

Chaganti Kamaraja Siddhartha

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1 Potential, Field and Charge distribution

Given

$$V(r) = A \frac{e^{-\lambda r}}{r} \quad (1)$$

Electric field, \vec{E} :

$$\vec{E} = -\vec{\nabla}V \quad (2)$$

$$\vec{E} = -\left(\frac{\partial}{\partial r}\hat{r} + \frac{1}{r\sin\phi}\frac{\partial}{\partial\phi}\hat{\phi} + \frac{1}{r}\frac{\partial}{\partial\phi}\hat{\phi}\right)V(r) \quad (3)$$

$$\vec{E} = -\frac{dV(r)}{dr}\hat{r} \quad (4)$$

$$\boxed{\vec{E} = A \frac{e^{-\lambda r}}{r^2}(1 + \lambda r)\hat{r}} \quad (5)$$

Charge distribution, ρ :

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_o} \quad (6)$$

$$\vec{\nabla} \cdot \left(A e^{-\lambda r} (1 + \lambda r) \frac{\hat{r}}{r^2} \right) = \frac{\rho}{\epsilon_o} \quad (7)$$

$$A e^{-\lambda r} (1 + \lambda r) \vec{\nabla} \cdot \left(\frac{\hat{r}}{r^2} \right) + \frac{\hat{r}}{r^2} \cdot \vec{\nabla} (A e^{-\lambda r} (1 + \lambda r)) = \frac{\rho}{\epsilon_o} \quad (8)$$

$$A e^{-\lambda r} (1 + \lambda r) (4\pi\delta^3(r)) + \frac{\hat{r}}{r^2} \cdot (A e^{-\lambda r} (-\lambda^2 r) \hat{r}) = \frac{\rho}{\epsilon_o} \quad (9)$$

$$\left(\because \vec{\nabla} \cdot \left(\frac{\hat{r}}{r^2} \right) = 4\pi\delta^3(r) \right)$$

$$\boxed{\rho = A\epsilon_o \left(4\pi\delta^3(r) - \lambda^2 \frac{e^{-\lambda r}}{r} \right)} \quad (\because f(x)\delta(x) = f(0)\delta(x)) \quad (10)$$

Total Charge, Q :

$$Q = \int_{-\infty}^{+\infty} \rho d\tau \quad (11)$$

$$Q = \int_{-\infty}^{+\infty} A\epsilon_o \left(4\pi\delta^3(r) - \lambda^2 \frac{e^{-\lambda r}}{r} \right) d\tau \quad (12)$$

$$Q = A\epsilon_o \int_{-\infty}^{+\infty} 4\pi\delta^3(r) d\tau - A\epsilon_o \int_{-\infty}^{+\infty} \lambda^2 \frac{e^{-\lambda r}}{r} d\tau \quad (13)$$

$$Q = A\epsilon_o(4\pi) - A\epsilon_o\lambda^2 4\pi \left(\frac{1}{\lambda^2} \right) \quad (14)$$

$$\boxed{Q = 0} \quad (15)$$

Answers

$$\boxed{\vec{E} = A \frac{e^{-\lambda r}}{r^2}(1 + \lambda r)\hat{r}},$$

$$\boxed{\rho = A\epsilon_o \left(4\pi\delta^3(r) - \lambda^2 \frac{e^{-\lambda r}}{r} \right)},$$

$$\boxed{Q = 0}.$$

2 Dipole

Considering Proton above $Z=0$ and electron below $Z=0$, $10^{-11}m \ll 13m$, so we can consider this charge distribution as dipole with dipole moment $\vec{p} = ed\hat{k}$

$$V(R) = \frac{q}{4\pi\epsilon_o} \left(\frac{1}{R_+} - \frac{1}{R_-} \right) \quad (16)$$

Law of cosines,

$$R_{\pm}^2 = R^2 + \left(\frac{d}{2}\right)^2 \mp Rd \cos \phi = R^2 \left(1 \mp \frac{d}{R} \cos \phi + \frac{d^2}{4R^2}\right) \quad (17)$$

$$\frac{1}{R_{\pm}} \approx \frac{1}{R} \left(1 \mp \frac{d}{R} \cos \phi\right)^{-\frac{1}{2}} \approx \frac{1}{R} \left(1 \pm \frac{d}{2R} \cos \phi\right) \quad (18)$$

$$\Rightarrow \frac{1}{R_+} - \frac{1}{R_-} \approx \frac{d}{R^2} \cos \phi \quad (19)$$

$$V(R) \cong \frac{1}{4\pi\epsilon_o} \frac{qd \cos \phi}{R^2} \quad (20)$$

$$\boxed{V(R) = \frac{1}{4\pi\epsilon_o} \frac{\vec{p} \cdot \hat{R}}{R^2}} \quad (21)$$

Electric Field, $\vec{E}(R)$ is

$$\vec{E}(R) = -\vec{\nabla}V(R) \quad (22)$$

$$\vec{E}(R) = -\left(\frac{\partial}{\partial R}\hat{R} + \frac{1}{R\sin\phi}\frac{\partial}{\partial\theta} + \frac{1}{R}\frac{\partial}{\partial\phi}\right)\left(\frac{1}{4\pi\epsilon_o}\frac{qd \cos \phi}{R^2}\right) \quad (23)$$

$$\vec{E}(R) = \frac{qd}{4\pi\epsilon_o R^3} \left(2 \cos \phi \hat{R} + \sin \phi \hat{\phi}\right) \quad (24)$$

$$\boxed{\vec{E}(R) = \frac{1}{4\pi\epsilon_o R^3} \left(3(\vec{p} \cdot \hat{R})\hat{R} - \vec{p}\right)} \quad (25)$$

Here, $\vec{p} = 10^{-11}e\hat{k}\text{Cm}$ and $\vec{R} = (3\hat{i} + 4\hat{j} + 12\hat{k})m$

$$V(R) = \frac{1}{4\pi\epsilon_o} \frac{12 \times 10^{-11}e}{13^3} V = 4.65 \times 10^{-25} V \quad (26)$$

$$\vec{E}(R) = \frac{1}{4\pi\epsilon_o 13^3} \left(3\left(\frac{12}{13} \times 10^{-11}e\right) \frac{3\hat{i} + 4\hat{j} + 12\hat{k}}{13} - 10^{-11}e\hat{k}\right) \quad (27)$$

$$\vec{E}(R) = (4.188 \times 10^{-24}\hat{i} + 5.585 \times 10^{-24}\hat{j} + 1.019 \times 10^{-23}\hat{k}) NC^{-1} \quad (28)$$

Answers

$$\boxed{V(R) = 4.65 \times 10^{-25} V}$$

$$\boxed{\vec{E}(R) = (4.188 \times 10^{-24}\hat{i} + 5.585 \times 10^{-24}\hat{j} + 1.019 \times 10^{-23}\hat{k}) NC^{-1}}$$

3 Magnetic Boundary

$$B_{1n} = B_{2n} \quad (29)$$

$$H_{1t} = H_{2t} \Rightarrow \frac{B_{1t}}{\mu_1} = \frac{B_{2t}}{\mu_2} \quad (30)$$

(i) $\vec{B}_1 = 0.5\hat{x} - 10\hat{y}(\text{mT})$

Boundary is $y = 0$ therefore, normal is \hat{y}

$$B_{1n} = B_{2n} \implies B_{2n} = -10mT \quad (31)$$

$$\frac{B_{1t}}{\mu_1} = \frac{B_{2t}}{\mu_2} \implies B_{2t} = \frac{\mu_2}{\mu_1} B_{1t} \quad (32)$$

$$B_{2t} = 2.5T \quad (33)$$

$$\text{Angle with interface is } \arctan(B_{2n}/B_{2t}) = \arctan\left(-\frac{10 * 10^{-3}}{2.5}\right) = 0.114^\circ \quad (34)$$

$$\boxed{\vec{B}_2 = 2.5T\hat{x} - 10mT\hat{y}, \text{ makes } 0.114^\circ \text{ with interface.}} \quad (35)$$

(ii) $\vec{B}_2 = 10\hat{x} + 0.5\hat{y}(\text{mT})$

Boundary is $y = 0$ therefore, normal is \hat{y}

$$B_{1n} = B_{2n} \implies B_{1n} = 0.5mT \quad (36)$$

$$\frac{B_{1t}}{\mu_1} = \frac{B_{2t}}{\mu_2} \implies B_{1t} = \frac{\mu_1}{\mu_2} B_{2t} \quad (37)$$

$$B_{1t} = \frac{10}{5000}mT = 2\mu T \quad (38)$$

$$\text{Angle with normal is } \arctan(B_{1t}/B_{1n}) = \arctan\left(\frac{0.002}{0.5}\right) = 0.229^\circ \quad (39)$$

$$\boxed{\vec{B}_1 = 2\mu T\hat{x} + 0.5mT\hat{y}, \text{ makes } 0.229^\circ \text{ with normal.}} \quad (40)$$

4 Maxwell's equations in scalar form

Maxwell equations

$$\vec{\nabla} \cdot \vec{D} = \rho \quad (41)$$

$$\vec{\nabla} \cdot \vec{B} = 0 \quad (42)$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad (43)$$

$$\vec{\nabla} \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \quad (44)$$

Linear Medium

$$\vec{D} = \epsilon \vec{E} \quad (45)$$

$$\vec{B} = \mu \vec{H} \quad (46)$$

(i) Cartesian coordinates

Scalar equations

$$\boxed{\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = \frac{\rho}{\epsilon}} \quad (47)$$

$$\boxed{\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} = 0} \quad (48)$$

$$\left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} \right) + \frac{\partial B_x}{\partial t} = 0 \quad (49)$$

$$\left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} \right) + \frac{\partial B_y}{\partial t} = 0 \quad (50)$$

$$\left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} \right) + \frac{\partial B_z}{\partial t} = 0 \quad (51)$$

$$\left(\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z} \right) = \mu J_x + \mu\epsilon \frac{\partial E_x}{\partial t} \quad (52)$$

$$\left(\frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x} \right) = \mu J_y + \mu\epsilon \frac{\partial E_y}{\partial t} \quad (53)$$

$$\left(\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} \right) = \mu J_z + \mu\epsilon \frac{\partial E_z}{\partial t} \quad (54)$$

(ii) Cylindrical Coordinates

Scalar Equations

$$\frac{1}{r} \frac{\partial(rE_r)}{\partial r} + \frac{1}{r} \frac{\partial E_\phi}{\partial \phi} + \frac{\partial E_z}{\partial z} = \frac{\rho}{\epsilon} \quad (55)$$

$$\frac{1}{r} \frac{\partial(rB_r)}{\partial r} + \frac{1}{r} \frac{\partial B_\phi}{\partial \phi} + \frac{\partial B_z}{\partial z} = 0 \quad (56)$$

$$\frac{1}{r} \frac{\partial E_z}{\partial \phi} - \frac{\partial E_\phi}{\partial z} + \frac{\partial B_r}{\partial t} = 0 \quad (57)$$

$$\frac{\partial E_r}{\partial z} - \frac{\partial E_z}{\partial r} + \frac{\partial B_\phi}{\partial t} = 0 \quad (58)$$

$$\frac{\partial(rE_\phi)}{\partial r} - \frac{\partial E_r}{\partial \phi} + \frac{\partial B_z}{\partial t} = 0 \quad (59)$$

$$\frac{1}{r} \frac{\partial B_z}{\partial \phi} - \frac{\partial B_\phi}{\partial z} = \mu J_r + \mu\epsilon \frac{\partial E_r}{\partial t} \quad (60)$$

$$\frac{\partial B_r}{\partial z} - \frac{\partial B_z}{\partial r} = \mu J_\phi + \mu\epsilon \frac{\partial E_\phi}{\partial t} \quad (61)$$

$$\frac{\partial(rB_\phi)}{\partial r} - \frac{\partial B_r}{\partial \phi} = \mu J_z + \mu\epsilon \frac{\partial E_z}{\partial t} \quad (62)$$

(iii) Spherical coordinates

Scalar equations

$$\frac{1}{r^2} \frac{\partial(r^2 E_r)}{\partial r} + \frac{1}{r \sin \theta} \frac{\partial E_\theta \sin \theta}{\partial \theta} + \frac{1}{r \sin \theta} \frac{\partial E_\phi}{\partial \phi} = \frac{\rho}{\epsilon} \quad (63)$$

$$\frac{1}{r^2} \frac{\partial(r^2 B_r)}{\partial r} + \frac{1}{r \sin \theta} \frac{\partial B_\theta \sin \theta}{\partial \theta} + \frac{1}{r \sin \theta} \frac{\partial B_\phi}{\partial \phi} = 0 \quad (64)$$

$$\frac{1}{r \sin \theta} \left(\frac{\partial E_\phi \sin \theta}{\partial \theta} - \frac{\partial E_\theta}{\partial \phi} \right) + \frac{\partial B_r}{\partial t} = 0 \quad (65)$$

$$\frac{1}{r} \left(\frac{1}{\sin \theta} \frac{\partial E_r}{\partial \phi} - \frac{\partial(r E_\phi)}{\partial r} \right) + \frac{\partial B_\theta}{\partial t} = 0 \quad (66)$$

$$\frac{1}{r} \left(\frac{\partial(r E_\theta)}{\partial r} - \frac{\partial E_r}{\partial \theta} \right) + \frac{\partial B_\phi}{\partial t} = 0 \quad (67)$$

$$\frac{1}{r \sin \theta} \left(\frac{\partial B_\phi \sin \theta}{\partial \theta} - \frac{\partial B_\theta}{\partial \phi} \right) = \mu J_r + \mu \epsilon \frac{\partial E_r}{\partial t} \quad (68)$$

$$\frac{1}{r} \left(\frac{1}{\sin \theta} \frac{\partial B_r}{\partial \phi} - \frac{\partial(r B_\phi)}{\partial r} \right) = \mu J_\theta + \mu \epsilon \frac{\partial E_\theta}{\partial t} \quad (69)$$

$$\frac{1}{r} \left(\frac{\partial(r B_\theta)}{\partial r} - \frac{\partial B_r}{\partial \theta} \right) = \mu J_\phi + \mu \epsilon \frac{\partial E_\phi}{\partial t} \quad (70)$$

5 Lorentz Condition and Equation of Continuity

A **corollary** of **Helmholtz Decomposition theorem** says that all physically realistic scalar fields obey a continuity equation. The theorem states that for any reasonable scalar field **S** and Vector field **C** there exists a vector field **F** such that $\vec{\nabla} \cdot \mathbf{F} = S$ and $\vec{\nabla} \times \mathbf{F} = \mathbf{C}$. [reference](#)

Lorentz Gauge:

$$\vec{\nabla} \cdot \vec{A} = -\frac{1}{c^2} \frac{\partial \phi}{\partial t} \quad (71)$$

from definition of Magnetic Vector potential, \vec{A}

$$\vec{\nabla} \times \vec{A} = \vec{B} \quad (72)$$

Considering $\mathbf{F} = \vec{A}$, $S = -\frac{1}{c^2} \frac{\partial \phi}{\partial t}$, $\mathbf{C} = \vec{B}$

Lorentz condition satisfy the condition for equation of continuity.

6 Homogenous wave equation

$$U = f(t \pm R\sqrt{\mu\epsilon}) \quad (73)$$

Let $x = t \pm R\sqrt{\mu\epsilon}$

$$\frac{\partial U}{\partial R} = \frac{df}{dx} \frac{\partial x}{\partial R} \quad (74)$$

$$\frac{\partial U}{\partial R} = \pm \sqrt{\mu\epsilon} \frac{df}{dx} \quad (75)$$

$$\boxed{\frac{\partial^2 U}{\partial R^2} = \mu\epsilon \frac{d^2 f}{dx^2}} \quad (76)$$

$$\frac{\partial U}{\partial t} = \frac{df}{dx} \frac{\partial x}{\partial t} \quad (77)$$

$$\frac{\partial U}{\partial t} = \frac{df}{dx} \quad (78)$$

$$\boxed{\frac{\partial^2 U}{\partial t^2} = \frac{d^2 f}{dx^2}} \quad (79)$$

$$\boxed{\frac{\partial^2 U}{\partial R^2} - \mu\epsilon \frac{\partial^2 U}{\partial t^2} = \mu\epsilon \frac{d^2 f}{dx^2} - \mu\epsilon \frac{d^2 f}{dx^2} = 0} \quad (80)$$

Therefore, any function of $t \pm R\sqrt{\mu\epsilon}$ satisfies the Homogenous wave equation.