proofs of Lemmas

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1 Introduction

1.0.1 Rotational Symmetries

In the rotational case, which is easier than the reflectional case, we can actually prove Lemma 7.3 directly. The first step of the rotational symmetry proofs is to assign ω to a $j-1^{th}$ root of unity. We then proceed to iterate ω over the polynomial $f(z) = z^j + c$. After a few iterations, we observe that each iterate rotates the previous one by ω.

Proof of Lemma 7.3 Case 4

$\underline{f(z) = z^4 + c}$

Let $\omega = e^{2\pi i/3}$
$f(z) = z^4 + \omega c$

$f(0) = 0^4 + \omega c = \omega c$

$f(\omega c) = (\omega c)^4 + \omega c$
$\omega^3 = 1 \rightarrow \omega^4 = \omega$
$\rightarrow f(\omega c) = \omega c^4 + \omega c$
$= \omega(c^4 + c)$

$f(\omega(c^4 + c)) = (\omega(c^4 + c))^4 + \omega c$
$= \omega^4(c^4 + c)^4 + \omega c$
$= \omega(c^4 + c)^4 + \omega c$
$= \omega[(c^4 + c)^4 + c]$

Proof of Lemma 7.3 Case 9

$\underline{f(z) = z^5 + c}$

Let $\omega = e^{2\pi i/4}$

$\rightarrow \omega = i$
$f(z) = z^5 + ic$

$f(0) = 0^5 + ic = ic$

$f(ic) = (ic)^5 + ic$
$i^4 = 1 \rightarrow i^5 = i$ or $\omega^4 = 1 \rightarrow \omega^5 = \omega$
$(ic)^5 = i^5 * c^5 = ic^5$
$\rightarrow f(ic) = ic^5 + ic$
$= i(c^5 + c)$

$f(i(c^5 + c)) = (i(c^5 + c))^5 + ic$
$= i^5(c^5 + c)^5 + ic$
$= i(c^5 + c)^5 + ic$
$= i[(c^5 + c)^5 + c]$

Proof of Lemma 7.3 Case 11

$\underline{f(z) = z^6 + c}$

Let $\omega = e^{2\pi i/5}$
$f(z) = z^6 + \omega c$

$f(0) = 0^6 + \omega c = \omega c$

$f(\omega c) = (\omega c)^6 + \omega c$
$\omega^5 = 1 \rightarrow \omega^6 = \omega$
$\rightarrow f(\omega c) = \omega c^6 + \omega c$
$= \omega(c^6 + c)$

$f(\omega(c^6 + c)) = (\omega(c^6 + c))^6 + \omega c$
$= \omega^6(c^6 + c)^6 + \omega c$
$= \omega(c^6 + c)^6 + \omega c$
$= \omega[(c^6 + c)^6 + c]$

Note : For rotations, Lemma 7.1 is immediately clear by taking absolute values.

Disclaimer : After completing our work, we were informed that Cases 4, 9, and 11 of this lemma are also proved in the blog of Inigo Quilez [12], who is known for their beautiful mathematical visualizations.

1.0.2 Reflectional Symmetries

The next 16 cases regarding reflectional symmetry (from Lemmas 7.1 and 7.2) are proved by writing the formula for the reflection R in (x, y) form. Using

Wolfram Alpha, we checked some of the longer computations in this section.

1.0.3 $f(z) = z^4 + c$

$\underline{y = -\sqrt{3}x}$

$R(x, y) = (\frac{(1-(-\sqrt{3})^2)x+2(-\sqrt{3})y}{1+(-\sqrt{3})^2}, \frac{((-\sqrt{3})^2-1)y+2(-\sqrt{3})x}{1+(-\sqrt{3})^2})$

$R(x, y) = (-\frac{1}{2}x - \frac{\sqrt{3}}{2}y, -\frac{\sqrt{3}}{2}x + \frac{1}{2}y)$

Proof of Lemma 7.1 Case 1

Let $z = a + bi; (a, b)$
$R(z)$
$= R(a, b)$
$= (-\frac{1}{2}a - \frac{\sqrt{3}}{2}b, -\frac{\sqrt{3}}{2}a + \frac{1}{2}b)$

$|R(z)|$
$= \sqrt{(-\frac{1}{2}a - \frac{\sqrt{3}}{2}b)^2 + (-\frac{\sqrt{3}}{2}a + \frac{1}{2}b)^2}$
$= \sqrt{a^2 + b^2}$
$= |z|$

$|R(z)| = |z|$

Proof of Lemma 7.2 Case 1

$z^4 = (a^4 - 6a^2b^2 + b^4, 4a^3b - 4ab^3)$

$R(z^4)$
$= R(a^4 - 6a^2b^2 + b^4, 4a^3b - 4ab^3)$
$= (-\frac{1}{2}(a^4-6a^2b^2+b^4) - \frac{\sqrt{3}}{2}(4a^3b-4ab^3), -\frac{\sqrt{3}}{2}(a^4-6a^2b^2+b^4) + \frac{1}{2}(4a^3b-4ab^3))$
$= [-\frac{1}{2}(a^4-6a^2b^2+b^4) - \frac{\sqrt{3}}{2}(4a^3b-4ab^3)] + [-\frac{\sqrt{3}}{2}(a^4-6a^2b^2+b^4) + \frac{1}{2}(4a^3b-4ab^3))]i$

$(R(z))^4$
$R(z)$
$= R(a, b)$
$= (-\frac{1}{2}a - \frac{\sqrt{3}}{2}b, -\frac{\sqrt{3}}{2}a + \frac{1}{2}b)$
$= [-\frac{1}{2}a - \frac{\sqrt{3}}{2}b] + [-\frac{\sqrt{3}}{2}a + \frac{1}{2}b]i$
$(R(z))^4$
$= ([-\frac{1}{2}a - \frac{\sqrt{3}}{2}b] + [-\frac{\sqrt{3}}{2}a + \frac{1}{2}b]i)^4$
$= [-\frac{1}{2}(a^4-6a^2b^2+b^4) - \frac{\sqrt{3}}{2}(4a^3b-4ab^3)] + [-\frac{\sqrt{3}}{2}(a^4-6a^2b^2+b^4) + \frac{1}{2}(4a^3b-4ab^3))]i$
$= R(z^4)$

$R(z^4) = (R(z))^4$

$\underline{y = \sqrt{3}x}$

$R(x, y) = (\frac{(1-(\sqrt{3})^2)x+2(\sqrt{3})y}{1+(\sqrt{3})^2}, \frac{((\sqrt{3})^2-1)y+2(\sqrt{3})x}{1+(\sqrt{3})^2})$

$R(x, y) = (-\frac{1}{2}x + \frac{\sqrt{3}}{2}y, \frac{\sqrt{3}}{2}x + \frac{1}{2}y)$

Proof of Lemma 7.1 Case 2

Let $z = a + bi; (a, b)$
$R(z)$
$= R(a, b)$
$= (-\frac{1}{2}a + \frac{\sqrt{3}}{2}b, \frac{\sqrt{3}}{2}a + \frac{1}{2}b)$

$|R(z)|$
$= \sqrt{(-\frac{1}{2}a + \frac{\sqrt{3}}{2}b)^2 + (\frac{\sqrt{3}}{2}a + \frac{1}{2}b)^2}$
$= \sqrt{a^2 + b^2}$
$= |z|$

$|R(z)| = |z|$

Proof of Lemma 7.2 Case 2

$z^4 = (a + bi)^4 = a^4 + 4a^3bi - 6a^2b^2 - 4ab^3i + b^4$
$= (a^4 - 6a^2b^2 + b^4) + (4a^3b - 4ab^3)i$
$= (a^4 - 6a^2b^2 + b^4, 4a^3b - 4ab^3)$

$R(z^4)$
$= R(a^4 - 6a^2b^2 + b^4, 4a^3b - 4ab^3)$
$= (-\frac{1}{2}(a^4 - 6a^2b^2 + b^4) + \frac{\sqrt{3}}{2}(4a^3b - 4ab^3), \frac{\sqrt{3}}{2}(a^4 - 6a^2b^2 + b^4) + \frac{1}{2}(4a^3b - 4ab^3))$
$= [-\frac{1}{2}(a^4-6a^2b^2+b^4)+\frac{\sqrt{3}}{2}(4a^3b-4ab^3)]+[\frac{\sqrt{3}}{2}(a^4-6a^2b^2+b^4)+\frac{1}{2}(4a^3b-4ab^3))]i$

$(R(z))^4$
$R(z)$
$= R(a, b)$
$= (-\frac{1}{2}a + \frac{\sqrt{3}}{2}b, \frac{\sqrt{3}}{2}a + \frac{1}{2}b)$
$= [-\frac{1}{2}a + \frac{\sqrt{3}}{2}b] + [\frac{\sqrt{3}}{2}a + \frac{1}{2}b]i$
$(R(z))^4$
$= ([-\frac{1}{2}a + \frac{\sqrt{3}}{2}b] + [\frac{\sqrt{3}}{2}a + \frac{1}{2}b]i)^4$
$= [-\frac{1}{2}(a^4-6a^2b^2+b^4)+\frac{\sqrt{3}}{2}(4a^3b-4ab^3)]+[\frac{\sqrt{3}}{2}(a^4-6a^2b^2+b^4)+\frac{1}{2}(4a^3b-4ab^3))]i$

$= R(z^4)$

$R(z^4) = (R(z))^4$

<u>$y = 0$</u>

$R(x, y) = (\frac{(1-(0)^2)x+2(0)y}{1+(0)^2}, \frac{((0)^2-1)y+2(0)x}{1+(0)^2})$

$R(x, y) = (x, -y)$

Proof of Lemma 7.1 Case 3

Let $z = a + bi; (a, b)$
$R(z) = R(a, b) = (a, -b)$

$|R(z)| = |(a, -b)|$
$= \sqrt{a^2 + (-b)^2}$
$= \sqrt{a^2 + b^2}$
$= |z|$

$|R(z)| = |z|$

Proof of Lemma 7.2 Case 3

$z^4 = (a^4 - 6a^2b^2 + b^4, 4a^3b - 4ab^3)$

$R(z^4)$
$= R(a^4 - 6a^2b^2 + b^4, 4a^3b - 4ab^3)$
$= (a^4 - 6a^2b^2 + b^4, -[4a^3b - 4ab^3])$
$= [a^4 - 6a^2b^2 + b^4] + [-(4a^3b - 4ab^3)]i$

$(R(z))^4$
$= [a + (-b)i]^4$
$= [a^4 - 6a^2b^2 + b^4] + [-(4a^3b - 4ab^3)]i$
$= R(z^4)$

$R(z^4) = (R(z))^4$

1.0.4 $f(z) = z^5 + c$

<u>$y = x$</u>

$R(x,y) = (\frac{(1-(1)^2)x+2(1)y}{1+(1)^2}, \frac{((1)^2-1)y+2(1)x}{1+(1)^2})$

$R(x,y) = (y,x)$

Proof of Lemma 7.1 Case 5

Let $z = a + bi; (a,b)$
$R(z)$
$= R(a,b)$
$= (b,a)$

$|R(z)|$
$= |(b,a)|$
$= \sqrt{b^2 + a^2}$
$= \sqrt{a^2 + b^2}$
$= |z|$

$|R(z)| = |z|$

Proof of Lemma 7.2 Case 5

$z^5 = (a+bi)^5$
$= a^5 + 5a^4bi - 10a^3b^2 - 10a^2b^3i + 5ab^4 + b^5i$
$= a^5 - 10a^3b^2 + 5ab^4 + 5a^4bi - 10a^2b^3i + b^5i$
$= [a^5 - 10a^3b^2 + 5ab^4] + [5a^4b - 10a^2b^3 + b^5]i$
$= (a^5 - 10a^3b^2 + 5ab^4, 5a^4b - 10a^2b^3 + b^5)$

$R(z^5)$
$= R(a^5 - 10a^3b^2 + 5ab^4, 5a^4b - 10a^2b^3 + b^5)$
$= (5a^4b - 10a^2b^3 + b^5, a^5 - 10a^3b^2 + 5ab^4)$
$= [5a^4b - 10a^2b^3 + b^5] + [a^5 - 10a^3b^2 + 5ab^4]i$

$(R(z))^5$
$= (b+ai)^5$
$= [5a^4b - 10a^2b^3 + b^5] + [a^5 - 10a^3b^2 + 5ab^4]i$
$= R(z^5)$

$R(z^5) = (R(z))^5$

$\underline{y = 0}$

$R(x,y) = (\frac{(1-(0)^2)x+2(0)y}{1+(0)^2}, \frac{((0)^2-1)y+2(0)x}{1+(0)^2})$

$R(x,y) = (x,-y)$

Proof of Lemma 7.1 Case 6

Let $z = a + bi; (a,b)$
$R(z)$
$= R(a,b)$
$= (a,-b)$

$|R(z)|$
$= |(a,-b)|$
$= \sqrt{a^2 + (-b)^2}$
$= \sqrt{a^2 + b^2}$
$= |z|$

$|R(z)| = |z|$

Proof of Lemma 7.2 Case 6

$z^5 = (a^5 - 10a^3b^2 + 5ab^4, 5a^4b - 10a^2b^3 + b^5)$

$R(z^5)$
$= R(a^5 - 10a^3b^2 + 5ab^4, 5a^4b - 10a^2b^3 + b^5)$
$= (a^5 - 10a^3b^2 + 5ab^4, -[5a^4b - 10a^2b^3 + b^5])$
$= [a^5 - 10a^3b^2 + 5ab^4] + (-[5a^4b - 10a^2b^3 + b^5])i$

$(R(z))^5$
$= [a + (-b)i]^5$
$= [a^5 - 10a^3b^2 + 5ab^4] + (-[5a^4b - 10a^2b^3 + b^5])i$
$= R(z^5)$

$R(z^5) = (R(z))^5$

$\underline{y = -x}$

$R(x,y) = (\frac{(1-(-1)^2)x+2(-1)y}{1+(-1)^2}, \frac{((-1)^2-1)y+2(-1)x}{1+(-1)^2})$

$R(x,y) = (-y,-x)$

Proof of Lemma 7.1 Case 7

Let $z = a + bi; (a,b)$
$R(z)$

$= R(a,b)$
$= (-b,-a)$

$|R(z)|$
$= |(-b,-a)|$
$= \sqrt{(-b)^2+(-a)^2}$
$= \sqrt{a^2+b^2}$
$= |z|$

$|R(z)| = |z|$

Proof of Lemma 7.2 Case 7

$z^5 = (a^5 - 10a^3b^2 + 5ab^4, 5a^4b - 10a^2b^3 + b^5)$

$R(z^5)$
$= R(a^5 - 10a^3b^2 + 5ab^4, 5a^4b - 10a^2b^3 + b^5)$
$= (-[5a^4b - 10a^2b^3 + b^5], -[a^5 - 10a^3b^2 + 5ab^4])$
$= -[5a^4b - 10a^2b^3 + b^5] + (-[a^5 - 10a^3b^2 + 5ab^4])i$

$(R(z))^5$
$= [(-b) + (-a)i]^5$
$= -[5a^4b - 10a^2b^3 + b^5] + (-[a^5 - 10a^3b^2 + 5ab^4])i$
$= R(z^5)$

$R(z^5) = (R(z))^5$

$\underline{x = 0}$

$R(x,y)$
$= \lim_{x\to\infty}(\frac{(1-(m)^2)x+2(m)y}{1+(m)^2}, \frac{((m)^2-1)y+2(m)x}{1+(m)^2})$
$= \lim_{x\to\infty}(\frac{(-(m)^2)x+2(m)y}{(m)^2}, \frac{((m)^2)y+2(m)x}{(m)^2})$
$= \lim_{x\to\infty}(-x + \frac{2y}{m}, y + \frac{2x}{m})$
$= (-x,y)$

$R(x,y) = (-x,y)$

Proof of Lemma 7.1 Case 8

Let $z = a + bi; (a,b)$
$R(z)$
$= R(a,b)$

$= (-a, b)$

$|R(z)|$
$= |(-a, b)|$
$= \sqrt{(-a)^2 + (b)^2}$
$= \sqrt{a^2 + b^2}$
$= |z|$

$|R(z)| = |z|$

Proof of Lemma 7.2 Case 8

$z^5 = (a^5 - 10a^3b^2 + 5ab^4, 5a^4b - 10a^2b^3 + b^5)$

$R(z^5)$
$= R(a^5 - 10a^3b^2 + 5ab^4, 5a^4b - 10a^2b^3 + b^5)$
$= (-[a^5 - 10a^3b^2 + 5ab^4], 5a^4b - 10a^2b^3 + b^5)$
$= -[a^5 - 10a^3b^2 + 5ab^4] + [5a^4b - 10a^2b^3 + b^5]i$

$(R(z))^5$
$= (-a + bi)^5$
$= -[a^5 - 10a^3b^2 + 5ab^4] + [5a^4b - 10a^2b^3 + b^5]i$
$= R(z^5)$

$R(z^5) = (R(z))^5$

1.0.5 $f(z) = z^6 + c$

$\underline{f(z) = z^6 + c, y = \tan(\frac{2\pi}{5})x}$

$R(x, y) = (\frac{[(1-\tan^2(\frac{2\pi}{5})]x+2\tan(\frac{2\pi}{5})y}{1+\tan^2(\frac{2\pi}{5})}, \frac{[\tan^2(\frac{2\pi}{5})-1]y+2\tan(\frac{2\pi}{5})x}{1+\tan^2(\frac{2\pi}{5})})$
$= (\frac{[(1-\tan^2(\frac{2\pi}{5})]x+2\tan(\frac{2\pi}{5})y}{\sec^2(\frac{2\pi}{5})}, \frac{[\tan^2(\frac{2\pi}{5})-1]y+2\tan(\frac{2\pi}{5})x}{\sec^2(\frac{2\pi}{5})})$

Proof of Lemma 7.1 Case 10

Let $z = a + bi; (a, b)$
$|R(z)|$
$= |R(a, b)|$
$= |(\frac{[(1-\tan^2(\frac{2\pi}{5})]a+2\tan(\frac{2\pi}{5})b}{\sec^2(\frac{2\pi}{5})}, \frac{[\tan^2(\frac{2\pi}{5})-1]b+2\tan(\frac{2\pi}{5})a}{\sec^2(\frac{2\pi}{5})})|$

$$\sqrt{(\frac{[(1-\tan^2(\frac{2\pi}{5})]a+2\tan(\frac{2\pi}{5})b}{\sec^2(\frac{2\pi}{5})})^2+(\frac{[\tan^2(\frac{2\pi}{5})-1]b+2\tan(\frac{2\pi}{5})a}{\sec^2(\frac{2\pi}{5})})^2}$$
$$= \sqrt{a^2+b^2}$$
$$= |z|$$

$$|R(z)| = |z|$$

Proof of Lemma 7.2 Case 10

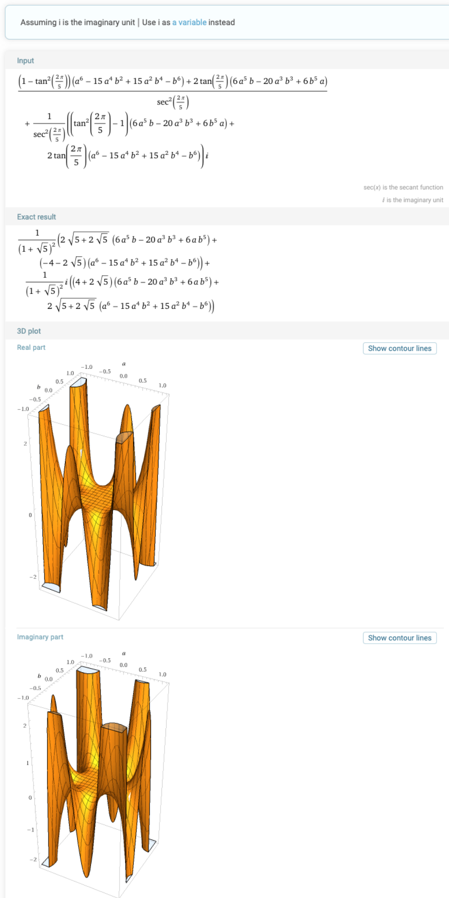
$$z^6 = (a+bi)^6$$
$$= a^6 + 6a^5bi - 15a^4b^2 - 20a^3b^3i + 15a^2b^4 + 6ab^5i - b^6$$
$$= a^6 - 15a^4b^2 + 15a^2b^4 - b^6 + 6a^5bi - 20a^3b^3i + 6ab^5i$$
$$= [a^6 - 15a^4b^2 + 15a^2b^4 - b^6] + [6a^5b - 20a^3b^3 + 6ab^5]i$$
$$= (a^6 - 15a^4b^2 + 15a^2b^4 - b^6, 6a^5b - 20a^3b^3 + 6ab^5)$$

$$R(z^6)$$
$$= R(a^6 - 15a^4b^2 + 15a^2b^4 - b^6, 6a^5b - 20a^3b^3 + 6ab^5)$$
$$= (\frac{[(1-\tan^2(\frac{2\pi}{5})](a^6-15a^4b^2+15a^2b^4-b^6)+2\tan(\frac{2\pi}{5})(6a^5b-20a^3b^3+6ab^5)}{\sec^2(\frac{2\pi}{5})},$$
$$\frac{[\tan^2(\frac{2\pi}{5})-1](6a^5b-20a^3b^3+6ab^5)+2\tan(\frac{2\pi}{5})(a^6-15a^4b^2+15a^2b^4-b^6)}{\sec^2(\frac{2\pi}{5})})$$
$$= \frac{[(1-\tan^2(\frac{2\pi}{5})](a^6-15a^4b^2+15a^2b^4-b^6)+2\tan(\frac{2\pi}{5})(6a^5b-20a^3b^3+6ab^5)}{\sec^2(\frac{2\pi}{5})}$$
$$+\frac{[\tan^2(\frac{2\pi}{5})-1](6a^5b-20a^3b^3+6ab^5)+2\tan(\frac{2\pi}{5})(a^6-15a^4b^2+15a^2b^4-b^6)}{\sec^2(\frac{2\pi}{5})}i$$

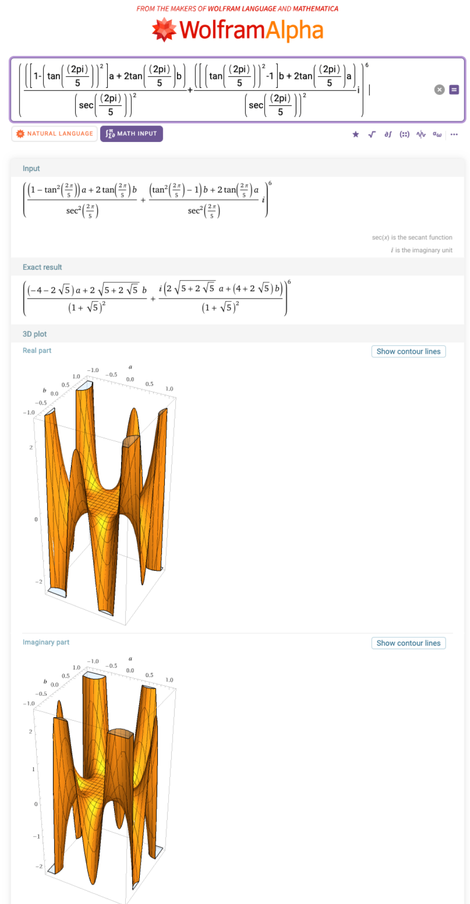
$$(R(z))^6$$
$$= (\frac{[(1-\tan^2(\frac{2\pi}{5})]a+2\tan(\frac{2\pi}{5})b}{\sec^2(\frac{2\pi}{5})}+\frac{[\tan^2(\frac{2\pi}{5})-1]b+2\tan(\frac{2\pi}{5})a}{\sec^2(\frac{2\pi}{5})}i)^6$$
$$= \frac{[(1-\tan^2(\frac{2\pi}{5})](a^6-15a^4b^2+15a^2b^4-b^6)+2\tan(\frac{2\pi}{5})(6a^5b-20a^3b^3+6ab^5)}{\sec^2(\frac{2\pi}{5})}$$
$$+\frac{[\tan^2(\frac{2\pi}{5})-1](6a^5b-20a^3b^3+6ab^5)+2\tan(\frac{2\pi}{5})(a^6-15a^4b^2+15a^2b^4-b^6)}{\sec^2(\frac{2\pi}{5})}i$$
$$= R(z^6)$$

$$R(z^6) = (R(z))^6$$

While equality may not seem immediately obvious due to its large computational nature, the corresponding contour plots of $R(z)^6$ and $(R(z))^6$ match, strongly suggesting equality.


Assuming i is the imaginary unit | Use i as a variable instead
Input
sec(x) is the secant function
i is the imaginary unit
Exact result
3D plot
Real part
Show contour lines
Imaginary part
Show contour lines

$R(z)^6$


FROM THE MAKERS OF WOLFRAM LANGUAGE AND MATHEMATICA
WolframAlpha
NATURAL LANGUAGE
MATH INPUT
Input
sec(x) is the secant function
i is the imaginary unit
Exact result
3D plot
Real part
Show contour lines
Imaginary part
Show contour lines

$(R(z))^6$