proofs of Lemmas

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1 Introduction

1.0.1 Rotational Symmetries

In the rotational case, which is easier than the reflectional case, we can actually prove Lemma 4 directly. The first step of the rotational symmetry proofs is to assign ω to a $j-1^{th}$ root of unity. We then proceed to iterate ω over the polynomial $f(z) = z^j + c$. After a few iterations, we observe that each iterate rotates the previous one by ω.

Proof of Lemma 4 Case 4

$\underline{f(z) = z^4 + c}$

Let $\omega = e^{2\pi i/3}$
$f(z) = z^4 + \omega c$

$f(0) = 0^4 + \omega c = \omega c$

$f(\omega c) = (\omega c)^4 + \omega c$
$\omega^3 = 1 \rightarrow \omega^4 = \omega$
$\rightarrow f(\omega c) = \omega c^4 + \omega c$
$= \omega(c^4 + c)$

$f(\omega(c^4 + c)) = (\omega(c^4 + c))^4 + \omega c$
$= \omega^4(c^4 + c)^4 + \omega c$
$= \omega(c^4 + c)^4 + \omega c$
$= \omega[(c^4 + c)^4 + c]$

Proof of Lemma 4 Case 9

$\underline{f(z) = z^5 + c}$

Let $\omega = e^{2\pi i/4}$

$\rightarrow \omega = i$
$f(z) = z^5 + ic$

$f(0) = 0^5 + ic = ic$

$f(ic) = (ic)^5 + ic$
$i^4 = 1 \rightarrow i^5 = i$ or $\omega^4 = 1 \rightarrow \omega^5 = \omega$
$(ic)^5 = i^5 * c^5 = ic^5$
$\rightarrow f(ic) = ic^5 + ic$
$= i(c^5 + c)$

$f(i(c^5 + c)) = (i(c^5 + c))^5 + ic$
$= i^5(c^5 + c)^5 + ic$
$= i(c^5 + c)^5 + ic$
$= i[(c^5 + c)^5 + c]$

Proof of Lemma 4 Case 11

$\underline{f(z) = z^6 + c}$

Let $\omega = e^{2\pi i/5}$
$f(z) = z^6 + \omega c$

$f(0) = 0^6 + \omega c = \omega c$

$f(\omega c) = (\omega c)^6 + \omega c$
$\omega^5 = 1 \rightarrow \omega^6 = \omega$
$\rightarrow f(\omega c) = \omega c^6 + \omega c$
$= \omega(c^6 + c)$

$f(\omega(c^6 + c)) = (\omega(c^6 + c))^6 + \omega c$
$= \omega^6(c^6 + c)^6 + \omega c$
$= \omega(c^6 + c)^6 + \omega c$
$= \omega[(c^6 + c)^6 + c]$

Note : For rotations, Lemma 2 is immediately clear by taking absolute values.

Disclaimer : After completing our work, we were informed that Cases 4, 9, and 11 of this lemma are also proved in the blog of Inigo Quilez [12], who is known for their beautiful mathematical visualizations.

1.0.2 Reflectional Symmetries

The next 16 cases regarding reflectional symmetry (from Lemmas 2 and 3) are proved by writing the formula for the reflection R in (x, y) form. Using Wolfram Alpha, we checked some of the longer computations in this section.

1.0.3 $f(z) = z^4 + c$

$\underline{y = -\sqrt{3}x}$

$R(x, y) = (\frac{(1-(-\sqrt{3})^2)x+2(-\sqrt{3})y}{1+(-\sqrt{3})^2}, \frac{((-\sqrt{3})^2-1)y+2(-\sqrt{3})x}{1+(-\sqrt{3})^2})$

$R(x, y) = (-\frac{1}{2}x - \frac{\sqrt{3}}{2}y, -\frac{\sqrt{3}}{2}x + \frac{1}{2}y)$

Proof of Lemma 2 Case 1

Let $z = a + bi; (a, b)$
$R(z)$
$= R(a, b)$
$= (-\frac{1}{2}a - \frac{\sqrt{3}}{2}b, -\frac{\sqrt{3}}{2}a + \frac{1}{2}b)$

$|R(z)|$
$= \sqrt{(-\frac{1}{2}a - \frac{\sqrt{3}}{2}b)^2 + (-\frac{\sqrt{3}}{2}a + \frac{1}{2}b)^2}$
$= \sqrt{a^2 + b^2}$
$= |z|$

$|R(z)| = |z|$

Proof of Lemma 3 Case 1

$z^4 = (a^4 - 6a^2b^2 + b^4, 4a^3b - 4ab^3)$

$R(z^4)$
$= R(a^4 - 6a^2b^2 + b^4, 4a^3b - 4ab^3)$
$= (-\frac{1}{2}(a^4 - 6a^2b^2 + b^4) - \frac{\sqrt{3}}{2}(4a^3b - 4ab^3), -\frac{\sqrt{3}}{2}(a^4 - 6a^2b^2 + b^4) + \frac{1}{2}(4a^3b - 4ab^3))$
$= [-\frac{1}{2}(a^4 - 6a^2b^2 + b^4) - \frac{\sqrt{3}}{2}(4a^3b - 4ab^3)] + [-\frac{\sqrt{3}}{2}(a^4 - 6a^2b^2 + b^4) + \frac{1}{2}(4a^3b - 4ab^3))]i$

$(R(z))^4$
$R(z)$
$= R(a, b)$
$= (-\frac{1}{2}a - \frac{\sqrt{3}}{2}b, -\frac{\sqrt{3}}{2}a + \frac{1}{2}b)$
$= [-\frac{1}{2}a - \frac{\sqrt{3}}{2}b] + [-\frac{\sqrt{3}}{2}a + \frac{1}{2}b]i$
$(R(z))^4$
$= ([-\frac{1}{2}a - \frac{\sqrt{3}}{2}b] + [-\frac{\sqrt{3}}{2}a + \frac{1}{2}b]i)^4$

$= [-\frac{1}{2}(a^4-6a^2b^2+b^4)-\frac{\sqrt{3}}{2}(4a^3b-4ab^3)]+[-\frac{\sqrt{3}}{2}(a^4-6a^2b^2+b^4)+\frac{1}{2}(4a^3b-4ab^3))]i$
$= R(z^4)$

$R(z^4) = (R(z))^4$

$\underline{y = \sqrt{3}x}$

$R(x, y) = (\frac{(1-(\sqrt{3})^2)x+2(\sqrt{3})y}{1+(\sqrt{3})^2}, \frac{((\sqrt{3})^2-1)y+2(\sqrt{3})x}{1+(\sqrt{3})^2})$

$R(x, y) = (-\frac{1}{2}x + \frac{\sqrt{3}}{2}y, \frac{\sqrt{3}}{2}x + \frac{1}{2}y)$

Proof of Lemma 2 Case 2

Let $z = a + bi; (a, b)$
$R(z)$
$= R(a, b)$
$= (-\frac{1}{2}a + \frac{\sqrt{3}}{2}b, \frac{\sqrt{3}}{2}a + \frac{1}{2}b)$

$|R(z)|$
$= \sqrt{(-\frac{1}{2}a + \frac{\sqrt{3}}{2}b)^2 + (\frac{\sqrt{3}}{2}a + \frac{1}{2}b)^2}$
$= \sqrt{a^2 + b^2}$
$= |z|$

$|R(z)| = |z|$

Proof of Lemma 3 Case 2

$z^4 = (a + bi)^4 = a^4 + 4a^3bi - 6a^2b^2 - 4ab^3i + b^4$
$= (a^4 - 6a^2b^2 + b^4) + (4a^3b - 4ab^3)i$
$= (a^4 - 6a^2b^2 + b^4, 4a^3b - 4ab^3)$

$R(z^4)$
$= R(a^4 - 6a^2b^2 + b^4, 4a^3b - 4ab^3)$
$= (-\frac{1}{2}(a^4 - 6a^2b^2 + b^4) + \frac{\sqrt{3}}{2}(4a^3b - 4ab^3), \frac{\sqrt{3}}{2}(a^4 - 6a^2b^2 + b^4) + \frac{1}{2}(4a^3b - 4ab^3))$
$= [-\frac{1}{2}(a^4-6a^2b^2+b^4)+\frac{\sqrt{3}}{2}(4a^3b-4ab^3)]+[\frac{\sqrt{3}}{2}(a^4-6a^2b^2+b^4)+\frac{1}{2}(4a^3b-4ab^3))]i$

$(R(z))^4$
$R(z)$
$= R(a, b)$
$= (-\frac{1}{2}a + \frac{\sqrt{3}}{2}b, \frac{\sqrt{3}}{2}a + \frac{1}{2}b)$
$= [-\frac{1}{2}a + \frac{\sqrt{3}}{2}b] + [\frac{\sqrt{3}}{2}a + \frac{1}{2}b]i$
$(R(z))^4$

$= ([-\frac{1}{2}a + \frac{\sqrt{3}}{2}b] + [\frac{\sqrt{3}}{2}a + \frac{1}{2}b]i)^4$
$= [-\frac{1}{2}(a^4 - 6a^2b^2 + b^4) + \frac{\sqrt{3}}{2}(4a^3b - 4ab^3)] + [\frac{\sqrt{3}}{2}(a^4 - 6a^2b^2 + b^4) + \frac{1}{2}(4a^3b - 4ab^3))]i$
$= R(z^4)$

$R(z^4) = (R(z))^4$

$\underline{y = 0}$

$R(x, y) = (\frac{(1-(0)^2)x + 2(0)y}{1+(0)^2}, \frac{((0)^2-1)y + 2(0)x}{1+(0)^2})$

$R(x, y) = (x, -y)$

Proof of Lemma 2 Case 3

Let $z = a + bi; (a, b)$
$R(z) = R(a, b) = (a, -b)$

$|R(z)| = |(a, -b)|$
$= \sqrt{a^2 + (-b)^2}$
$= \sqrt{a^2 + b^2}$
$= |z|$

$|R(z)| = |z|$

Proof of Lemma 3 Case 3

$z^4 = (a^4 - 6a^2b^2 + b^4, 4a^3b - 4ab^3)$

$R(z^4)$
$= R(a^4 - 6a^2b^2 + b^4, 4a^3b - 4ab^3)$
$= (a^4 - 6a^2b^2 + b^4, -[4a^3b - 4ab^3])$
$= [a^4 - 6a^2b^2 + b^4] + [-(4a^3b - 4ab^3)]i$

$(R(z))^4$
$= [a + (-b)i]^4$
$= [a^4 - 6a^2b^2 + b^4] + [-(4a^3b - 4ab^3)]i$
$= R(z^4)$

$R(z^4) = (R(z))^4$

1.0.4 $f(z) = z^5 + c$

$\underline{y = x}$

$R(x,y) = (\frac{(1-(1)^2)x+2(1)y}{1+(1)^2}, \frac{((1)^2-1)y+2(1)x}{1+(1)^2})$

$R(x,y) = (y,x)$

Proof of Lemma 2 Case 5

Let $z = a + bi; (a,b)$
$R(z)$
$= R(a,b)$
$= (b,a)$

$|R(z)|$
$= |(b,a)|$
$= \sqrt{b^2 + a^2}$
$= \sqrt{a^2 + b^2}$
$= |z|$

$|R(z)| = |z|$

Proof of Lemma 3 Case 5

$z^5 = (a + bi)^5$
$= a^5 + 5a^4bi - 10a^3b^2 - 10a^2b^3i + 5ab^4 + b^5i$
$= a^5 - 10a^3b^2 + 5ab^4 + 5a^4bi - 10a^2b^3i + b^5i$
$= [a^5 - 10a^3b^2 + 5ab^4] + [5a^4b - 10a^2b^3 + b^5]i$
$= (a^5 - 10a^3b^2 + 5ab^4, 5a^4b - 10a^2b^3 + b^5)$

$R(z^5)$
$= R(a^5 - 10a^3b^2 + 5ab^4, 5a^4b - 10a^2b^3 + b^5)$
$= (5a^4b - 10a^2b^3 + b^5, a^5 - 10a^3b^2 + 5ab^4)$
$= [5a^4b - 10a^2b^3 + b^5] + [a^5 - 10a^3b^2 + 5ab^4]i$

$(R(z))^5$
$= (b + ai)^5$
$= [5a^4b - 10a^2b^3 + b^5] + [a^5 - 10a^3b^2 + 5ab^4]i$
$= R(z^5)$

$R(z^5) = (R(z))^5$

$\underline{y = 0}$

$R(x,y) = (\frac{(1-(0)^2)x+2(0)y}{1+(0)^2}, \frac{((0)^2-1)y+2(0)x}{1+(0)^2})$

$R(x,y) = (x, -y)$

Proof of Lemma 2 Case 6

Let $z = a + bi; (a, b)$
$R(z)$
$= R(a, b)$
$= (a, -b)$

$|R(z)|$
$= |(a, -b)|$
$= \sqrt{a^2 + (-b)^2}$
$= \sqrt{a^2 + b^2}$
$= |z|$

$|R(z)| = |z|$

Proof of Lemma 3 Case 6

$z^5 = (a^5 - 10a^3b^2 + 5ab^4, 5a^4b - 10a^2b^3 + b^5)$

$R(z^5)$
$= R(a^5 - 10a^3b^2 + 5ab^4, 5a^4b - 10a^2b^3 + b^5)$
$= (a^5 - 10a^3b^2 + 5ab^4, -[5a^4b - 10a^2b^3 + b^5])$
$= [a^5 - 10a^3b^2 + 5ab^4] + (-[5a^4b - 10a^2b^3 + b^5])i$

$(R(z))^5$
$= [a + (-b)i]^5$
$= [a^5 - 10a^3b^2 + 5ab^4] + (-[5a^4b - 10a^2b^3 + b^5])i$
$= R(z^5)$

$R(z^5) = (R(z))^5$

$\underline{y = -x}$

$R(x,y) = (\frac{(1-(-1)^2)x+2(-1)y}{1+(-1)^2}, \frac{((-1)^2-1)y+2(-1)x}{1+(-1)^2})$

$R(x,y) = (-y, -x)$

Proof of Lemma 2 Case 7

Let $z = a + bi; (a, b)$
$R(z)$
$= R(a, b)$
$= (-b, -a)$

$|R(z)|$
$= |(-b, -a)|$
$= \sqrt{(-b)^2 + (-a)^2}$
$= \sqrt{a^2 + b^2}$
$= |z|$

$|R(z)| = |z|$

Proof of Lemma 3 Case 7

$z^5 = (a^5 - 10a^3b^2 + 5ab^4, 5a^4b - 10a^2b^3 + b^5)$

$R(z^5)$
$= R(a^5 - 10a^3b^2 + 5ab^4, 5a^4b - 10a^2b^3 + b^5)$
$= (-[5a^4b - 10a^2b^3 + b^5], -[a^5 - 10a^3b^2 + 5ab^4])$
$= -[5a^4b - 10a^2b^3 + b^5] + (-[a^5 - 10a^3b^2 + 5ab^4])i$

$(R(z))^5$
$= [(-b) + (-a)i]^5$
$= -[5a^4b - 10a^2b^3 + b^5] + (-[a^5 - 10a^3b^2 + 5ab^4])i$
$= R(z^5)$

$R(z^5) = (R(z))^5$

$\underline{x = 0}$

$R(x, y)$
$= \lim_{x\to\infty}(\frac{(1-(m)^2)x+2(m)y}{1+(m)^2}, \frac{((m)^2-1)y+2(m)x}{1+(m)^2})$
$= \lim_{x\to\infty}(\frac{(-(m)^2)x+2(m)y}{(m)^2}, \frac{((m)^2)y+2(m)x}{(m)^2})$
$= \lim_{x\to\infty}(-x + \frac{2y}{m}, y + \frac{2x}{m})$
$= (-x, y)$

$R(x, y) = (-x, y)$

Proof of Lemma 2 Case 8

Let $z = a + bi; (a, b)$
$R(z)$
$= R(a, b)$
$= (-a, b)$

$|R(z)|$
$= |(-a, b)|$
$= \sqrt{(-a)^2 + (b)^2}$
$= \sqrt{a^2 + b^2}$
$= |z|$

$|R(z)| = |z|$

Proof of Lemma 3 Case 8

$z^5 = (a^5 - 10a^3b^2 + 5ab^4, 5a^4b - 10a^2b^3 + b^5)$

$R(z^5)$
$= R(a^5 - 10a^3b^2 + 5ab^4, 5a^4b - 10a^2b^3 + b^5)$
$= (-[a^5 - 10a^3b^2 + 5ab^4], 5a^4b - 10a^2b^3 + b^5)$
$= -[a^5 - 10a^3b^2 + 5ab^4] + [5a^4b - 10a^2b^3 + b^5]i$

$(R(z))^5$
$= (-a + bi)^5$
$= -[a^5 - 10a^3b^2 + 5ab^4] + [5a^4b - 10a^2b^3 + b^5]i$
$= R(z^5)$

$R(z^5) = (R(z))^5$

1.0.5 $f(z) = z^6 + c$

$\underline{f(z) = z^6 + c, y = \tan(\frac{2\pi}{5})x}$

$R(x, y) = (\frac{[(1-\tan^2(\frac{2\pi}{5})]x+2\tan(\frac{2\pi}{5})y}{1+\tan^2(\frac{2\pi}{5})}, \frac{[\tan^2(\frac{2\pi}{5})-1]y+2\tan(\frac{2\pi}{5})x}{1+\tan^2(\frac{2\pi}{5})})$
$= (\frac{[(1-\tan^2(\frac{2\pi}{5})]x+2\tan(\frac{2\pi}{5})y}{\sec^2(\frac{2\pi}{5})}, \frac{[\tan^2(\frac{2\pi}{5})-1]y+2\tan(\frac{2\pi}{5})x}{\sec^2(\frac{2\pi}{5})})$

Proof of Lemma 2 Case 10

Let $z = a + bi; (a, b)$
$|R(z)|$

$= |R(a,b)|$
$= |(\frac{[(1-\tan^2(\frac{2\pi}{5})]a+2\tan(\frac{2\pi}{5})b}{\sec^2(\frac{2\pi}{5})}, \frac{[\tan^2(\frac{2\pi}{5})-1]b+2\tan(\frac{2\pi}{5})a}{\sec^2(\frac{2\pi}{5})})|$
$\sqrt{(\frac{[(1-\tan^2(\frac{2\pi}{5})]a+2\tan(\frac{2\pi}{5})b}{\sec^2(\frac{2\pi}{5})})^2 + (\frac{[\tan^2(\frac{2\pi}{5})-1]b+2\tan(\frac{2\pi}{5})a}{\sec^2(\frac{2\pi}{5})})^2}$
$= \sqrt{a^2+b^2}$
$= |z|$

$|R(z)| = |z|$

Proof of Lemma 3 Case 10

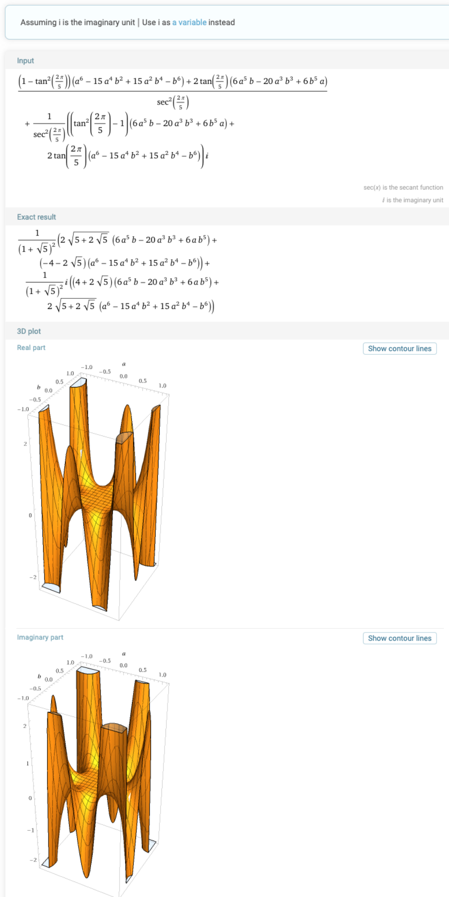
$z^6 = (a+bi)^6$
$= a^6 + 6a^5bi - 15a^4b^2 - 20a^3b^3i + 15a^2b^4 + 6ab^5i - b^6$
$= a^6 - 15a^4b^2 + 15a^2b^4 - b^6 + 6a^5bi - 20a^3b^3i + 6ab^5i$
$= [a^6 - 15a^4b^2 + 15a^2b^4 - b^6] + [6a^5b - 20a^3b^3 + 6ab^5]i$
$= (a^6 - 15a^4b^2 + 15a^2b^4 - b^6, 6a^5b - 20a^3b^3 + 6ab^5)$

$R(z^6)$
$= R(a^6 - 15a^4b^2 + 15a^2b^4 - b^6, 6a^5b - 20a^3b^3 + 6ab^5)$
$= (\frac{[(1-\tan^2(\frac{2\pi}{5})](a^6-15a^4b^2+15a^2b^4-b^6)+2\tan(\frac{2\pi}{5})(6a^5b-20a^3b^3+6ab^5)}{\sec^2(\frac{2\pi}{5})},$
$\frac{[\tan^2(\frac{2\pi}{5})-1](6a^5b-20a^3b^3+6ab^5)+2\tan(\frac{2\pi}{5})(a^6-15a^4b^2+15a^2b^4-b^6)}{\sec^2(\frac{2\pi}{5})})$
$= \frac{[(1-\tan^2(\frac{2\pi}{5})](a^6-15a^4b^2+15a^2b^4-b^6)+2\tan(\frac{2\pi}{5})(6a^5b-20a^3b^3+6ab^5)}{\sec^2(\frac{2\pi}{5})}$
$+ \frac{[\tan^2(\frac{2\pi}{5})-1](6a^5b-20a^3b^3+6ab^5)+2\tan(\frac{2\pi}{5})(a^6-15a^4b^2+15a^2b^4-b^6)}{\sec^2(\frac{2\pi}{5})}i$

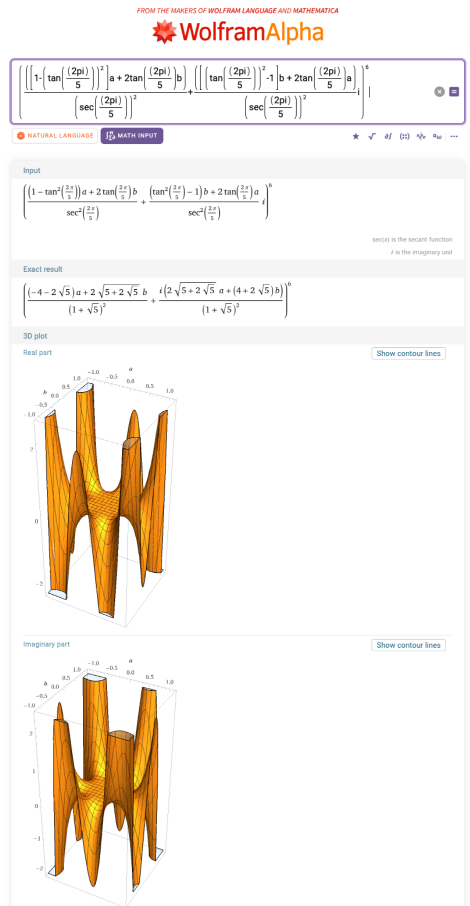
$(R(z))^6$
$= (\frac{[(1-\tan^2(\frac{2\pi}{5})]a+2\tan(\frac{2\pi}{5})b}{\sec^2(\frac{2\pi}{5})} + \frac{[\tan^2(\frac{2\pi}{5})-1]b+2\tan(\frac{2\pi}{5})a}{\sec^2(\frac{2\pi}{5})}i)^6$
$= \frac{[(1-\tan^2(\frac{2\pi}{5})](a^6-15a^4b^2+15a^2b^4-b^6)+2\tan(\frac{2\pi}{5})(6a^5b-20a^3b^3+6ab^5)}{\sec^2(\frac{2\pi}{5})}$
$+ \frac{[\tan^2(\frac{2\pi}{5})-1](6a^5b-20a^3b^3+6ab^5)+2\tan(\frac{2\pi}{5})(a^6-15a^4b^2+15a^2b^4-b^6)}{\sec^2(\frac{2\pi}{5})}i$
$= R(z^6)$

$R(z^6) = (R(z))^6$

While equality may not seem immediately obvious due to its large computational nature, the corresponding contour plots of $R(z)^6$ and $(R(z))^6$ match, strongly suggesting equality.


Assuming i is the imaginary unit | Use i as a variable instead
Input
sec(x) is the secant function
i is the imaginary unit
Exact result
3D plot
Real part
Show contour lines
Imaginary part
Show contour lines

$R(z)^6$


FROM THE MAKERS OF WOLFRAM LANGUAGE AND MATHEMATICA
WolframAlpha
NATURAL LANGUAGE
MATH INPUT
Input
sec(x) is the secant function
i is the imaginary unit
Exact result
3D plot
Real part
Show contour lines
Imaginary part
Show contour lines

$(R(z))^6$