Representation of Floating Point Numbers in Single Precision IEEE 754 Standard

Value =
$$N = (-1)^S \times 2^{E-127} \times (1.M)$$

0 < E < 255Actual exponent is: e = E - 127

8 sign

> exponent: excess 127 binary integer added

mantissa:

23

М

sign + magnitude, normalized binary significand with a hidden integer bit: 1.M

Magnitude of numbers that can be represented is in the range:

Which is approximately:

$$1.8 \times 10^{-38}$$
 to 3.40×10^{38}

Representation of Floating Point Numbers in Double Precision IEEE 754 Standard

Value =
$$N = (-1)^S \times 2^{E-1023} \times (1.M)$$

0 < E < 2047**Actual exponent is:** e = E - 1023

11 **52** sign М

> exponent: excess 1023 binary integer added

mantissa: sign + magnitude, normalized binary significand with

a hidden integer bit: 1.M

Magnitude of numbers that can be represented is in the range:

Which is approximately:

$$2.23 \times 10^{-308}$$
 to 1.8×10^{-308}

IEEE 754 Format Parameters

| | Single Precision | Double Precision |
|------------------------------------|------------------|------------------|
| p (bits of precision) | 24 | 53 |
| Unbiased exponent e _{max} | 127 | 1023 |
| Unbiased exponent e _{min} | -126 | -1022 |
| Exponent bias | 127 | 1023 |

IEEE 754 Special Number Representation

| Single Precision | | Double Precision | | Number Represented |
|------------------|-------------|------------------|-------------|----------------------------------|
| Exponent | Significand | Exponent | Significand | |
| 0 | 0 | 0 | 0 | 0 |
| 0 | nonzero | 0 | nonzero | Denormalized number ¹ |
| 1 to 254 | anything | 1 to 2046 | anything | Floating Point Number |
| 255 | 0 | 2047 | 0 | Infinity ² |
| 255 | nonzero | 2047 | nonzero | NaN (Not A Number) ³ |

¹ May be returned as a result of underflow in multiplication

² Positive divided by zero yields "infinity"

³ Zero divide by zero yields NaN "not a number"

Floating Point Conversion Example

• The decimal number $.75_{10}$ is to be represented in the *IEEE 754* 32-bit single precision format:

-2345.125₁₀ =
$$0.11_2$$
 (converted to a binary number)
= 1.1×2^{-1} (normalized a binary number)

Hidden <

• The mantissa is positive so the sign S is given by:

$$S = 0$$

• The biased exponent E is given by E = e + 127

$$E = -1 + 127 = 126_{10} = 011111110_2$$

• Fractional part of mantissa M:

The IEEE 754 single precision representation is given by:

| 0 | 01111110 | 100000000000000000000000000000000000000 | |
|-------|----------|---|-------|
| S | E | M | |
| 1 bit | 8 bits | 23 bits | FFCC2 |

Floating Point Conversion Example

The decimal number -2345.125_{10} is to be represented in the IEEE 754 32-bit single precision format:

$$-2345.125_{10} = -100100101001.001_2$$
 (converted to binary)
= $-1.00100101001001 \times 2^{11}$ (normalized binary)

• The mantissa is negative so the sign S is given by:

$$S = 1$$

• The biased exponent E is given by E = e + 127

$$E = 11 + 127 = 138_{10} = 10001010_2$$

• Fractional part of mantissa M:

The IEEE 754 single precision representation is given by:

| 1 | 10001010 | 0010010100100100000000 | |
|-------|----------|------------------------|-------|
| S | E | M | |
| 1 bit | 8 bits | 23 bits | FFCC2 |

Basic Floating Point Addition Algorithm

Assuming that the operands are already in the IEEE 754 format, performing floating point addition: Result = $X + Y = (Xm \ x \ 2^{Xe}) + (Ym \ x \ 2^{Ye})$ involves the following steps:

- (1) Align binary point:
 - Initial result exponent: the larger of Xe, Ye
 - Compute exponent difference: Ye Xe
 - If Ye > Xe Right shift Xm that many positions to form Xm 2 Xe-Ye
 - If Xe > Ye Right shift Ym that many positions to form Ym 2 Ye-Xe
- (2) Compute sum of aligned *mantissas*:

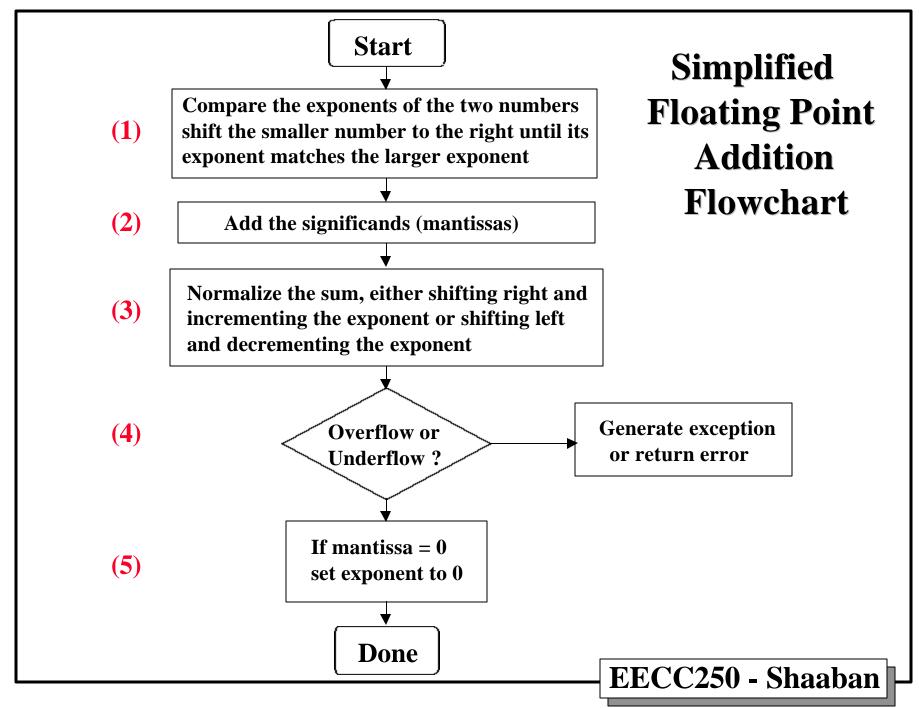
i.e $Xm2^{Xe-Ye} + Ym$ or $Xm + Xm2^{Ye-Xe}$

- (3) If normalization of result is needed, then a normalization step follows:
 - Left shift result, decrement result exponent (e.g., if result is 0.001xx...) or
 - Right shift result, increment result exponent (e.g., if result is 10.1xx...)

Continue until MSB of data is 1 (NOTE: Hidden bit in IEEE Standard)

- (4) Check result exponent:
 - If larger than maximum exponent allowed return exponent overflow
 - If smaller than minimum exponent allowed return exponent underflow
- (5) If result mantissa is 0, may need to set the exponent to zero by a special step to return a proper zero.

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Floating Point Addition Example

Add the following two numbers represented in the IEEE 754 single precision format: $X = 2345.125_{10}$ represented as:

> 10001010 001001010010010000000000

to $Y = .75_{10}$ represented as:

01111110

- (1) Align binary point:
 - Xe > Ye initial result exponent = $Ye = 10001010 = 138_{10}$
 - $Xe Ye = 10001010 011111110 = 00000110 = 12_{10}$
 - Shift Ym 12_{10} postions to the right to form

 $Ym \ 2^{Ye-Xe} = Ym \ 2^{-12} = 0.0000000000110000000000$

(2) Add mantissas:

 $Xm + Ym 2^{-12} = 1.00100101001001000000000$ 1. 00100101001111000000000

- (3) Normailzed? Yes
- (4) Overflow? No. Underflow? No. (5) zero result? No.

Result

0

10001010 | 00100101001111000000000

IEEE 754 Single precision Addition Notes

- If the exponents differ by more than 24, the smaller number will be shifted right entirely out of the mantissa field, producing a zero mantissa.
 - The sum will then equal the larger number.
 - Such truncation errors occur when the numbers differ by a factor of more than 2^{24} , which is approximately 1.6 x 10^7 .
 - Thus, the precision of IEEE single precision floating point arithmetic is approximately 7 decimal digits.
- Negative mantissas are handled by first converting to 2's complement and then performing the addition.
 - After the addition is performed, the result is converted back to sign-magnitude form.
- When adding numbers of opposite sign, cancellation may occur, resulting in a sum which is arbitrarily small, or even zero if the numbers are equal in magnitude.
 - Normalization in this case may require shifting by the total number of bits in the mantissa, resulting in a large loss of accuracy.
- Floating point subtraction is achieved simply by inverting the sign bit and performing addition of signed mantissas as outlined above.

Basic Floating Point Subtraction Algorithm

Assuming that the operands are already in the IEEE 754 format, performing floating point addition: Result = $X - Y = (Xm \ x \ 2^{Xe}) - (Ym \ x \ 2^{Ye})$ involves the following steps:

- (1) Align binary point:
 - Initial result exponent: the larger of Xe, Ye
 - Compute exponent difference: Ye Xe
 - If Ye > Xe Right shift Xm that many positions to form Xm 2 Xe-Ye
 - If Xe > Ye Right shift Ym that many positions to form Ym 2 Ye-Xe
- (2) Subtract the aligned *mantissas*:

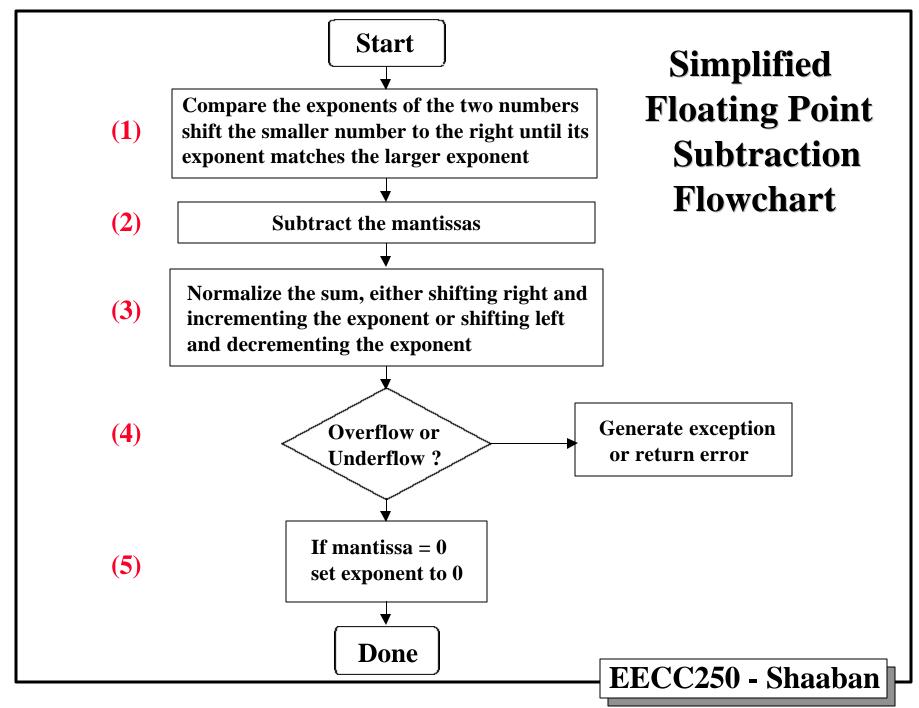
i.e $Xm2^{Xe-Ye} - Ym$ or $Xm - Xm2^{Ye-Xe}$

- (3) If normalization of result is needed, then a normalization step follows:
 - Left shift result, decrement result exponent (e.g., if result is 0.001xx...) or
 - Right shift result, increment result exponent (e.g., if result is 10.1xx...)

Continue until MSB of data is 1 (NOTE: Hidden bit in IEEE Standard)

- (4) Check result exponent:
 - If larger than maximum exponent allowed return exponent overflow
 - If smaller than minimum exponent allowed return exponent underflow
- (5) If result mantissa is 0, may need to set the exponent to zero by a special step to return a proper zero.

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Basic Floating Point Multiplication Algorithm

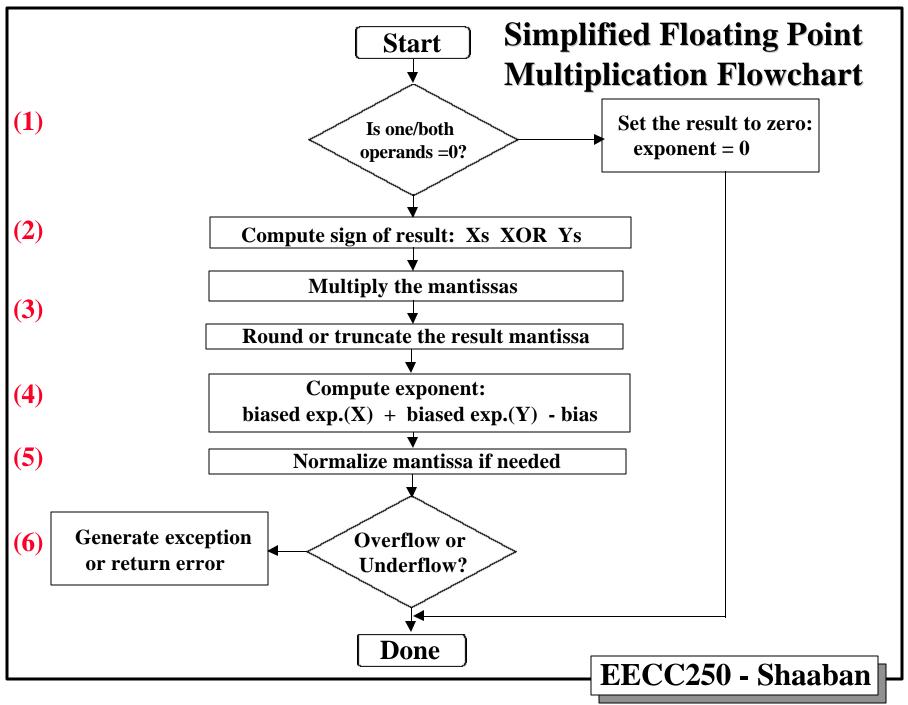
Assuming that the operands are already in the IEEE 754 format, performing floating point multiplication:

Result = $R = X * Y = (-1)^{Xs} (Xm x 2^{Xe}) * (-1)^{Ys} (Ym x 2^{Ye})$ involves the following steps:

- (1) If one or both operands is equal to zero, return the result as zero, otherwise:
- (2) Compute the sign of the result Xs XOR Ys
- (3) Compute the mantissa of the result:
 - Multiply the mantissas: Xm * Ym
 - Round the result to the allowed number of mantissa bits
- (4) Compute the exponent of the result:

Result exponent = biased exponent (X) + biased exponent (Y) - bias

- (5) Normalize if needed, by shifting mantissa right, incrementing result exponent.
- (6) Check result exponent for overflow/underflow:
 - If larger than maximum exponent allowed return exponent overflow
 - If smaller than minimum exponent allowed return exponent underflow



Floating Point Multiplication Example

• Multiply the following two numbers represented in the *IEEE 754* single precision format: $X = -18_{10}$ represented as:

1 10000011 0010000000000000000000

and $Y = 9.5_{10}$ represented as:

0 10000010 00110000000000000000000

- (1) Value of one or both operands = 0? No, continue with step 2
- (2) Compute the sign: $S = Xs \times XOR \times Ys = 1 \times XOR \times 0 = 1$
- (3) Multiply the mantissas: The product of the 24 bit mantissas is 48 bits with two bits to the left of the binary point:

(01).0101011000000....000000

Truncate to 24 bits:

(4) Compute exponent of result:

 $Xe + Ye - 127_{10} = 1000\ 0011 + 1000\ 0010 - 01111111 = 1000\ 0110$

- (5) Result mantissa needs normalization? No
- (6) Overflow? No. Underflow? No

IEEE 754 Single precision Multiplication Notes

- Rounding occurs in floating point multiplication when the mantissa of the product is reduced from 48 bits to 24 bits.
 - The least significant 24 bits are discarded.
- Overflow occurs when the sum of the exponents exceeds 127, the largest value which is defined in bias-127 exponent representation.
 - When this occurs, the exponent is set to 128 (E=255) and the mantissa is set to zero indicating + or infinity.
- Underflow occurs when the sum of the exponents is more negative than 126, the most negative value which is defined in bias-127 exponent representation.
 - When this occurs, the exponent is set to -127 (E = 0).
 - If M = 0, the number is exactly zero.
 - If M is not zero, then a denormalized number is indicated which has an exponent of -127 and a hidden bit of 0.
 - The smallest such number which is not zero is 2⁻¹⁴⁹. This number retains only a single bit of precision in the rightmost bit of the mantissa.

Basic Floating Point Division Algorithm

Assuming that the operands are already in the IEEE 754 format, performing floating point multiplication:

Result = $R = X / Y = (-1)^{X_S} (Xm \times 2^{X_E}) / (-1)^{Y_S} (Ym \times 2^{Y_E})$ involves the following steps:

- (1) If the divisor Y is zero return "Infinity", if both are zero return "NaN"
- (2) Compute the sign of the result Xs XOR Ys
- (3) Compute the mantissa of the result:
 - The dividend mantissa is extended to 48 bits by adding 0's to the right of the least significant bit.
 - When divided by a 24 bit divisor Ym, a 24 bit quotient is produced.
- (4) Compute the exponent of the result:

Result exponent = [biased exponent (X) - biased exponent (Y)] + bias

- (5) Normalize if needed, by shifting mantissa left, decrementing result exponent.
- (6) Check result exponent for overflow/underflow:
 - If larger than maximum exponent allowed return exponent overflow
 - If smaller than minimum exponent allowed return exponent underflow

IEEE 754 Error Rounding

- In integer arithmetic, the result of an operation is well-defined:
 - Either the exact result is obtained or overflow occurs and the result cannot be represented.
- In floating point arithmetic, rounding errors occur as a result of the limited precision of the mantissa. For example, consider the average of two floating point numbers with identical exponents, but mantissas which differ by 1. Although the mathematical operation is well-defined and the result is within the range of representable numbers, the average of two adjacent floating point values cannot be represented exactly.
- The IEEE FPS defines four rounding rules for choosing the closest floating point when a rounding error occurs:
 - RN Round to Nearest. Break ties by choosing the least significant bit = 0.
 - RZ Round toward Zero. Same as truncation in sign-magnitude.
 - RP Round toward Positive infinity.
 - RM Round toward Minus infinity. Same as truncation in integer 2's complement arithmetic.
- RN is generally preferred and introduces less systematic error than the other rules.

Floating Point Error Rounding Observations

- The absolute error introduced by rounding is the actual difference between the exact value and the floating point representation.
- The size of the absolute error is proportional to the magnitude of the number.
 - For numbers in single Precision IEEE 754 format, the absolute error is less than 2^{-24} .
 - The largest absolute rounding error occurs when the exponent is 127 and is approximately 10^{31} since $2^{-24} \cdot 2^{127} = 10^{31}$
- The relative error is the absolute error divided by the magnitude of the number which is approximated. For normalized floating point numbers, the relative error is approximately 10^{-7}
- Rounding errors affect the outcome of floating point computations in several ways:
 - Exact comparison of floating point variables often produces incorrect results.
 Floating variables should not be used as loop counters or loop increments.
 - Operations performed in different orders may give different results. On many computers, a+b may differ from b+a and (a+b)+c may differ from a+(b+c).
 - Errors accumulate over time. While the relative error for a single operation in single precision floating point is about 10⁻⁷, algorithms which iterate many times may experience an accumulation of errors which is much larger.

68000 FLOATING POINT ADD/SUBTRACT (FFPADD/FFPSUB) Subroutine

* FFPADD/FFPSUB FAST FLOATING POINT ADD/SUBTRACT FFPADD/FFPSUB - FAST FLOATING POINT ADD AND SUBTRACT INPUT: FFPADD D6 - FLOATING POINT ADDEND D7 - FLOATING POINT ADDER **FFPSUB** D6 - FLOATING POINT SUBTRAHEND D7 - FLOATING POINT MINUEND OUTPUT: D7 - FLOATING POINT ADD RESULT CONDITION CODES: N - RESULT IS NEGATIVE Z - RESULT IS ZERO V - OVERFLOW HAS OCCURED C - UNDEFINED X - UNDEFINED EECC250 - Shaaban

```
REGISTERS D3 THRU D5 ARE VOLATILE
CODE SIZE: 228 BYTES STACK WORK AREA: 0 BYTES
NOTES:
 1) ADDEND/SUBTRAHEND UNALTERED (D6).
 2) UNDERFLOW RETURNS ZERO AND IS UNFLAGGED.
 3) OVERFLOW RETURNS THE HIGHEST VALUE WITH THE
    CORRECT SIGN AND THE 'V' BIT SET IN THE CCR.
TIME: (8 MHZ NO WAIT STATES ASSUMED)
       COMPOSITE AVERAGE 20.625 MICROSECONDS
                     7.75 MICROSECONDS
       ARG1=0
ADD:
          ARG2=0
                       5.25 MICROSECONDS
      LIKE SIGNS 14.50 - 26.00 MICROSECONDS
               AVERAGE 18.00 MICROSECONDS
      UNLIKE SIGNS 20.13 - 54.38 MICROCECONDS
               AVERAGE 22.00 MICROSECONDS
SUBTRACT: ARG1=0 4.25 MICROSECONDS
           ARG2=0
                     9.88 MICROSECONDS
      LIKE SIGNS 15.75 - 27.25 MICROSECONDS
               AVERAGE 19.25 MICROSECONDS
      UNLIKE SIGNS 21.38 - 55.63 MICROSECONDS
               AVERAGE 23.25 MICROSECONDS
                                   EECC250 - Shaaban
```

```
*******
* SUBTRACT ENTRY POINT *
*******
FFPSUB
      MOVE.B D6,D4 TEST ARG1
       BEQ.S FPART2 RETURN ARG2 IF ARG1 ZERO
       EOR.B #$80,D4 INVERT COPIED SIGN OF ARG1
       BMI.S FPAMI1 BRANCH ARG1 MINUS
* + ARG1
       MOVE.B D7,D5 COPY AND TEST ARG2
       BMI.S FPAMS BRANCH ARG2 MINUS
       BNE.S FPALS BRANCH POSITIVE NOT ZERO
       BRA.S FPART1 RETURN ARG1 SINCE ARG2 IS ZERO
******
* ADD ENTRY POINT *
******
FFPADD MOVE.B D6,D4 TEST ARGUMENT1
       BMI.S FPAMI1 BRANCH IF ARG1 MINUS
       BEO.S FPART2 RETURN ARG2 IF ZERO
* + ARG1
       MOVE.B D7,D5 TEST ARGUMENT2
       BMI.S FPAMS BRANCH IF MIXED SIGNS
       BEQ.S FPART1 ZERO SO RETURN ARGUMENT1
```

```
* +ARG1 +ARG2
```

* -ARG1 -ARG2

FPALS SUB.B D4,D5 TEST EXPONENT MAGNITUDES

BMI.S FPA2LT BRANCH ARG1 GREATER

MOVE.B D7,D4 SETUP STRONGER S+EXP IN D4

* ARG1EXP <= ARG2EXP

| CMP.B | #24 D5 | OVERBEARING | STZE |
|-------|----------------------------|-------------|---------------------|
| | $\pi \Delta \pi \cdot D J$ | | $O \perp L \perp L$ |

BCC.S FPART2 BRANCH YES, RETURN ARG2

MOVE.L D6,D3 COPY ARG1

CLR.B D3 CLEAN OFF SIGN+EXPONENT

LSR.L D5,D3 SHIFT TO SAME MAGNITUDE

MOVE.B #\$80,D7 FORCE CARRY IF LSB-1 ON

ADD.L D3,D7 ADD ARGUMENTS

BCS.S FPA2GC BRANCH IF CARRY PRODUCED

FPARSR MOVE.B D4,D7 RESTORE SIGN/EXPONENT

RTS RETURN TO CALLER

* ADD SAME SIGN OVERFLOW NORMALIZATION

FPA2GC ROXR.L #1,D7 SHIFT CARRY BACK INTO RESULT

ADD.B #1,D4 ADD ONE TO EXPONENT

BVS.S FPA2OS BRANCH OVERFLOW

BCC.S FPARSR BRANCH IF NO EXPONENT OVERFLOW

FPA2OS MOVEQ #-1,D7 CREATE ALL ONES

SUB.B #1,D4 BACK TO HIGHEST EXPONENT+SIGN

MOVE.B D4,D7 REPLACE IN RESULT

* OR.B #\$02,CCR SHOW OVERFLOW OCCURRED

DC.L \$003C0002 ****ASSEMBLER ERROR****

RTS RETURN TO CALLER

* RETURN ARGUMENT1

FPART1 MOVE.L D6,D7 MOVE IN AS RESULT

MOVE.B D4,D7 MOVE IN PREPARED SIGN+EXPONENT

RTS RETURN TO CALLER

* RETURN ARGUMENT2

FPART2 TST.B D7 TEST FOR RETURNED VALUE

RTS RETURN TO CALLER

```
* +ARG1EXP > +ARG2EXP

FPA2LT CMP.B #-24,D5 ? ARGUMENTS WITHIN RANGE
BLE.S FPART1 NOPE, RETURN LARGER
NEG.B D5 CHANGE DIFFERENCE TO POSITIVE
MOVE.L D6,D3 SETUP LARGER VALUE
CLR.B D7 CLEAN OFF SIGN+EXPONENT
LSR.L D5,D7 SHIFT TO SAME MAGNITUDE
MOVE.B #$80,D3 FORCE CARRY IF LSB-1 ON
ADD.L D3,D7 ADD ARGUMENTS
BCS.S FPA2GC BRANCH IF CARRY PRODUCED
MOVE.B D4,D7 RESTORE SIGN/EXPONENT
RTS RETURN TO CALLER
```

* -ARG1

* -ARG1EXP > -ARG2EXP

FPAMI1 MOVE.B D7,D5 TEST ARG2'S SIGN

BMI.S FPALS BRANCH FOR LIKE SIGNS

BEQ.S FPART1 IF ZERO RETURN ARGUMENT1

```
* -ARG1 +ARG2
* +ARG1 -ARG2
FPAMS
       MOVEQ #-128,D3 CREATE A CARRY MASK ($80)
        EOR.B D3,D5
                       STRIP SIGN OFF ARG2 S+EXP COPY
        SUB.B D4,D5 COMPARE MAGNITUDES
        BEQ.S FPAEQ BRANCH EQUAL MAGNITUDES
        BMI.S FPATLT BRANCH IF ARG1 LARGER
* ARG1 <= ARG2
        CMP.B #24,D5 COMPARE MAGNITUDE DIFFERENCE
        BCC.S FPART2 BRANCH ARG2 MUCH BIGGER
        MOVE.B D7,D4 ARG2 S+EXP DOMINATES
        MOVE.B D3,D7 SETUP CARRY ON ARG2
        MOVE.L D6,D3 COPY ARG1
       CLR.B D3 CLEAR EXTRANEOUS BITS
FPAMSS
        LSR.L D5,D3 ADJUST FOR MAGNITUDE
        SUB.L D3,D7 SUBTRACT SMALLER FROM LARGER
        BMI.S FPARSR RETURN FINAL RESULT IF NO
  OVERFLOW
```

* MIXED SIGNS NORMALIZE

FPANOR MOVE.B D4,D5 SAVE CORRECT SIGN

FPANRM CLR.B D7 CLEAR SUBTRACT RESIDUE

SUB.B #1,D4 MAKE UP FOR FIRST SHIFT

CMP.L #\$00007FFF,D7 ? SMALL ENOUGH FOR SWAP

BHI.S FPAXON BRANCH NOPE

SWAP.W D7 SHIFT LEFT 16 BITS REAL FAST

SUB.B #16,D4 MAKE UP FOR 16 BIT SHIFT

FPAXQN ADD.L D7,D7 SHIFT UP ONE BIT

DBMI D4, FPAXON DECREMENT AND BRANCH IF POSITIVE

EOR.B D4,D5 ? SAME SIGN

BMI.S FPAZRO BRANCH UNDERFLOW TO ZERO

MOVE.B D4,D7 RESTORE SIGN/EXPONENT

BEQ.S FPAZRO RETURN ZERO IF EXPONENT

UNDERFLOWED

RTS RETURN TO CALLER

* EXPONENT UNDERFLOWED - RETURN ZERO

FPAZRO MOVEQ.L #0,D7 CREATE A TRUE ZERO

RTS RETURN TO THE CALLER

```
* ARG1 > ARG2
        CMP.B \#-24,D5 ? ARG1 >> ARG2
FPATLT
        BLE.S FPART1
                       RETURN IT IF SO
        NEG.B
               D5
                       ABSOLUTIZE DIFFERENCE
        MOVE.L D7,D3 MOVE ARG2 AS LOWER VALUE
        MOVE.L D6,D7 SETUP ARG1 AS HIGH
        MOVE.B #$80,D7 SETUP ROUNDING BIT
        BRA.S FPAMSS PERFORM THE ADDITION
* EQUAL MAGNITUDES
FPAEQ
      MOVE.B D7,D5 SAVE ARG1 SIGN
        EXG.L D5,D4 SWAP ARG2 WITH ARG1 S+EXP
        MOVE.B D6,D7 INSURE SAME LOW BYTE
               D6,D7 OBTAIN DIFFERENCE
        SUB.L
        BEQ.S FPAZRO RETURN ZERO IF IDENTICAL
        BPL.S FPANOR BRANCH IF ARG2 BIGGER
        NEG.L D7
                       CORRECT DIFFERENCE TO POSITIVE
        MOVE.B D5,D4 USE ARG2'S SIGN+EXPONENT
        BRA.S FPANRM AND GO NORMALIZE
```

END