EP 3120: Statistical Physics and Applications

Tutorial 1
Aug 6, 2022

1. **Moments and Cumulants:** Recall that the moments of a probability distribution function $p(x)$ can be obtained from the characteristic function $\tilde{p}(k)$ of the distribution, as,

$$(-i)^n \langle x^n \rangle = \left(\frac{\partial^n \tilde{p}(k)}{\partial k^n} \right)_{k=0}.$$

The cumulants of the distribution $p(x)$ are defined via the cumulant generating function $\ln \tilde{p}(k)$, as,

$$(-i)^n \langle x^n \rangle_c = \left(\frac{\partial^n \ln \tilde{p}(k)}{\partial k^n} \right)_{k=0}.$$

Using these definitions, show that the first three moments and cumulants can be related as follows.

$$\begin{aligned} \langle x \rangle_c &= \langle x \rangle \quad \text{(Mean)}, \\ \langle x^2 \rangle_c &= \langle x^2 \rangle - \langle x \rangle^2 \quad \text{(Variance)}, \\ \langle x^3 \rangle_c &= \langle x^3 \rangle - 3\langle x^2 \rangle \langle x \rangle + 2\langle x \rangle^3 \quad \text{(Skewness)}. \end{aligned} \tag{1}$$

2. **Characteristic Functions:** Calculate the characteristic function, the mean, variance and skewness of the following distributions. Also, make a qualitative sketch of each of the distributions, noting the mean and mode (most probable value) in each plot.

   (a) Uniform distribution: $p(x) = \frac{1}{a}$ for $-a < x < a$, and $p(x) = 0$ otherwise.

   (b) Laplace distribution: $p(x) = \frac{1}{a} \exp(-\frac{|x|}{a})$.

   (c) Wigner semi-circle distribution: $p(x) = \frac{2}{\pi R^2} \sqrt{R^2 - x^2}$, for $x \in [-R, R]$.
   Hint: The Bessel functions $J_n(z)$ of integer order n are defined by [1]:

$$J_n(z) = \frac{(\frac{1}{2}z)^n}{\sqrt{\pi}\Gamma(n + \frac{1}{2})} \int_0^\pi \cos(z \cos\theta) \sin^{2n}\theta \, d\theta. \tag{2}$$

   (d) Gaussian distribution (also called the *normal* distribution): $p(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/2\sigma^2}$.

[1] Reference: Abramowitz and Stegun, Pg. 360

3. The energy levels of certain complex systems, such as complex nuclei, or protein molecules, or certain glasses, can be viewed as random. Consider n energy levels $\epsilon_1, \epsilon_2, \ldots, \epsilon_n$ and assume that they are independently drawn from the same probability distribution $p(\epsilon)$. Suppose E is the lowest of all levels $\{\epsilon\}$ (ground state); of course, E in general will be different for different realizations of the random set $\{\epsilon\}$.

   (a) Write down the probability distribution of E, call it $W_n(E)$. Check that it normalized!

   (b) Obtain the distribution $W_n(E)$ when $p(\epsilon)$ is a uniform distribution between 0 and 1. Describe its behavior in the limit of large n.

   (c) Find average and variance of E when $p(\epsilon)$ is a uniform distribution between 0 and $\mathcal{E}$. How do these quantities behave with increasing n?

4. **Poisson Distribution:** The classical example of a Poisson process is radioactive decay. Consider a piece of radioactive material undergoing radioactive deacy over a time interval T. We can imagine that the time interval T is subdivided into many small intervals of length Δt. Then, the following two properties are observed:
(i) the probability of a decay event during time Δt is completely *independent* of decay events occurring at other times.
(ii) Furthermore, Δt is imagined to be small enough so that the probability of more than one decay event occurring in time Δt is negligibly small. The probability one (and only one) decay in the interval $[t, t+\Delta t]$ is proportional to Δt as $\Delta t \to 0$. Say this probability is equal to $\alpha \Delta t$.

   (a) The probability of observing exactly n trials in the interval T is given by the Poisson distribution. Show that this probability distribution function is

   $$p_{\alpha T}(n) = e^{-\alpha T} \frac{(\alpha T)^n}{n!}. \tag{3}$$

   Hint: This can be obtained as a limit of the binomial distribution by subdividing T into $N = \frac{T}{\Delta t} >> 1$.

   (b) Show that the corresponding characteristic function is given by $\tilde{p}(k) = \exp[\alpha T(e^{-ik} - 1)]$. Hence obtain the mean and variance of the Poisson distribution.

5. **Sum of Random Variables:** Consider the random variable $\mathcal{X} = \{x = \sum_{i=1}^{N} x_i\}$, where the N random variables x_i are distributed according to the *joint* probability distribution function $p(\vec{x})$.

   (a) Write down the probability distribution function $p_{\mathcal{X}}(x)$ of the variable cX, in terms of $p(\vec{x})$.

   (b) Evaluate the characteristic function $\tilde{p}_{\mathcal{X}}(k)$ in terms of the characteristic function $\tilde{p}(\vec{k})$ corresponding to the joint probability distribution. Hence show that the first two cumulants of the sum variable $\mathcal{X}$ are given in terms of the cumulants of the individual variables x_i, as follows:

$$\langle \mathcal{X} \rangle_c = \sum_{i=1}^{N} \langle x_i \rangle_c \; ; \; \langle \mathcal{X}^2 \rangle_c = \sum_{i,j} \langle x_i x_j \rangle_c .$$

(c) Assume that the N random variables x_i are independent, that is,

$$p(\vec{x}) = p_1(x_1) p_2(x_2) \dots p_N(x_N),$$

where $p_i(x_i)$ is the distribution corresponding to the i^{th} random variable. In this case, show that the n^{th} cumulant of the sum variable satisfies,

$$\langle \mathcal{X}^n \rangle_c = \sum_i \langle x_i^n \rangle_c .$$

Thus, if $p_i(x_i) = p(x)$ for all $i = 1, 2, \dots, N$, $\langle \mathcal{X}^n \rangle_c = N \langle x^n$, for the n^{th} cumulant.

(d) Consider the random variable

$$y = \frac{\mathcal{X} - N\langle x \rangle_c}{\sqrt{N}} .$$

Show that $\langle y \rangle = 0$, and the n^{th} cumulant falls off as,

$$\langle y^n \rangle_c \propto \frac{1}{N^{n/2-1}} .$$

Hence argue that $p(y)$ is a Gaussian distribution in the large-N limit, that is,

$$\lim_{N \to \infty} p(y) = \frac{1}{\sqrt{2\pi \langle x^2 \rangle_c}} e^{-\frac{y^2}{2\langle x^2 \rangle_c}} . \tag{4}$$

You have just proved the **Central Limit Theorem**! The sum of a set of *independent* and *identically distributed* random variables $\{x_i\}$, $i = 1, 2, \dots, N$ converges to the normal distribution in the limit of $N \to \infty$.

6. **Two-level systems:** Consider a system of N free particles in which the energy of each particle can assume *only* two distinct values: $E_0 = 0$ and $E_1 = E (E > 0)$. Let n_0 and n_1 denote the *occupation numbers* of the two level, that is, the number of particles in levels E_0 and E_1 respectively. Let U denote the total energy of the system.

   (a) Find the entropy S of the system.

   (b) Find the most probable values of n_0 and n_1, as well as their mean square fluctuations.

   (c) Find the temperature as a function of U, and show that it can be negative. Are there physical systems where just negative absolute temperatures are possible?

   (d) Find the heat-capacity of this system as a function of temperature.