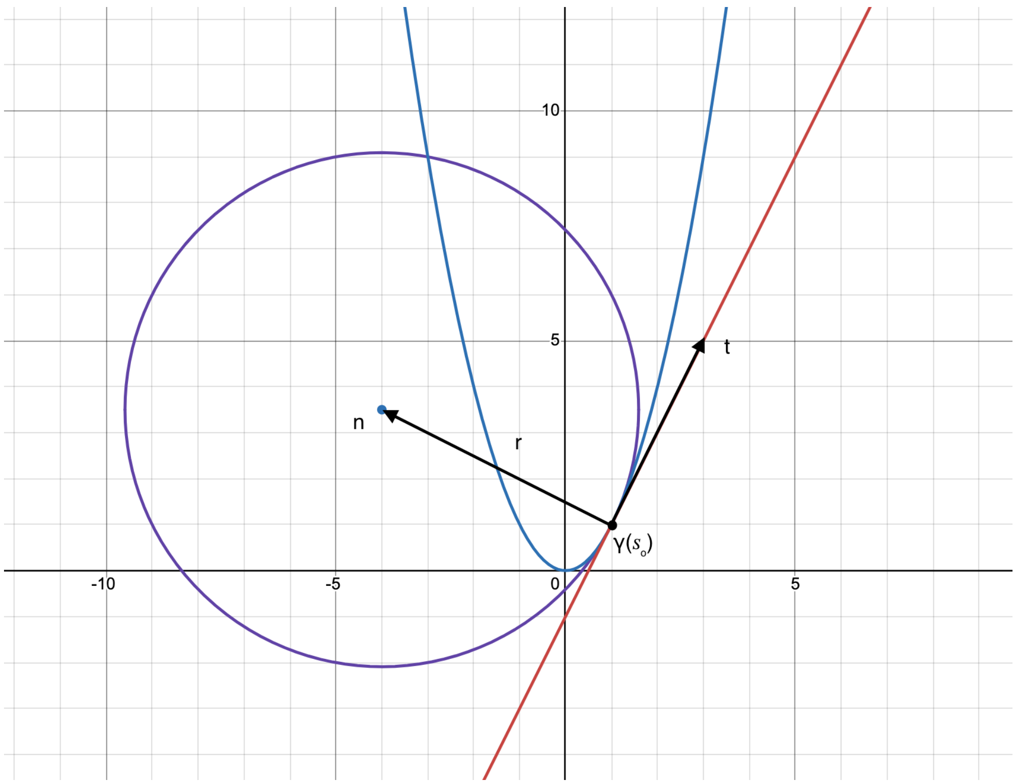

Figure 4: Illustration of Tangency Between Curve and Osculating Circle

We define γ as a unit-speed curve, and κ as the signed curvature. $\mathrm{n} = \ddot{\gamma}$ is the unit normal vector orthogonal to the tangent vector $t = \dot{\gamma}$. Let s_0 be the point where the osculating circle fits γ.

In order to reach the center, we start from $\gamma(s_0)$ and add the distance of the radius $r = \frac{1}{|\kappa|}$ in the direction of the normal vector n.

$$C = \gamma(s_0) + \frac{1}{|\kappa|} n$$

And thus, given a curve and a point, we have derived a formula to calculate the center of an osculating circle, from which we can easily find the equation of the circle. Understanding centers of osculating circles is crucial in areas like kinematics. Equations relating the centers of curvature of wheel trajectories to the vehicle's center of rotation form the basis for understanding offtracking and optimal steering. The Tait-Kneser theorem complements this by guaranteeing that, for curves with monotonic curvature, these centers of curvature nest without overlap, ensuring smooth and predictable vehicle motion.

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