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EP20B012

ASSIGNMENT I — JULY-NOV 2022

Due date: August 11th, before midnight

Contents

1 Potential, Field and Charge distribution

Given

$$V(r) = A\frac{e^{-\lambda r}}{r} \tag{1}$$

Electric field, $\vec{E}$:

$$\vec{E} = -\vec{\nabla}V \tag{2}$$

$$\vec{E} = -\left(\frac{\partial}{\partial r}\hat{r} + \frac{1}{r\sin\phi}\frac{\partial}{\partial\phi}\hat{\phi} + \frac{1}{r}\frac{\partial}{\partial\phi}\hat{\phi}\right)V(r) \tag{3}$$

$$\vec{E} = -\frac{\mathrm{d}V(r)}{\mathrm{d}r}\hat{r} \tag{4}$$

$$\boxed{\vec{E} = A\frac{e^{-\lambda r}}{r^2}(1+\lambda r)\hat{r}} \tag{5}$$

Charge distribution, ρ :

$$\vec{\nabla}.\vec{E} = \frac{\rho}{\epsilon_o} \tag{6}$$

$$\vec{\nabla}.\left(Ae^{-\lambda r}(1+\lambda r)\frac{\hat{r}}{r^2}\right) = \frac{\rho}{\epsilon_o} \tag{7}$$

$$Ae^{-\lambda r}(1+\lambda r)\vec{\nabla}.\left(\frac{\hat{r}}{r^2}\right) + \frac{\hat{r}}{r^2}.\vec{\nabla}(Ae^{-\lambda r}(1+\lambda r)) = \frac{\rho}{\epsilon_o} \tag{8}$$

$$Ae^{-\lambda r}(1+\lambda r)(4\pi\delta^3(r)) + \frac{\hat{r}}{r^2}.(Ae^{-\lambda r}(-\lambda^2 r)\hat{r}) = \frac{\rho}{\epsilon_o} \tag{9}$$

$$\left(\because \vec{\nabla}.\left(\frac{\hat{r}}{r^2}\right) = 4\pi\delta^3(r)\right)$$

$$\boxed{\rho = A\epsilon_o\left(4\pi\delta^3(r) - \lambda^2\frac{e^{-\lambda r}}{r}\right)}(\because f(x)\delta(x) = f(0)\delta(x)) \tag{10}$$

Total Charge, Q:

$$Q = \int_{-\infty}^{+\infty}\rho d\tau \tag{11}$$

$$Q = \int_{-\infty}^{+\infty}A\epsilon_o\left(4\pi\delta^3(r) - \lambda^2\frac{e^{-\lambda r}}{r}\right)d\tau \tag{12}$$

$$Q = A\epsilon_o\int_{-\infty}^{+\infty}4\pi\delta^3(r)d\tau - A\epsilon_o\int_{-\infty}^{+\infty}\lambda^2\frac{e^{-\lambda r}}{r}d\tau \tag{13}$$

$$Q = A\epsilon_o(4\pi) - A\epsilon_o\lambda^2 4\pi(\frac{1}{\lambda^2}) \tag{14}$$

$$\boxed{Q = 0} \tag{15}$$

Answers:- $\vec{E} = A\frac{e^{-\lambda r}}{r^2}(1+\lambda r)\hat{r}$, $\rho = A\epsilon_o\left(4\pi\delta^3(r) - \lambda^2\frac{e^{-\lambda r}}{r}\right)$, $Q = 0$.

2 Dipole

Considering Proton above Z=0 and electron below Z=0, $10^{-11}m \ll 13m$, so we can consider this charge distribution as dipole with dipole moment $\vec{p} = ed\hat{k}$

$$V(R) = \frac{q}{4\pi\epsilon_o}\left(\frac{1}{R_+} - \frac{1}{R_-}\right) \tag{16}$$

Law of cosines,

$$R_\pm^2 = R^2 + (\frac{d}{2})^2 \mp Rd\cos\phi = R^2(1 \mp \frac{d}{R}\cos\phi + \frac{d^2}{4R^2}) \tag{17}$$

$$\frac{1}{R_\pm} \approx \frac{1}{R}\left(1 \mp \frac{d}{R}\cos\phi\right)^{-\frac{1}{2}} \approx \frac{1}{R}\left(1 \pm \frac{d}{2R}\cos\phi\right) \tag{18}$$

$$\implies \frac{1}{R_+} - \frac{1}{R_-} \approx \frac{d}{R^2}\cos\phi \tag{19}$$

$$V(R) \cong \frac{1}{4\pi\epsilon_o}\frac{qd\cos\phi}{R^2} \tag{20}$$

$$\boxed{V(R) = \frac{1}{4\pi\epsilon_o}\frac{\vec{p}.\hat{R}}{R^2}} \tag{21}$$

Electric Field, $\vec{E}(R)$ is

$$\vec{E}(R) = -\vec{\nabla}V(R) \tag{22}$$

$$\vec{E}(R) = -\left(\frac{\partial}{\partial R}\hat{R} + \frac{1}{R\sin\phi}\frac{\partial}{\partial\theta} + \frac{1}{R}\frac{\partial}{\partial\phi}\right)\left(\frac{1}{4\pi\epsilon_o}\frac{qd\cos\phi}{R^2}\right) \tag{23}$$

$$\vec{E}(R) = \frac{qd}{4\pi\epsilon_o R^3}\left(2\cos\phi\hat{R} + \sin\phi\hat{\phi}\right) \tag{24}$$

$$\boxed{\vec{E}(R) = \frac{1}{4\pi\epsilon_o R^3}\left(3(\vec{p}.\hat{r})\hat{r} - \vec{p}\right)} \tag{25}$$

Here, $\vec{p} = 10^{-11}e\hat{k}$Cm and $\vec{R} = (3\hat{\imath} + 4\hat{\imath} + 12\hat{k})m$

$$V(R) = \frac{1}{4\pi\epsilon_o}\frac{12X10^{-11}e}{13^2}V = 6.05X10^{-24}V \tag{26}$$

$$\vec{E}(R) = \frac{1}{4\pi\epsilon_o 13^3}(3(\frac{12}{13}X10^{-11}e)\frac{3\hat{\imath} + 4\hat{\jmath} + 12\hat{k}}{13} - 10^{-11}e\hat{k}) \tag{27}$$

$$\vec{E}(R) = (4.188X10^{-24}\hat{\imath} + 5.585X10^{-24}\hat{\jmath} + 1.019X10^{-23}\hat{k})NC^{-1} \tag{28}$$

Answers:- $\boxed{V(R) = 6.05X10^{-24}V, \vec{E}(R) = (4.188X10^{-24}\hat{\imath} + 5.585X10^{-24}\hat{\jmath} + 1.019X10^{-23}\hat{k})NC^{-1}}$

3 Lorentz Condition and Equation of Continuity

A **corollary** of **Helmholtz Decomposition theorem** says that all physically realistic scalar fields obey a continuity equation. The theorem states that for any reasonable scalar field S and Vector field **C** there exists a vector field **F** such that $\vec{\nabla}$.**F** = S and$\vec{\nabla}$X**F** = **C**. ref

Lorentz Gauge:

$$\vec{\nabla}.\vec{A} = -\frac{1}{c^2}\frac{\partial \phi}{\partial t} \tag{29}$$

from definition of Magnetic Vector potential, $\vec{A}$

$$\vec{\nabla} X \vec{A} = \vec{B} \tag{30}$$

Considering **F** = $\vec{A}$, S = $-\frac{1}{c^2}\frac{\partial \phi}{\partial t}$, **C** = $\vec{B}$
Lorentz condition satisfy the condition for equation of continuity.

4 Homogenous wave equation

$$U = f(t \pm R\sqrt{\mu\epsilon}) \tag{31}$$

Let $x = t \pm R\sqrt{\mu\epsilon}$

$$\frac{\partial U}{\partial R} = \frac{\mathrm{d}f}{\mathrm{d}x}\frac{\partial x}{\partial R} \tag{32}$$

$$\frac{\partial U}{\partial R} = \pm\sqrt{\mu\epsilon}\frac{\mathrm{d}f}{\mathrm{d}x} \tag{33}$$

$$\boxed{\frac{\partial^2 U}{\partial R^2} = \mu\epsilon\frac{d^2 f}{dx^2}} \tag{34}$$

$$\frac{\partial U}{\partial t} = \frac{\mathrm{d}f}{\mathrm{d}x}\frac{\partial x}{\partial t} \tag{35}$$

$$\frac{\partial U}{\partial t} = \frac{\mathrm{d}f}{\mathrm{d}x} \tag{36}$$

$$\boxed{\frac{\partial^2 U}{\partial t^2} = \frac{d^2 f}{dx^2}} \tag{37}$$

$$\boxed{\frac{\partial^2 U}{\partial R^2} - \frac{\partial^2 U}{\partial t^2} = \mu\epsilon\frac{d^2 f}{dx^2} - \mu\epsilon\frac{d^2 f}{dx^2} = 0} \tag{38}$$

Therefore, any function of $t \pm R\sqrt{\mu\epsilon}$ satisfies the Homogenous wave equation.