Project 1 - How Much Does that House Really Cost?

Due in Gradescope Tuesday, February 17, 2026 at 11:59 PM MST

1 Introduction

A pair of close friends is currently in the market to buy a house in Boulder. Both have obtained engineering degrees from CU and have an understanding of differential equations but lack the skills necessary to decide what type of mortgage to take out on their new house. In the following sections, you will find some research they have done on commonly used mortgage structures. However, they have become busy and cannot continue their analysis. Since you are in Differential Equations this semester, they have decided to enlist your help in writing a report to understand their options.

You need to read the following sections and complete the tasks found in Section 3. Your friends were nice enough to list various problems they would like solved. However, you should be careful to write your responses in a cohesive report, not just a list of answers to their questions. Round all monetary values to the nearest cent and all time values to the nearest hundredth of a year.

2 Background Information

2.1 Compounding Interest

Often the interest on a loan is expressed as the annual rate, that is, the percentage of the outstanding balance that is charged as interest over a year. However, the frequency with which the rate is applied to the current balance may vary. This frequency is how often the loan *compounds*. If the interest is compounded annually, the formula to calculate the amount of money owed after the first year is

$$A(1) = (1 + r)A(0),$$

where $A(t)$ is the outstanding balance after t years, and r is the annual interest rate. Note that the interest rate is usually discussed as a percent but in these formulas it is expressed as a decimal. For example an interest rate of 5% would correspond to $r = 0.05$.

How would this change if, instead, the loan compounded semiannually (twice a year)? In this case, half the interest rate would be applied to the loan value every 6 months.

$$A(0.5) = \left(1 + \frac{r}{2}\right) A(0)$$

$$A(1) = \left(1 + \frac{r}{2}\right) A(0.5) = \left(1 + \frac{r}{2}\right)\left(1 + \frac{r}{2}\right) A(0) = \left(1 + \frac{r}{2}\right)^2 A(0)$$

This pattern continues for any frequency of compounding. That is, if the interest is compounded n times per year, the value of the loan after one year is

$$A(1) = \left(1 + \frac{r}{n}\right)^n A(0)$$

and after 2 years

$$A(2) = \left(1 + \frac{r}{n}\right)^n A(1) = \left(1 + \frac{r}{n}\right)^n \left[\left(1 + \frac{r}{n}\right)^n A(0)\right] = \left(1 + \frac{r}{n}\right)^{2n} A(0)$$

More generally, the amount of a loan at time t years, compounded n times per year, is

$$A(t) = A(0)\left(1 + \frac{r}{n}\right)^{nt} \tag{1}$$

The more frequently that a loan compounds, the higher the value at any time t. In the limit as the number of compounding periods increases without bound, which models a *continuously compounding* loan, we have

$$A(t) = A(0)\lim_{n\to\infty}\left(1 + \frac{r}{n}\right)^{nt} = A(0)e^{rt} \tag{2}$$

This model can also be expressed as an initial value problem

$$\frac{\mathrm{d}A}{\mathrm{d}t} = A' = rA,\ A(0) = A_0$$

If the borrower makes a monthly payment of p dollars, the model becomes, assuming that the payments are distributed continuously throughout the year,

$$A' = rA - 12p,\ A(0) = A_0 \tag{3}$$

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2.2 Adjustable rate (ARM) versus fixed rate mortgages

Mortgages began using a fixed rate, the r in the model. The interest rate which a borrower receives depends on his/her credit score, the value of the house, the borrower's income, the duration of the loan, and several other factors. Fixed rate loans are sold with a minimum monthly payment, which ensures that the loan will be paid off within the lifetime of the loan. The bank will get all of the money back sooner with a shorter duration loan, so the interest rate is lower. The advantage of this type of loan is that it is extremely predictable; one can calculate exactly how much the loan will cost at the outset, and how much will be due each month.

Adjustable rate mortgages start at a lower interest rate than a fixed rate loan but, after a certain period of time, the interest rate increases, and becomes tied to one of several public indexes. This results in the possibility of the interest rate increasing above the rate that a fixed-rate mortgage would have for the second portion of the loan. Common fixed-rate periods are 3, 5, 7, or 10 years. Mathematically, this results in r being a function of time $r(t)$ rather than a constant and the model becomes

$$A' = r(t)A - 12p, \; A(0) = A_0 \tag{4}$$

This type of mortgage is ideal for people who plan to move after a short period of time, so the early low interest rate outweighs the increased (and unpredictable) later rate.

3 Questions

Your friends recently found a house on the market and need to borrow \$750,000 from the bank in order to purchase it. Your friends are very careful when reading mathematics so in your analysis you should include as much relevant work as necessary so your friends can follow. They provided you a list of questions that they need answered.

3.1 Analysis of Fixed Rate Mortgages

Your friends have various options when choosing a mortgage structure. The following points should help you know what to include in your analysis.

1. Examine the effect of continuous compounding on the value of a loan. Assuming that the interest rate is 3% ($r = 0.03$) and the original loan is \$750,000, compute the total cost of the loan after 5 years for loans compounded 1, 2, 4, and 12 times per year, without any payments, using Equation (1). Use Equation (2) to compare these values to the value of the loan compounded continuously. On the same graph, plot the value of the loan as a function of time compounded 4 times a year and 12 times a year as well as the value of the loan when the interest is compounded continuously for $0 \leq t \leq 30$ years.
2. Next, gain a broad understanding of the behavior of the loan value by determining whether there any equilibrium solutions to Eq. (3). If so, what are they, and what is their stability? What do these equilibria represent in real-world terms?
3. Determine the exact behavior of the loan in your friends' situation by solving (3) with $A(0) = A_0$ and r and p arbitrary. Be sure to show your work so that your friends are confident that you have the correct solution.
4. The size of the monthly payment p that your friends are willing to make plays a large role in deciding the type of loan they should choose. Use the solution to (3) to find the correct p to pay off a 10-year fixed rate mortgage with rate of 3% and initial debt of \$750,000. Do the same for a 30-year fixed rate mortgage with an interest rate of 5%. Hint: you want to find p such that $A(t_l) = 0$, were t_l is the duration (years) of your mortgage. Find this analytically, not numerically using a root finding routine.
5. While having a low monthly payment is nice, you should warn your friends that there is quite literally a price to pay for this convenience. We can determine the total amount paid by summing each monthly payment over the duration of the loan. How much interest is paid in the 30-year fixed rate mortgage? The 10-year?
6. Buyers often choose to pay as much of the cost as they can up front (make a down payment) so that they don't have to borrow quite so much. Might this option be worth it for your friends? How much money would they save in each case if they paid \$100,000 down on the house, *i.e.*, the original loan amount was \$650,000? Use the interest rates and loan periods from part (4).
7. What are the advantages and disadvantages of taking out a 30-year fixed rate mortgage as opposed to a 10-year mortgage?

3.2 Numerical Solutions

Often, the differential equations we wish to solve will be difficult or impossible to solve by hand (analytically) so we enlist the help of a numerical scheme. Here we will use Euler's method. First we will use it to approximate the solution of Eq. (3) in order to compare its results with the known analytic solution you found in item 3 of Section 3.1. Once we are satisfied that the method is working in this known case, we will use it in the case where the interest rate is variable. (This technique of checking software against known information is good coding practice). Recall that Euler's method is $A_{n+1} = A_n + hf(t_n, A_n), \; n = 0, 1, 2, \ldots$.

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3.2.1 Fixed rate mortgage

Consider a mortgage for $750,000 with a constant interest rate of 5% ($r = 0.05$) and a monthly payment $p = \$4000$.

1. Implement Euler's method for Eq. (3) with step size $h = 0.5$. Run the method until the mortgage is paid off and determine when it is paid off. Note: in reality, the mortgage is paid off when its value is zero. However, due to errors in the computations (both discretization and roundoff), it is likely that Euler's method will not produce an exact value of 0 for the mortgage value for any time. To account for this, consider the mortgage to be paid off when its value first becomes negative.

2. Plot the numerical solution $A(t)$ and the true solution to Eq. (3) with the parameters given here on the same graph and compare the two.

3. Repeat the previous item for a step size $h = 0.01$ and comment on the difference.

3.2.2 Adjustable rate mortgage

Now we turn to the adjustable rate mortgages. Suppose that for the same $750,000 mortgage a bank offers an adjustable rate mortgage, which starts with an initial lower fixed rate of 3% ($r = 0.03$) for the first 5 years and is tied to credit markets after that. Let's assume that after the first 5 years the rate increases as $r(t) = 0.03 + 0.015\sqrt{t-5}$, so

$$r(t) = \begin{cases} 0.03 & t \leq 5 \\ 0.03 + 0.015\sqrt{t-5} & t > 5 \end{cases} \tag{5}$$

Use Euler's method with $h = 0.01$ to answer the following.

1. Suppose your friends pay $4000 per month. How long will it take them to pay off the mortgage?

2. What about if they pay $4500 per month?

3. How much interest is paid in each case?

4. Plot the numerical solution $A(t)$ for both scenarios on the same graph. How does the variable interest rate affect the graph, compared to the fixed rate? How do the different payment sizes affect the graph?

4 Conclusion

Your friends want a complete report on your findings, so be sure to write full sentences explaining the questions posed and your responses. Don't simply number your responses to individual questions. Be sure to discuss things like monthly payments, total amounts paid with the various mortgage options, how long your friends may live in the house, advantages and disadvantages of short term versus long term loans, the effects of down payments and interest rates, *etc.*

Your group's report needs to be submitted to Gradescope as a single pdf file. Any `MATLAB` code that you include should be included within the single pdf as an appendix. Be sure to include all group members' names in the report as well as in the Gradescope submission.