proofs of escape criterion for j=2,3,1000

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1 Introduction

Proof for j = 2

Assume $|z_n| = 2 + a, a > 0$ ($|z_n| > 2$)
$f(z) = z^2 + c \rightarrow z_{n+1} = z_n^2 + c$

1) Let $|c| \leq 2$:
$|z_{n+1}| = |z_n^2 + c| \geq |z_n|^2 - |c| = (2+a)^2 - 2 = 2 + 2a + a^2 > 2 + 2a$ (triangle inequality)
By induction, $|z_{n+k}| > 2 + ak \rightarrow \infty$ as $k \rightarrow \infty$
$\rightarrow |z_{n+1}| > |z_n|$
$\therefore z_n$ diverges to infinity if $|z_n| > 2$.

2) Let $|c| > 2$:
For $n = 0 : |z_{n+1}| = |z_{0+1}| = |z_0^2 + c| = 0^2 + c = |c|$
$\rightarrow |z_1| > 2$, which already guarantees divergence.

Proof for j = 3

Assume $|z_n| = 2 + a, a > 0$ ($|z_n| > 2$)
$f(z) = z^3 + c \rightarrow z_{n+1} = z_n^3 + c$
As proved earlier, for $|c| > 2$, divergence is guaranteed, so we will proceed for $|c| \leq 2$.
$|z_{n+1}| = |z_n^3 + c| \geq |z_n|^3 - |c| = (2+a)^3 - 2 = 6 + 12a + 6a^2 + a^3 > 6 + 12a$ (triangle inequality)
By induction, $|z_{n+k}| > 6 + ak \rightarrow \infty$ as $k \rightarrow \infty$
$\rightarrow |z_{n+1}| > |z_n|$
$\therefore z_n$ diverges to infinity if $|z_n| > 2$.
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Proof for j = 1000

Assume $|z_n| = 2 + a, a > 0$ ($|z_n| > 2$)
$f(z) = z^{1000} + c \rightarrow z_{n+1} = z_n^{1000} + c$
$|z_{n+1}| = |z_n^{1000} + c| \geq |z_n|^{1000} - |c| = (2 + a)^{1000} - 2$ (triangle inequality)

$(2 + a)^{1000} = 2^{1000} + 1000(2^{999})(a) + ...$
$\rightarrow (2 + a)^{1000} \geq 2^{1000} + 1000(2^{999})(a)$
$\rightarrow (2 + a)^{1000} - 2 \geq 2^{1000} + 1000(2^{999})(a) - 2$
$\rightarrow |z_{n+1}| \geq 2^{1000} + 1000(2^{999})(a) - 2$
By induction, $|z_{n+k}| > 2^{1000} + ak \rightarrow \infty$ as $k \rightarrow \infty$
$\rightarrow |z_{n+1}| > |z_n|$
$\therefore z_n$ diverges to infinity if $|z_n| > 2$.