IIT Madras
Physics

EP3110 Electromagnetics and Applications

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Contents

1 Potential, Field and Charge distribution

Given

$$V(r) = A\frac{e^{-\lambda r}}{r} \tag{1}$$

Electric field, $\vec{E}$:

$$\vec{E} = -\vec{\nabla}V \tag{2}$$

$$\vec{E} = -\left(\frac{\partial}{\partial r}\hat{r} + \frac{1}{r\sin\phi}\frac{\partial}{\partial\phi}\hat{\phi} + \frac{1}{r}\frac{\partial}{\partial\phi}\hat{\phi}\right)V(r) \tag{3}$$

$$\vec{E} = -\frac{\mathrm{d}V(r)}{\mathrm{d}r}\hat{r} \tag{4}$$

$$\boxed{\vec{E} = A\frac{e^{-\lambda r}}{r^2}(1+\lambda r)\hat{r}} \tag{5}$$

Charge distribution, ρ :

$$\vec{\nabla}.\vec{E} = \frac{\rho}{\epsilon_o} \tag{6}$$

$$\vec{\nabla}.\left(Ae^{-\lambda r}(1+\lambda r)\frac{\hat{r}}{r^2}\right) = \frac{\rho}{\epsilon_o} \tag{7}$$

$$Ae^{-\lambda r}(1+\lambda r)\vec{\nabla}.\left(\frac{\hat{r}}{r^2}\right) + \frac{\hat{r}}{r^2}.\vec{\nabla}(Ae^{-\lambda r}(1+\lambda r)) = \frac{\rho}{\epsilon_o} \tag{8}$$

$$Ae^{-\lambda r}(1+\lambda r)(4\pi\delta^3(r)) + \frac{\hat{r}}{r^2}.(Ae^{-\lambda r}(-\lambda^2 r)\hat{r}) = \frac{\rho}{\epsilon_o} \tag{9}$$

$$\left(\because \vec{\nabla}.\left(\frac{\hat{r}}{r^2}\right) = 4\pi\delta^3(r)\right)$$

$$\boxed{\rho = A\epsilon_o\left(4\pi\delta^3(r) - \lambda^2\frac{e^{-\lambda r}}{r}\right)}(\because f(x)\delta(x) = f(0)\delta(x)) \tag{10}$$

Total Charge, Q:

$$Q = \int_{-\infty}^{+\infty}\rho d\tau \tag{11}$$

$$Q = \int_{-\infty}^{+\infty}A\epsilon_o\left(4\pi\delta^3(r) - \lambda^2\frac{e^{-\lambda r}}{r}\right)d\tau \tag{12}$$

$$Q = A\epsilon_o\int_{-\infty}^{+\infty}4\pi\delta^3(r)d\tau - A\epsilon_o\int_{-\infty}^{+\infty}\lambda^2\frac{e^{-\lambda r}}{r}d\tau \tag{13}$$

$$Q = A\epsilon_o(4\pi) - A\epsilon_o\lambda^2 4\pi(\frac{1}{\lambda^2}) \tag{14}$$

$$\boxed{Q = 0} \tag{15}$$

Answers

$\boxed{\vec{E} = A\frac{e^{-\lambda r}}{r^2}(1+\lambda r)\hat{r}}$,

$\boxed{\rho = A\epsilon_o\left(4\pi\delta^3(r) - \lambda^2\frac{e^{-\lambda r}}{r}\right)}$,

$\boxed{Q = 0}$.

2 Dipole

Considering Proton above Z=0 and electron below Z=0, $10^{-11}m \ll 13m$, so we can consider this charge distribution as dipole with dipole moment $\vec{p} = ed\hat{k}$

$$V(R) = \frac{q}{4\pi\epsilon_o}\left(\frac{1}{R_+} - \frac{1}{R_-}\right) \tag{16}$$

Law of cosines,

$$R_\pm^2 = R^2 + (\frac{d}{2})^2 \mp Rd\cos\phi = R^2(1 \mp \frac{d}{R}\cos\phi + \frac{d^2}{4R^2}) \tag{17}$$

$$\frac{1}{R_\pm} \approx \frac{1}{R}\left(1 \mp \frac{d}{R}\cos\phi\right)^{-\frac{1}{2}} \approx \frac{1}{R}\left(1 \pm \frac{d}{2R}\cos\phi\right) \tag{18}$$

$$\implies \frac{1}{R_+} - \frac{1}{R_-} \approx \frac{d}{R^2}\cos\phi \tag{19}$$

$$V(R) \cong \frac{1}{4\pi\epsilon_o}\frac{qd\cos\phi}{R^2} \tag{20}$$

$$\boxed{V(R) = \frac{1}{4\pi\epsilon_o}\frac{\vec{p}.\hat{R}}{R^2}} \tag{21}$$

Electric Field, $\vec{E}(R)$ is

$$\vec{E}(R) = -\vec{\nabla}V(R) \tag{22}$$

$$\vec{E}(R) = -\left(\frac{\partial}{\partial R}\hat{R} + \frac{1}{R\sin\phi}\frac{\partial}{\partial\theta} + \frac{1}{R}\frac{\partial}{\partial\phi}\right)\left(\frac{1}{4\pi\epsilon_o}\frac{qd\cos\phi}{R^2}\right) \tag{23}$$

$$\vec{E}(R) = \frac{qd}{4\pi\epsilon_o R^3}\left(2\cos\phi\hat{R} + \sin\phi\hat{\phi}\right) \tag{24}$$

$$\boxed{\vec{E}(R) = \frac{1}{4\pi\epsilon_o R^3}\left(3(\vec{p}.\hat{R})\hat{R} - \vec{p}\right)} \tag{25}$$

Here, $\vec{p} = 10^{-11}e\hat{k}$Cm and $\vec{R} = (3\hat{i} + 4\hat{i} + 12\hat{k})m$

$$V(R) = \frac{1}{4\pi\epsilon_o}\frac{12X10^{-11}e}{13^3}V = 4.65X10^{-25}V \tag{26}$$

$$\vec{E}(R) = \frac{1}{4\pi\epsilon_o 13^3}(3(\frac{12}{13}X10^{-11}e)\frac{3\hat{i} + 4\hat{j} + 12\hat{k}}{13} - 10^{-11}e\hat{k}) \tag{27}$$

$$\vec{E}(R) = (4.188X10^{-24}\hat{i} + 5.585X10^{-24}\hat{j} + 1.019X10^{-23}\hat{k})NC^{-1} \tag{28}$$

Answers

$\boxed{V(R) = 4.65X10^{-25}V}$

$\boxed{\vec{E}(R) = (4.188X10^{-24}\hat{i} + 5.585X10^{-24}\hat{j} + 1.019X10^{-23}\hat{k})NC^{-1}}$

3 Magnetic Boundary

$$B_{1n} = B_{2n} \tag{29}$$

$$H_{1t} = H_{2t} \implies \frac{B_{1t}}{\mu_1} = \frac{B_{2t}}{\mu_2} \tag{30}$$

(i) $\tilde{\mathbf{B_1}} = \mathbf{0.5\hat{x} - 10\hat{y}(mT)}$

Boundary is y = 0 therefore, normal is $\hat{y}$

$$B_{1n} = B_{2n} \implies B_{2n} = -10mT \tag{31}$$
$$\frac{B_{1t}}{\mu_1} = \frac{B_{2t}}{\mu_2} \implies B_{2t} = \frac{\mu_2}{\mu_1}B_{1t} \tag{32}$$
$$B_{2t} = 2.5T \tag{33}$$
$$\text{Angle with interface is} \arctan(B_{2n}/B_{2t}) = \arctan(-\frac{10 * 10^{-3}}{2.5}) = 0.114^\circ \tag{34}$$

$$\boxed{\vec{B_2} = 2.5T\hat{x} - 10mT\hat{y}, \text{makes } 0.114^\circ \text{ with interface.}} \tag{35}$$

(ii) $\tilde{\mathbf{B_2}} = \mathbf{10\hat{x} + 0.5\hat{y}(mT)}$

Boundary is y = 0 therefore, normal is $\hat{y}$

$$B_{1n} = B_{2n} \implies B_{1n} = 0.5mT \tag{36}$$
$$\frac{B_{1t}}{\mu_1} = \frac{B_{2t}}{\mu_2} \implies B_{1t} = \frac{\mu_1}{\mu_2}B_{2t} \tag{37}$$
$$B_{1t} = \frac{10}{5000}mT = 2\mu T \tag{38}$$
$$\text{Angle with normal is} \arctan(B_{1t}/B_{1n}) = \arctan(\frac{0.002}{0.5}) = 0.229^\circ \tag{39}$$

$$\boxed{\vec{B_1} = 2\mu T\hat{x} + 0.5mT\hat{y}, \text{makes } 0.229^\circ \text{ with normal.}} \tag{40}$$

4 Maxwell's equations in scalar form

Maxwell equations

$$\vec{\nabla}.\vec{D} = \rho \tag{41}$$
$$\vec{\nabla}.\vec{B} = 0 \tag{42}$$
$$\vec{\nabla}X\vec{E} = -\frac{\partial B}{\partial t} \tag{43}$$
$$\vec{\nabla}X\vec{H} = \vec{J} + \frac{\partial D}{\partial t} \tag{44}$$

Linear Medium

$$\vec{D} = \epsilon\vec{E} \tag{45}$$
$$\vec{B} = \mu\vec{H} \tag{46}$$

(i) Cartesian coordinates

Scalar equations

$$\boxed{\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = \frac{\rho}{\epsilon}} \tag{47}$$

$$\boxed{\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} = 0} \tag{48}$$

$$\boxed{\left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}\right) + \frac{\partial B_x}{\partial t} = 0} \tag{49}$$

$$\boxed{\left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}\right) + \frac{\partial B_y}{\partial t} = 0} \tag{50}$$

$$\boxed{\left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right) + \frac{\partial B_z}{\partial t} = 0} \tag{51}$$

$$\boxed{\left(\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z}\right) = \mu J_x + \mu\epsilon\frac{\partial E_x}{\partial t}} \tag{52}$$

$$\boxed{\left(\frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x}\right) = \mu J_y + \mu\epsilon\frac{\partial E_y}{\partial t}} \tag{53}$$

$$\boxed{\left(\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y}\right) = \mu J_z + \mu\epsilon\frac{\partial E_z}{\partial t}} \tag{54}$$

(ii) Cylindrical Coordinates

Scalar Equations

$$\boxed{\frac{1}{r}\frac{\partial (rE_r)}{\partial r} + \frac{1}{r}\frac{\partial E_\phi}{\partial \phi} + \frac{\partial E_z}{\partial z} = \frac{\rho}{\epsilon}} \tag{55}$$

$$\boxed{\frac{1}{r}\frac{\partial (rB_r)}{\partial r} + \frac{1}{r}\frac{\partial B_\phi}{\partial \phi} + \frac{\partial B_z}{\partial z} = 0} \tag{56}$$

$$\boxed{\frac{1}{r}\frac{\partial E_z}{\partial \phi} - \frac{\partial E_\phi}{\partial z} + \frac{\partial B_r}{\partial t} = 0} \tag{57}$$

$$\boxed{\frac{\partial E_r}{\partial z} - \frac{\partial E_z}{\partial r} + \frac{\partial B_\phi}{\partial t} = 0} \tag{58}$$

$$\boxed{\frac{\partial (rE_\phi)}{\partial r} - \frac{\partial E_r}{\partial \phi} + \frac{\partial B_z}{\partial t} = 0} \tag{59}$$

$$\boxed{\frac{1}{r}\frac{\partial B_z}{\partial \phi} - \frac{\partial B_\phi}{\partial z} = \mu J_r + \mu\epsilon\frac{\partial E_r}{\partial t}} \tag{60}$$

$$\boxed{\frac{\partial B_r}{\partial z} - \frac{\partial B_z}{\partial r} = \mu J_\phi + \mu\epsilon\frac{\partial E_\phi}{\partial t}} \tag{61}$$

$$\boxed{\frac{\partial (rB_\phi)}{\partial r} - \frac{\partial B_r}{\partial \phi} = \mu J_z + \mu\epsilon\frac{\partial E_z}{\partial t}} \tag{62}$$

(iii) Spherical coordinates

Scalar equations

$$\boxed{\frac{1}{r^2}\frac{\partial(r^2E_r)}{\partial r}+\frac{1}{r\sin\theta}\frac{\partial E_\theta\sin\theta}{\partial\theta}+\frac{1}{r\sin\theta}\frac{\partial E_\phi}{\partial\phi}=\frac{\rho}{\epsilon}} \tag{63}$$

$$\boxed{\frac{1}{r^2}\frac{\partial(r^2B_r)}{\partial r}+\frac{1}{r\sin\theta}\frac{\partial B_\theta\sin\theta}{\partial\theta}+\frac{1}{r\sin\theta}\frac{\partial B_\phi}{\partial\phi}=0} \tag{64}$$

$$\boxed{\frac{1}{r\sin\theta}\left(\frac{\partial E_\phi\sin\theta}{\partial\theta}-\frac{\partial E_\theta}{\partial\phi}\right)+\frac{\partial B_r}{\partial t}=0} \tag{65}$$

$$\boxed{\frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial E_r}{\partial\phi}-\frac{\partial(rE_\phi)}{\partial r}\right)+\frac{\partial B_\theta}{\partial t}=0} \tag{66}$$

$$\boxed{\frac{1}{r}\left(\frac{\partial(rE_\theta)}{\partial r}-\frac{\partial E_r}{\partial\theta}\right)+\frac{\partial B_\phi}{\partial t}=0} \tag{67}$$

$$\boxed{\frac{1}{r\sin\theta}\left(\frac{\partial B_\phi\sin\theta}{\partial\theta}-\frac{\partial B_\theta}{\partial\phi}\right)=\mu J_r+\mu\epsilon\frac{\partial E_r}{\partial t}} \tag{68}$$

$$\boxed{\frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial B_r}{\partial\phi}-\frac{\partial(rB_\phi)}{\partial r}\right)=\mu J_\theta+\mu\epsilon\frac{\partial E_\theta}{\partial t}} \tag{69}$$

$$\boxed{\frac{1}{r}\left(\frac{\partial(rB_\theta)}{\partial r}-\frac{\partial B_r}{\partial\theta}\right)=\mu J_\phi+\mu\epsilon\frac{\partial E_\phi}{\partial t}} \tag{70}$$

5 Lorentz Condition and Equation of Continuity

A **corollary** of **Helmholtz Decomposition theorem** says that all physically realistic scalar fields obey a continuity equation. The theorem states that for any reasonable scalar field S and Vector field **C** there exists a vector field **F** such that $\vec{\nabla}$.**F** = S and$\vec{\nabla}$X**F** = **C**. reference
Lorentz Gauge:

$$\vec{\nabla}.\vec{A}=-\frac{1}{c^2}\frac{\partial\phi}{\partial t} \tag{71}$$

from definition of Magnetic Vector potential, $\vec{A}$

$$\vec{\nabla}X\vec{A}=\vec{B} \tag{72}$$

Considering **F** = $\vec{A}$, S = $-\frac{1}{c^2}\frac{\partial\phi}{\partial t}$, **C** = $\vec{B}$
Lorentz condition satisfy the condition for equation of continuity.

6 Homogenous wave equation

$$U=f(t\pm R\sqrt{\mu\epsilon}) \tag{73}$$

Let $x=t\pm R\sqrt{\mu\epsilon}$

$$\frac{\partial U}{\partial R}=\frac{\mathrm{d}f}{\mathrm{d}x}\frac{\partial x}{\partial R} \tag{74}$$

$$\frac{\partial U}{\partial R}=\pm\sqrt{\mu\epsilon}\frac{\mathrm{d}f}{\mathrm{d}x} \tag{75}$$

$$\boxed{\frac{\partial^2 U}{\partial R^2} = \mu\epsilon \frac{d^2 f}{dx^2}} \tag{76}$$

$$\frac{\partial U}{\partial t} = \frac{\mathrm{d}f}{\mathrm{d}x}\frac{\partial x}{\partial t} \tag{77}$$

$$\frac{\partial U}{\partial t} = \frac{\mathrm{d}f}{\mathrm{d}x} \tag{78}$$

$$\boxed{\frac{\partial^2 U}{\partial t^2} = \frac{d^2 f}{dx^2}} \tag{79}$$

$$\boxed{\frac{\partial^2 U}{\partial R^2} - \mu\epsilon \frac{\partial^2 U}{\partial t^2} = \mu\epsilon \frac{d^2 f}{dx^2} - \mu\epsilon \frac{d^2 f}{dx^2} = 0} \tag{80}$$

Therefore, any function of $t \pm R\sqrt{\mu\epsilon}$ satisfies the Homogenous wave equation.