Problem Set 7

CMSC 27230: Honors Theory of Algorithms

Assigned February 27, Due March 3

Problem 1: Bundles (30 points)

Let's say that you are an artist selling n paintings $p_1, \ldots, p_n$. You receive orders from m customers. However, most of your customers do not just want one painting. Instead, each customer j wants a subset $S_j \subseteq \{p_1, \ldots, p_n\}$ of your paintings. If they cannot buy all of the paintings in their bundle S_j then they will not buy any paintings. Is it possible to satisfy at least k of the customers?

We can express this problem mathematically as follows. Given subsets $S_1, \ldots, S_m \subseteq [n]$, is there a set $I \subseteq [m]$ such that $|I| = k$ and for all distinct $i, j \in I$, $S_i \cap S_j = \emptyset$ (no painting is sold more than once)?

Prove that this problem is NP-complete. Hint is available.

Problem 2: Tech Tree (30 points)

You are playing a game with a technology tree and there is a set of technologies you want to research. However, to research a technology, you need to research at least one of the technologies which lead to it. Also, you only have a limited number of scientists so you can only research a total of m technologies. Given this, is it possible to research all of the technologies you would like to research?

We model this problem as follows. You have an acyclic directed graph G where the vertices $V = \{v_1, \ldots, v_n\}$ are the technologies and you have the edges $E = \{(v_i, v_j) : \text{ technology } i \text{ unlocks technology } j\}$. You also have a set $T \subseteq [2, n]$ of target technologies which you want to research (you start with the technology v_1). The problem is now as follows: Is there a set $R \subseteq [2, n]$ of technologies you can research which satisfies the following conditions?

1. For every $i \in R$, there is a path P from v_1 to v_i such that for every vertex v_j in P, $j \in R \cup \{1\}$ (Given the starting technology v_1, you can iteratively research all of the technologies $\{v_i : i \in R\}$ without needing any other technologies).

2. $T \subseteq R$ (you research all of the desired technologies)

3. $|R| \leq m$ (you research at most m technologies)

Prove that this problem is NP-complete. Hint is available.

Problem 3: Currency Trader (40 points)

You are a currency trader specializing in two currencies, currency A and currency B. You start with some amount a_{start} of currency A and some amount b_{start} of currency B. You then receive n offers to trade some amount of one currency for the other currency. For each trade, you can either accept the trade or reject it. If you somehow knew all of the offers you would receive in advance, would it be possible to end with at least a_{target} of currency A and b_{target} of currency B for some target values a_{target} and b_{target}?

We can express this problem mathematically as follows. Given starting values (a_{start}, b_{start}), target values (a_{target}, b_{target}), and n pairs $\{(a_i, b_i) : i \in [n]\}$ representing the possible trades (where $(-x, y)$ represents trading x units of currency A for y units of currency B and $(x, -y)$ represents trading y units of currency B for x units of currency A), can we find a subset $S \subseteq [n]$ such that

1. $a_{start} + \sum_{i \in S} a_i \geq a_{target}$ and $b_{start} + \sum_{i \in S} b_i \geq b_{target}$ (we achieve our targets)
2. For all $j \in [n]$, $a_{start} + \sum_{i \in S \cap [j]} a_i \geq 0$ and $b_{start} + \sum_{i \in S \cap [j]} b_i \geq 0$ (we can make all of the trades without going into debt for either currency).

For this problem, assume that $a_{start} \geq 0$, $b_{start} \geq 0$, and all of the values a_{start}, b_{start}, a_{target}, b_{target}, $\{a_i : i \in [n]\}$, and $\{b_i : i \in [n]\}$ are integers.

(a) 20 points: Let $B = \max\{a_{start}, b_{start}, a_{target}, b_{target}, \max_{i \in [n]}\{|a_i|\}, \max_{i \in [n]}\{|b_i|\}\}$. Give a $poly(n, B)$ time algorithm for this problem, explain why your algorithm is correct, and analyze its runtime.

(b) 20 points: Show that if B can be exponentially large then this problem is NP-complete. Hint is available.

Note: Similar to the knapsack problem, we take the size of the input to be $poly(n, log(B))$. We want that B is at most $2^{poly(n)}$ so that the input size is polynomial in n.

Problem 4: Minimum Maximal Non-crossing Matching (10 points extra credit)

Let G be an undirected graph on the set of vertices $V(G) = \{v_1, \ldots, v_n\}$

Recall that a matching $M \subseteq E(G)$ is non-crossing if M does not contain two edges $e_1 = \{v_i, v_j\}$ and $e_2 = \{v_k, v_l\}$ such that $i < k < j < l$ or $k < i < j < l$. We say that $M \subseteq E(G)$ is a maximal non-crossing matching if M is a non-crossing matching and there is no edge $e \in E(G) \setminus M$ such that $M \cup \{e\}$ is also a non-crossing matching.

The minimum maximal non-crossing matching problem is as follows. Given a graph G, what is the minimum possible size of a maximal non-crossing matching of G?

Show that this problem is NP-hard. Hint is available.

Hints

1. Reducing either 3-SAT or independent set to this problem will work well.

2. Reducing either 3-SAT or vertex cover to this problem will work well.

3b. Reducing knapsack to this problem works well.

4. One way to reduce 3-SAT to the minimum maximal non-crossing matching problem is as follows:

1. Choose a target size for the maximal non-crossing matching which can only be achieved if the 3-SAT problem has a solution.
2. Create a gadget such that in any maximal non-crossing matching of the target size, you must choose between two edges corresponding to x_i and $\neg x_i$.
3. For each clause C, create a gadget such that in any maximal matching of the target size, you must choose two out of three edges where the edge which is not picked must correspond to one of the literals satisfying C.