Computer Graphics

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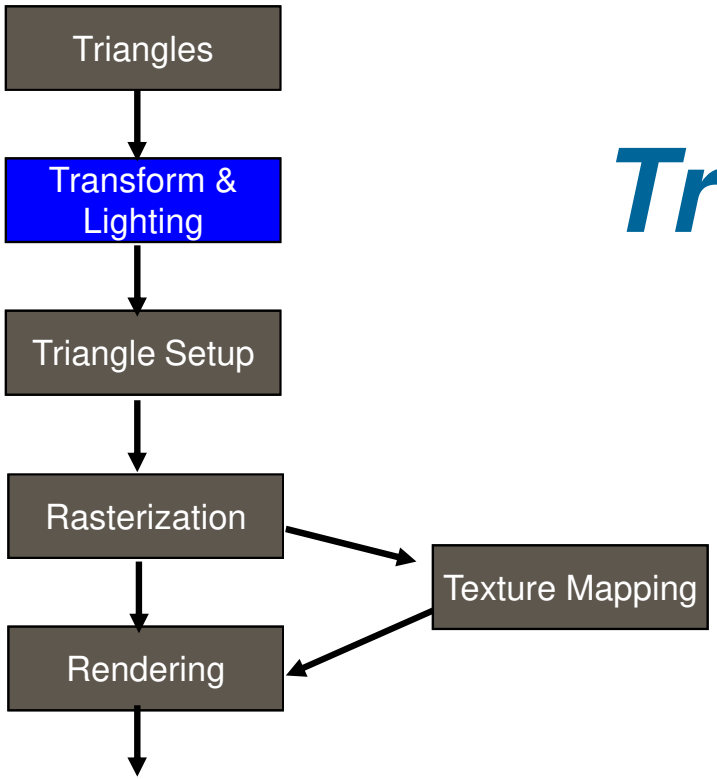
ICL/ITRI

Wrap up from last class

- Primitives
- Rasterization – Generate Pixels that Constitute the Primitive
  - Line Rasterization
  - Triangle Rasterization
- Deriving Attributes (Colors, Depths, Texture Coordinates) of Pixels inside a Triangle

Part I:
Conventional 3D Graphics Pipeline



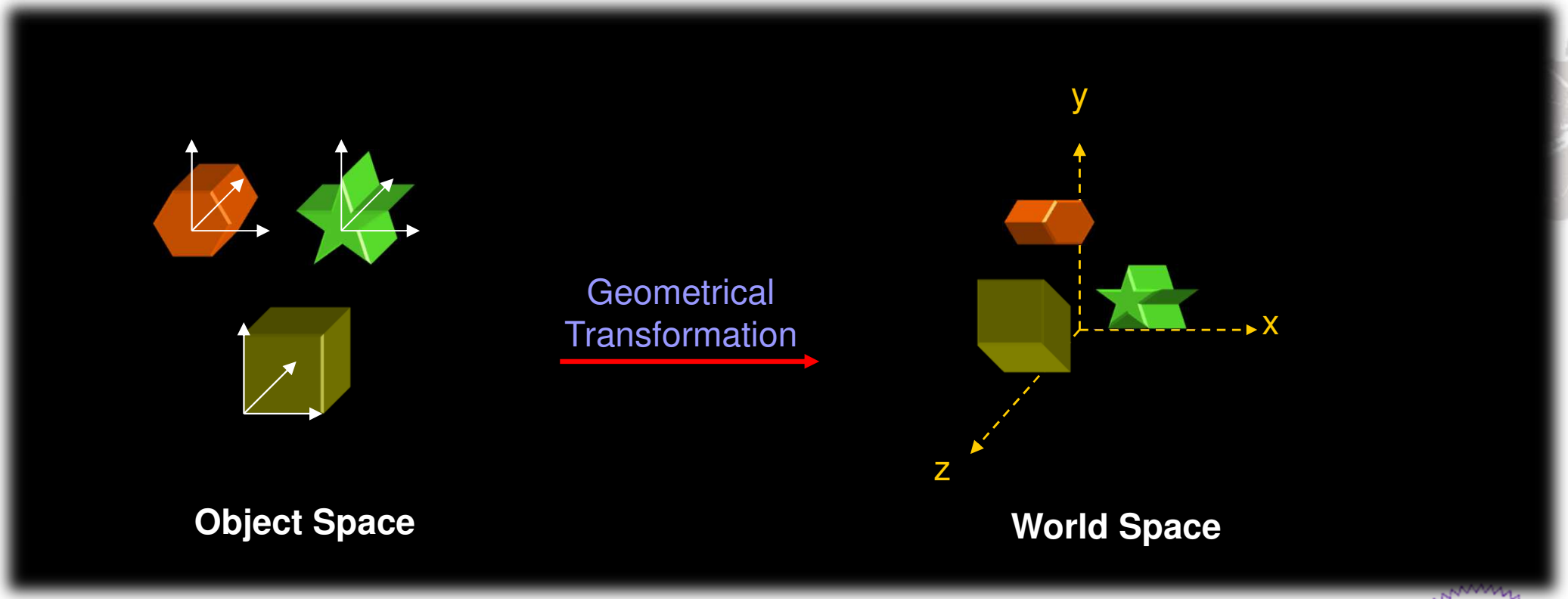
Conventional 3D Graphics Pipeline

Transformation

3D Viewing Process

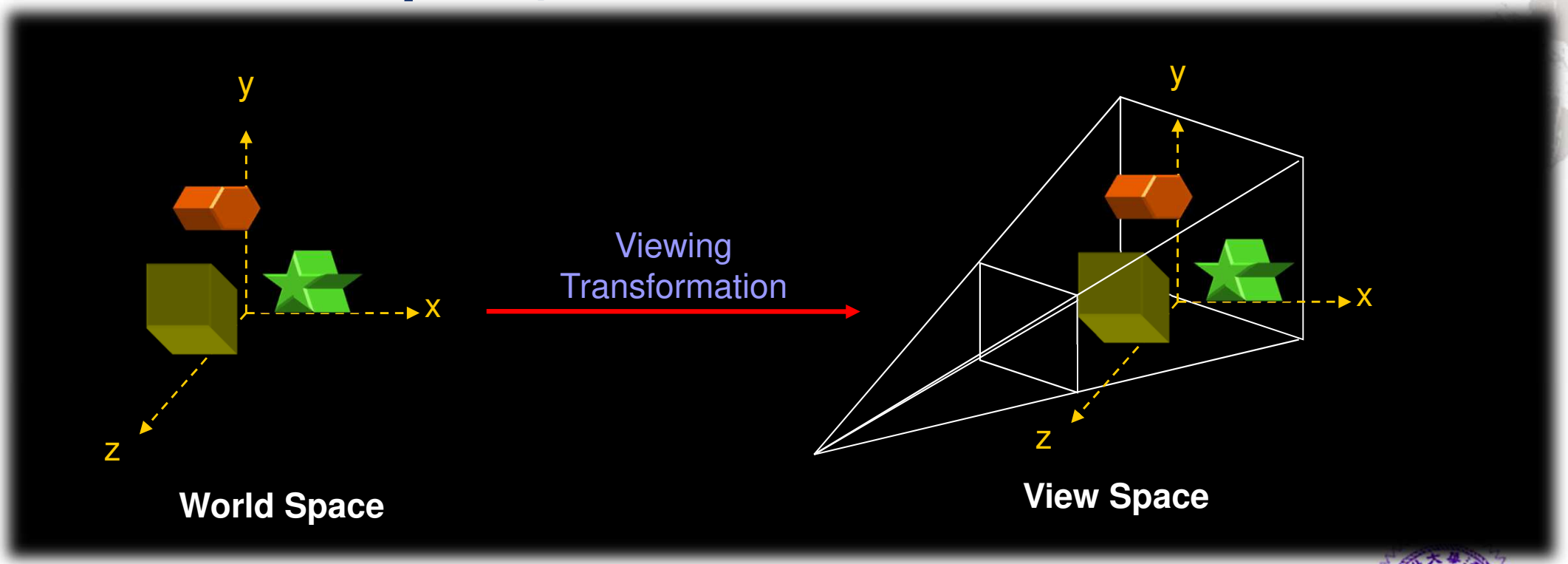
3D Viewing Process

- Geometrical Transformation
  - From Object Space to World Space



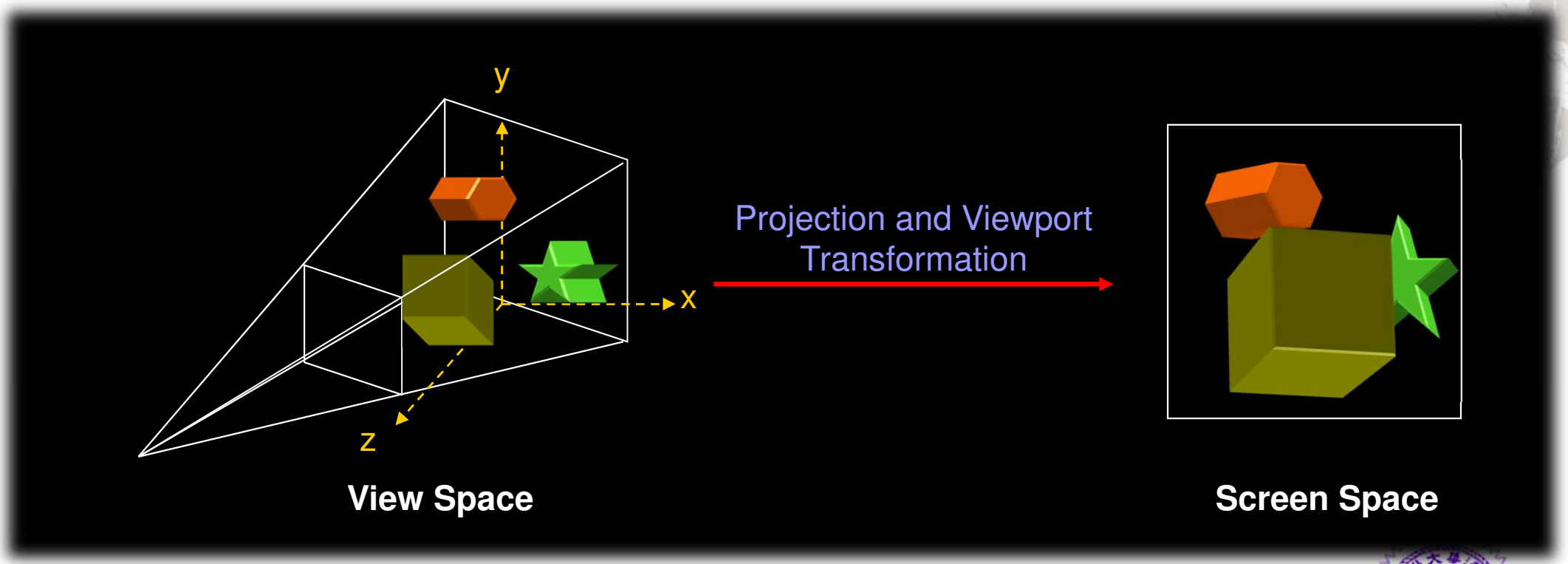
3D Viewing Process

- Viewing Transformation
  - From World Space to View Space (Eye Space or Camera Space)



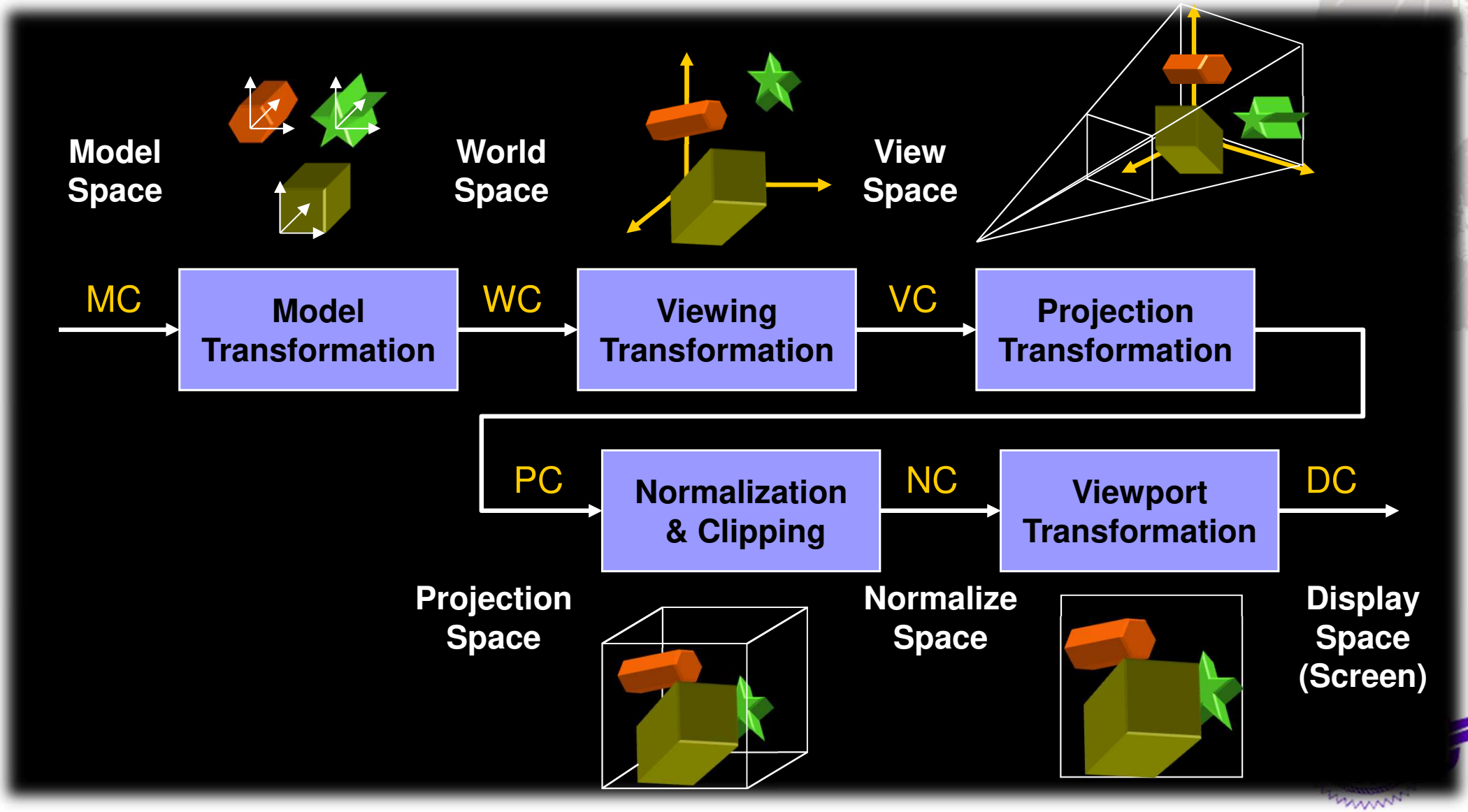
3D Viewing Process

- **Projection and Viewport Transformation**
  - **From View Space (3D) to Screen Space (2D)**



3D Viewing Process

- Coordinates Transformation

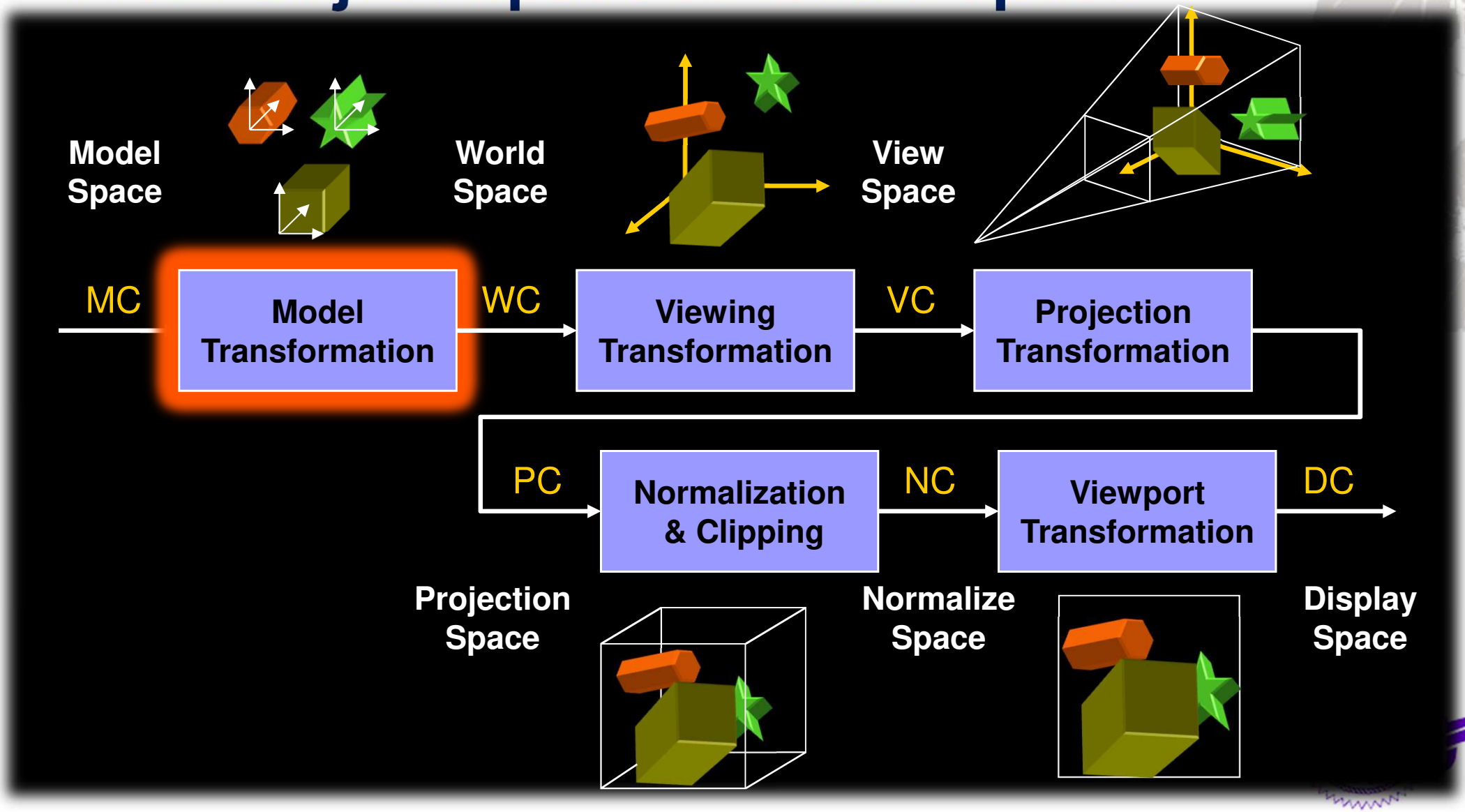


Geometrical Transformations

From Object Space to World Space

Geometrical Transformation

- From Object Space to World Space



2D Transformation

- **Translation**

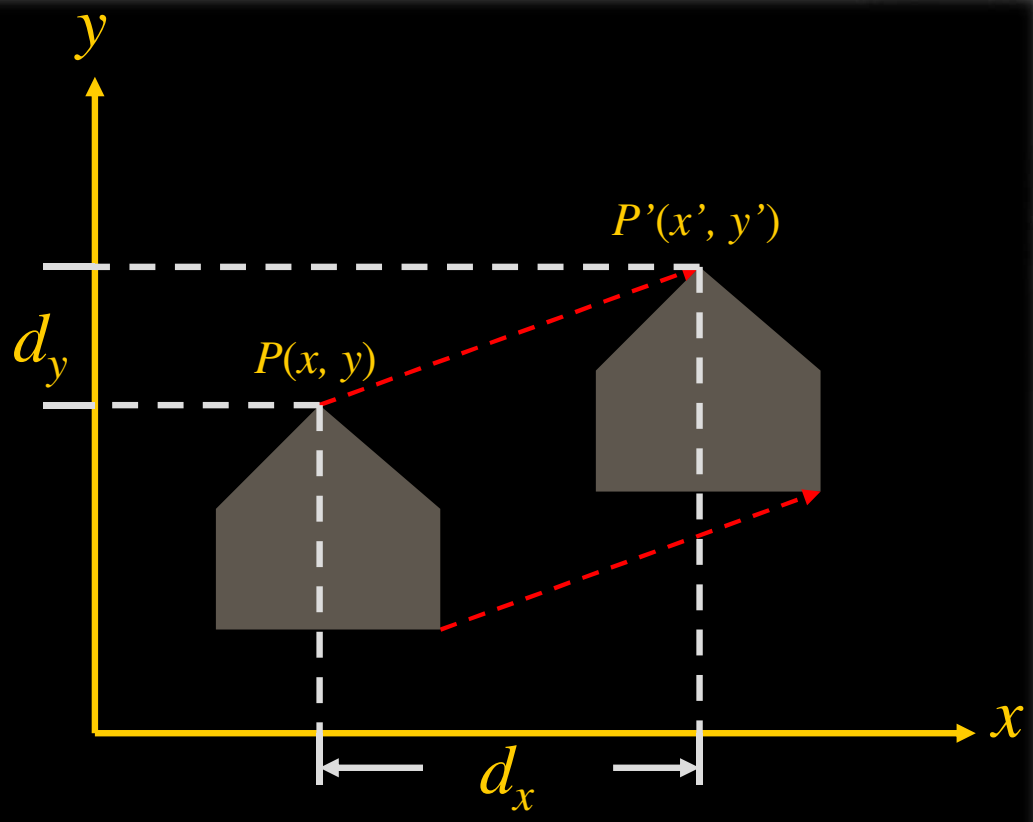
$P(x, y)$ move to $P'(x', y')$

$$x' = x + d_x$$

$$y' = y + d_y$$

$$P = \begin{bmatrix} x \\ y \end{bmatrix},\ P' = \begin{bmatrix} x' \\ y' \end{bmatrix},\ T = \begin{bmatrix} d_x \\ d_y \end{bmatrix}$$

$$P' = P + T(d_x, d_y)$$



2D Transformation

Scaling

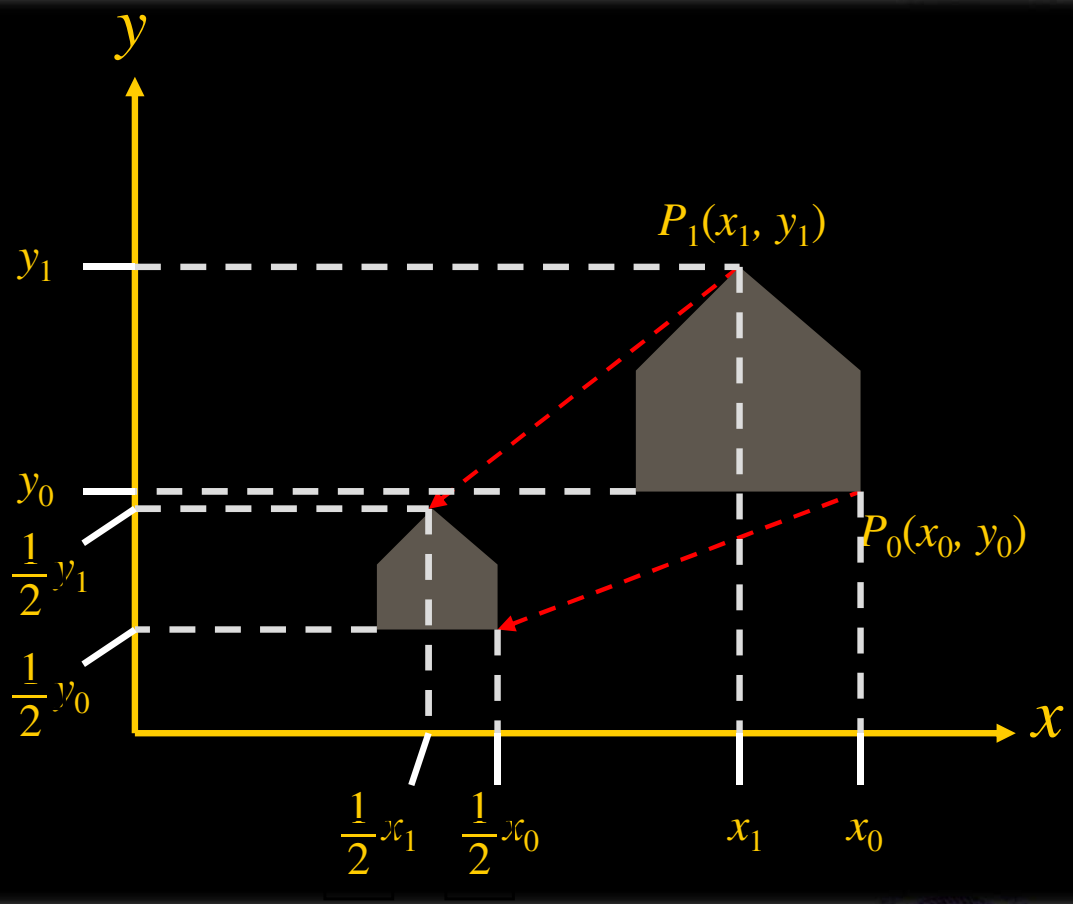
Scale by s_x along the x axis and by s_y along the y axis

$$x' = s_x \cdot x$$

$$y' = s_y \cdot y$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix}$$

$$P' = S(s_x, s_y) \cdot P$$



2D Transformation

Rotation

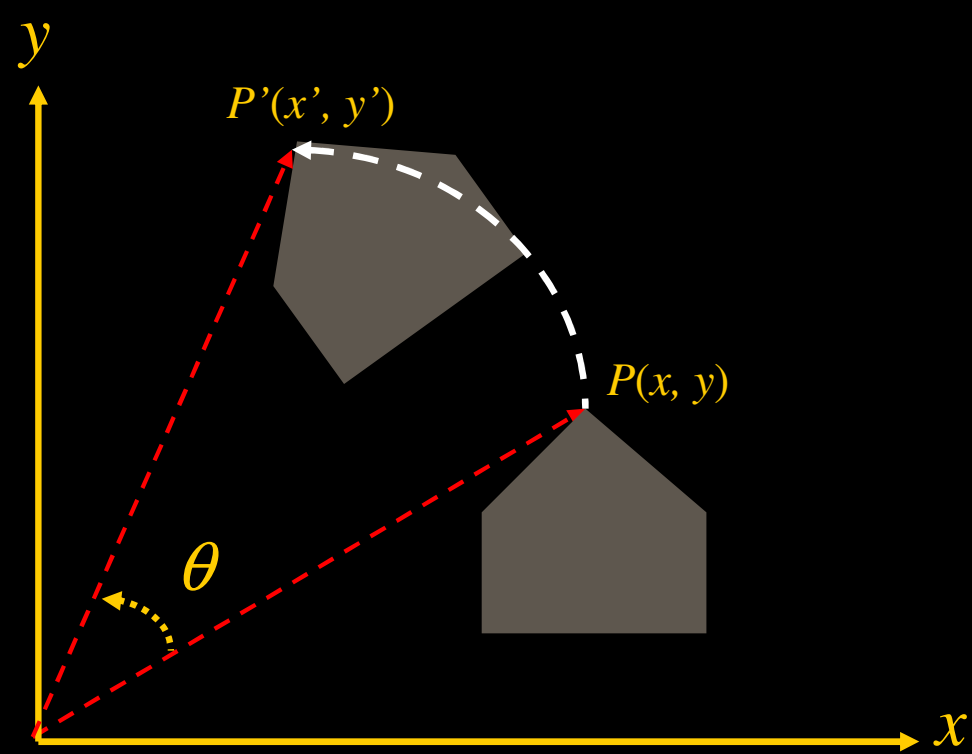
Rotate by an angle θ with respect to the origin

$$x' = x \cdot \cos\theta - y \cdot \sin\theta$$

$$y' = x \cdot \sin\theta + y \cdot \cos\theta$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix}$$

$$P' = R(\theta) \cdot P$$



2D Transformation

- **Derivation of the rotation equation**

$$x = r \cdot \cos\phi$$

$$y = r \cdot \sin\phi$$

$$x' = r \cdot \cos(\phi + \theta)$$

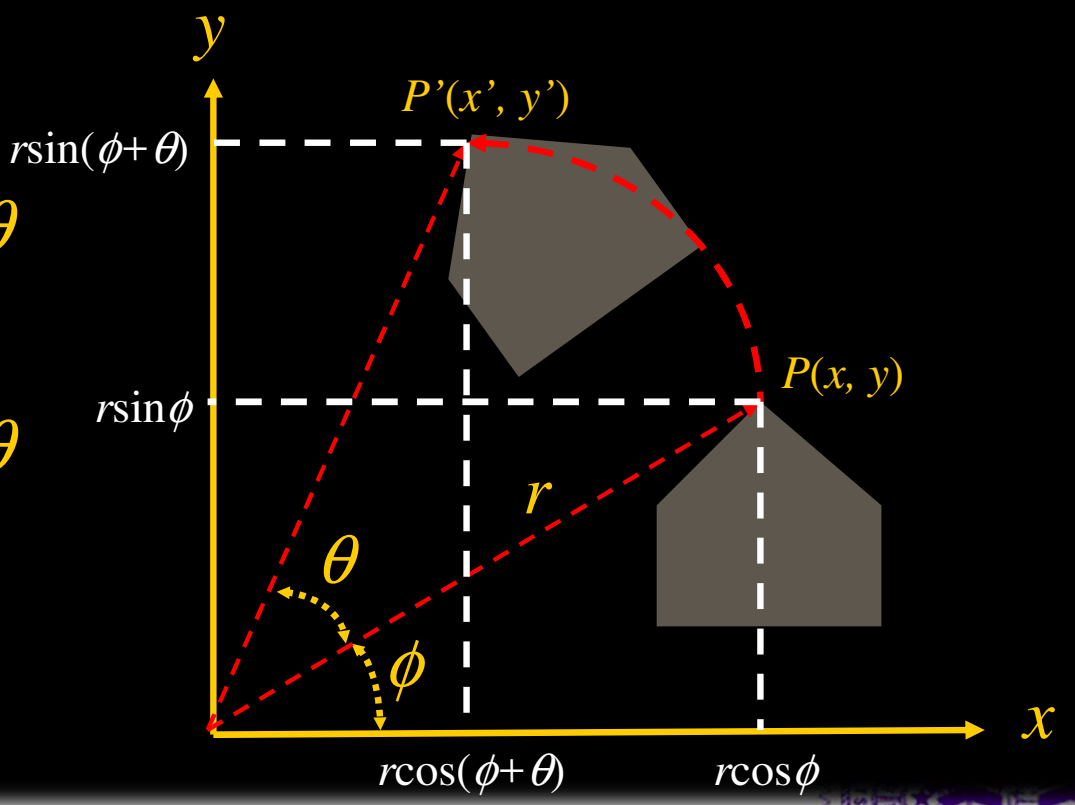
$$= r \cdot \cos\phi \cdot \cos\theta - r \cdot \sin\phi \cdot \sin\theta$$

$$y' = r \cdot \sin(\phi + \theta)$$

$$= r \cdot \cos\phi \cdot \sin\theta + r \cdot \sin\phi \cdot \cos\theta$$

$$x' = x \cdot \cos\theta - y \cdot \sin\theta$$

$$y' = x \cdot \sin\theta + y \cdot \cos\theta$$



Homogeneous Coordinates

- **Matrix representations for translation, scaling, and rotation**

$$P' = T + P \qquad \text{Translation}$$

$$P' = S \cdot P \qquad \text{Scaling}$$

$$P' = R \cdot P \qquad \text{Rotation}$$

- **How to derive a consistent way to represent all the transformations?**

Homogeneous coordinate representation

Homogeneous Coordinates

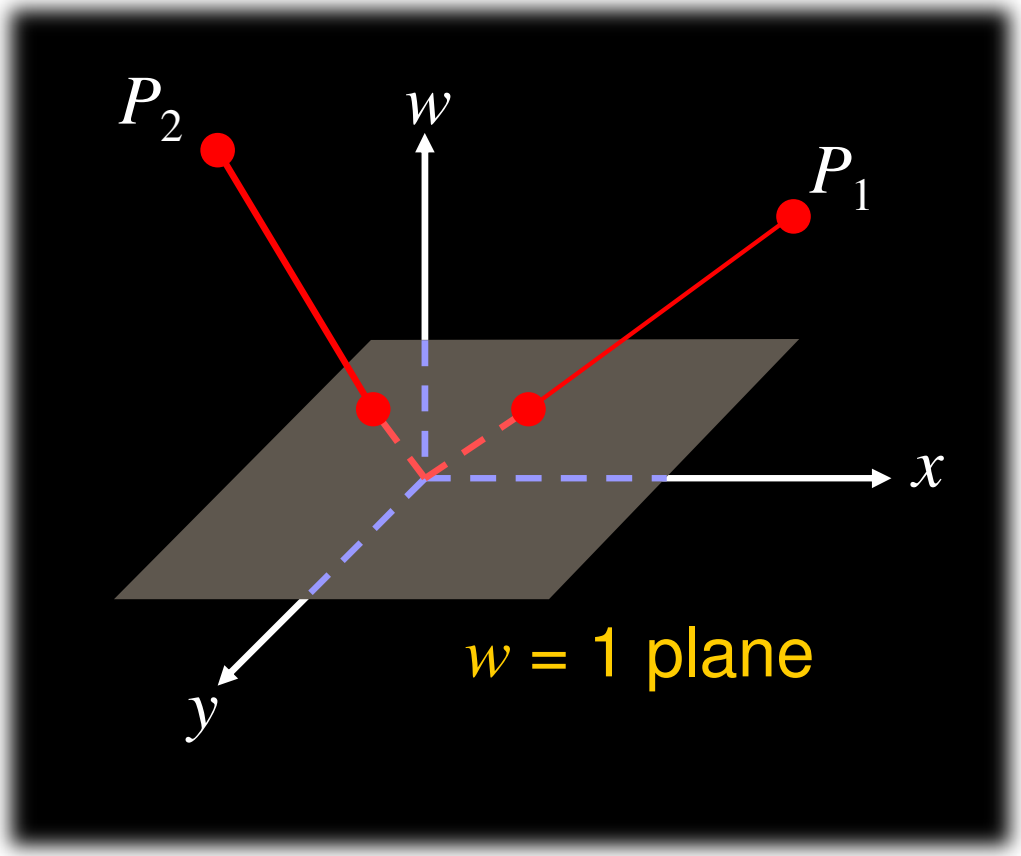
- A point (x, y) in 2D coordinates is represented by (x, y, W) in homogeneous coordinates
- Two homogeneous coordinates, (x, y, W) and (x', y', W'), represent the same point if and only if $(x', y', W') = (tx, ty, tW)$ for some $t \neq 0$
- Homogeneous coordinates to Cartesian Coordinates ($W \neq 0$)

$$(x, y, W) = (x/W, y/W, 1)$$

$$(x', y', W') = (x'/W', y'/W', 1) = (x/W, y/W, 1)$$

Homogeneous Coordinates

- All triples of the form (tx, ty, tW), with $t \neq 0$, get a line in homogeneous coordinates



Translation in Homogeneous Rep.

- 3x3 matrix form

$$\begin{aligned} x' &= x + d_x \\ y' &= y + d_y \end{aligned} \dashrightarrow \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} d_x \\ d_y \end{bmatrix} + \begin{bmatrix} x \\ y \end{bmatrix} \dashrightarrow \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & d_x \\ 0 & 1 & d_y \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$P' = T(d_x, d_y) \cdot P, \qquad T(d_x, d_y) = \begin{bmatrix} 1 & 0 & d_x \\ 0 & 1 & d_y \\ 0 & 0 & 1 \end{bmatrix}$$

Scaling in Homogeneous Rep.

- 3x3 matrix form

$$x' = s_x \cdot x \qquad y' = s_y \cdot y$$

$$\longrightarrow \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} \longrightarrow \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$P' = S(s_x, s_y) \cdot P, \qquad S(s_x, s_y) = \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Rotation in Homogeneous Rep.

- 3x3 matrix form

$$x' = x \cdot \cos\theta - y \cdot \sin\theta$$
$$y' = x \cdot \sin\theta + y \cdot \cos\theta$$

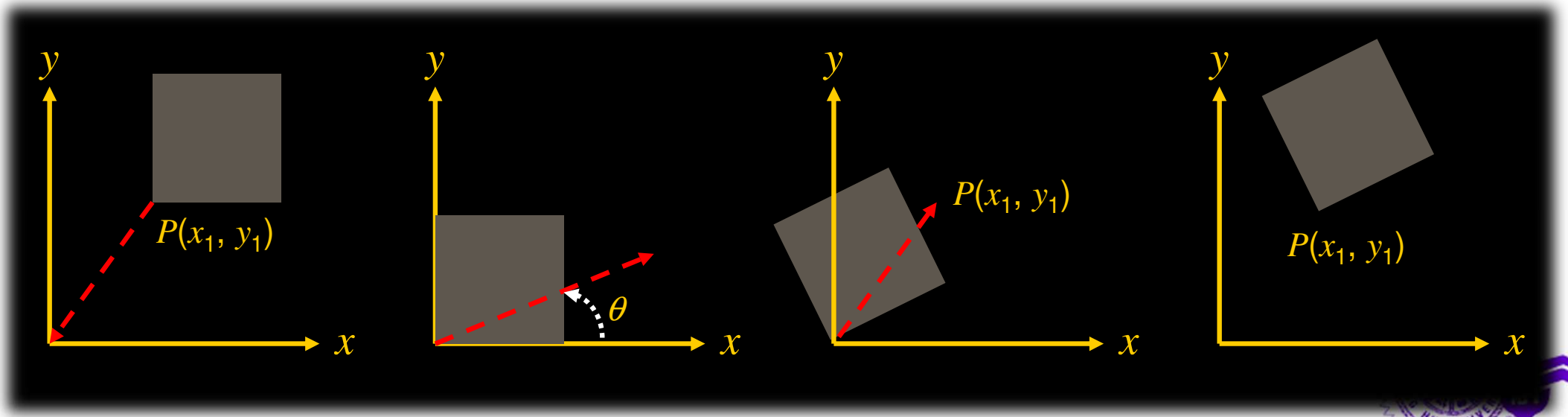
$$\dashrightarrow \quad \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\dashrightarrow \quad \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$P' = R(\theta) \cdot P, \qquad R(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Composition of 2D Transformations

- Rotate an object about an arbitrary point $P = (x_1, y_1)$
  - Translate from P to origin
  - Rotate by θ degree
  - Translate back from origin to P



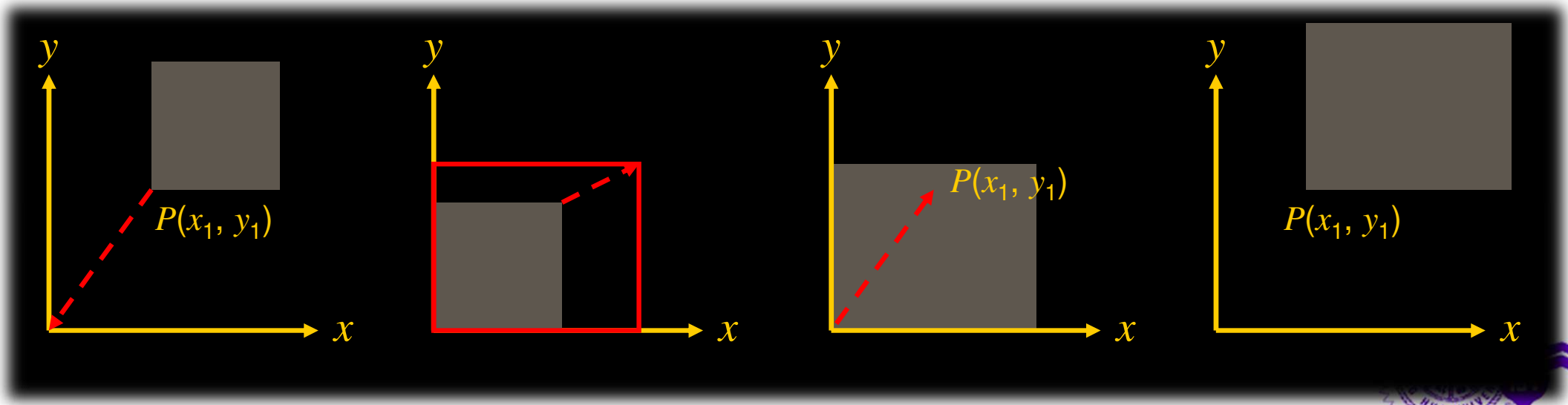
Composition of 2D Transformations

- Rotate an object about an arbitrary point $P = (x_1, y_1)$
  - Translate from P to origin, $T(-x_1, -y_1)$
  - Rotate by θ degree, $R(\theta)$
  - Translate back from origin to P, $T(x_1, y_1)$

$$T(x_1, y_1) \cdot R(\theta) \cdot T(-x_1, -y_1) = \begin{bmatrix} 1 & 0 & x_1 \\ 0 & 1 & y_1 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & -x_1 \\ 0 & 1 & -y_1 \\ 0 & 0 & 1 \end{bmatrix}$$

Composition of 2D Transformations

- Scale an object about an arbitrary point $P = (x_1, y_1)$
  - Translate from P to origin
  - Scale by (s_x, s_y)
  - Translate back from origin to P



Composition of 2D Transformations

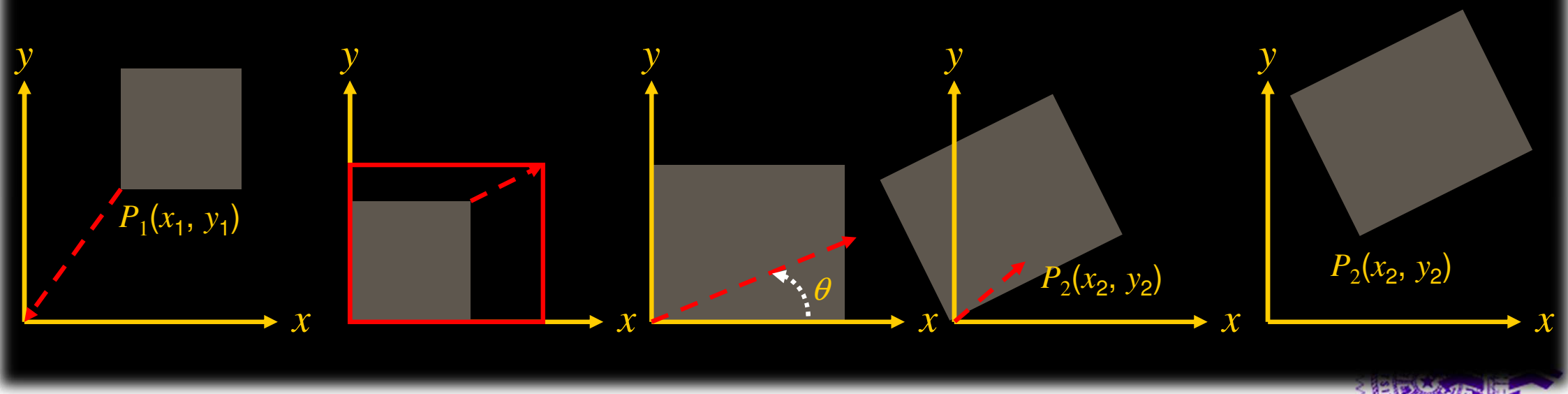
- Scale an object about an arbitrary point $P = (x_1, y_1)$
  - Translate from P to origin, $T(-x_1, -y_1)$
  - Scale by (s_x, s_y), $S(s_x, s_y)$
  - Translate back from origin to P, $T(x_1, y_1)$

$$T(x_1, y_1) \cdot S(s_x, s_y) \cdot T(-x_1, -y_1) = \begin{bmatrix} 1 & 0 & x_1 \\ 0 & 1 & y_1 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & -x_1 \\ 0 & 1 & -y_1 \\ 0 & 0 & 1 \end{bmatrix}$$

Composition of 2D Transformations

- Scale and rotate about $P_1 = (x_1, y_1)$ and translate to point $P_2 = (x_2, y_2)$

$$T(x_2, y_2) \cdot R(\theta) \cdot S(s_x, s_y) \cdot T(-x_1, -y_1)$$



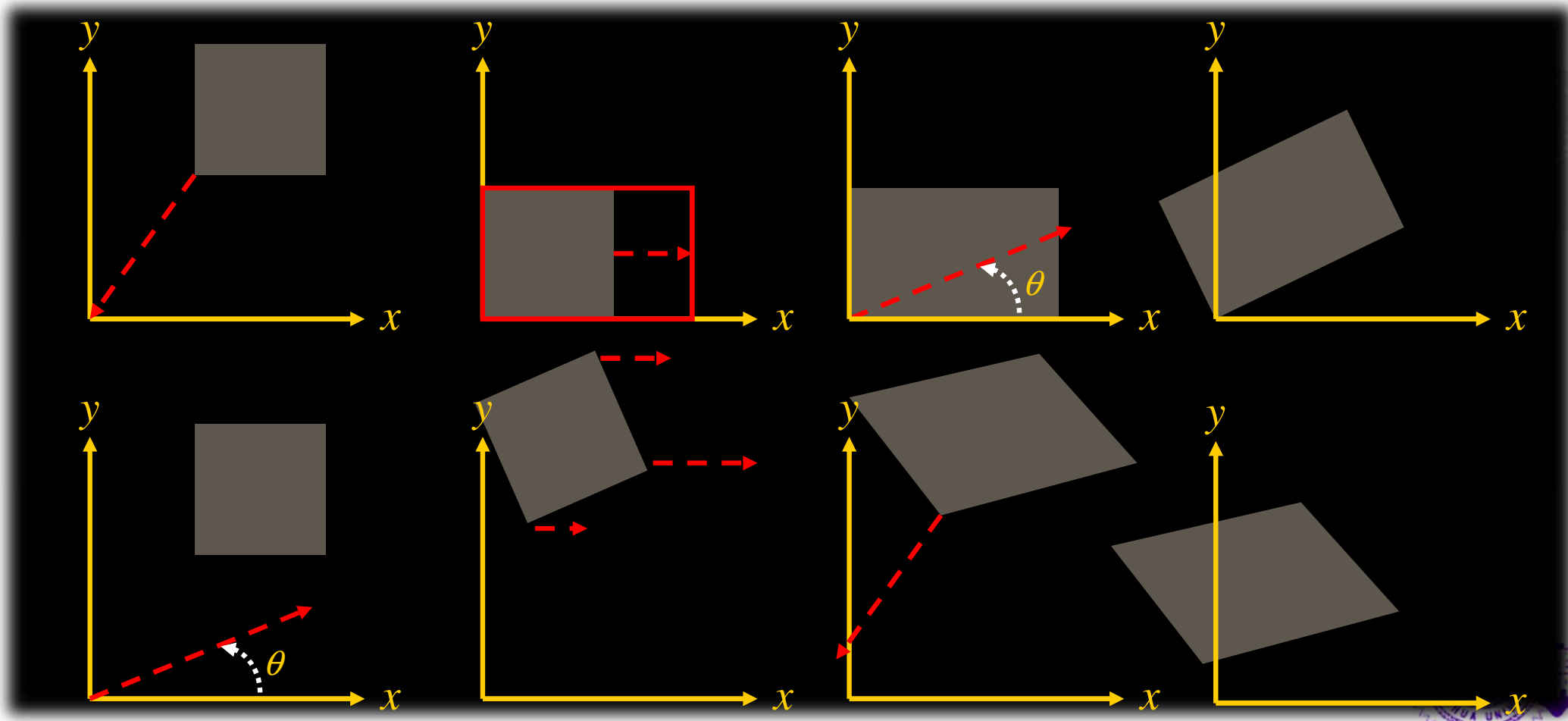
Composition of 2D Transformations

- Scale and rotate about $P_1 = (x_1, y_1)$ and translate to point $P_2 = (x_2, y_2)$
  - Translate from P_1 to origin, $T(-x_1, -y_1)$
  - Scale by (s_x, s_y), $S(s_x, s_y)$
  - Rotate by θ degree, $R(\theta)$
  - Translate from origin to P_2, $T(x_2, y_2)$

$$T(x_2, y_2) \cdot R(\theta) \cdot S(s_x, s_y) \cdot T(-x_1, -y_1)$$

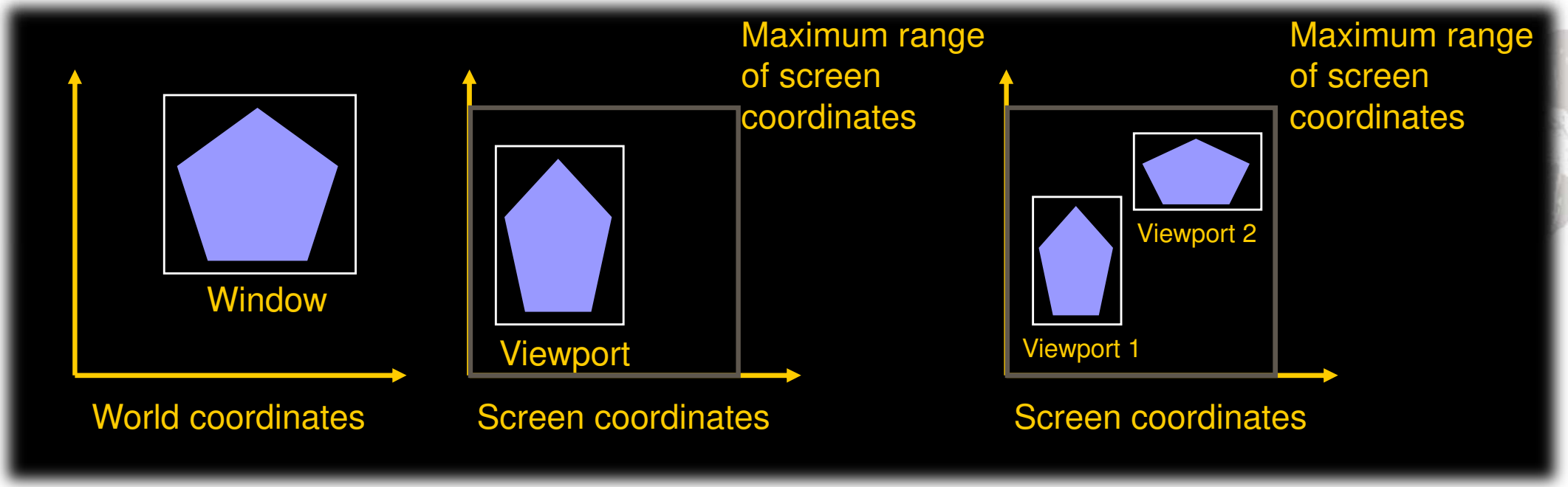
Composition of 2D Transformations

- **Commutativity**
  - $R(\theta) \cdot S(s_x, s_y) \cdot T(d_x, d_y) \neq T(d_x, d_y) \cdot S(s_x, s_y) \cdot R(\theta)$



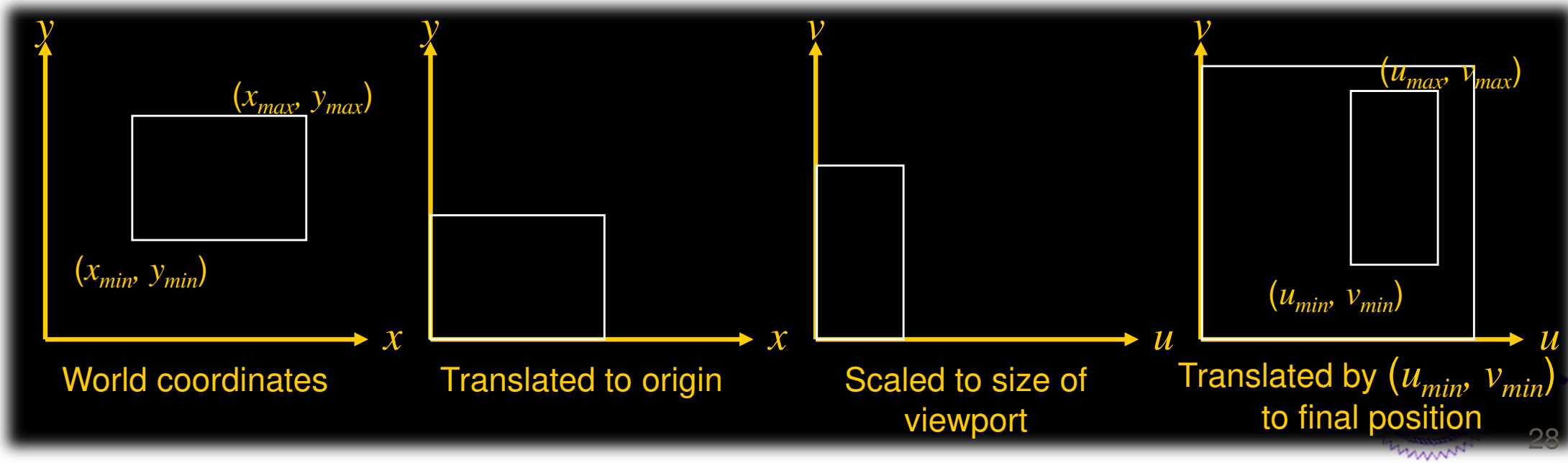
Window to Viewport Transformation

- **Application of 2D Transformation**



Window to Viewport Transformation

- Steps in transforming a world-coordinate window into a viewport
  - Window translated to origin
  - Window scaled to size of viewport
  - Translated by (u_{min}, v_{min}) to final position



Window to Viewport Transformation

- Composition matrix in transforming a world coordinate window into a viewport

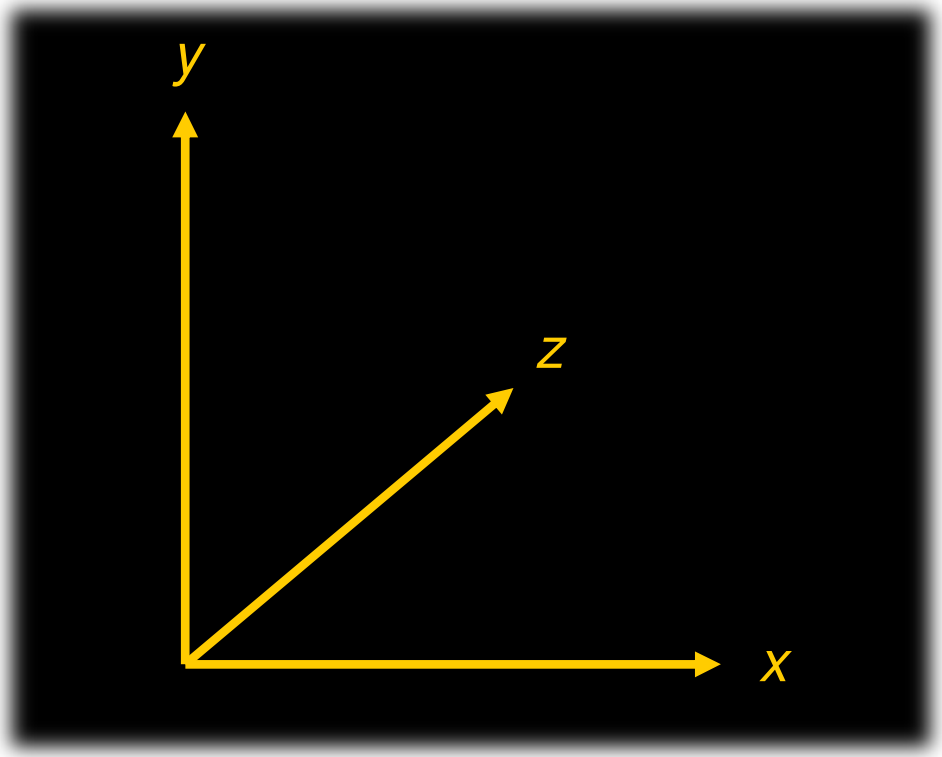
$$M_{WV} = T(u_{\min}, v_{\min}) \cdot S(\frac{u_{\max} - u_{\min}}{x_{\max} - x_{\min}}, \frac{v_{\max} - v_{\min}}{y_{\max} - y_{\min}}) \cdot T(-x_{\min}, -y_{\min})$$

3D Geometrical Transformations

Similar to 2D Geometrical Transformations

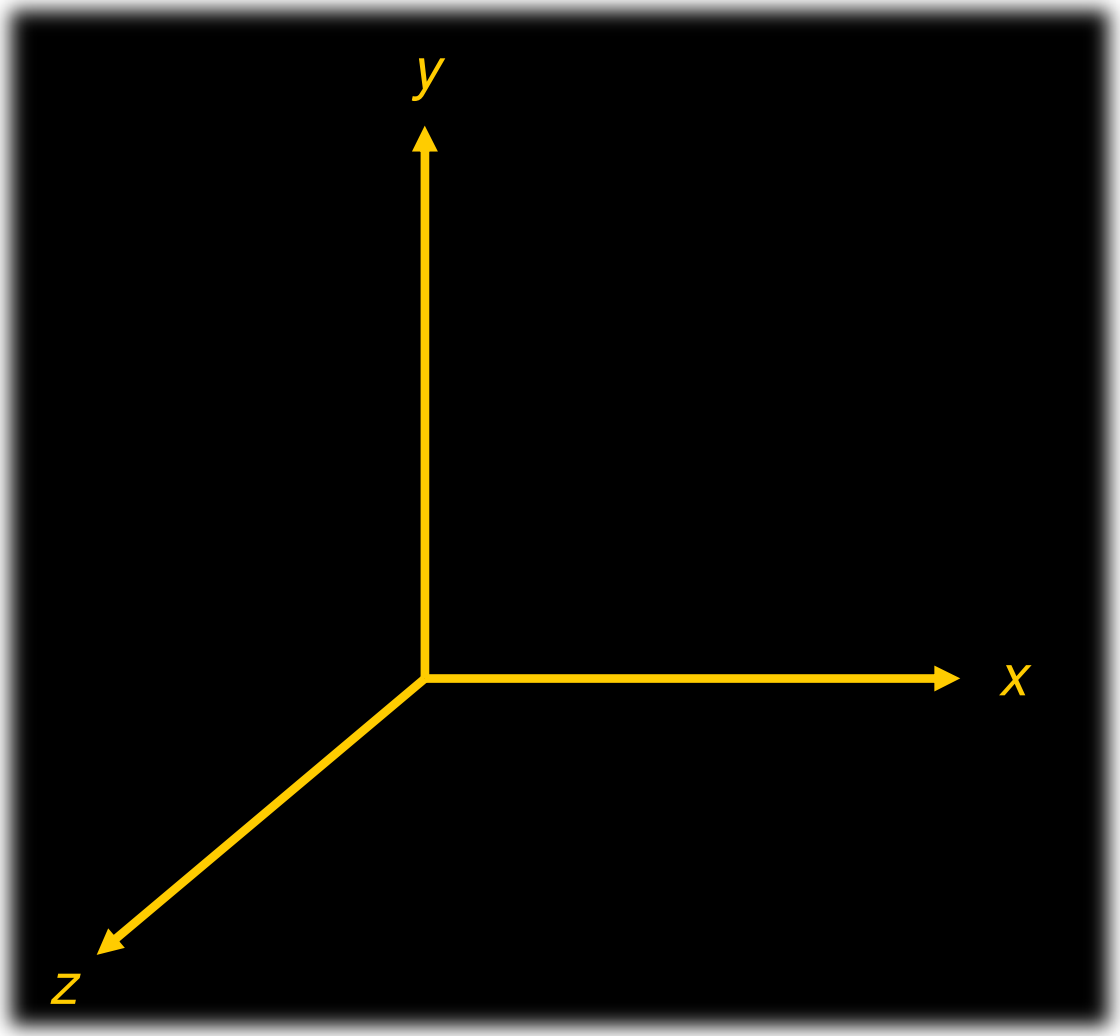
3D Coordinate Systems

- **Left-hand Coordinate Systems**
  - **Direct3D**



3D Coordinate Systems

- Right-hand Coordinate Systems
  - OpenGL



3D Transformation

- **Translation**

$P(x, y, z)$ move to $P'(x', y', z')$

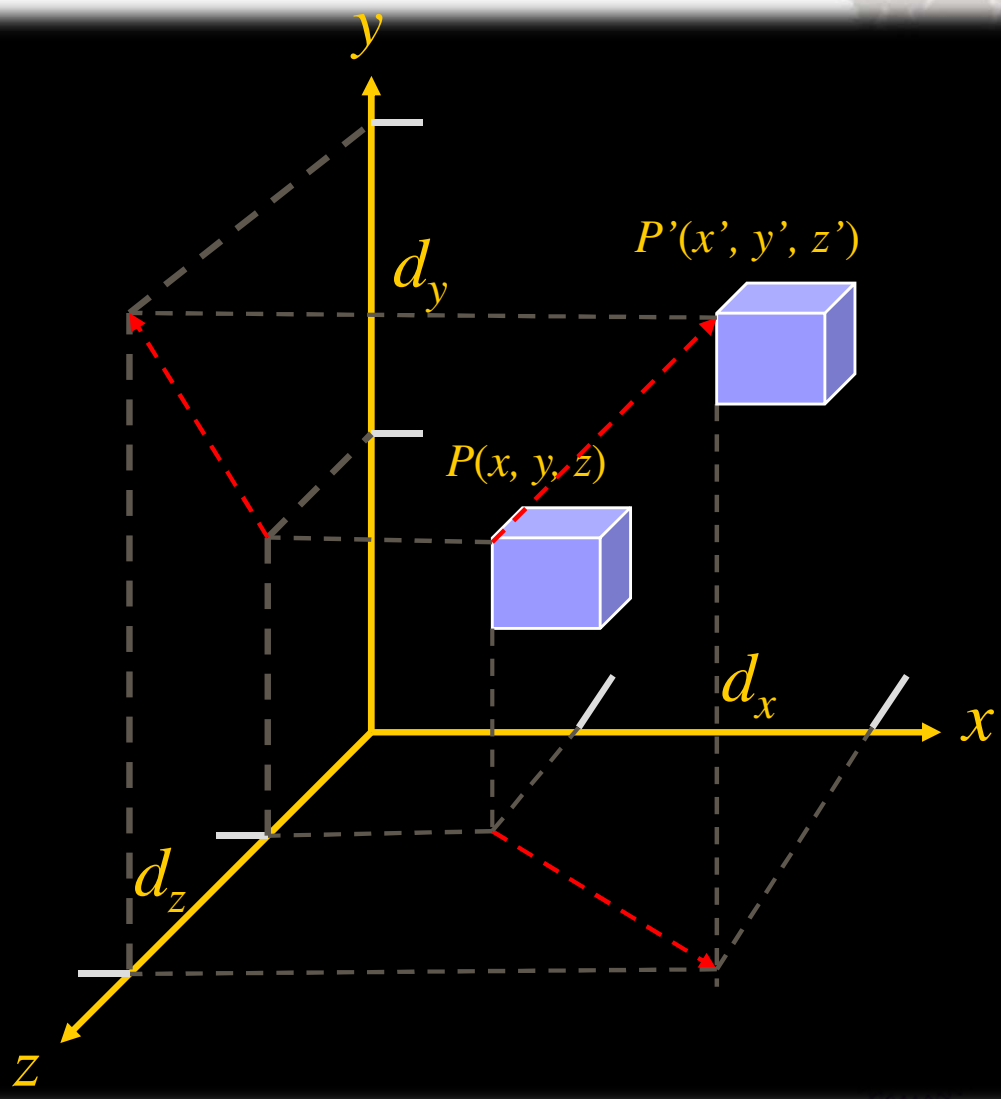
$$x' = x + d_x$$

$$y' = y + d_y$$

$$z' = z + d_z$$

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} d_x \\ d_y \\ d_z \end{bmatrix} + \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$P' = T(d_x, d_y, d_z) + P$$



3D Transformation

Scaling

Scale by s_x, s_y, and s_z with respect to the x axis, y axis, and z axis

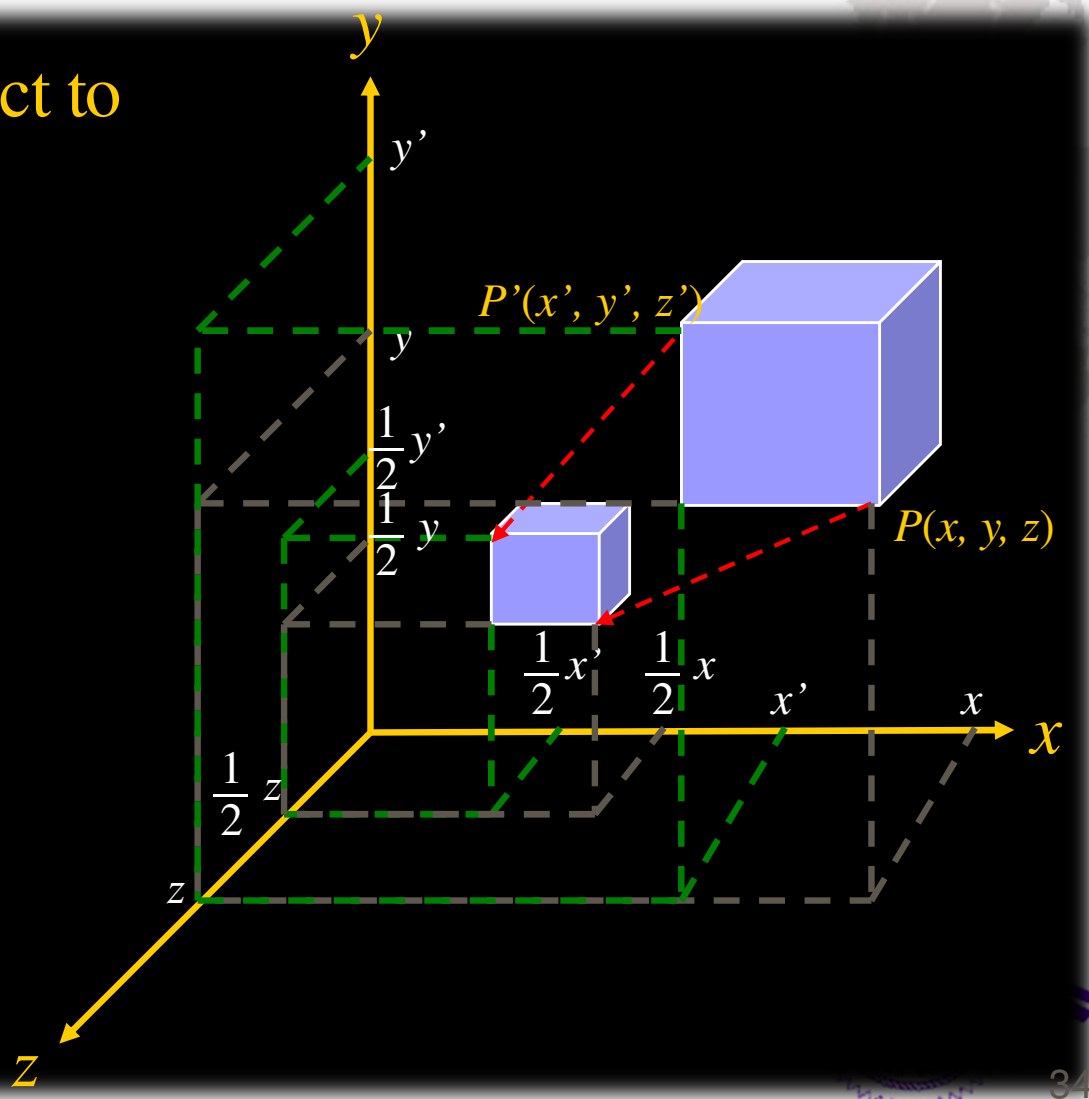
$$x' = s_x \cdot x$$

$$y' = s_y \cdot y$$

$$z' = s_z \cdot z$$

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & s_z \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$P' = S(s_x, s_y, s_z) \cdot P$$



3D Transformation

Rotation in x-axis

Rotate an angle θ in x-axis

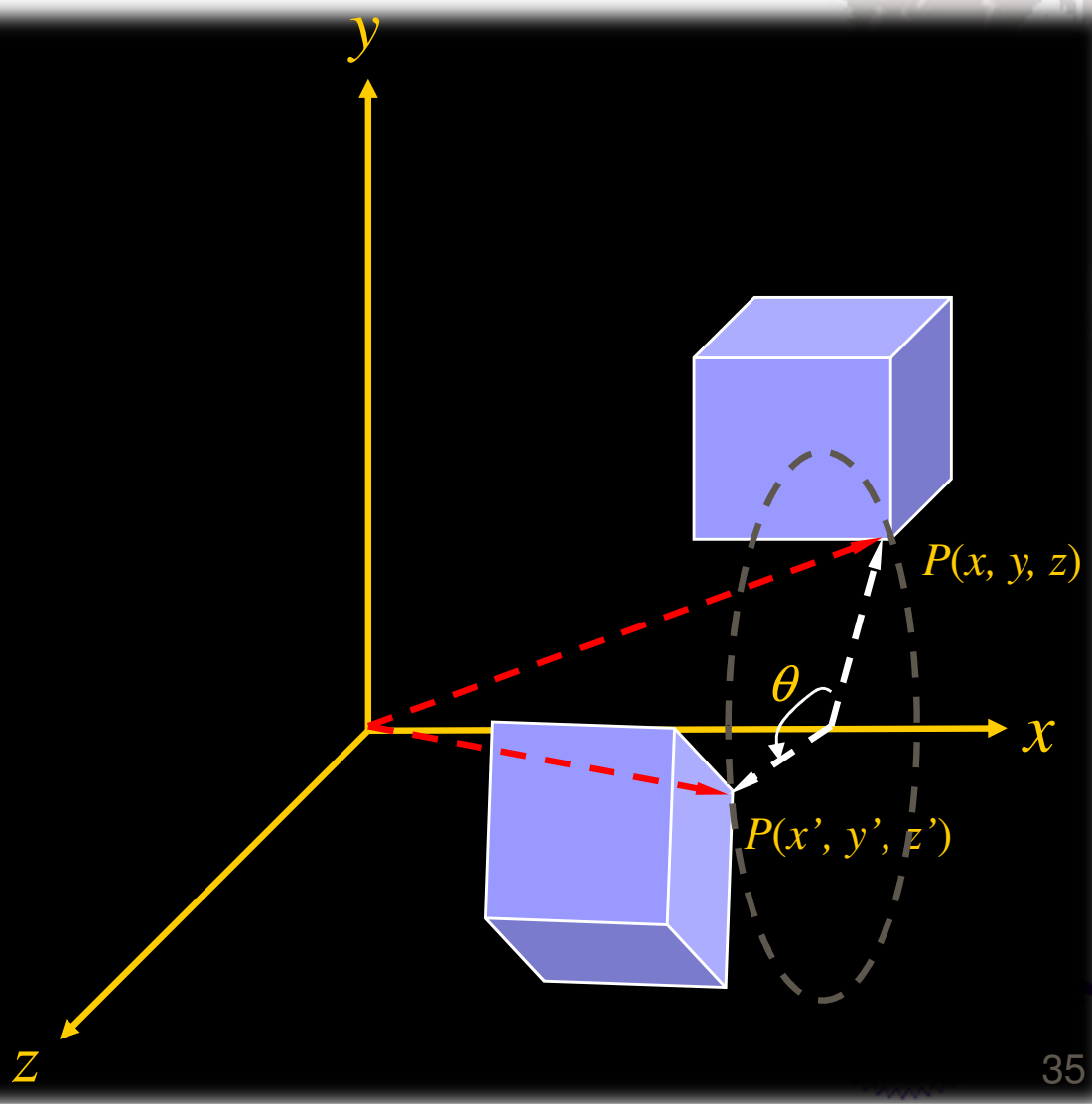
$$x' = x$$

$$y' = y \cdot \cos\theta - z \cdot \sin\theta$$

$$z' = y \cdot \sin\theta + z \cdot \cos\theta$$

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$P' = R_x(\theta) \cdot P$$



3D Transformation

- **Rotation in y-axis**

Rotate an angle θ in y-axis

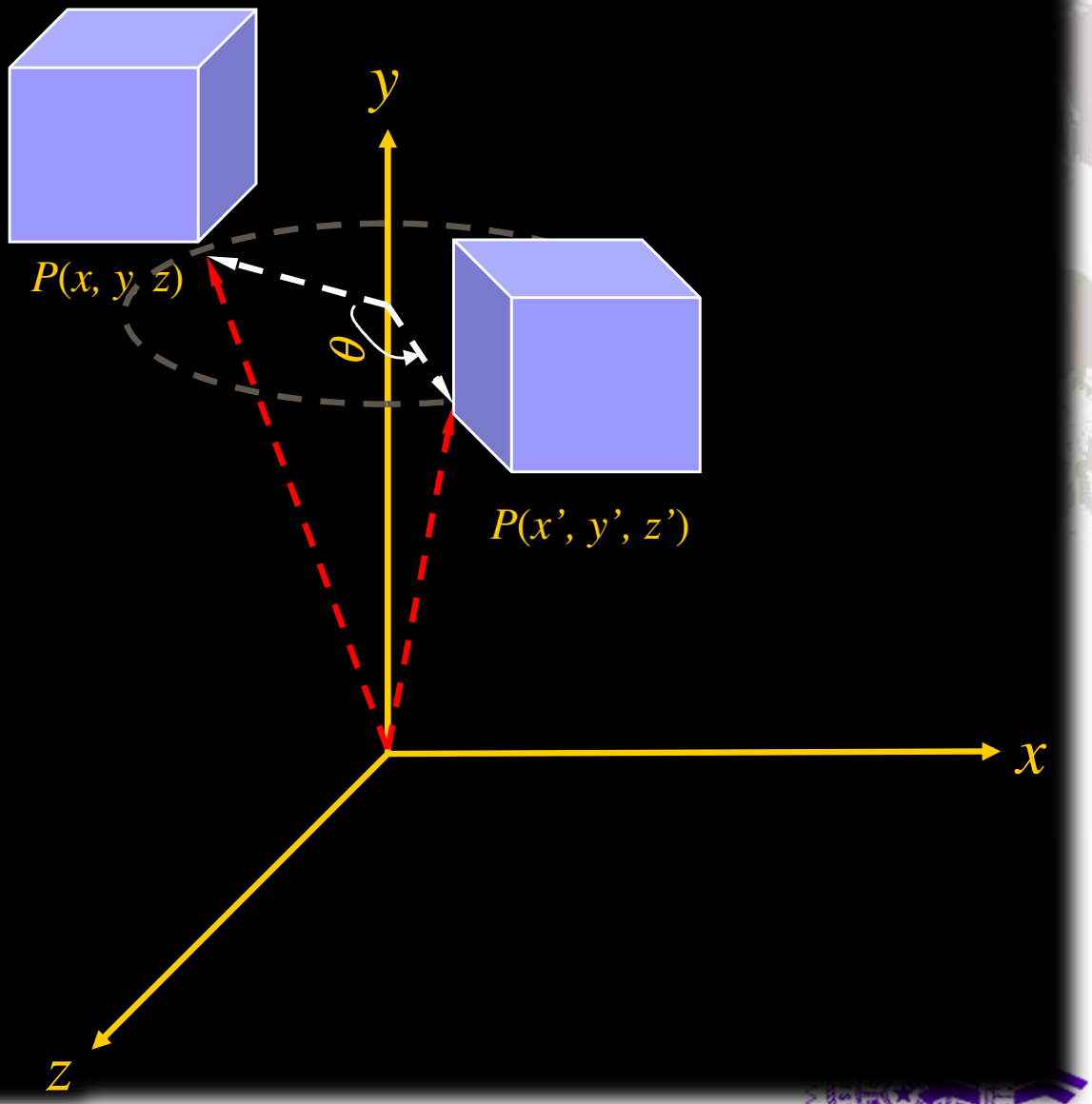
$$x' = x \cdot \cos\theta + z \cdot \sin\theta$$

$$y' = y$$

$$z' = -x \cdot \sin\theta + z \cdot \cos\theta$$

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$P' = R_y(\theta) \cdot P$$



3D Transformation

- **Rotation in z-axis**

Rotate an angle θ in z-axis

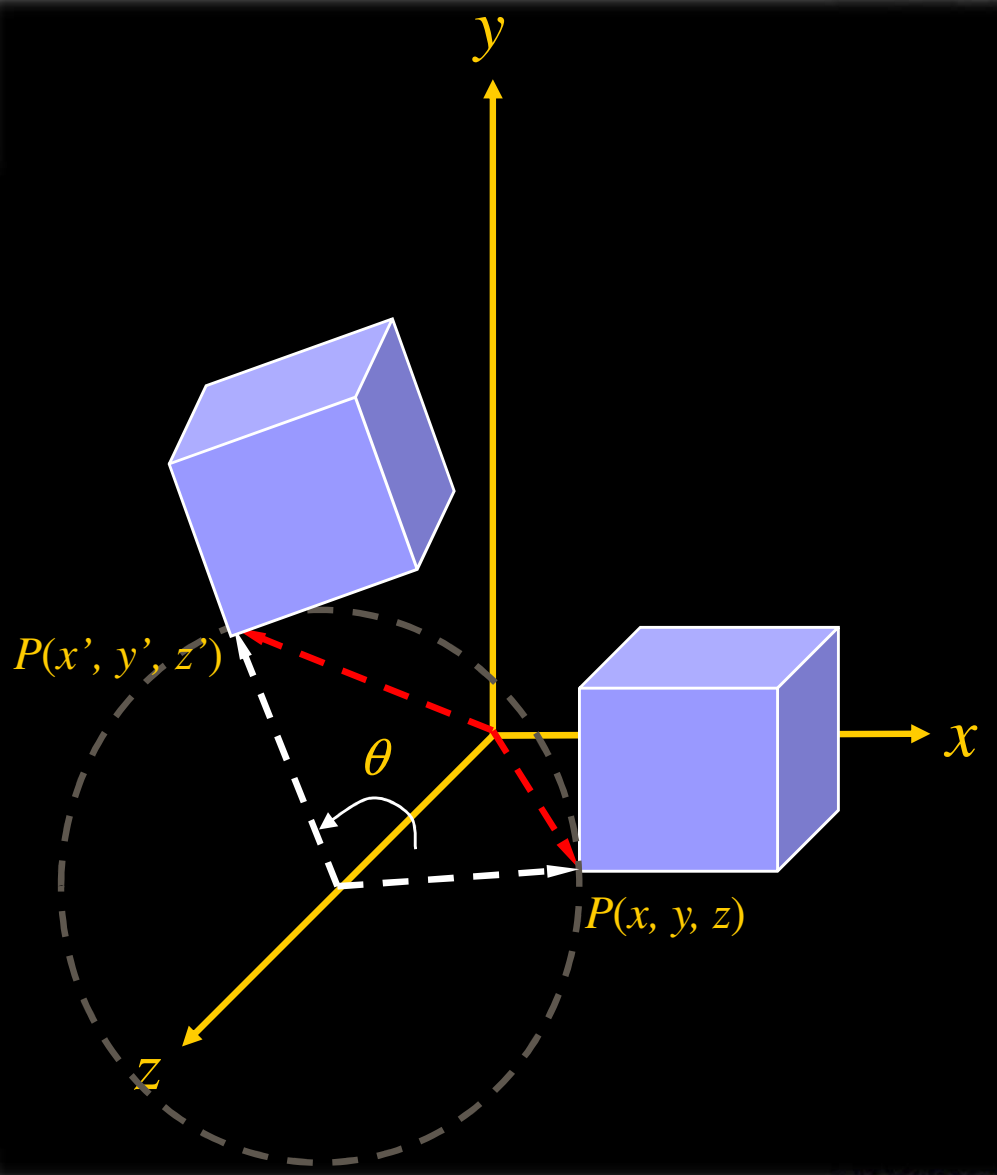
$$x' = x \cdot \cos\theta - y \cdot \sin\theta$$

$$y' = x \cdot \sin\theta + y \cdot \cos\theta$$

$$z' = z$$

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$P' = R_z(\theta) \cdot P$$



Homogeneous Coordinates

- A point (x, y, z) in 3D coordinates is represented by (x, y, z, W) in homogeneous coordinates
- Two homogeneous coordinates (x, y, z, W) and (x', y', z', W') represent the same point if and only if $(x', y', z', W') = (tx, ty, tz, tW)$ for some $t \neq 0$
- Cartesian Coordinates

$$(x, y, z, W) = (x/W, y/W, z/W, 1)$$

$$(x', y', z', W') = (x'/W', y'/W', z'/W', 1) = (x/W, y/W, z/W, 1)$$

Translation in Homogeneous Rep.

- 4x4 matrix form

$$x' = x + d_x \qquad y' = y + d_y \qquad z' = z + f_z$$

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} d_x \\ d_y \\ d_z \end{bmatrix} + \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & d_x \\ 0 & 1 & 0 & d_y \\ 0 & 0 & 1 & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$$P' = T(d_x, d_y, d_z) \cdot P, \qquad T(d_x, d_y, d_z) = \begin{bmatrix} 1 & 0 & 0 & d_x \\ 0 & 1 & 0 & d_y \\ 0 & 0 & 1 & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Scaling in Homogeneous Rep.

- 4x4 matrix form

$$x' = s_x \cdot x \qquad y' = s_y \cdot y \qquad z' = s_z \cdot z$$

$$\dashrightarrow \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & s_z \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\dashrightarrow \begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$$P' = S(s_x, s_y, s_z) \cdot P$$

Rotation in Homogeneous Rep.

- 4x4 matrix form (Rotate in x-axis)

$$x' = x$$
$$y' = y \cdot \cos\theta - z \cdot \sin\theta$$
$$z' = y \cdot \sin\theta + z \cdot \cos\theta$$

$$\Longrightarrow \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\Longrightarrow \begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$$P' = R_x(\theta) \cdot P$$

Rotation in Homogeneous Rep.

- 4x4 matrix form (Rotate in y-axis)

$$x' = x \cdot \cos\theta + z \cdot \sin\theta$$
$$y' = y$$
$$z' = -x \cdot \sin\theta + z \cdot \cos\theta$$

$$\dashrightarrow \quad \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\dashrightarrow \quad \begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$$P' = R_y(\theta) \cdot P$$

Rotation in Homogeneous Rep.

- 4x4 matrix form (Rotate in z-axis)

$$x' = x \cdot \cos\theta - y \cdot \sin\theta$$
$$y' = x \cdot \sin\theta + y \cdot \cos\theta$$
$$z' = z$$

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$$P' = R_z(\theta) \cdot P$$

Composition of 3D Transformations

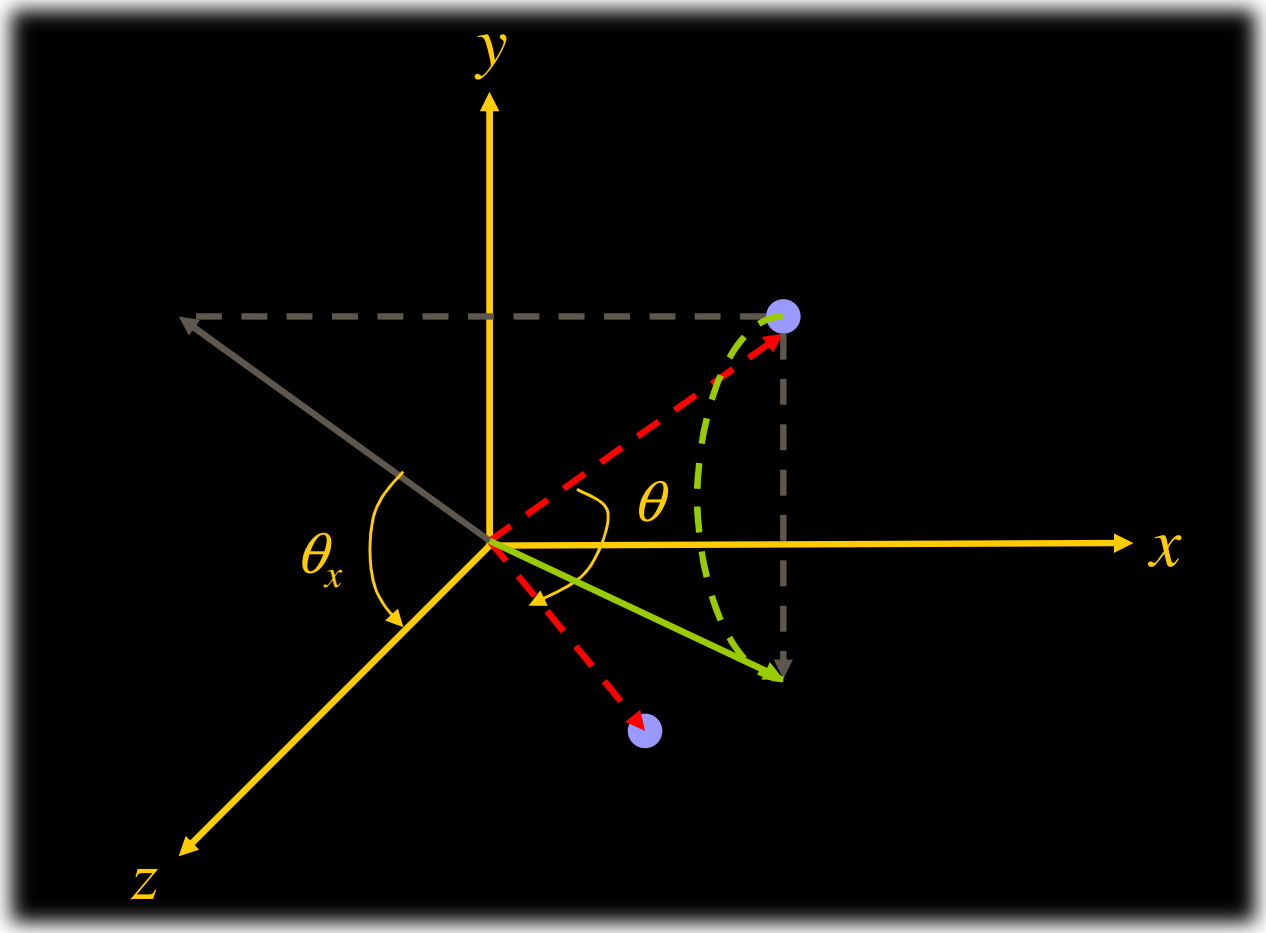
Rotation about the origin

Rotation about an arbitrary point

Rotation about an arbitrary axis

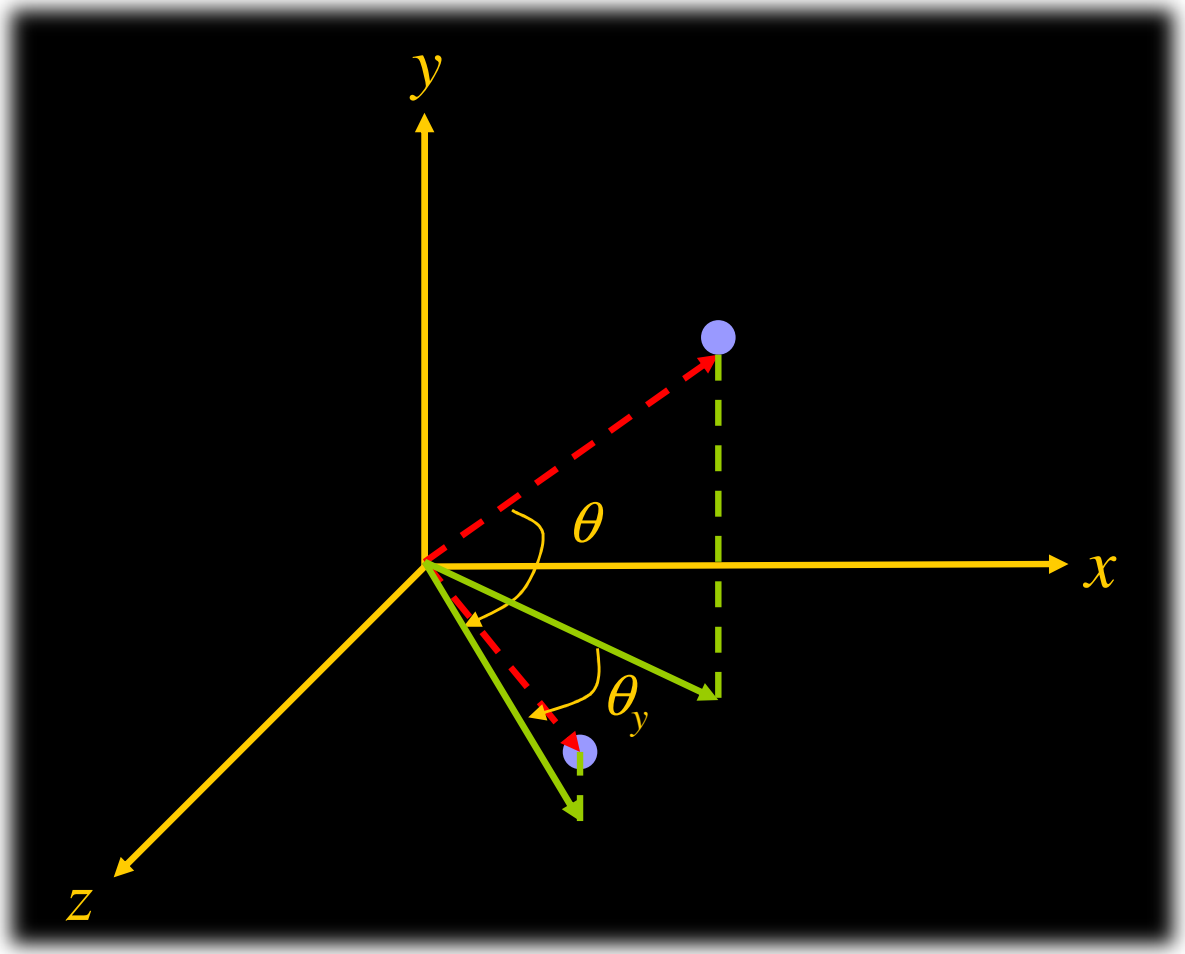
Composition of 3D Transformations

- **Rotation about the origin by θ degree**
  - **Step 1: Rotate in x-axis by θ_x degree**



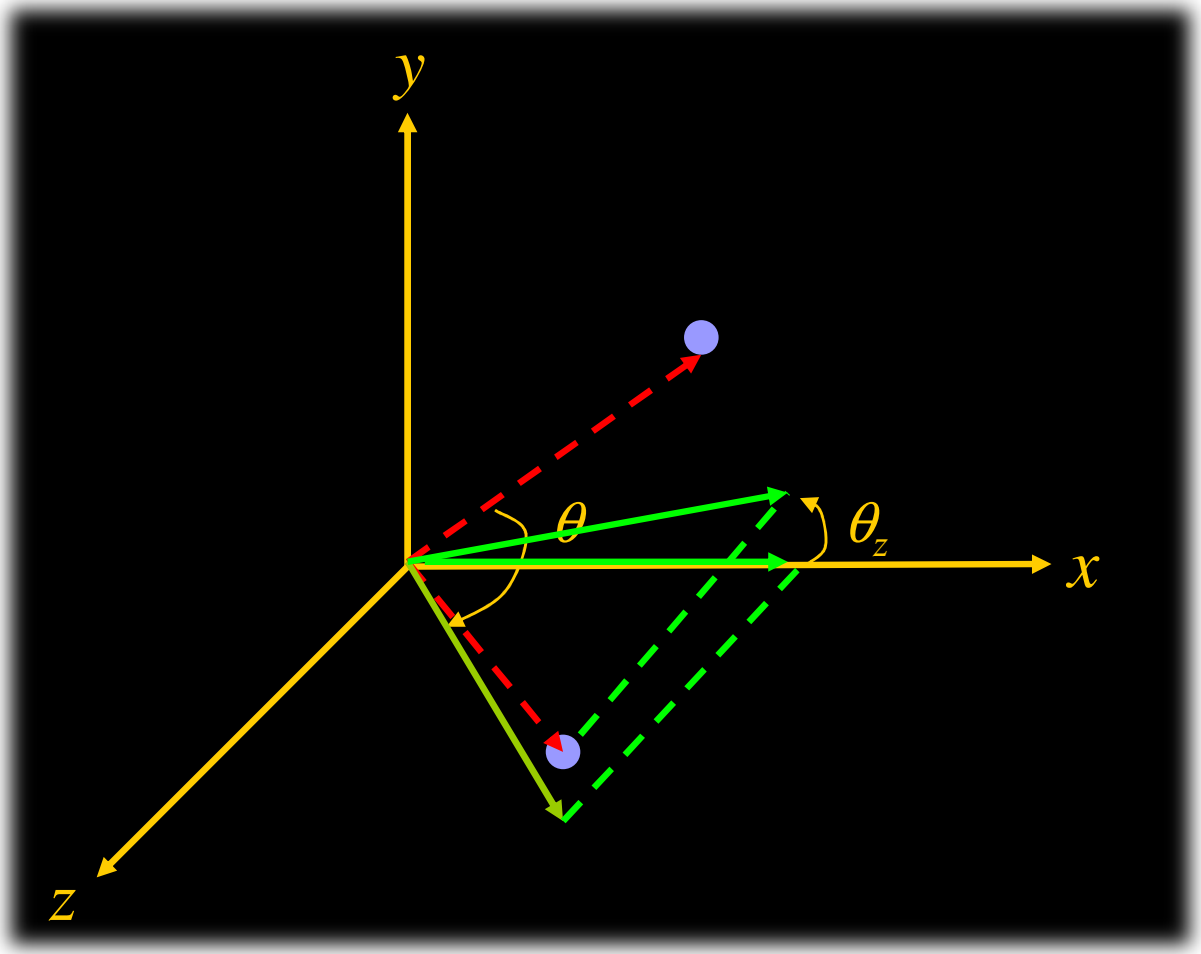
Composition of 3D Transformations

- Rotation about the origin by θ degree
  - Step 2: Rotate in y-axis by θ_y degree



Composition of 3D Transformations

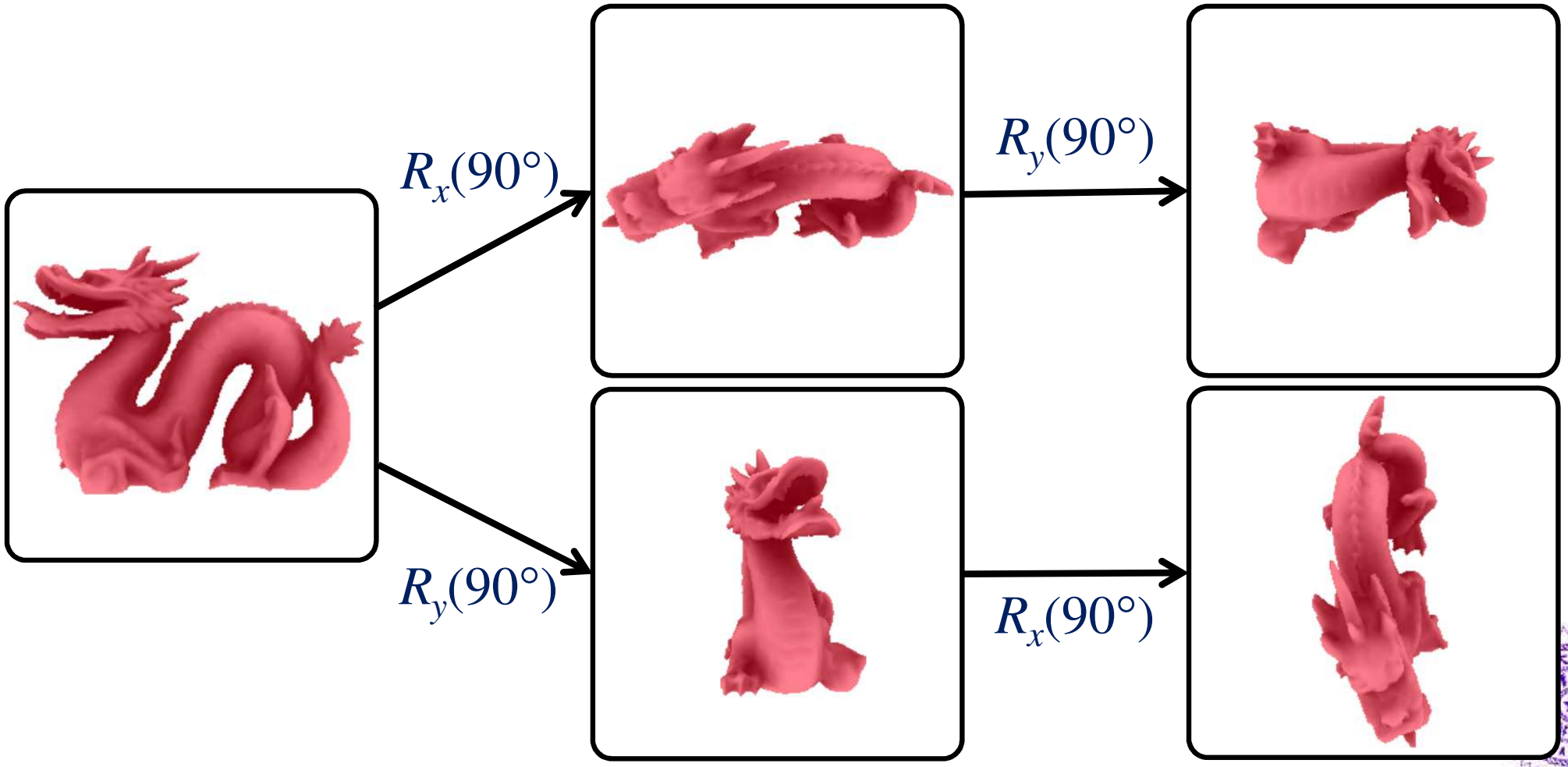
- Rotation about the origin by θ degree
  - Step 3: Rotate in z-axis by θ_z degree



Composition of 3D Transformations

- Commutative property is not held for rotation

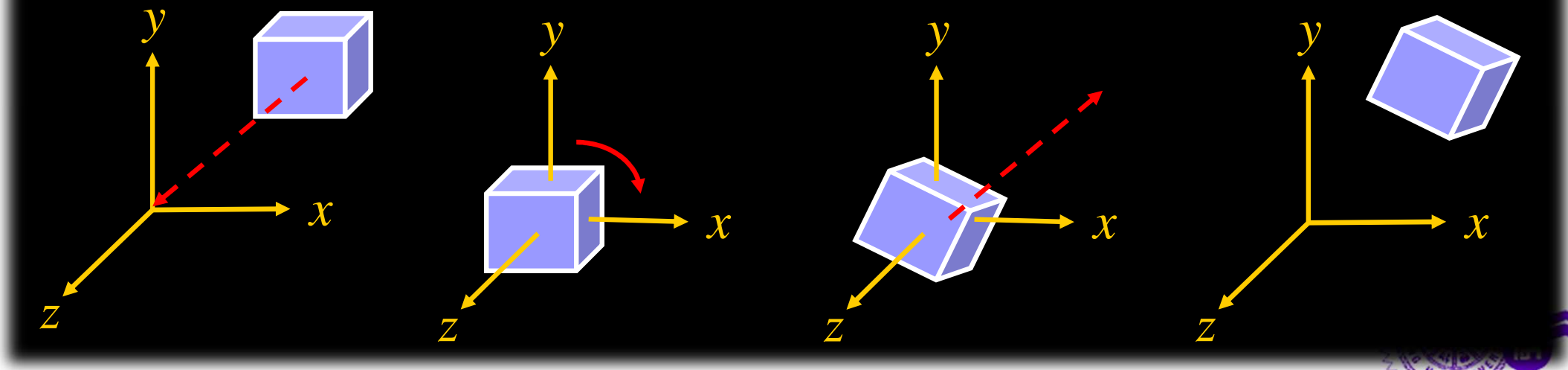
$$R_y(\theta_y) \cdot R_x(\theta_x) \neq R_x(\theta_x) \cdot R_y(\theta_y)$$



Composition of 3D Transformations

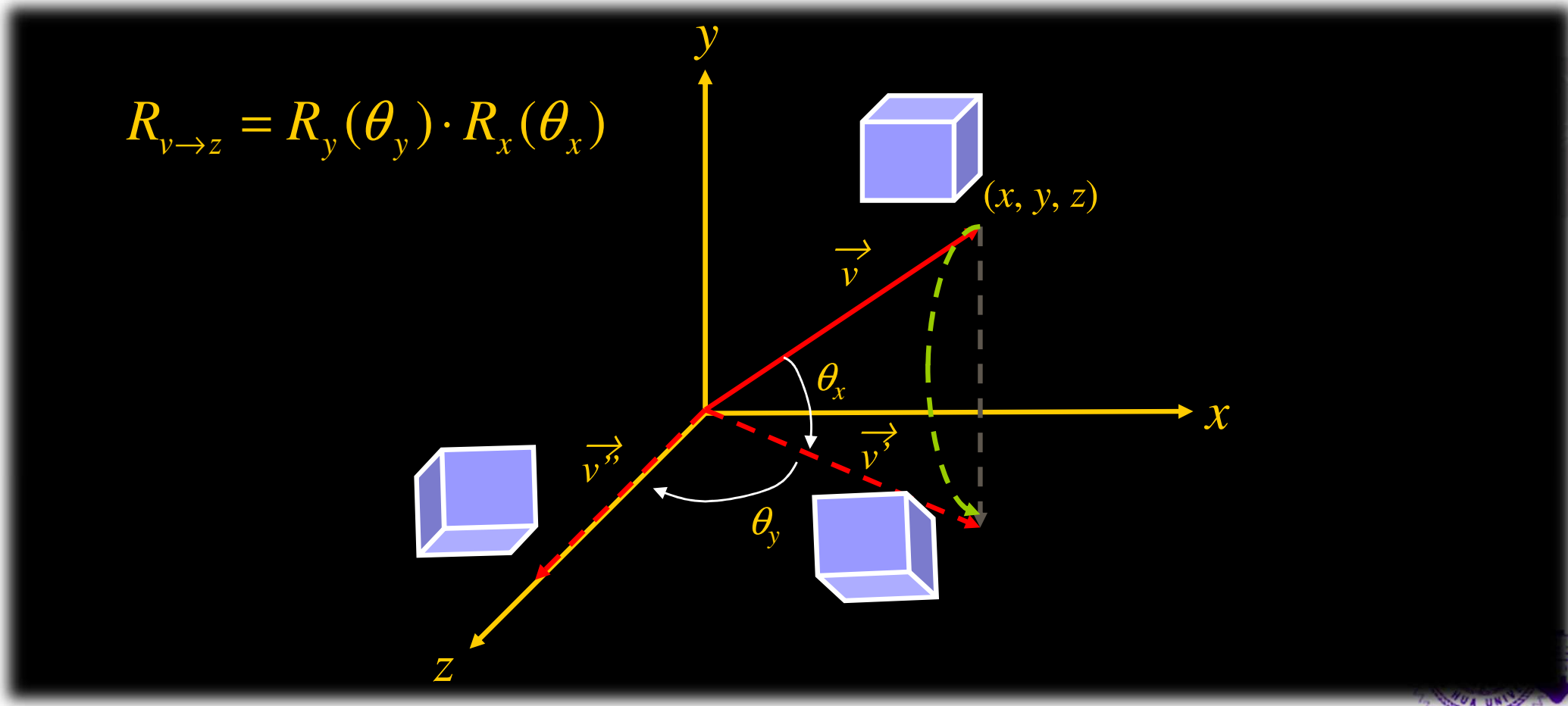
- Rotation about an arbitrary point $P_0(x, y, z)$
  - Translate from P_0 to origin
  - Rotate by θ degree // $R(\theta) = R_x(\theta_x) \cdot R_y(\theta_y) \cdot R_z(\theta_z)$
  - Translate from origin back to P_0

$$P' = (T(x, y, z) \cdot R(\theta) \cdot T(-x, -y, -z)) \cdot P$$



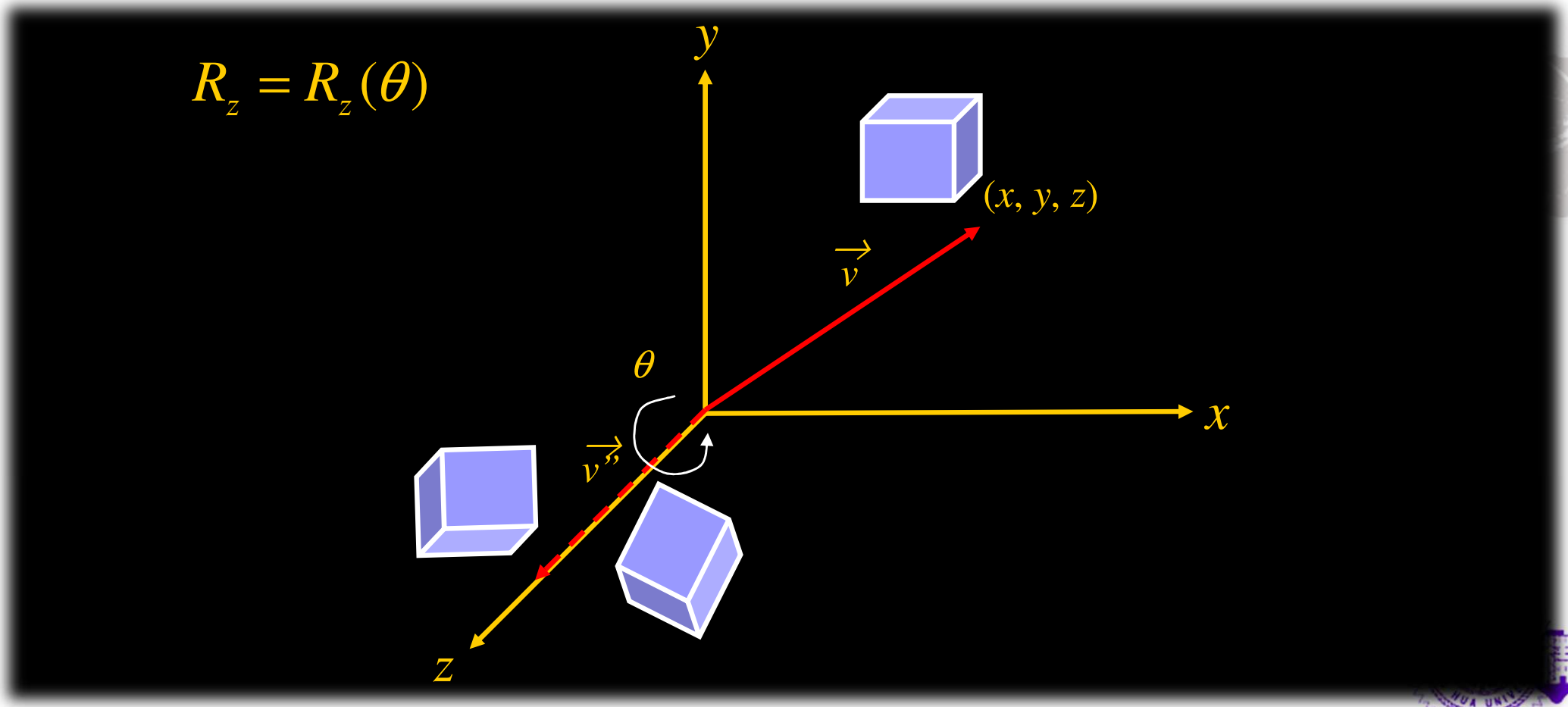
Composition of 3D Transformations

- Rotation about an arbitrary vector v
  - Step 1: Rotate the vector v to align with z-axis



Composition of 3D Transformations

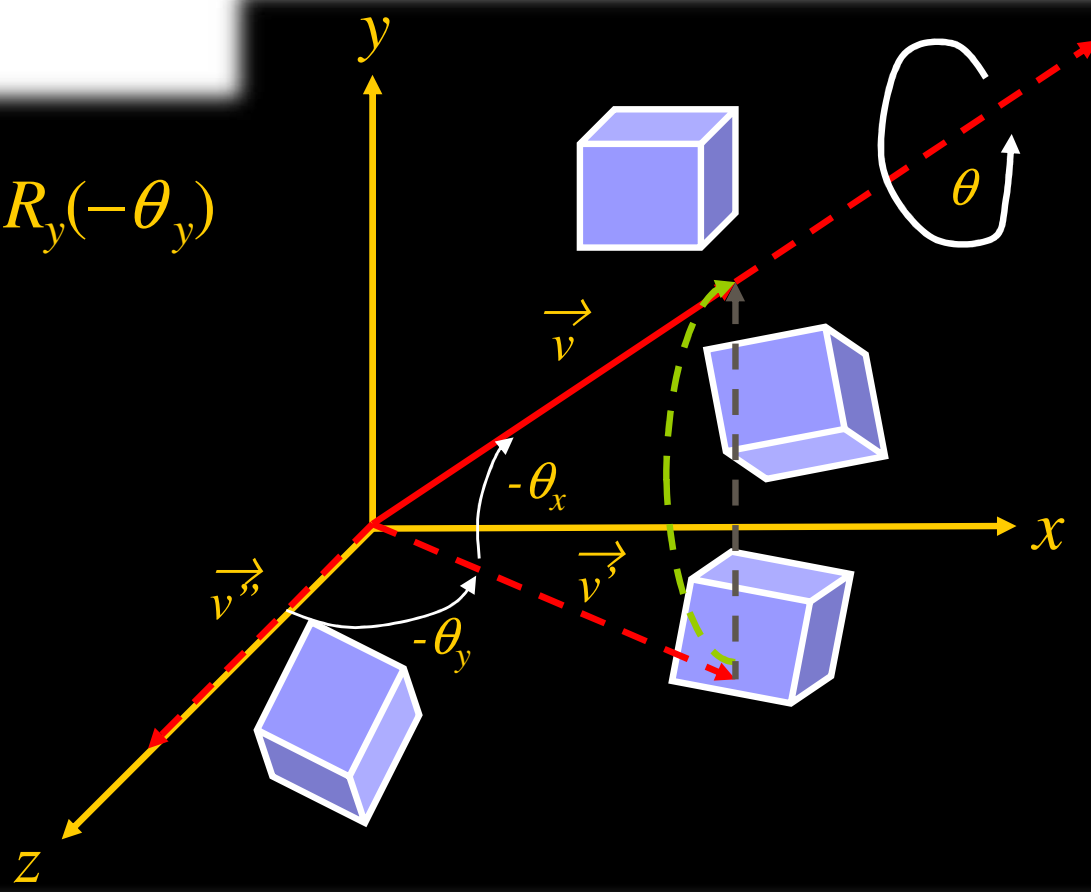
- Rotation about an arbitrary vector v
  - Step 2: Rotate in z by θ degree



Composition of 3D Transformations

- Rotation about an arbitrary vector v
  - Step 3: Rotate the z-axis back to align with vector v

$$R_{z \to v} = R_x(-\theta_x) \cdot R_y(-\theta_y)$$



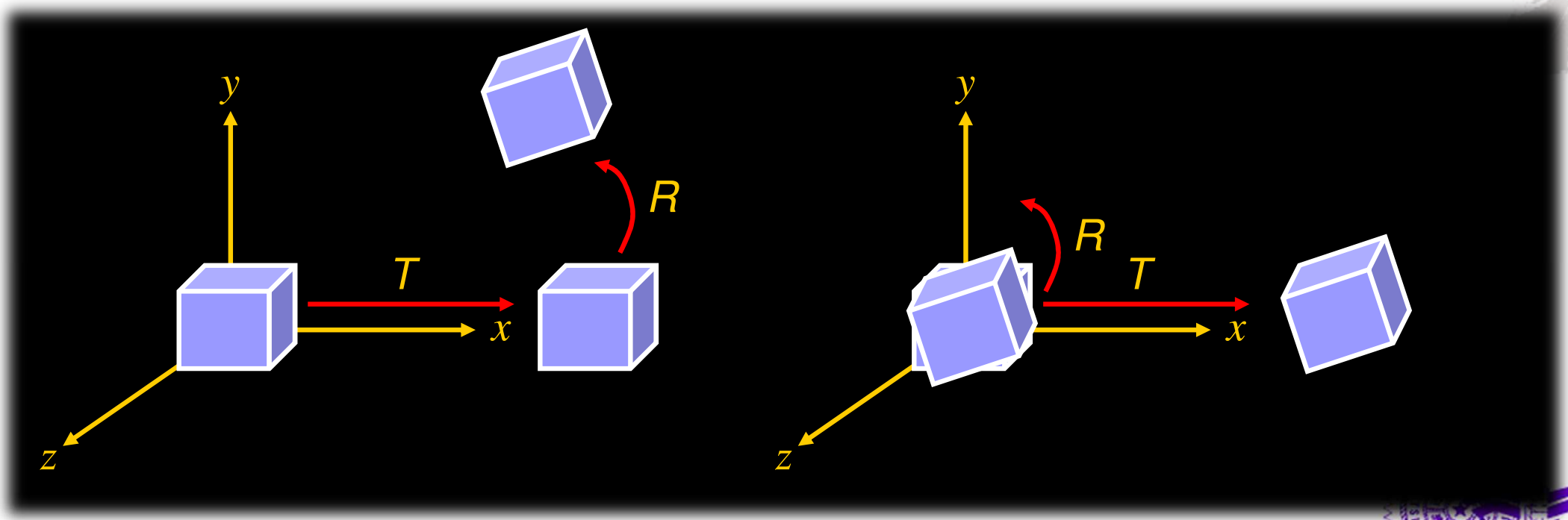
Composition of 3D Transformations

- Rotation about an arbitrary vector v
  - Step 1: Rotate the vector v to align with z-axis
  - Step 2: Rotate in z by θ degree
  - Step 3: Rotate the z-axis back to align with vector v

$$P' = (\underbrace{((R_x(-\theta_x) \cdot R_y(-\theta_y))}_{\text{Step 3}} \cdot \underbrace{R_z(\theta)}_{\text{Step 2}} \cdot \underbrace{(R_y(\theta_y) \cdot R_x(\theta_x))}_{\text{Step 1}}) \cdot P$$

Composition of 3D Transformations

- **Commutative Property**
  - Commutative property is not held for matrix composition
  - e.g. $T \cdot R \neq R \cdot T$



Application Order of Transformation

- **Think of the *model* transformation in reverse order to get the right sequence of transformation in OpenGL**
  - ***An example of OpenGL matrix multiplication***

```
glMatrixMode(GL_MODELVIEW);
glLoadIdentity();
glMultMatrixf(N);                  /* apply transformation N */
glMultMatrixf(M);                  /* apply transformation M */
glMultMatrixf(L);                  /* apply transformation L */
glBegin(GL_POINTS);
  glVertex3f(v);                   /* draw transformed vertex v */
glEnd();
```

The vertex transformation is $I(N(M(Lv))) = (INML)v = (NML)v$

Application Order of Transformation

- **If you are not thinking in reverse order, than all the transformation matrices derived should be transposed**
  - ***For example , the Direct3D matrix representation is applied in this order***

The previous transformation example becomes

$$(((vL^T)M^T)N^T)I = v(L^TM^TN^TI) = v(L^TM^TN^T)$$

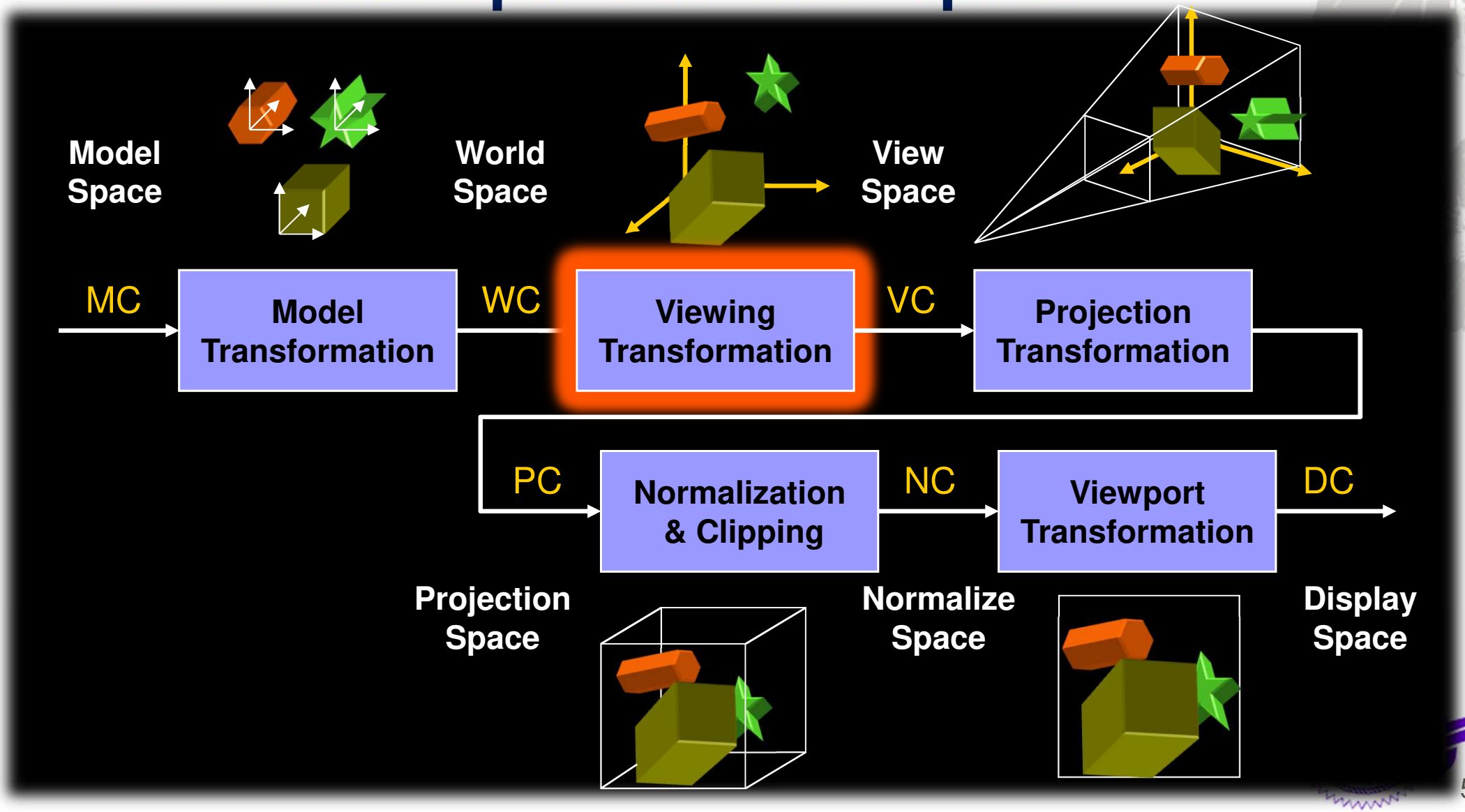
If L, M, N, I is represented as transformation matrix in OpenGL

Viewing Transformation

From World Space to View Space

3D Viewing Process

- From World Space to View Space

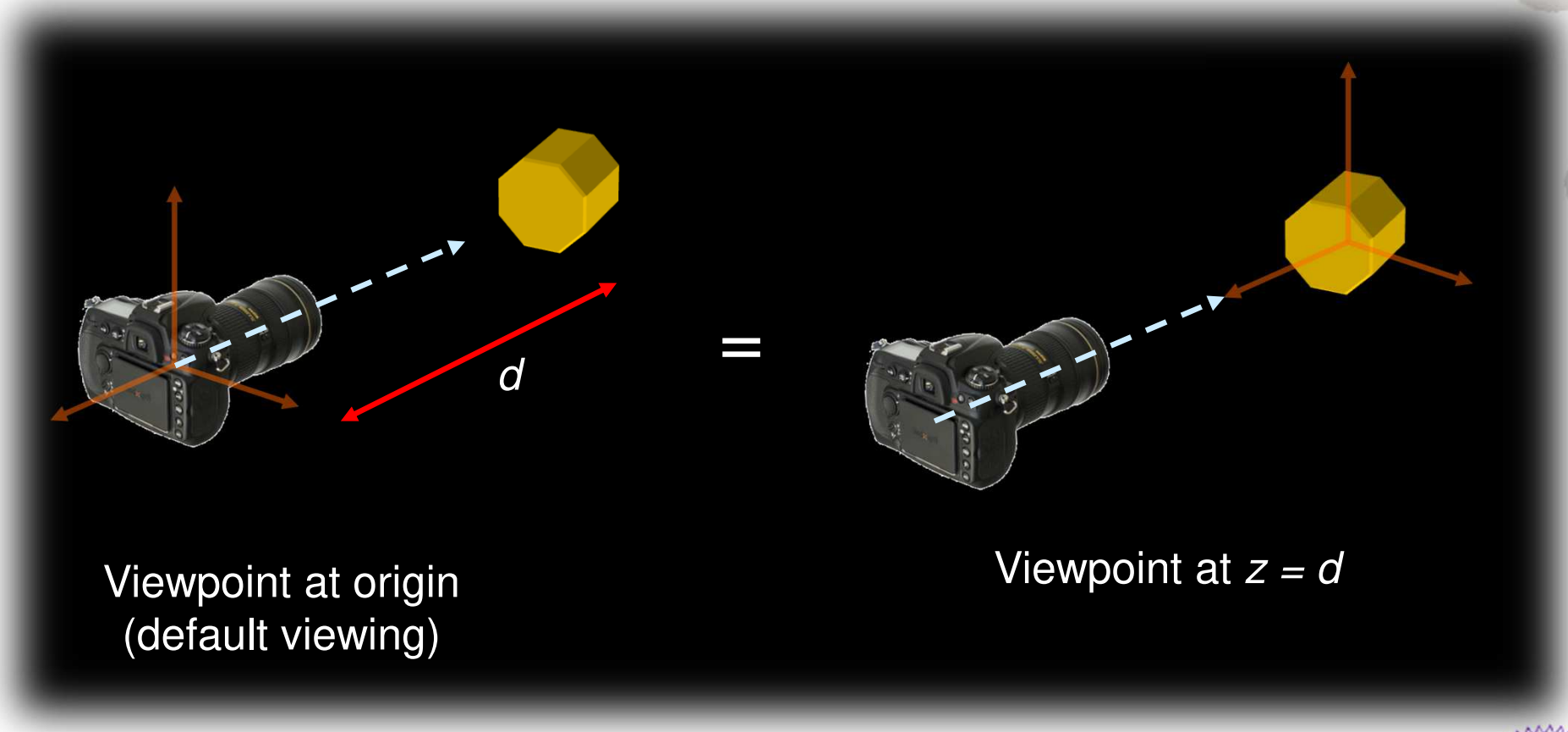


Viewing Transformation

- It defines the position and orientation of the viewpoint
- It is composed of a sequence of translation and rotation transformations
- In OpenGL, the default viewpoint is located at the origin and looking toward the negative z-axis direction

Viewing Transformation

- **Example**

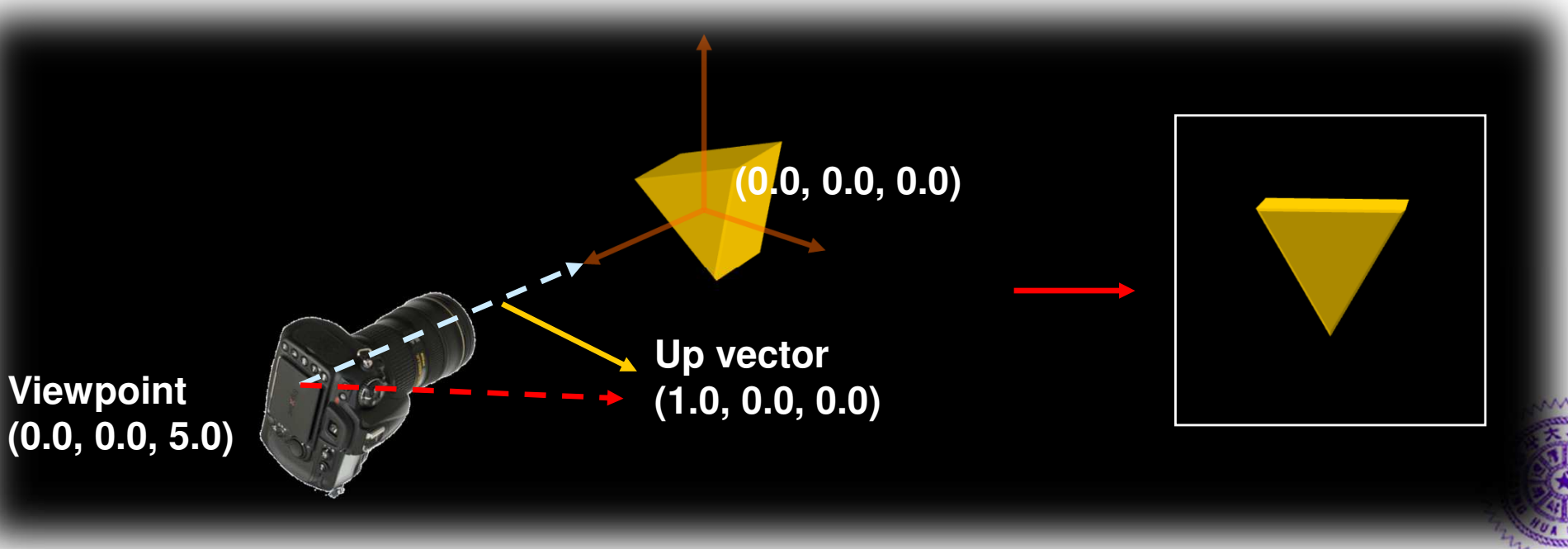


Specify the Viewing Parameters

- You need to define the viewpoint, viewing direction, and the pose when you are looking through a specific direction
- gluLookat(*ex*, *ey*, *ez*, // eye coordinates
  cx, *cy*, *cz*, // view center coordinates
  ux, *uy*, *uz*) // up vector
  - Viewpoint is at (*ex*, *ey*, *ez*)
  - Viewing direction is from (*ex*, *ey*, *ez*) toward (*cx*, *cy*, *cz*)
  - Up direction is from (0, 0, 0) to (*ux*, *uy*, *uz*)

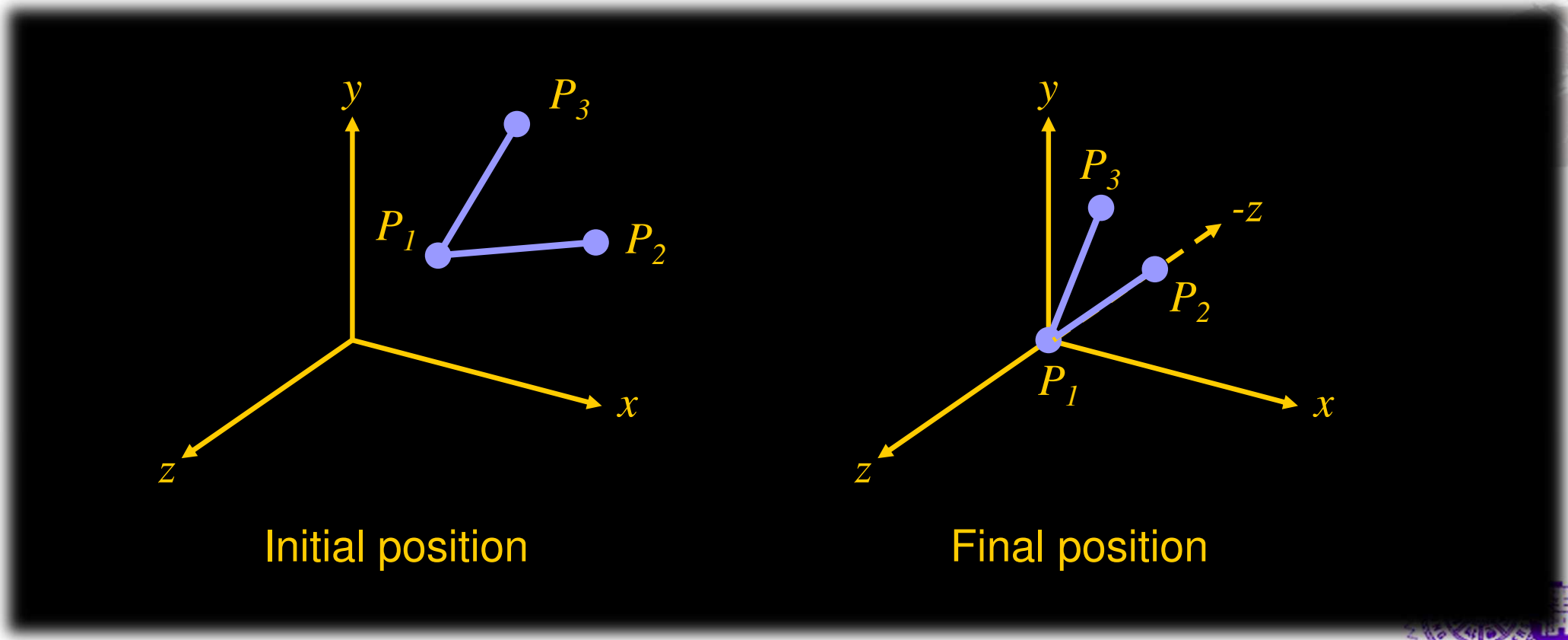
Specify the Viewing Parameters

- gluLookat(0.0, 0.0, 5.0, 0.0, 0.0, 0.0, 1.0, 0.0, 0.0)
  - Viewpoint is at (0.0, 0.0, 5.0)
  - Viewing direction is from (0.0, 0.0, 5.0) toward (0.0, 0.0, 0.0)
  - Up direction is (1.0, 0.0, 0.0)



Concept of Viewing Transformation

- Transforming P_1, P_2, and P_3 such that $\overrightarrow{P_1P_2}$ aligns with the **negative** *z*-axis and $\overrightarrow{P_1P_3}$ lies on the *yz*-plane



Concept of Viewing Transformation

- Transforming P_1, P_2, and P_3 from their initial position to their final position
  - Step 1: Translate P_1 to the origin
  - Step 2: Rotate about the *y*-axis such that $\overrightarrow{P_1P_2}$ lies on the *yz*-plane
  - Step 3: Rotate about the *x*-axis such that $\overrightarrow{P_1P_2}$ align with the negative *z*-axis
  - Step 4: Rotate about the *z*-axis such that $\overrightarrow{P_1P_3}$ lies on the *yz*-plane

Concept of Viewing Transformation

- Transforming P_1, P_2, and P_3 from their initial position to their final position
  - Step 1: Translate P_1 to the origin

$$P'_1 = T(-x_1,-y_1,-z_1)\cdot P_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$P'_2 = T(-x_1,-y_1,-z_1)\cdot P_2 = \begin{bmatrix} x_2 - x_1 \\ y_2 - y_1 \\ z_2 - z_1 \\ 1 \end{bmatrix}$$

$$P'_3 = T(-x_1,-y_1,-z_1)\cdot P_3 = \begin{bmatrix} x_3 - x_1 \\ y_3 - y_1 \\ z_3 - z_1 \\ 1 \end{bmatrix}$$

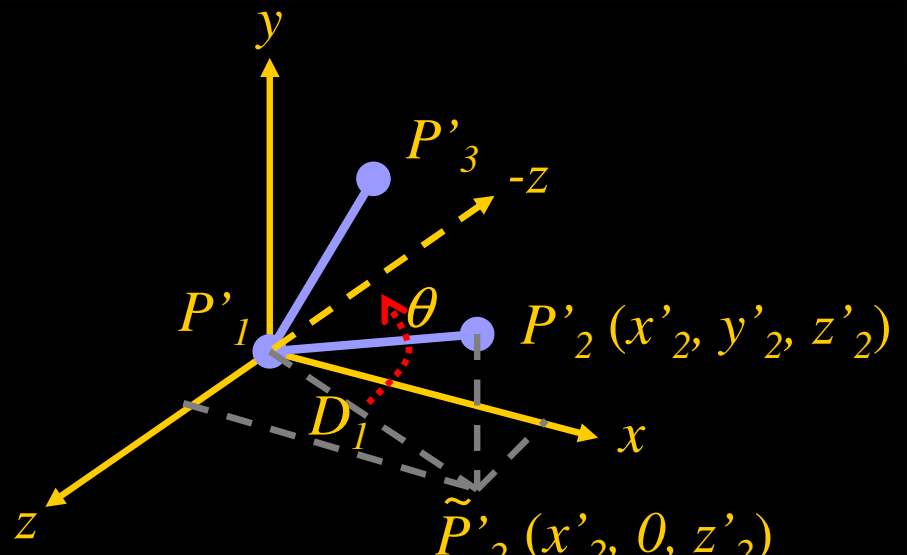
Concept of Viewing Transformation

- Transforming P_1, P_2, and P_3 from their initial position to their final position
  - Step 2: Rotate about the y-axis such that $\overrightarrow{P_1P_2}$ lies in the (y, z) plane

$$P''_2 = R_y(\theta) \cdot P'_2 = \begin{bmatrix} 0 \\ y'_2 - y'_1 \\ -D_1 \\ 1 \end{bmatrix}$$

where $D_1 = \left|\overline{P'_1 \widetilde{P}'_2}\right|$

Besides, $P''_1 = R_y(\theta) \cdot P'_1$ and $P''_3 = R_y(\theta) \cdot P'_3$

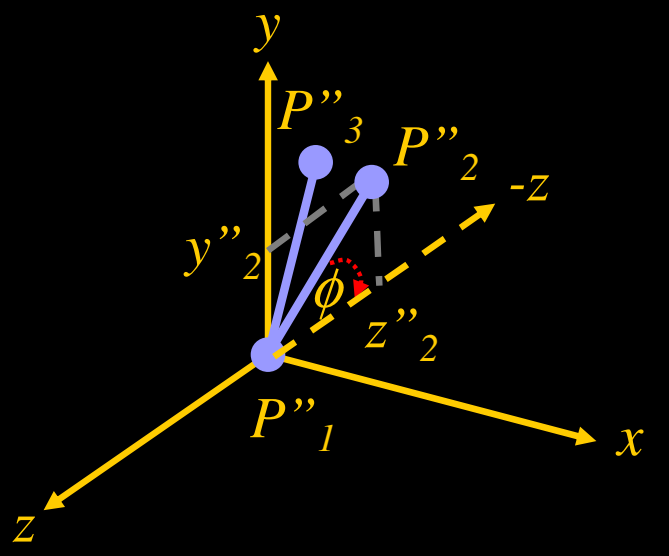


Concept of Viewing Transformation

- Transforming P_1, P_2, and P_3 from their initial position to their final position
  - Step 3: Rotate about the x-axis such that $\overrightarrow{P_1P_2}$ lies on the z-axis

$$P'''_2 = R_x(-\emptyset) \cdot P''_2$$
$$= R_x(-\emptyset) \cdot R_y(\theta) \cdot P'_2$$
$$= R_x(-\emptyset) \cdot R_y(\theta) \cdot T \cdot P_2$$

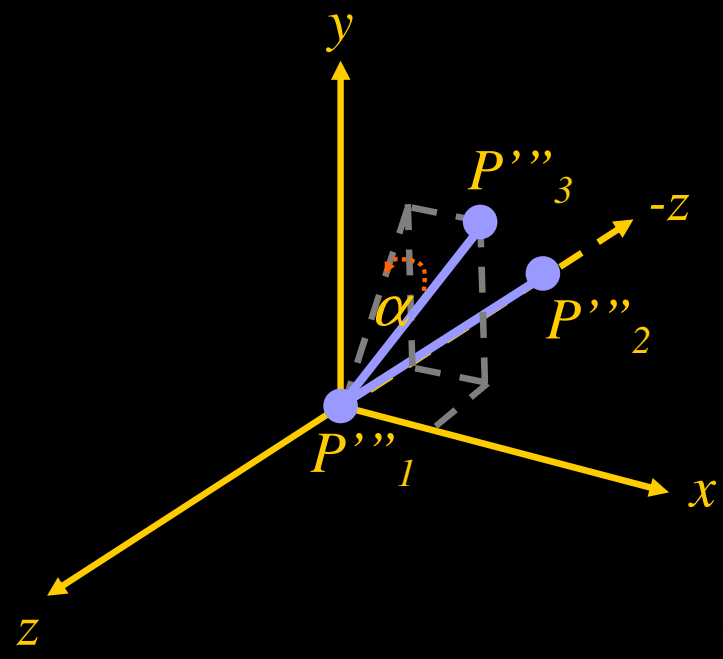
$$P'''_1 = R_x(-\emptyset) \cdot R_y(\theta) \cdot T \cdot P_1$$
$$P'''_3 = R_x(-\emptyset) \cdot R_y(\theta) \cdot T \cdot P_3$$



Concept of Viewing Transformation

- Transforming P_1, P_2, and P_3 from their initial position to their final position
  - Step 4: Rotate about the z-axis such that $\overrightarrow{P_1P_3}$ lies on the yz-plane

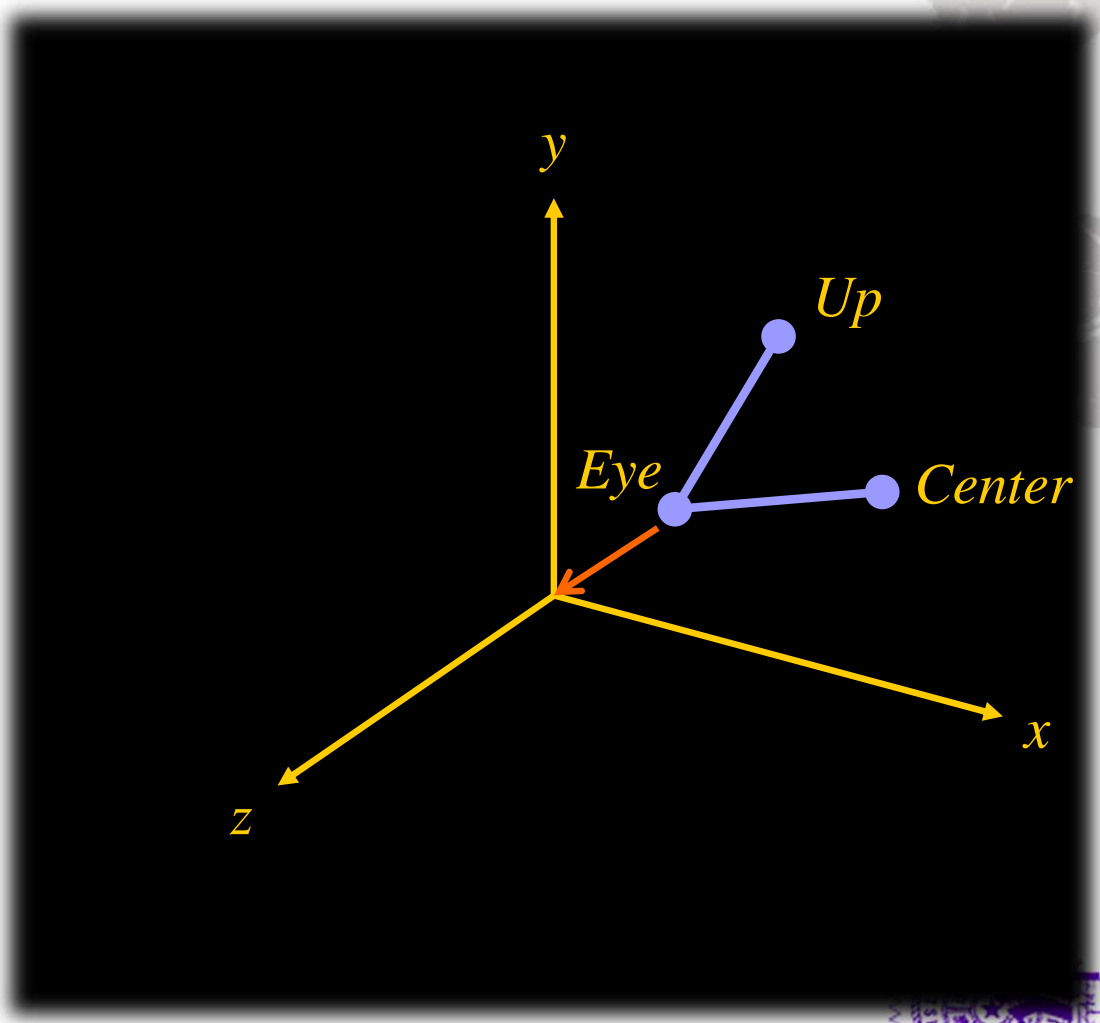
$$P''''_3 = R_z(\alpha) \cdot P'''_3$$
$$= R_z(\alpha) \cdot R_x(-\emptyset) \cdot P''_3$$
$$= R_z(\alpha) \cdot R_x(-\emptyset) \cdot R_y(\theta) \cdot P'_3$$
$$= R_z(\alpha) \cdot R_x(-\emptyset) \cdot R_y(\theta) \cdot T \cdot P_3$$
$$P''''_1 = R_z(\alpha) \cdot R_x(-\emptyset) \cdot R_y(\theta) \cdot T \cdot P_1$$
$$P''''_2 = R_z(\alpha) \cdot R_x(-\emptyset) \cdot R_y(\theta) \cdot T \cdot P_2$$



Fast Viewing Matrix Derivation

- Translate eye position to the origin

$$T = \begin{bmatrix} 1 & 0 & 0 & -eye_x \\ 0 & 1 & 0 & -eye_y \\ 0 & 0 & 1 & -eye_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



Fast Viewing Matrix Derivation

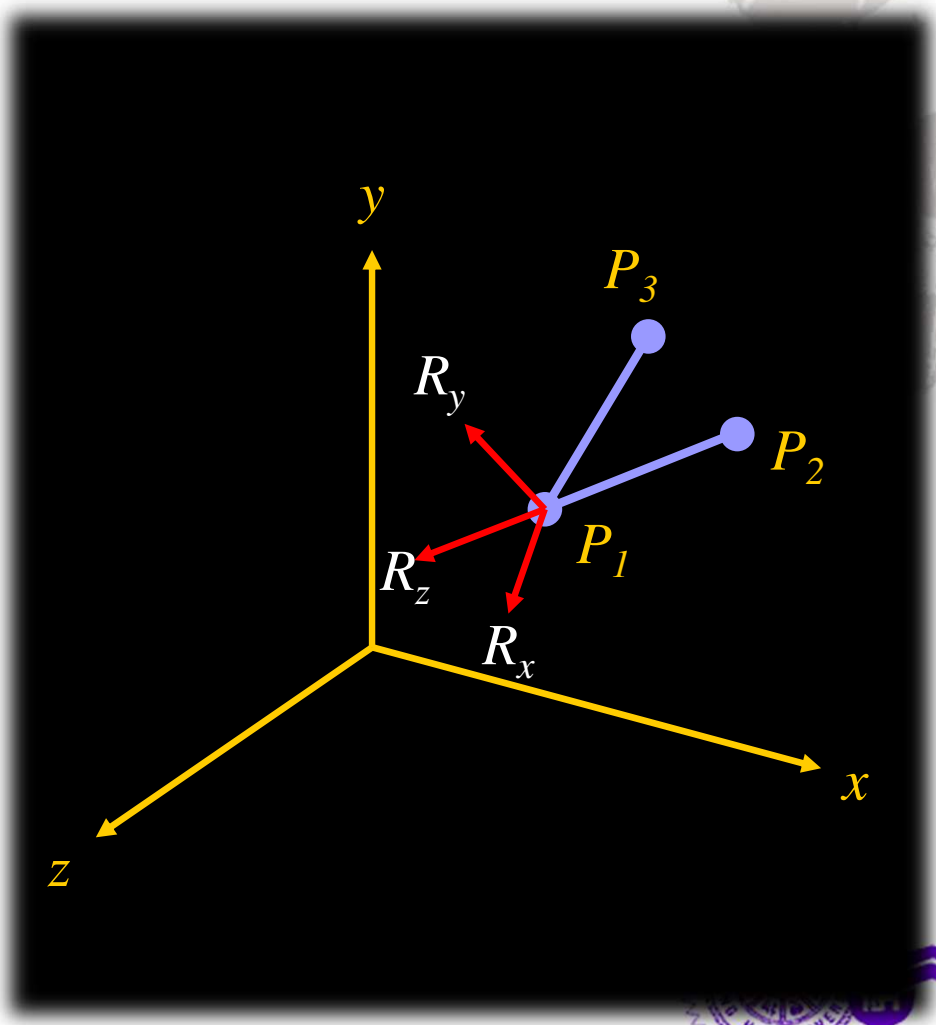
- Find the bases with respect to the new space

$$R = \begin{bmatrix} r_{1x} & r_{2x} & r_{3x} \\ r_{1y} & r_{2y} & r_{3y} \\ r_{1z} & r_{2z} & r_{3z} \end{bmatrix}$$

$$R_z = [r_{1z} \quad r_{2z} \quad r_{3z}]^T = \frac{-\overrightarrow{P_1P_2}}{|\overrightarrow{P_1P_2}|}$$

$$R_x = [r_{1x} \quad r_{2x} \quad r_{3x}]^T = \frac{\overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3}}{|\overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3}|}$$

$$R_y = [r_{1y} \quad r_{2y} \quad r_{3y}]^T = R_z \times R_x$$



Fast Viewing Matrix Derivation

- **Coordinates Conversion**

$$x' \times (r_{1x}, r_{2x}, r_{3x}) + y' \times (r_{1y}, r_{2y}, r_{3y}) + z' \times (r_{1z}, r_{2z}, r_{3z})$$
$$= x \times (1, 0, 0) + y \times (0, 1, 0) + z \times (0, 0, 1)$$

$$\begin{bmatrix} r_{1x} & r_{1y} & r_{1z} \\ r_{2x} & r_{2y} & r_{2z} \\ r_{3x} & r_{3y} & r_{3z} \end{bmatrix} \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Inverse matrix

$$\begin{bmatrix} r_{1x} & r_{2x} & r_{3x} \\ r_{1y} & r_{2y} & r_{3y} \\ r_{1z} & r_{2z} & r_{3z} \end{bmatrix} \begin{bmatrix} r_{1x} & r_{1y} & r_{1z} \\ r_{2x} & r_{2y} & r_{2z} \\ r_{3x} & r_{3y} & r_{3z} \end{bmatrix} \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} r_{1x} & r_{2x} & r_{3x} \\ r_{1y} & r_{2y} & r_{3y} \\ r_{1z} & r_{2z} & r_{3z} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} r_{1x} & r_{2x} & r_{3x} \\ r_{1y} & r_{2y} & r_{3y} \\ r_{1z} & r_{2z} & r_{3z} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Fast Viewing Matrix Derivation

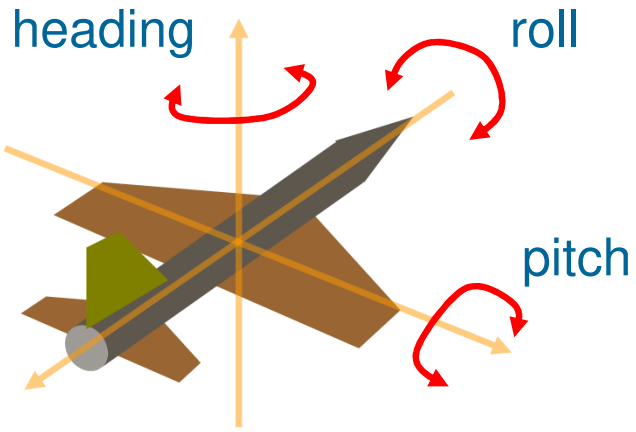
- **Final Viewing Matrix is defined as**

$$M_{view} = R \bullet T = \begin{bmatrix} r_{1x} & r_{2x} & r_{3x} & 0 \\ r_{1y} & r_{2y} & r_{3y} & 0 \\ r_{1z} & r_{2z} & r_{3z} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & -eye_x \\ 0 & 1 & 0 & -eye_y \\ 0 & 0 & 1 & -eye_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Custom Viewing

- **Pilot view**

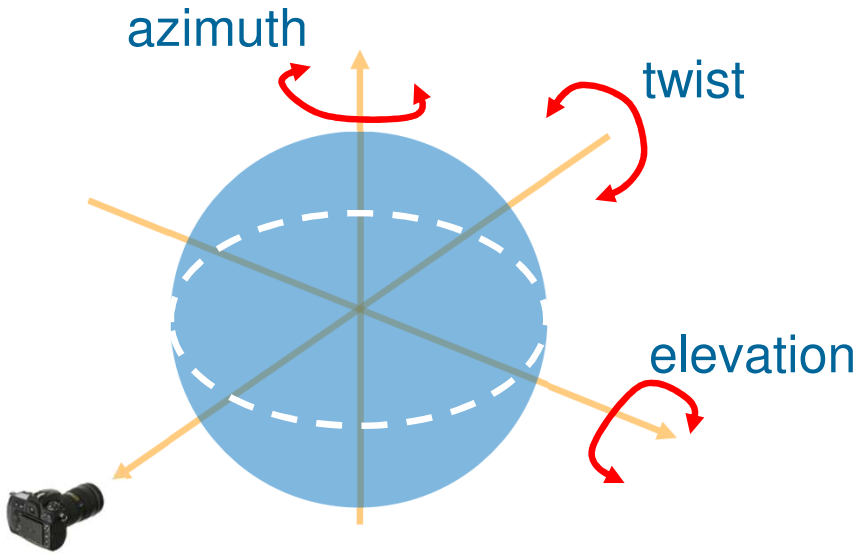
```
void pilotView(GLdouble planex, GLdouble planey,
               GLdouble planez, GLdouble roll,
               GLdouble pitch, GLdouble heading)
{
      glRotated(roll, 0.0, 0.0, 1.0);
      glRotated(pitch, 1.0, 0.0, 0.0);
      glRotated(heading, 0.0, 1.0, 0.0);  // yaw
      glTranslated(-planex, -planey, -planez);
}
```



Custom Viewing

- **Polar view**

```
void polarView(GLdouble distance, GLdouble twist,
               GLdouble elevation, GLdouble azimuth)
{
      glTranslate(0.0, 0.0, -distance);
      glRotated(twist, 0.0, 0.0, 1.0);
      glRotated(elevation, 1.0, 0.0, 0.0);
      glRotated(azimuth, 0.0, 1.0, 0.0);
}
```



Transformation in Reverse Order

- Each vertex is doing modeling transformation first and then viewing transformation

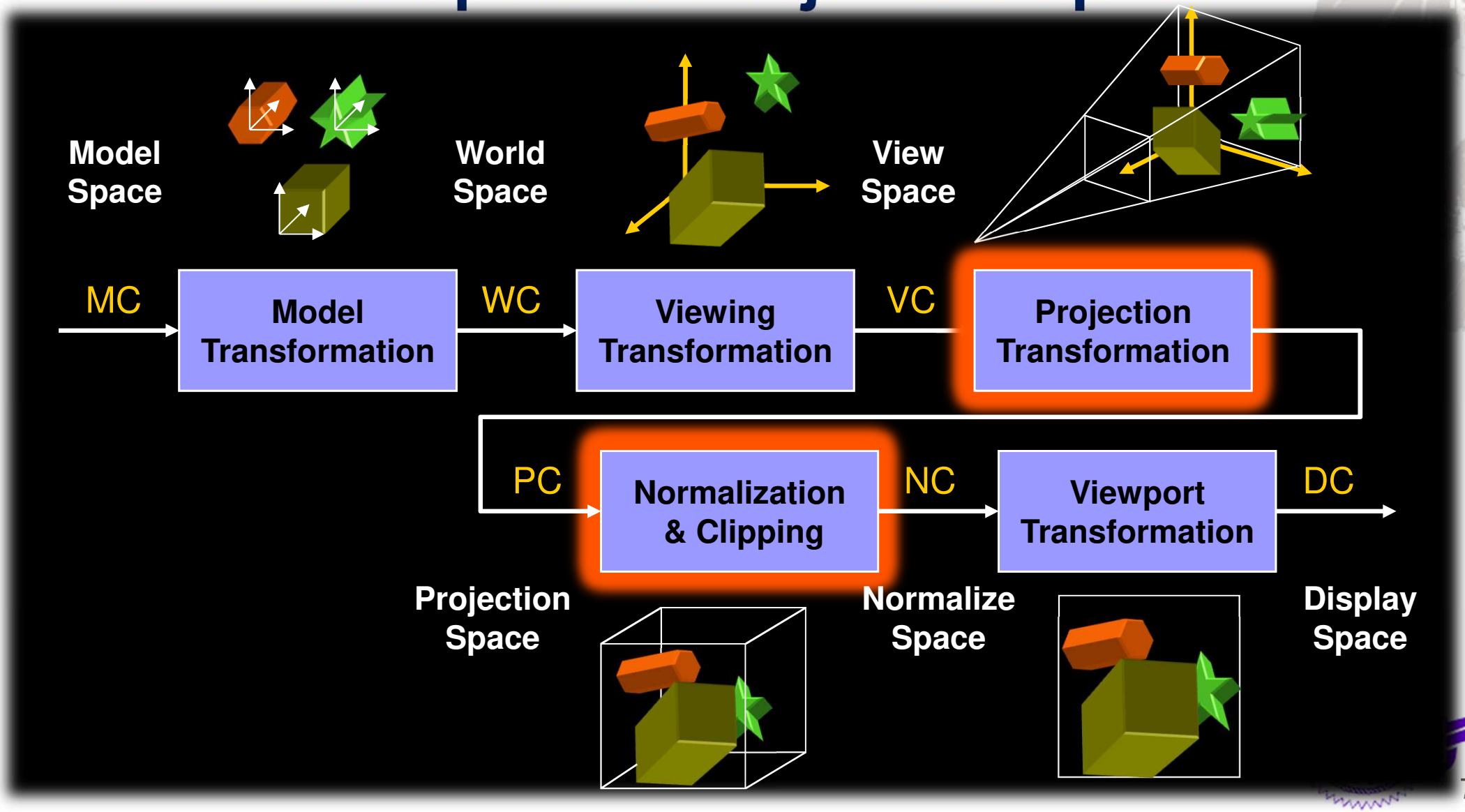
$$VMv = V(Mv) = V(M_n M_{n-1} \cdots M_2 M_1 M_0 v)$$
$$= V(M_n(M_{n-1}(\cdots(M_2(M_1(M_0 v)))\cdots)))$$

Projection

From View Space to Projection Space

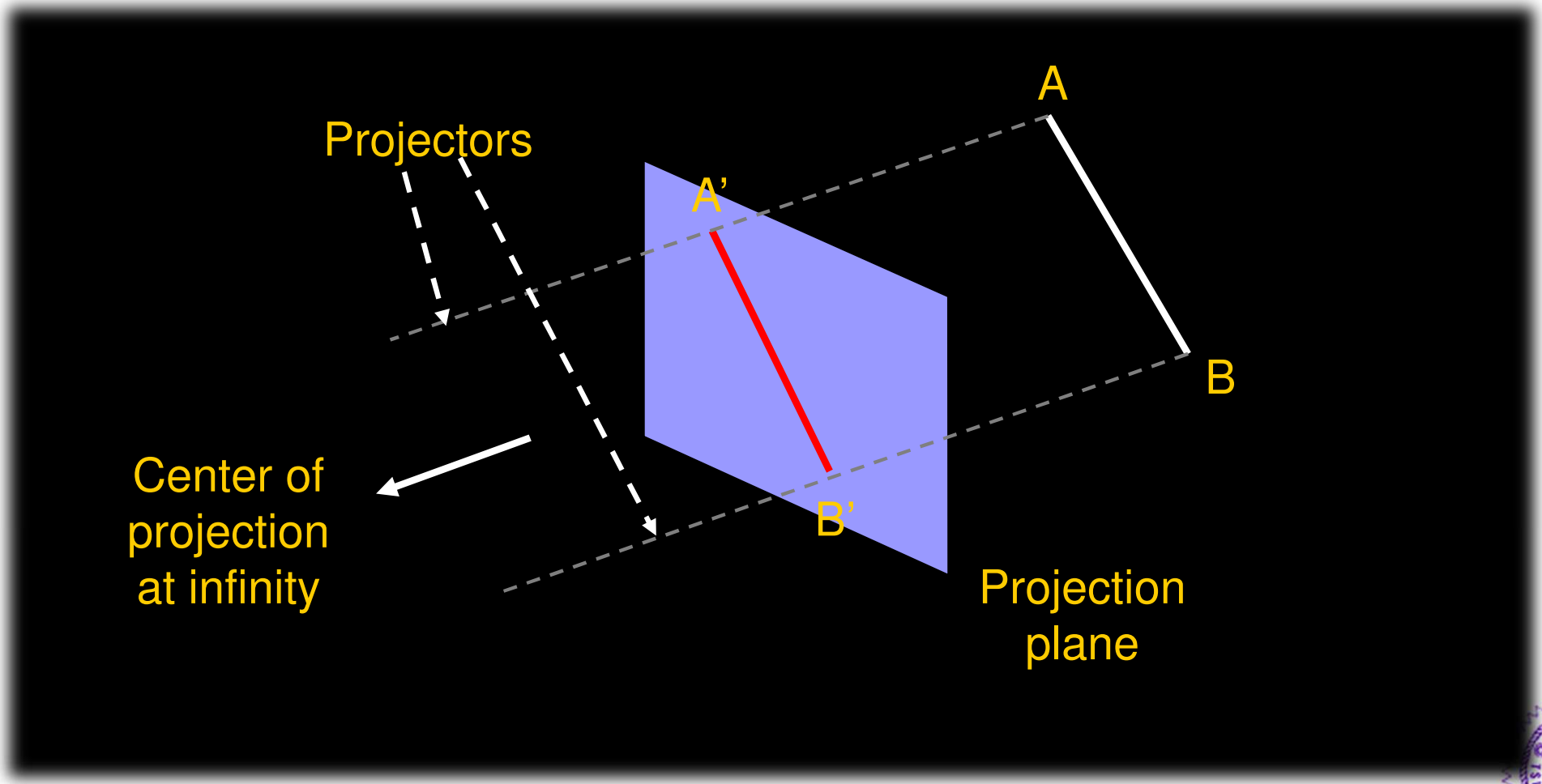
3D Viewing Process

- From View Space to Projection Space



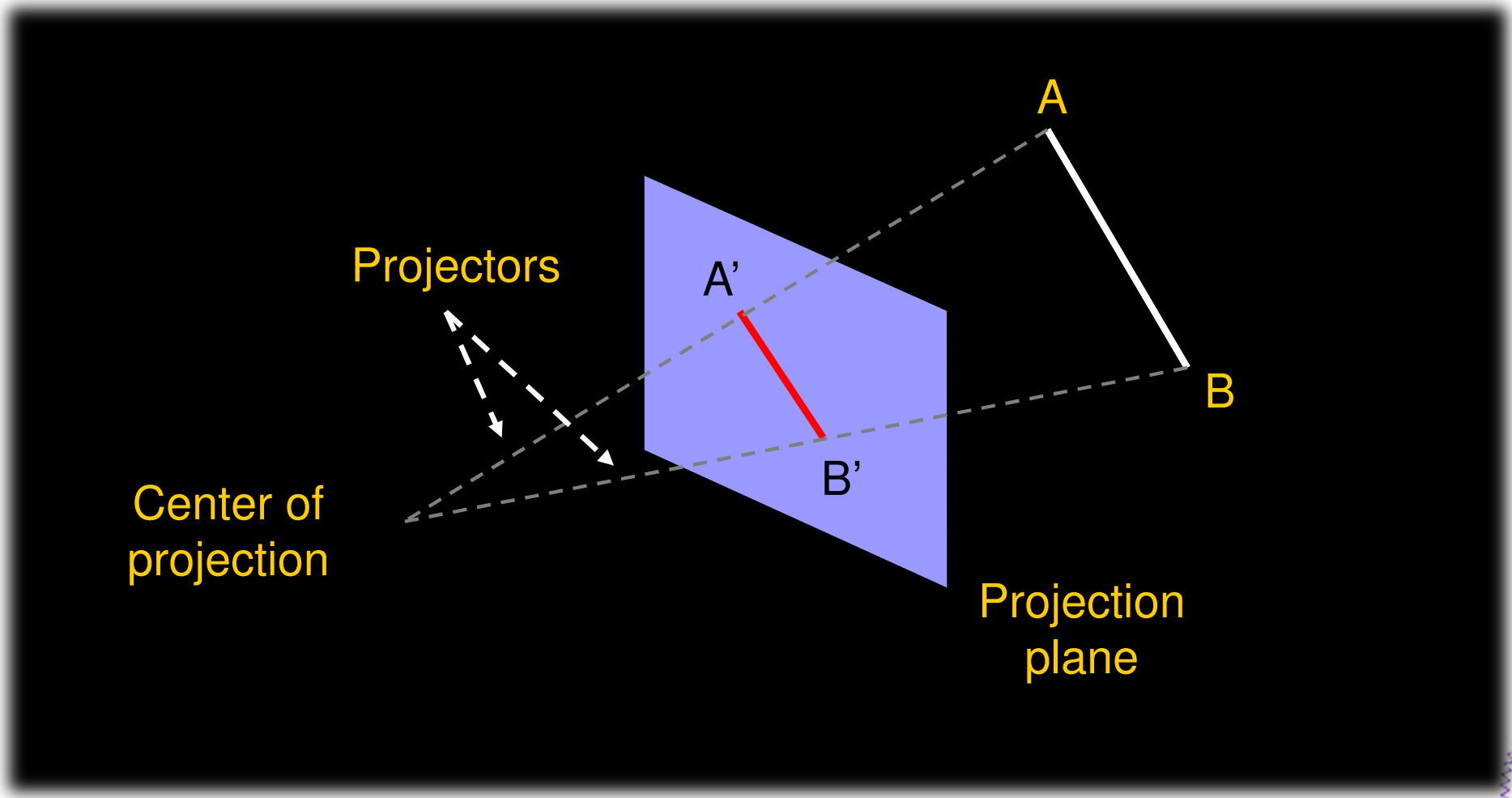
Projections

- **Parallel projection**



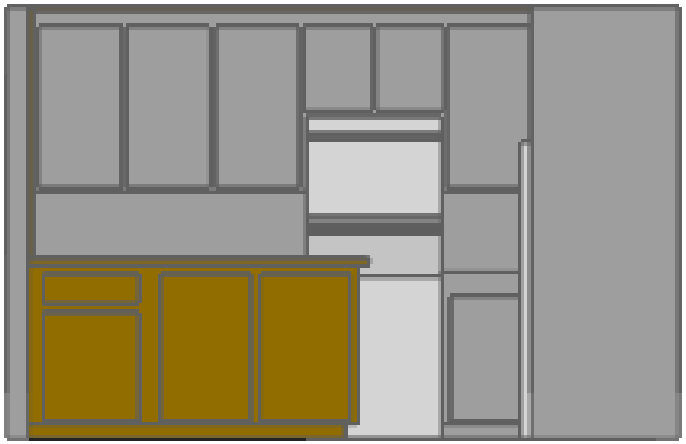
Projections

- **Perspective projection**

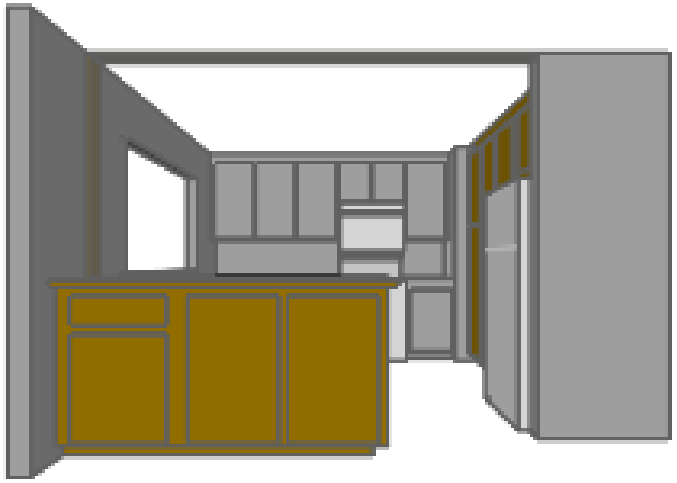


Projections

- **Parallel projection vs. perspective projection**



Parallel projection



Perspective projection

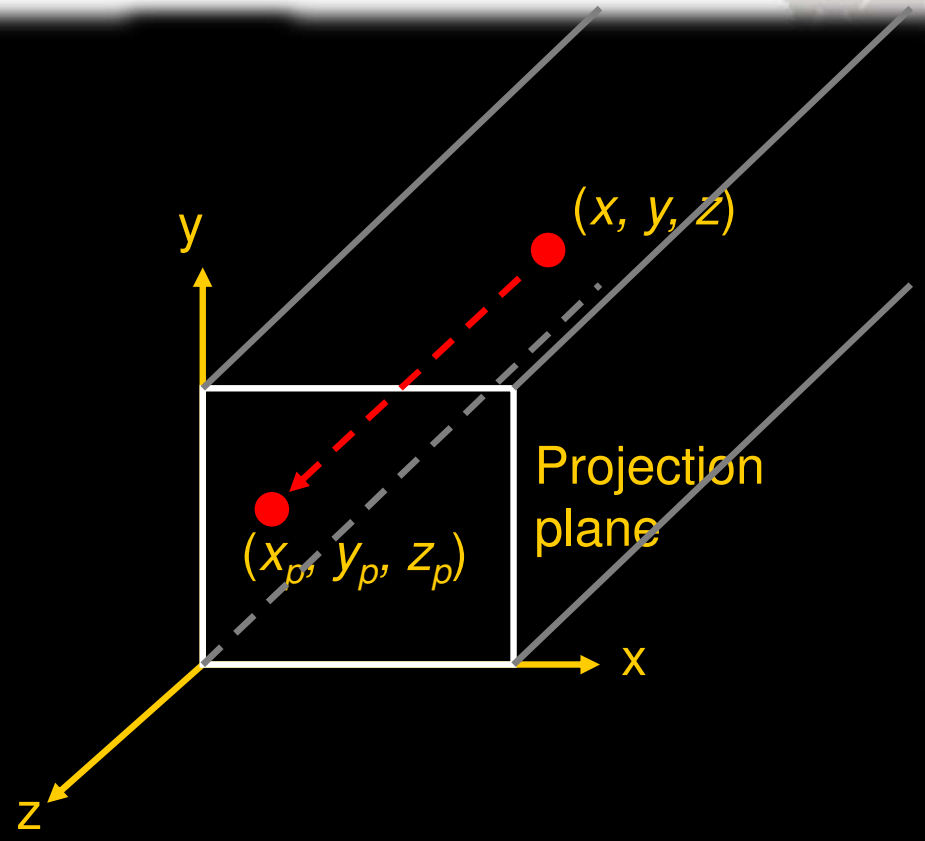
Parallel Projection

- Projection onto a projection plane at z = 0

$$x_p = x,\ y_p = y,\ z_p = 0$$

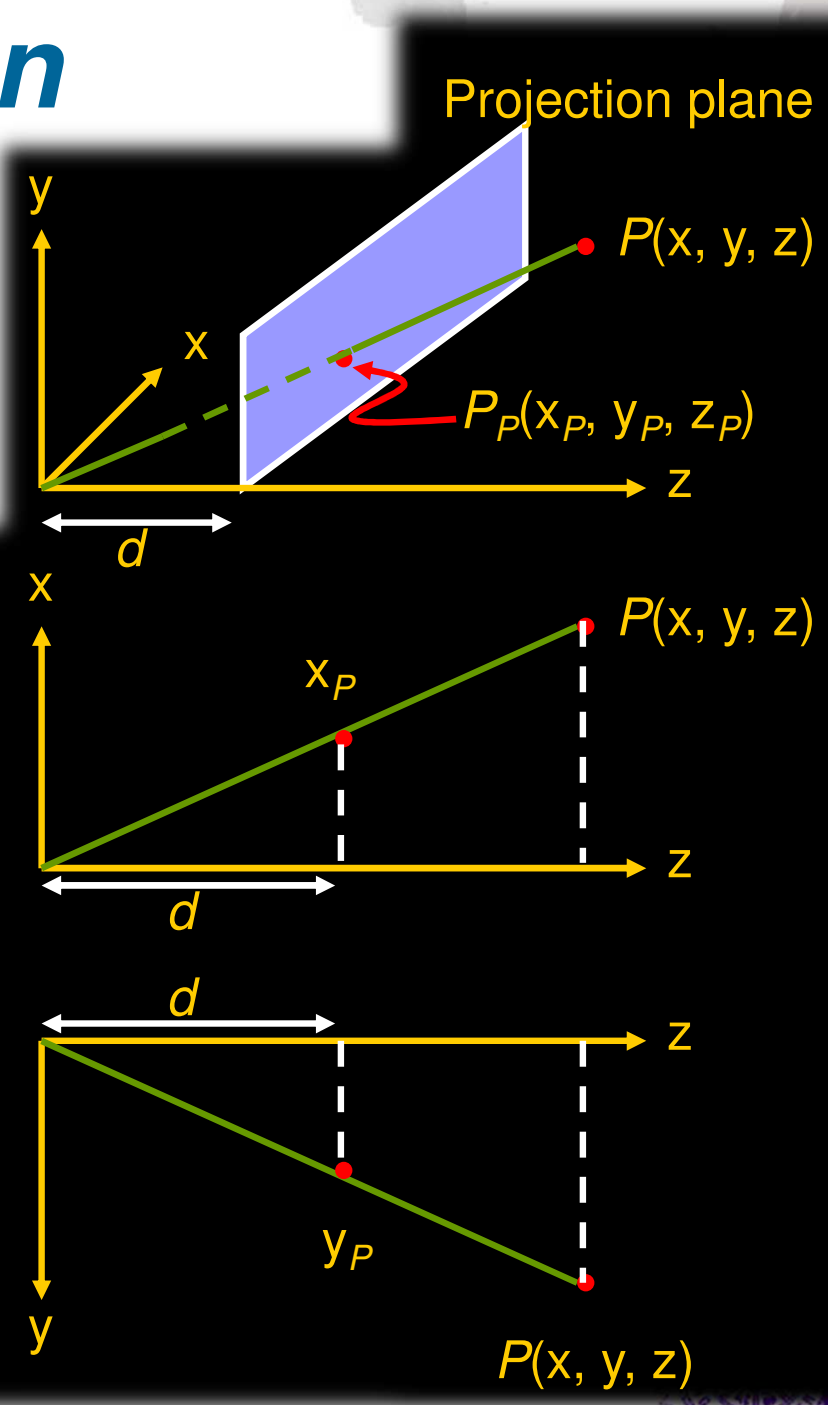
$$\begin{bmatrix} x_p \\ y_p \\ z_p \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ 0 \\ 1 \end{bmatrix}$$

$$= M_{ort} \cdot P$$



Perspective Projection

- **Projection onto a plane at z = *d***



$$\frac{x_p}{d} = \frac{x}{z}, \frac{y_p}{d} = \frac{y}{z}$$

$$x_p = \frac{x}{z/d}, y_p = \frac{y}{z/d}$$

$$M_{per} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1/d & 0 \end{bmatrix}$$

Perspective Projection

$$\begin{bmatrix} X \\ Y \\ Z \\ W \end{bmatrix} = M_{per} \cdot P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1/d & 0 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} X & Y & Z & W \end{bmatrix}^T = \begin{bmatrix} x & y & z & \frac{z}{d} \end{bmatrix}^T$$

From homogeneous coordinates to Cartesian coordinate

$$(\frac{X}{W}, \frac{Y}{W}, \frac{Z}{W}) = (x_p, y_p, z_p) = (\frac{x}{z/d}, \frac{y}{z/d}, d)$$

Perspective Projection

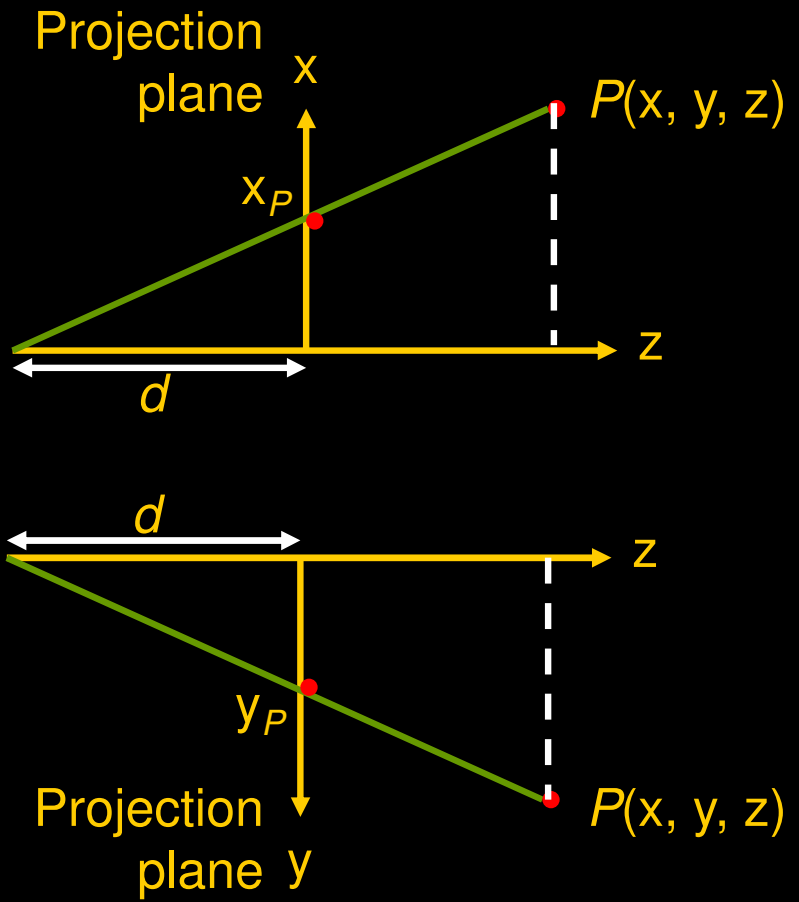
- **Alternative Perspective Projection**

$$\frac{x_p}{d} = \frac{x}{z+d}, \frac{y_p}{d} = \frac{y}{z+d}$$

$$x_p = \frac{d \cdot x}{z+d} = \frac{x}{(z/d)+1}$$

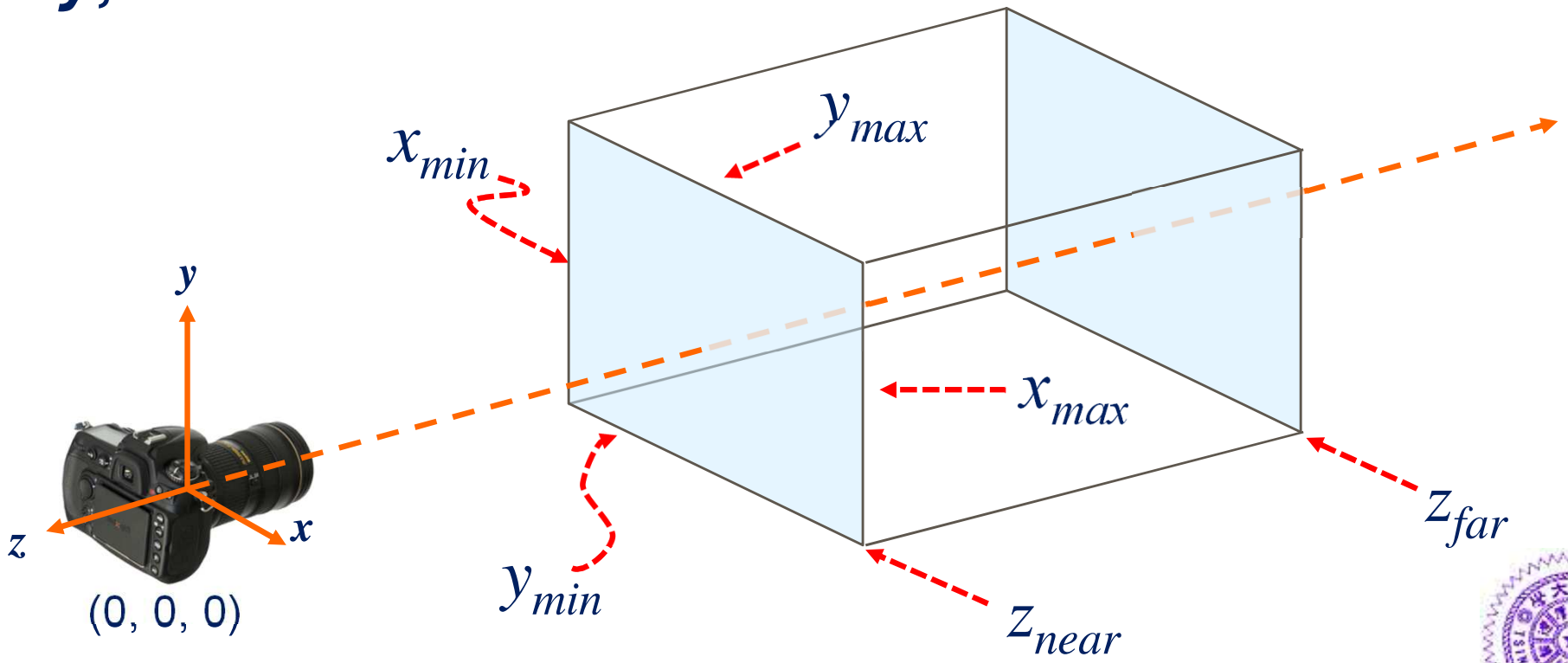
$$y_p = \frac{d \cdot y}{z+d} = \frac{y}{(z/d)+1}$$

$$M'_{per} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1/d & 1 \end{bmatrix}$$



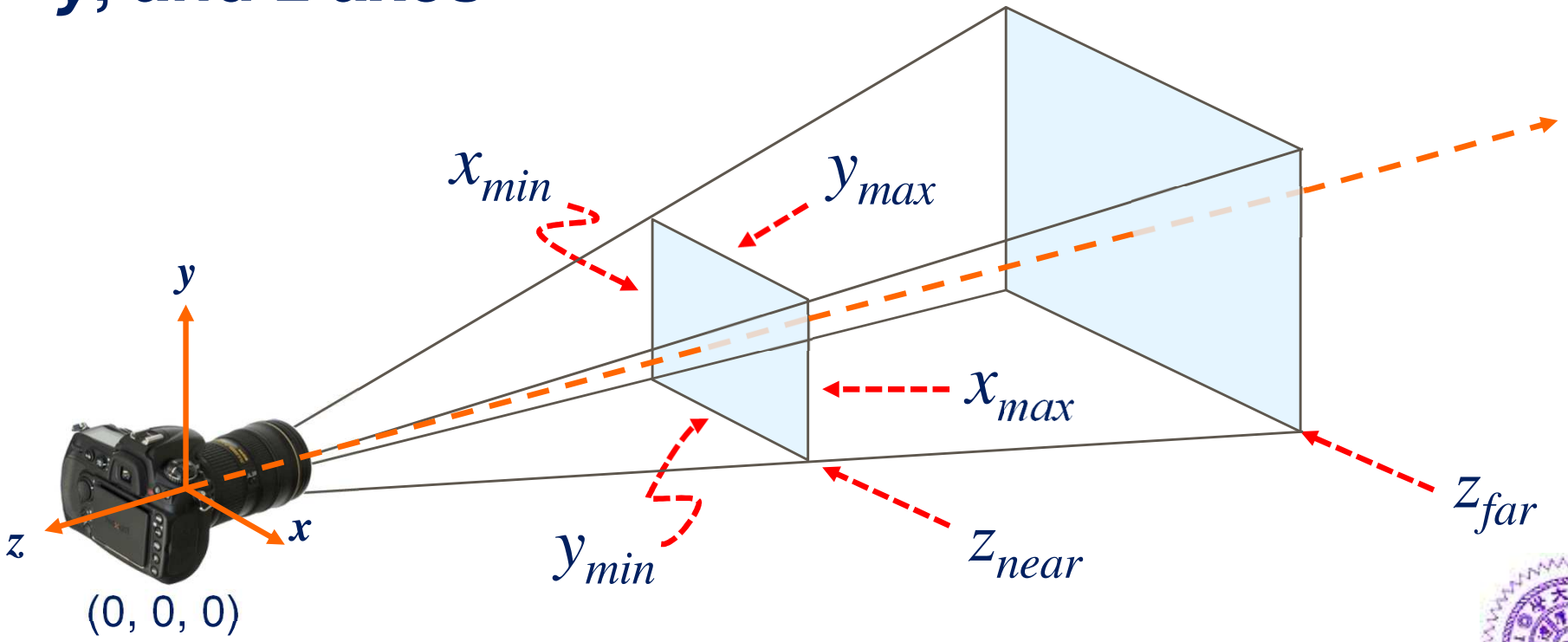
View Volume

- **View Volume in Orthographic Projection**
  - **Assume the eye is at the origin, then specify x_{max}, x_{min}, y_{max}, y_{min}, z_{near}, z_{far} as the boundaries of *x*, *y*, and *z* axes**



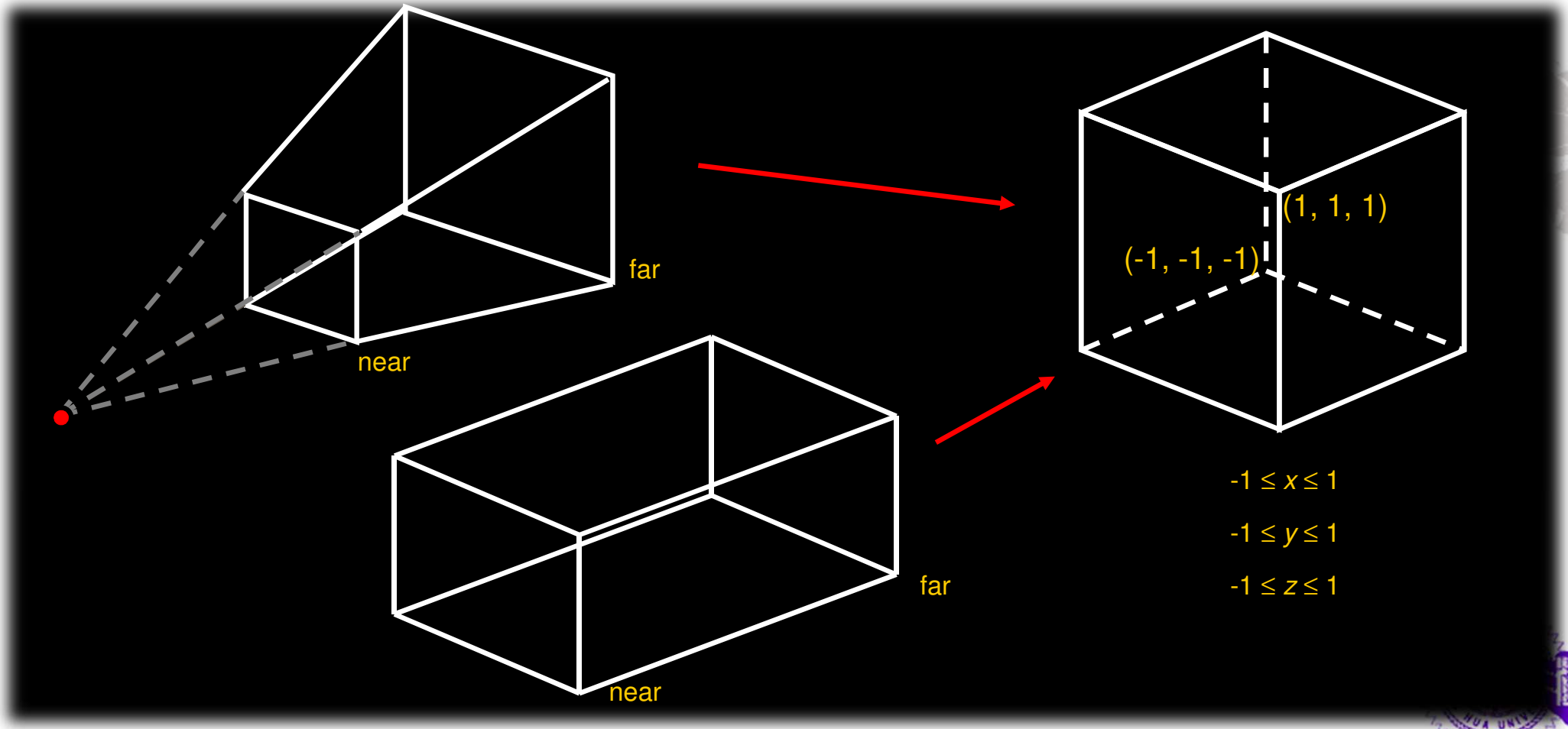
View Volume

- **View Volume in Perspective Projection**
  - **Assume the eye is at the origin, then specify x_{max}, x_{min}, y_{max}, y_{min}, z_{near}, z_{far} as the boundaries of *x*, *y*, and *z* axes**



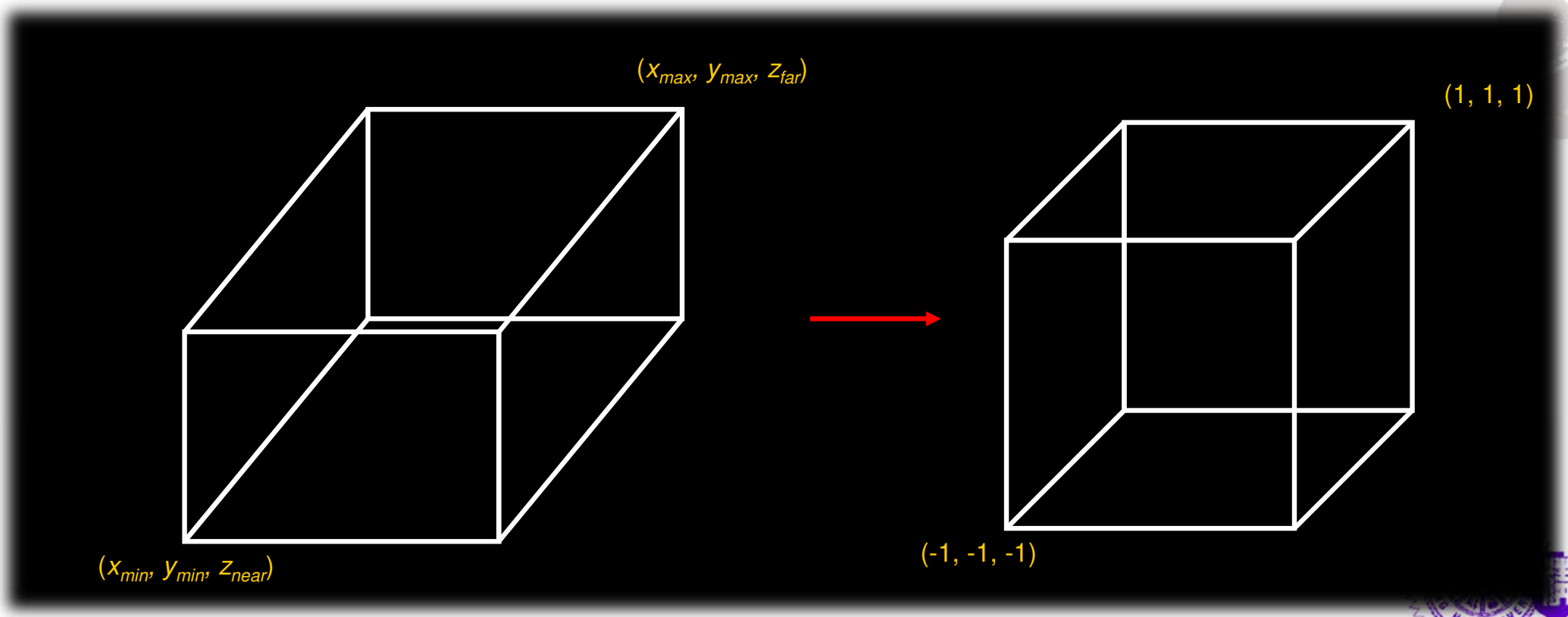
View Volume Normalization

- Make the view volume clipping easy under normalized clipping volume



Orthogonal Normalization

- Composition of orthogonal normalization
  - Translate the view volume center to origin
  - Scale to have each side of length 2



Orthogonal Normalization

- Composition of orthogonal normalization

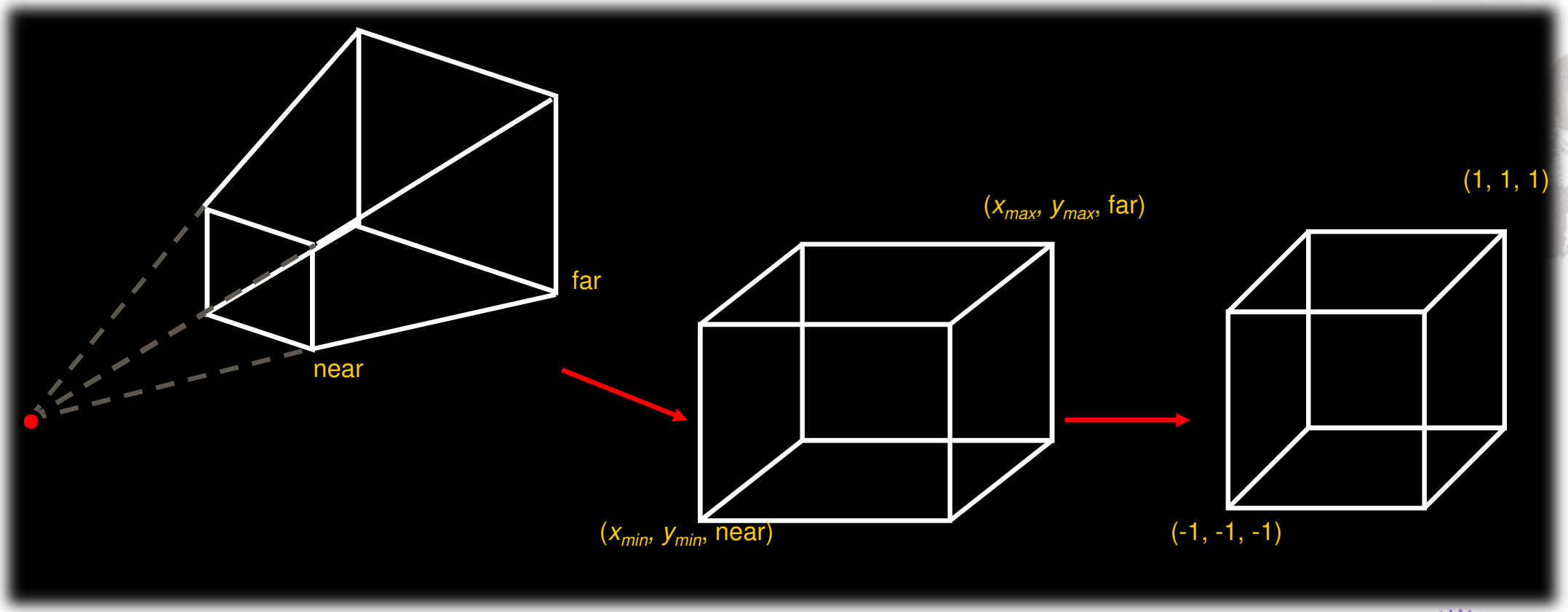
$$M_{orthonorm} = S\left(\frac{2}{x_{\max} - x_{\min}}, \frac{2}{y_{\max} - y_{\min}}, \frac{2}{z_{far} - z_{near}}\right)$$

$$\cdot T\left(-\frac{x_{\max} + x_{\min}}{2}, -\frac{y_{\max} + y_{\min}}{2}, -\frac{z_{far} + z_{near}}{2}\right)$$

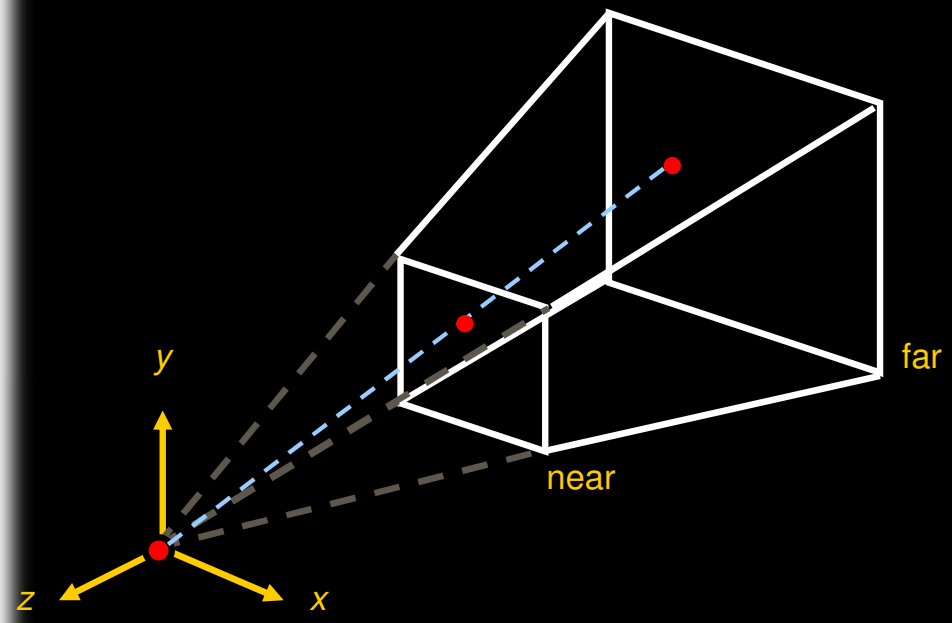
$$= \begin{bmatrix} \frac{2}{x_{\max} - x_{\min}} & 0 & 0 & -\frac{x_{\max} + x_{\min}}{x_{\max} - x_{\min}} \\ 0 & \frac{2}{y_{\max} - y_{\min}} & 0 & -\frac{y_{\max} + y_{\min}}{y_{\max} - y_{\min}} \\ 0 & 0 & \frac{2}{z_{far} - z_{near}} & -\frac{z_{far} + z_{near}}{z_{far} - z_{near}} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Perspective Normalization

- Composition of perspective normalization

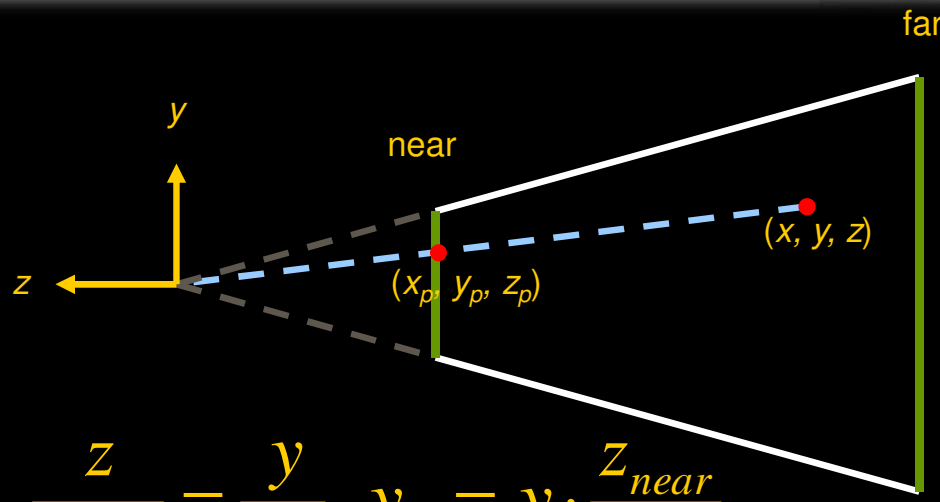


Perspective Normalization

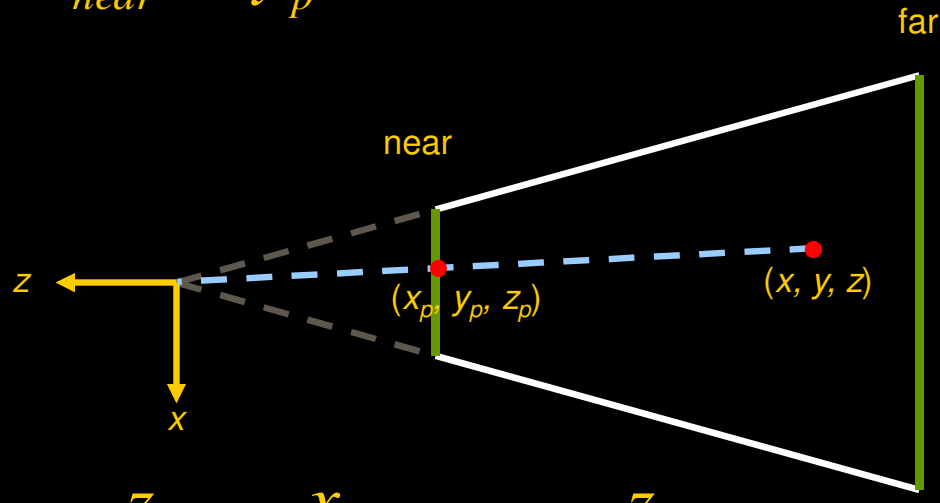


$z_p = z$

Keep z intact

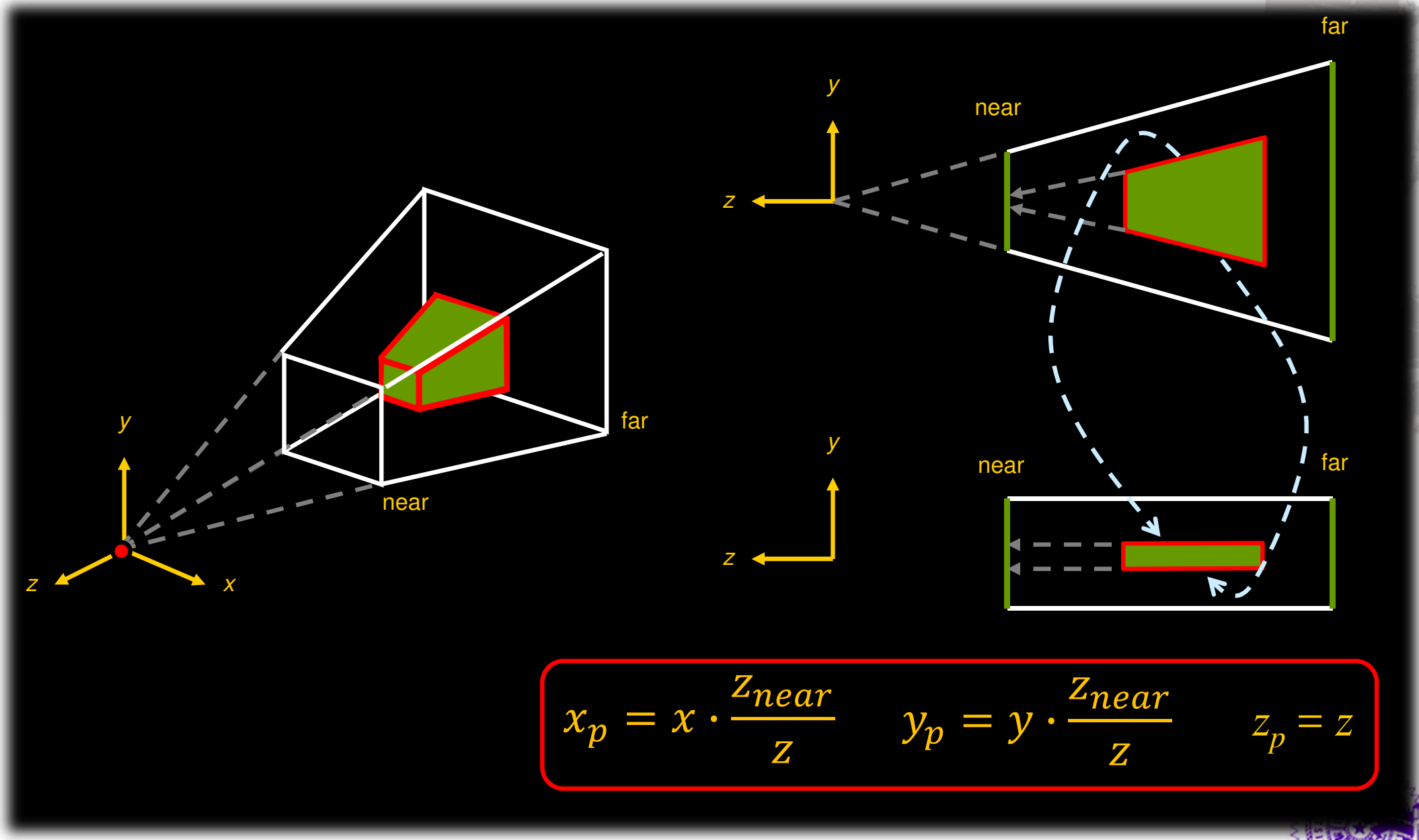


$$\frac{z}{z_{near}} = \frac{y}{y_p}, y_p = y \cdot \frac{z_{near}}{z}$$



$$\frac{z}{z_{near}} = \frac{x}{x_p}, x_p = x \cdot \frac{z_{near}}{z}$$

Perspective Normalization

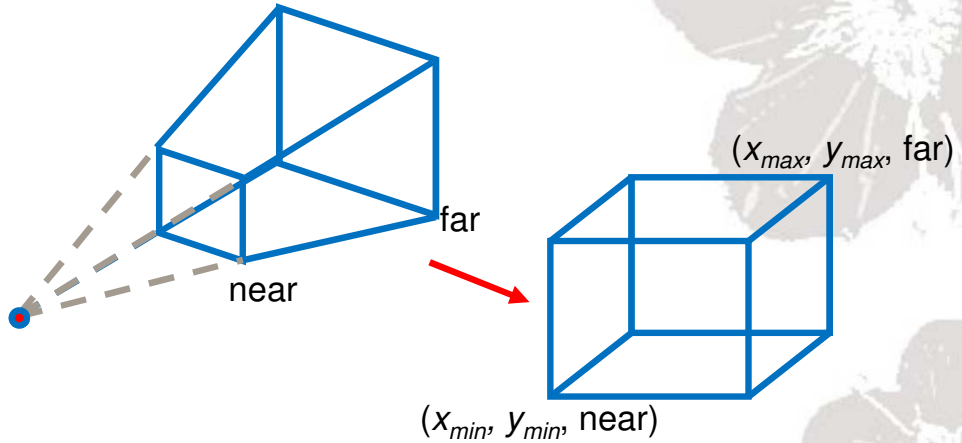


$$x_p = x \cdot \frac{z_{near}}{z} \qquad y_p = y \cdot \frac{z_{near}}{z} \qquad z_p = z$$

Perspective Normalization

- **From Frustum to Cuboid**

$$\begin{bmatrix} x' \\ y' \\ z' \\ w' \end{bmatrix} = \begin{bmatrix} z_{near} & 0 & 0 & 0 \\ 0 & z_{near} & 0 & 0 \\ 0 & 0 & S_z & T_z \\ 0 & 0 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$



$$x_p = \frac{x'}{w'} = x \cdot \frac{z_{near}}{z} \qquad y_p = \frac{y'}{w'} = y \cdot \frac{z_{near}}{z} \qquad z_p = \frac{z'}{w'} = \frac{S_z \cdot z + T_z}{z}$$

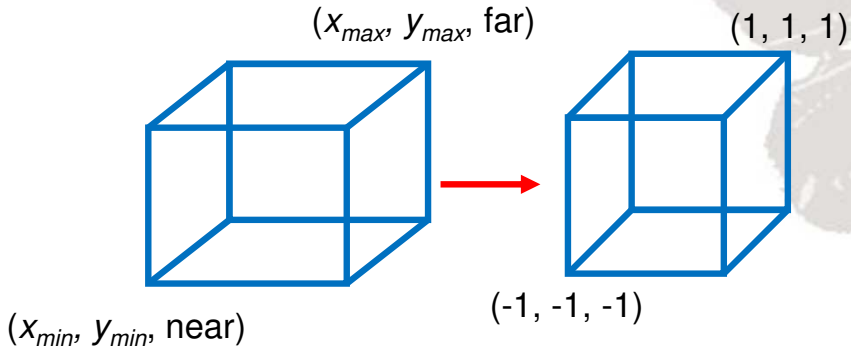
$$\begin{cases} z_p = z_{near}, \text{if } z = z_{near} \\ z_p = z_{far}, \quad \text{if } z = z_{far} \end{cases}$$

$$\begin{cases} \dfrac{S_z \cdot z_{near} + T_z}{z_{near}} = z_{near} \\ \dfrac{S_z \cdot z_{far} + T_z}{z_{far}} = z_{far} \end{cases}$$

$$\begin{cases} S_z = z_{far} + z_{near} \\ T_z = -z_{far} \cdot z_{near} \end{cases}$$

Perspective Normalization

- From Cuboid to Cube
  - Similar to orthogonal normalization



$$M_{orthonorm} = S(\frac{2}{x_{\max} - x_{\min}}, \frac{2}{y_{\max} - y_{\min}}, \frac{2}{z_{far} - z_{near}})$$

$$\cdot T(-\frac{x_{\max} + x_{\min}}{2}, -\frac{y_{\max} + y_{\min}}{2}, -\frac{z_{far} + z_{near}}{2})$$

Perspective Normalization

From cuboid to cube

From frustum to cuboid

$$M_{persnorm} = \begin{bmatrix} \frac{2}{x_{\max}-x_{\min}} & 0 & 0 & -\frac{x_{\max}+x_{\min}}{x_{\max}-x_{\min}} \\ 0 & \frac{2}{y_{\max}-y_{\min}} & 0 & -\frac{y_{\max}+y_{\min}}{y_{\max}-y_{\min}} \\ 0 & 0 & \frac{2}{z_{far}-z_{near}} & -\frac{z_{far}+z_{near}}{z_{far}-z_{near}} \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} z_{near} & 0 & 0 & 0 \\ 0 & z_{near} & 0 & 0 \\ 0 & 0 & z_{far}+z_{near} & -z_{far}z_{near} \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{2\cdot z_{near}}{x_{\max}-x_{\min}} & 0 & -\frac{x_{\max}+x_{\min}}{x_{\max}-x_{\min}} & 0 \\ 0 & \frac{2\cdot z_{near}}{y_{\max}-y_{\min}} & -\frac{y_{\max}+y_{\min}}{y_{\max}-y_{\min}} & 0 \\ 0 & 0 & \frac{z_{far}+z_{near}}{z_{far}-z_{near}} & \frac{-2z_{far}z_{near}}{z_{far}-z_{near}} \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Final perspective normalization matrix

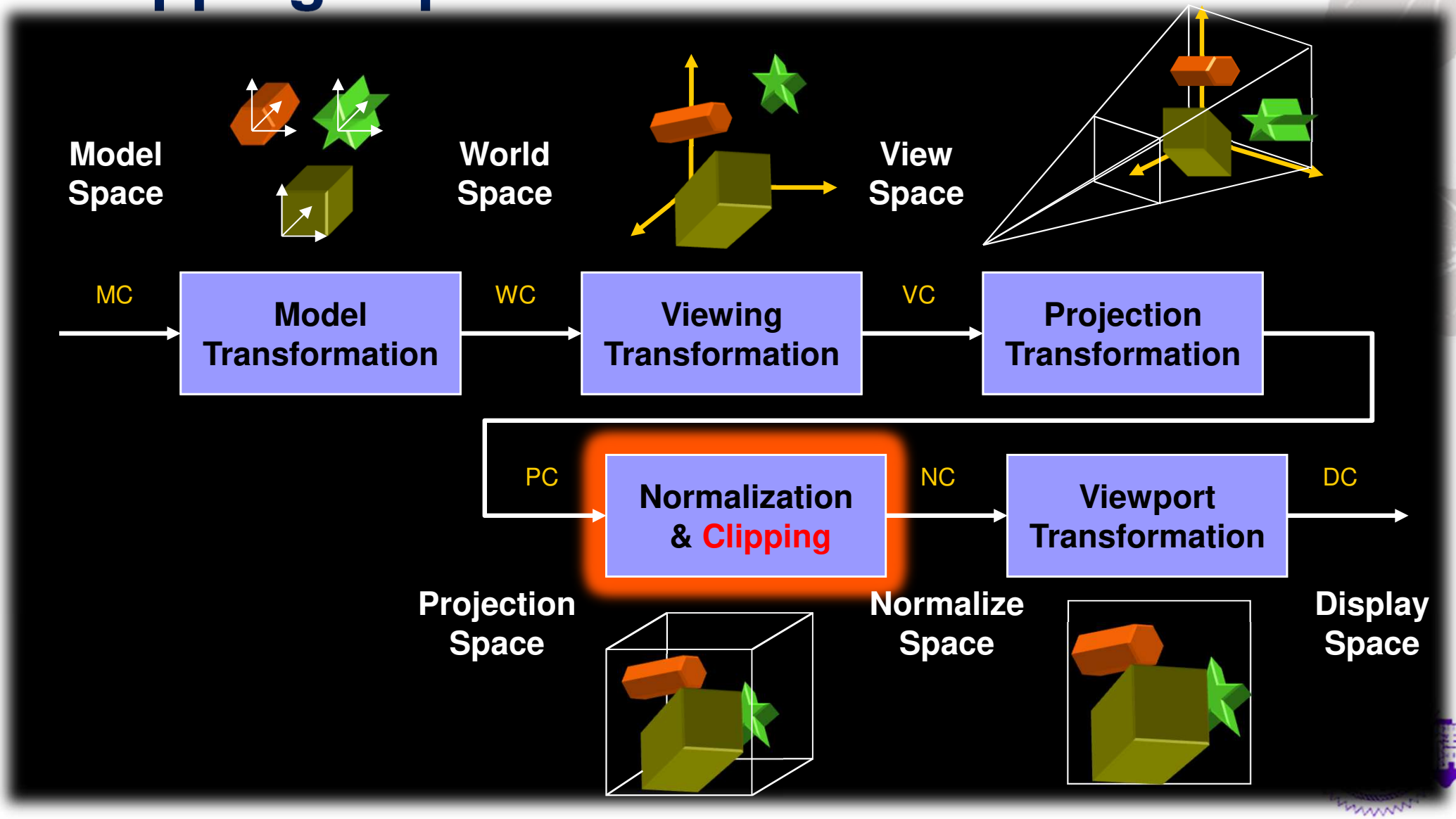
Clipping

2D Clipping

View Volume Clipping

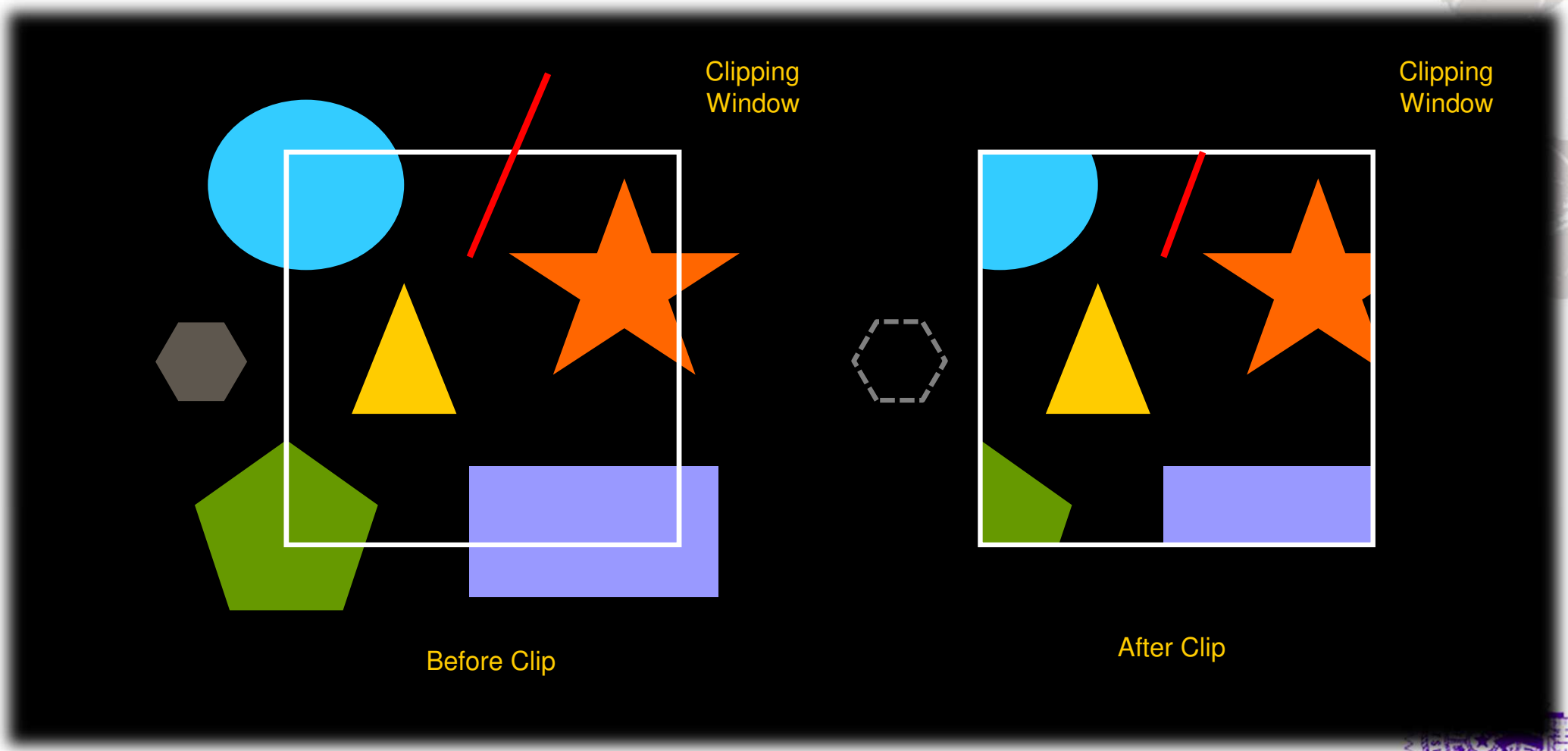
Review of 3D Viewing Process

- Clipping is performed after normalization



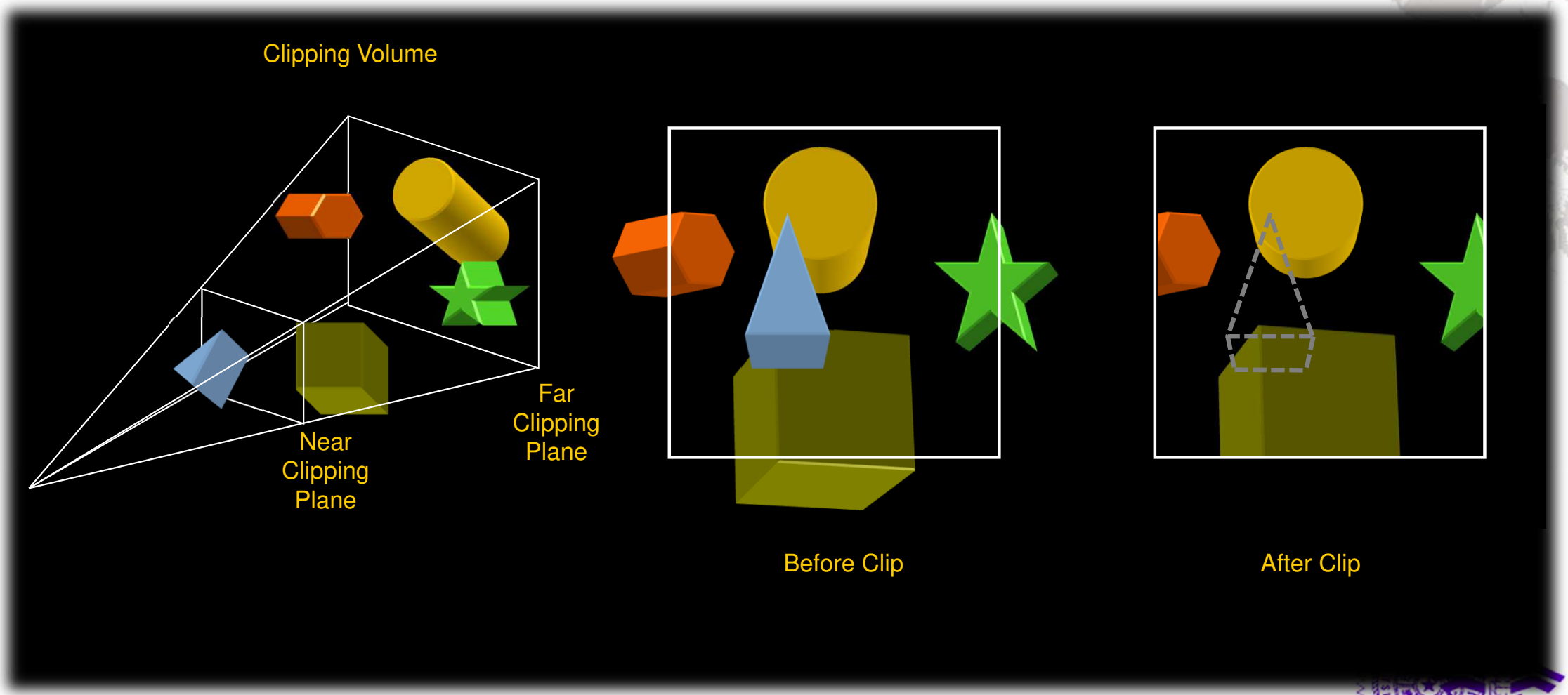
2D Clipping

- Clipping against 2D clipping window



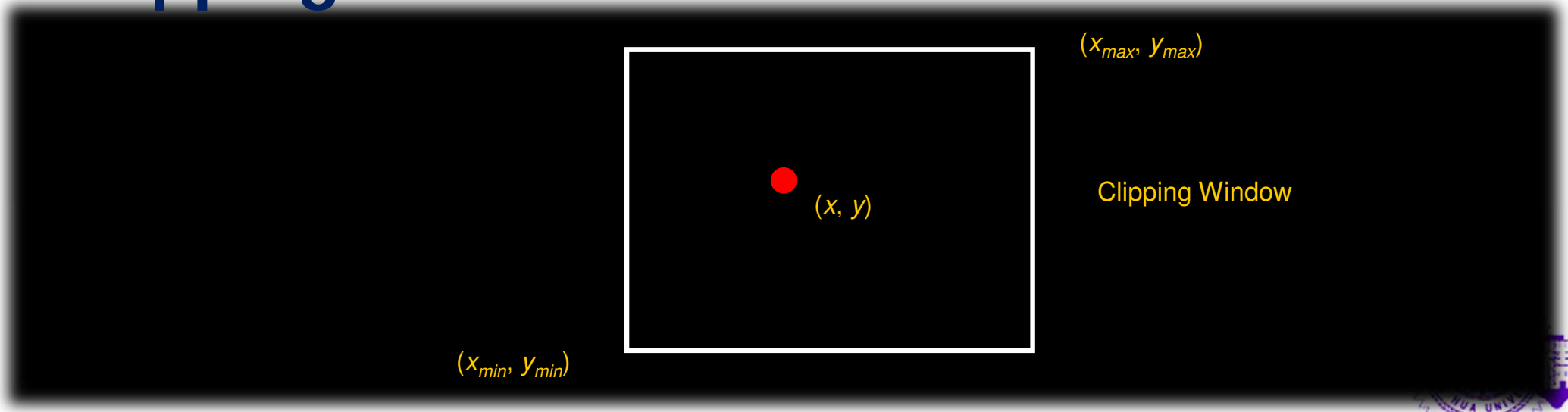
3D Clipping

- Clipping against 3D clipping volume



2D Clipping

- Define a clipping window with (x_{min}, y_{min}) and (x_{max}, y_{max}) as the lower-left and top-right coordinates, respectively.
- $x_{min} \leq x \leq x_{max}$ and $y_{min} \leq y \leq y_{max}$ must be satisfied for a point (x, y) to be inside the clipping window

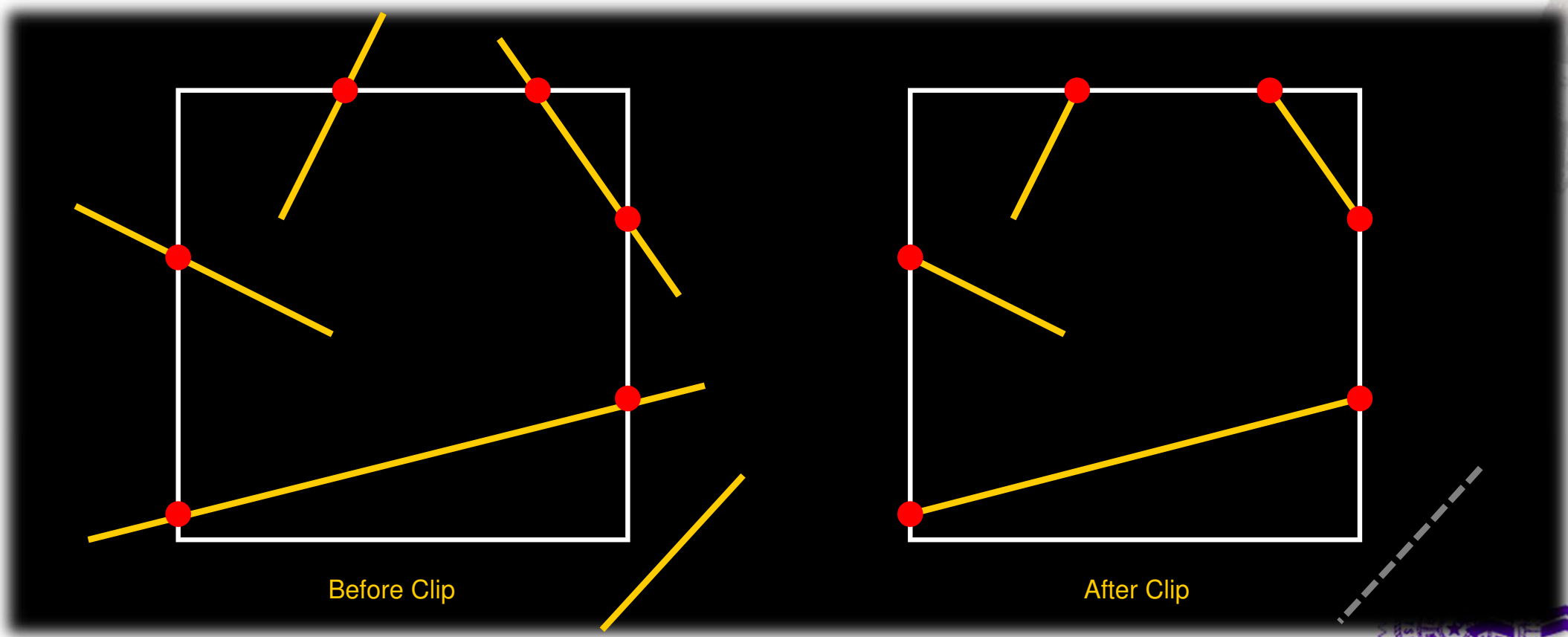


2D Clipping

- Clipping line segments
- Clipping polygons
- Other primitives can be converted into line segments or polygons before clipping

Clipping 2D Line Segments

- Compute intersections with each side of clipping window



Clipping 2D Line Segments

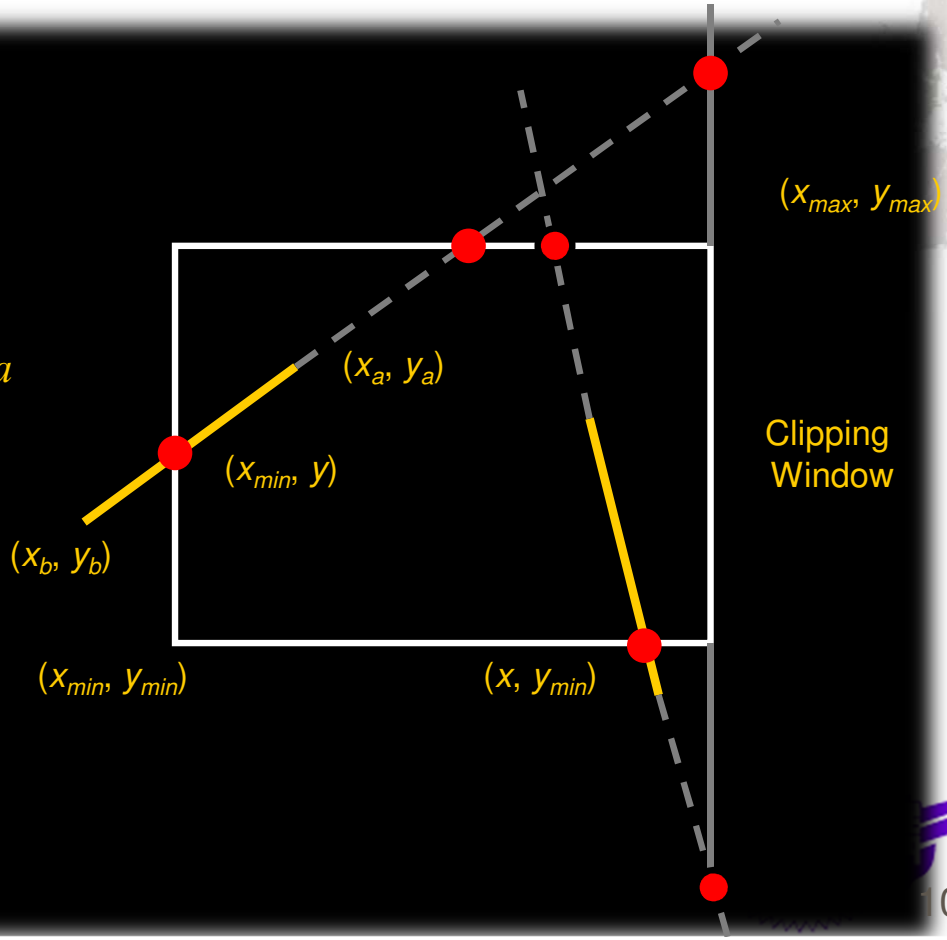
- For each line segment, $y = mx+b$, compute the intersection with each side of clipping window

$$\begin{cases} y = mx+b \\ x = x_{\min} \end{cases} \quad y_{\min} \le y \le y_{\max}$$

$$\begin{cases} y = mx+b \\ x = x_{\max} \end{cases} \quad y_b \le y \le y_a \text{ if } y_b \le y_a$$

$$\begin{cases} y = mx+b \\ y = y_{\min} \end{cases} \quad x_{\min} \le x \le x_{\max}$$

$$\begin{cases} y = mx+b \\ y = y_{\max} \end{cases} \quad x_b \le x \le x_a \text{ if } x_b \le x_a$$



Clipping 2D Line Segments

- **Compute intersection point using parametric equations**
  - **Example for intersection with $y = y_{min}$**

$$\begin{cases} x = x_a + t_0(x_b - x_a) \\ y = y_a + t_0(y_b - y_a) \end{cases} \qquad \begin{cases} x = x_{\min} + t_1(x_{\max} - x_{\min}) \\ y = y_{\min} \end{cases}$$

$$\begin{cases} x = x_a + t_0(x_b - x_a) = x_{\min} + t_1(x_{\max} - x_{\min}) \\ y = y_a + t_0(y_b - y_a) = y_{\min} \end{cases}$$

Solving t_0 and t_1, assume $t_0 = a$ and $t_1 = b$,

if $0 \le a \le 1$ and $0 \le b \le 1$, then

$x = x_{\min} + b(x_{\max} - x_{\min}), y = y_{\min}$ is the intersection point

Clipping 2D Line Segments

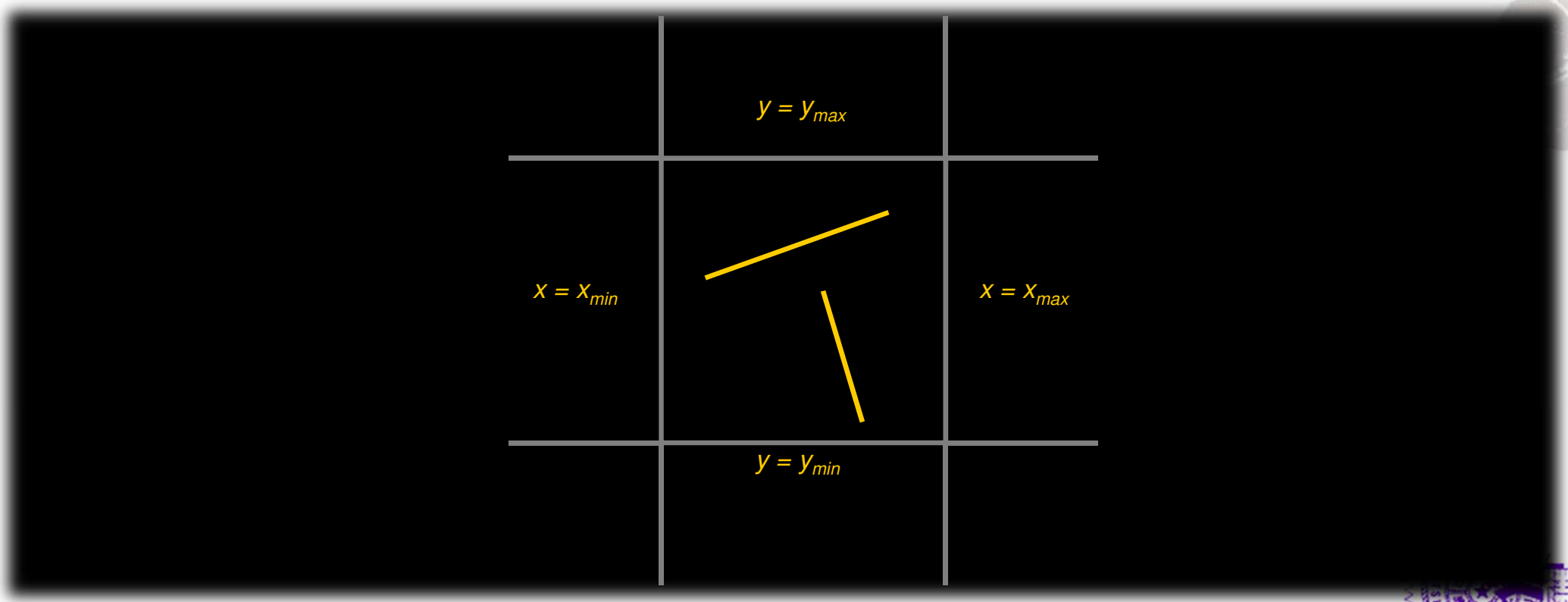
- Intersection derivation costs a lot of computations, such as division and multiplication.
- Reduce the number of intersection calculation can greatly speed up the clipping process.
- Do the intersection calculation only when it is necessary

Clipping 2D Line Segments

- Cohen-Sutherland line clipping algorithm
  - Step 1: check for trivially accepted
  - Step 2: Region check for trivially rejected
  - Step 3: Subdivide the line segment, repeat Step 1 and Step 2 until what remains is complete inside the clip rectangle or can be trivially rejected

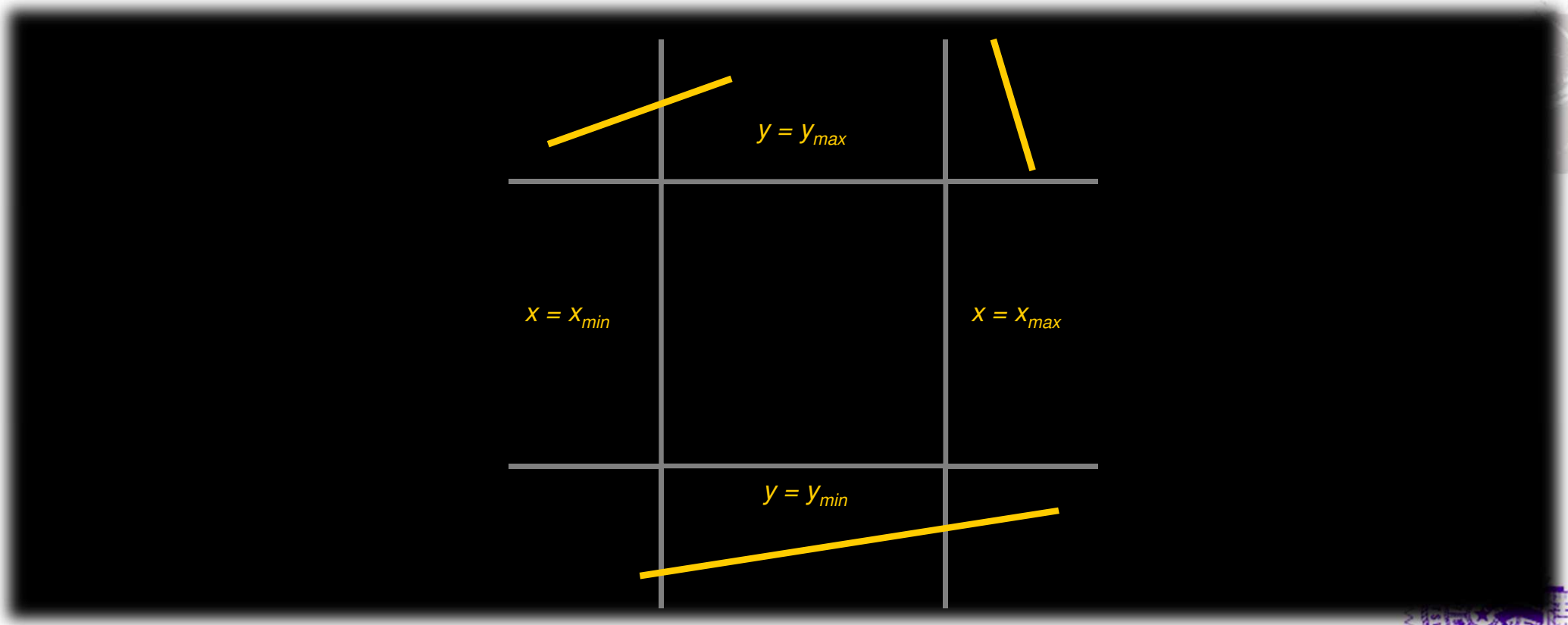
Cohen-Sutherland Algorithm

- Case for trivially accepted
  - Both endpoints of a line segments lie inside the clipping rectangle



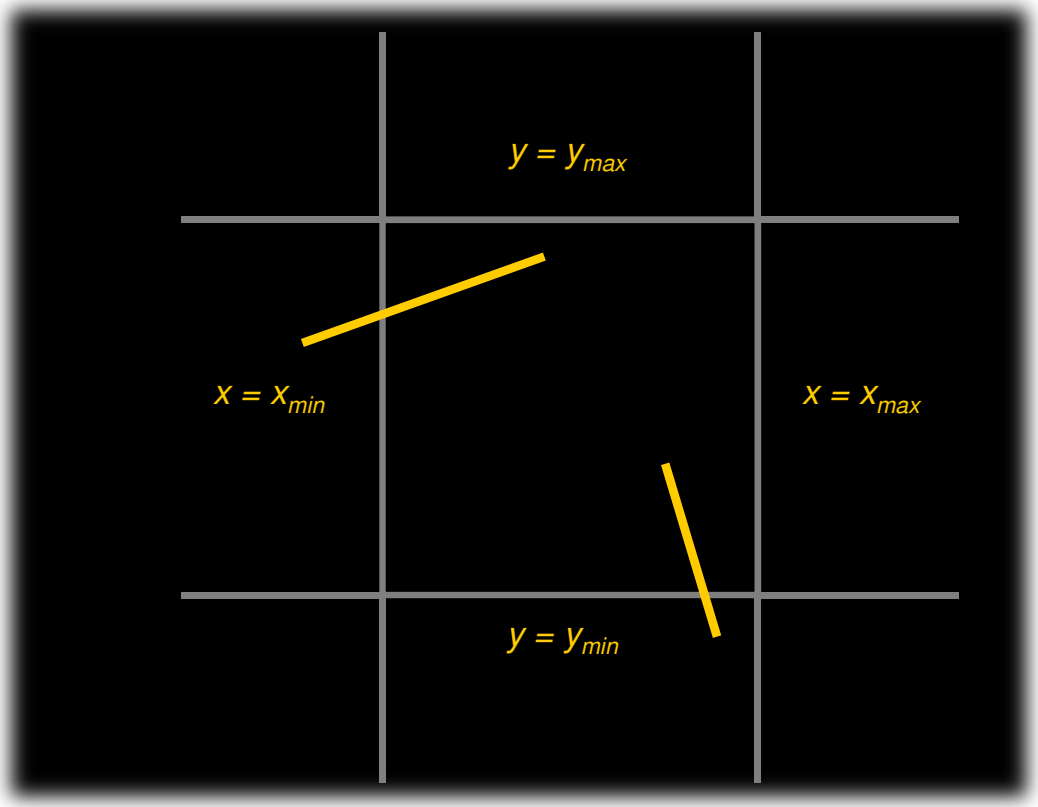
Cohen-Sutherland Algorithm

- Case for trivially rejected
  - Both endpoints of a line segments lie outside one of the clipping boundary



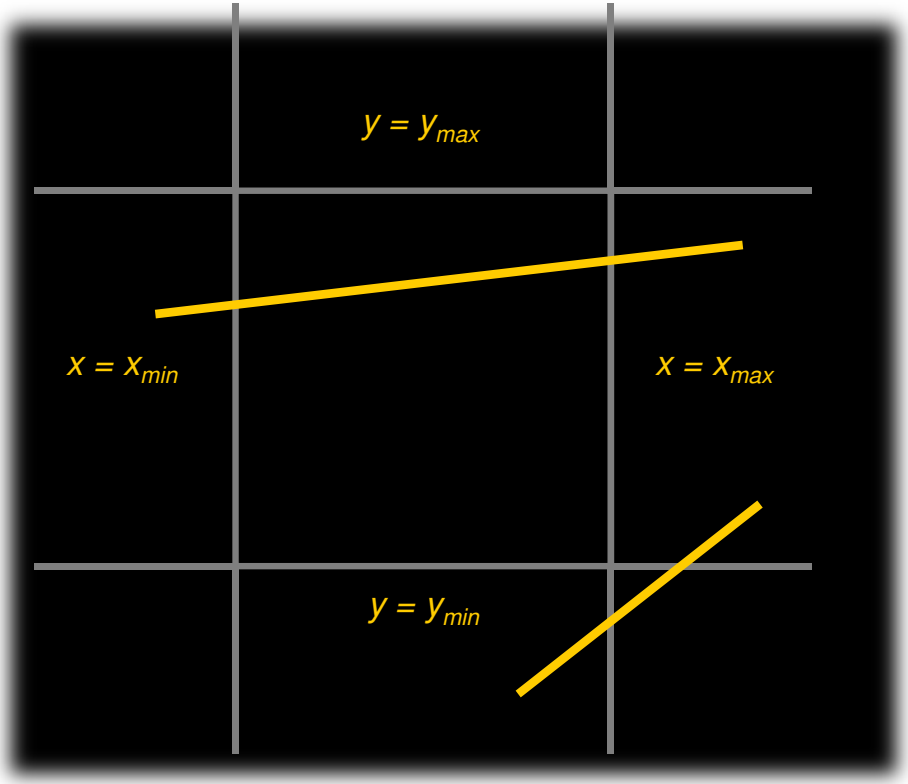
Cohen-Sutherland Algorithm

- Cases that require intersection calculation
  - One endpoint of a line segments lies inside the clipping rectangle and the other endpoint is outside



Cohen-Sutherland Algorithm

- Cases that require intersection calculation
  - Both endpoints of a line segments lie outside the clipping rectangle and the endpoints are not on the same side of a clipping boundary



Cohen-Sutherland Algorithm

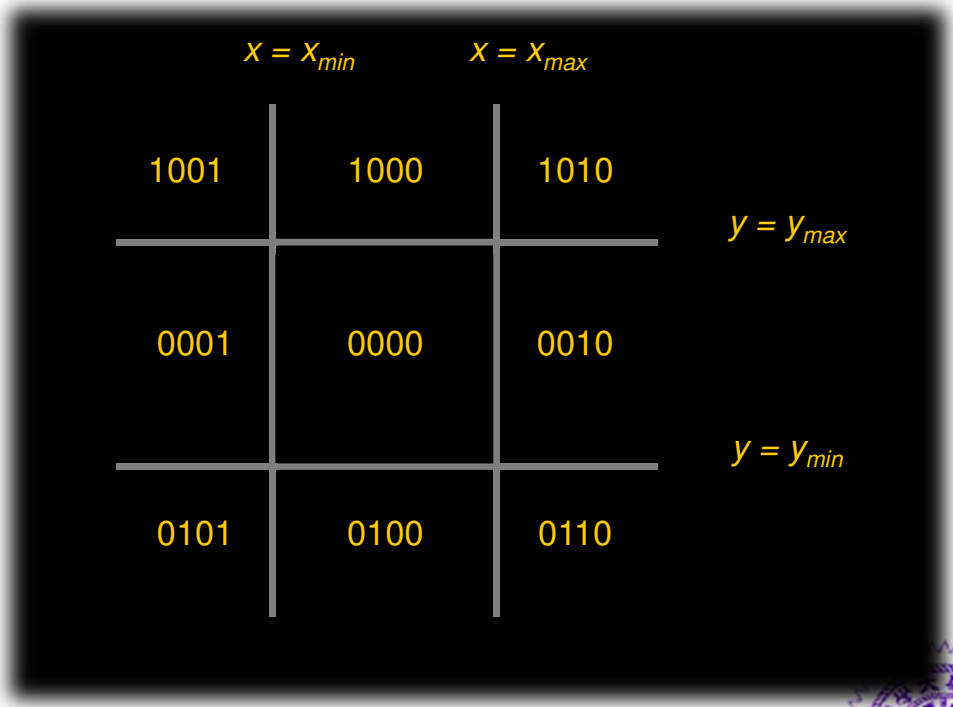
- **Outcodes**
  - For each endpoint (x, y) of a line segment, an outcode (b_0, b_1, b_2, b_3) is defined as

$b_0 = 1$, if $y > y_{max}$

$b_1 = 1$, if $y < y_{min}$

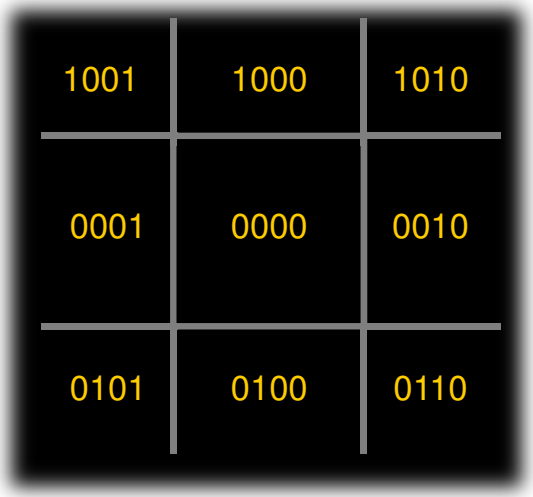
$b_2 = 1$, if $x > x_{max}$

$b_3 = 1$, if $x < x_{min}$



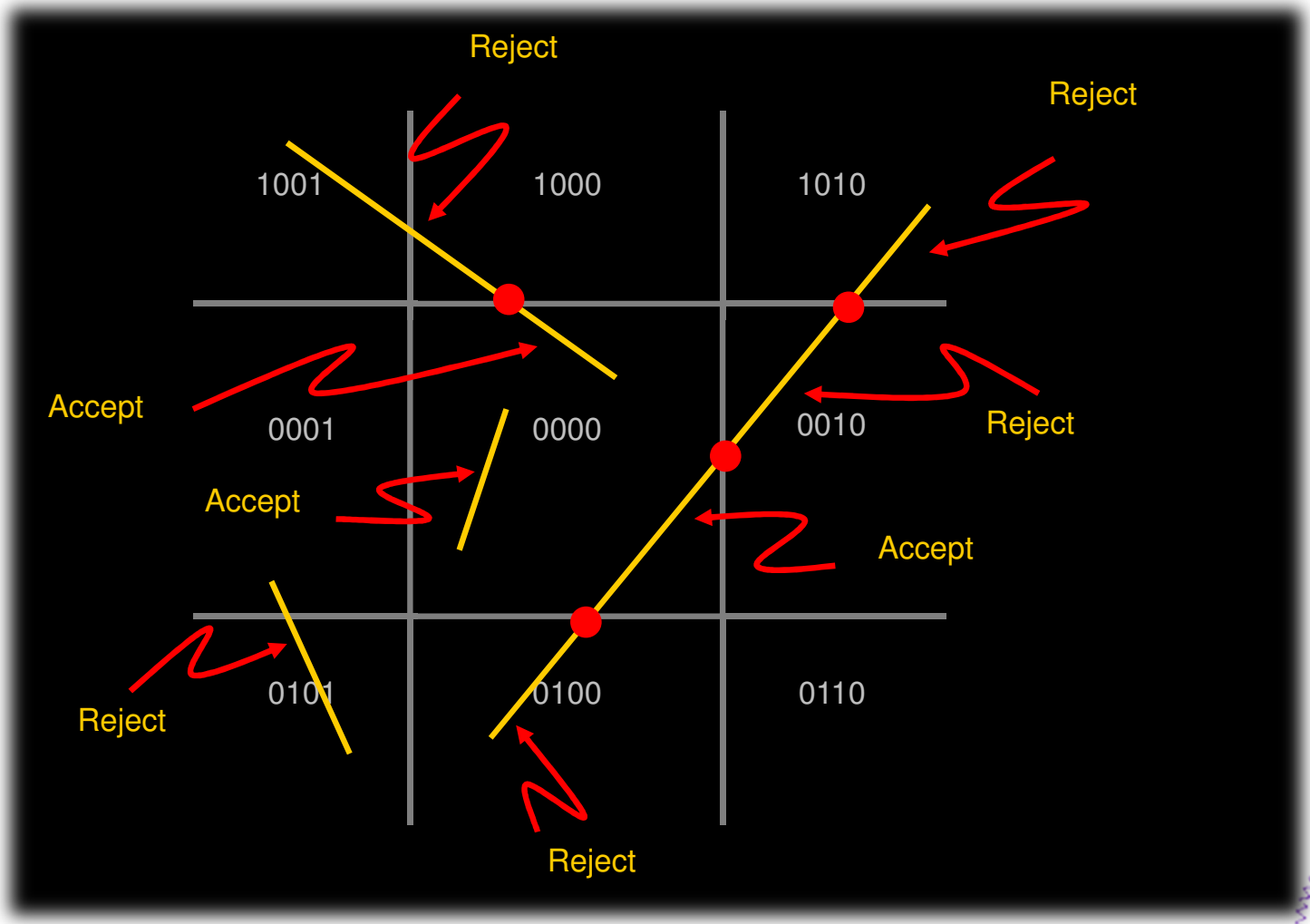
Cohen-Sutherland Algorithm

- Assume $(b_0, b_1, b_2, b_3)_A$ and $(b_0, b_1, b_2, b_3)_B$ are the outcodes of endpoints (A, B) of a line segment, respectively
  - If $((b_0, b_1, b_2, b_3)_A)$ & $((b_0, b_1, b_2, b_3)_B) \neq 0000$ implies trivially rejected
  - if $(b_0, b_1, b_2, b_3)_A = (b_0, b_1, b_2, b_3)_B = 0000$ implies trivially accepted



Cohen-Sutherland Algorithm

- **Examples**

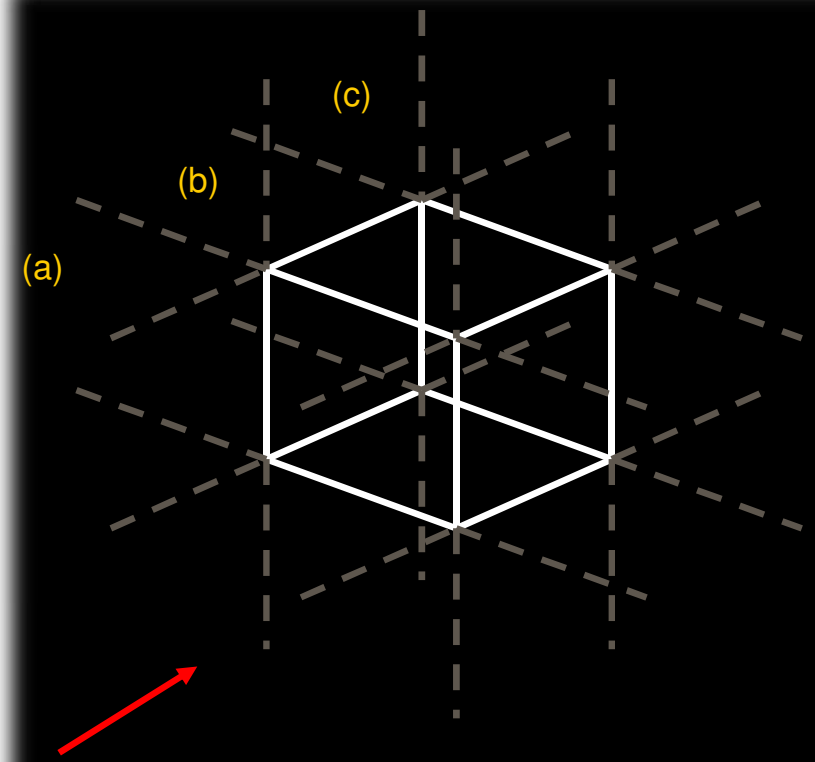


Cohen-Sutherland Algorithm

- Advantages
  - Most line segments can be easily accepted or rejected without any intersection calculation
  - Less intersection calculations are required in compare to brute force approach

Clipping in 3D

- Extension of Cohen-Sutherland algorithm
  - 6-bit outcode



101001	101000	101010
100001	100000	100010
100101	100100	100110

(a)

001001	001000	001010
000001	000000	000010
000101	000100	000110

(b)

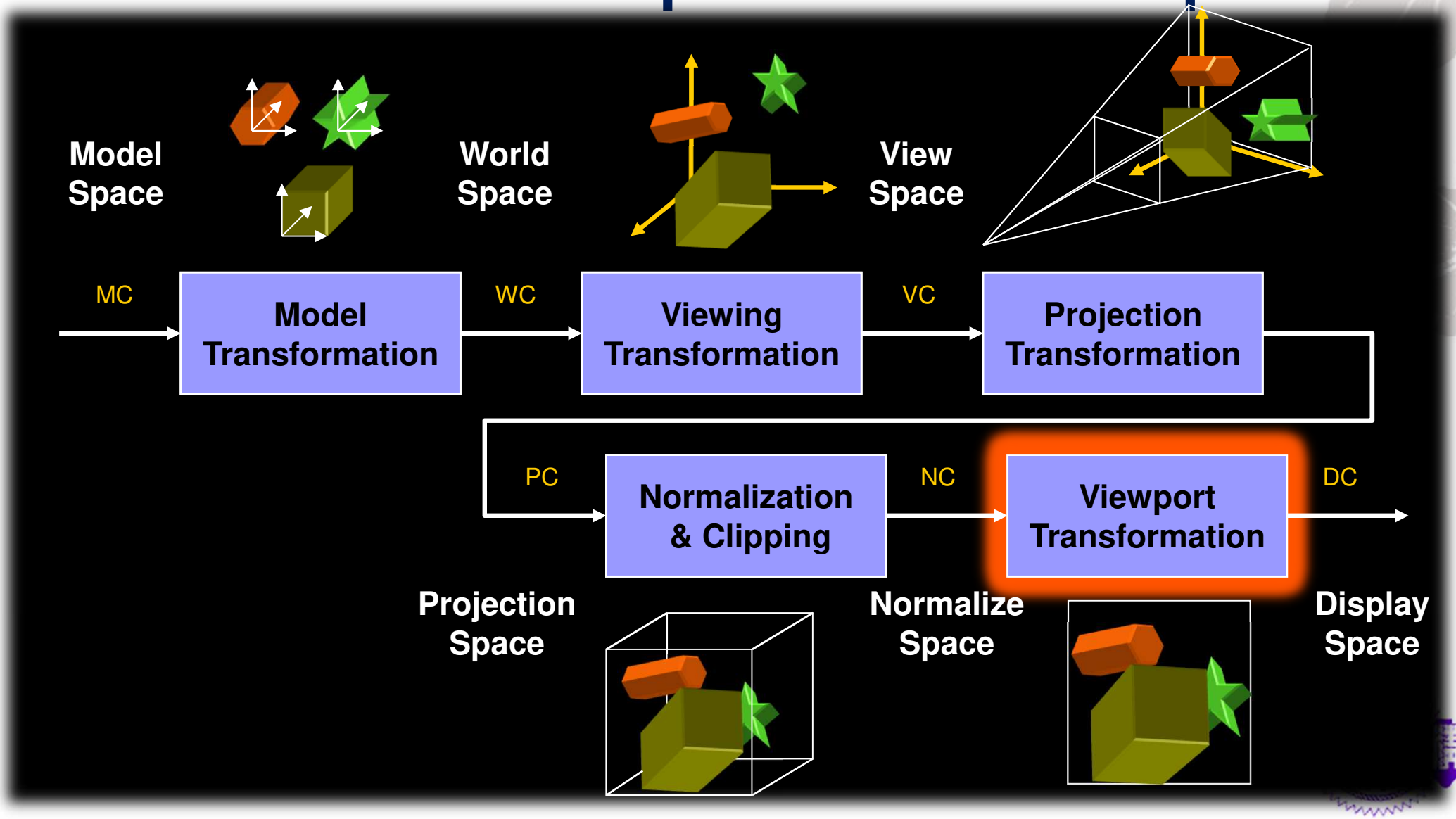
011001	011000	011010
010001	010000	010010
010101	010100	010110

(c)

Viewport Transformation

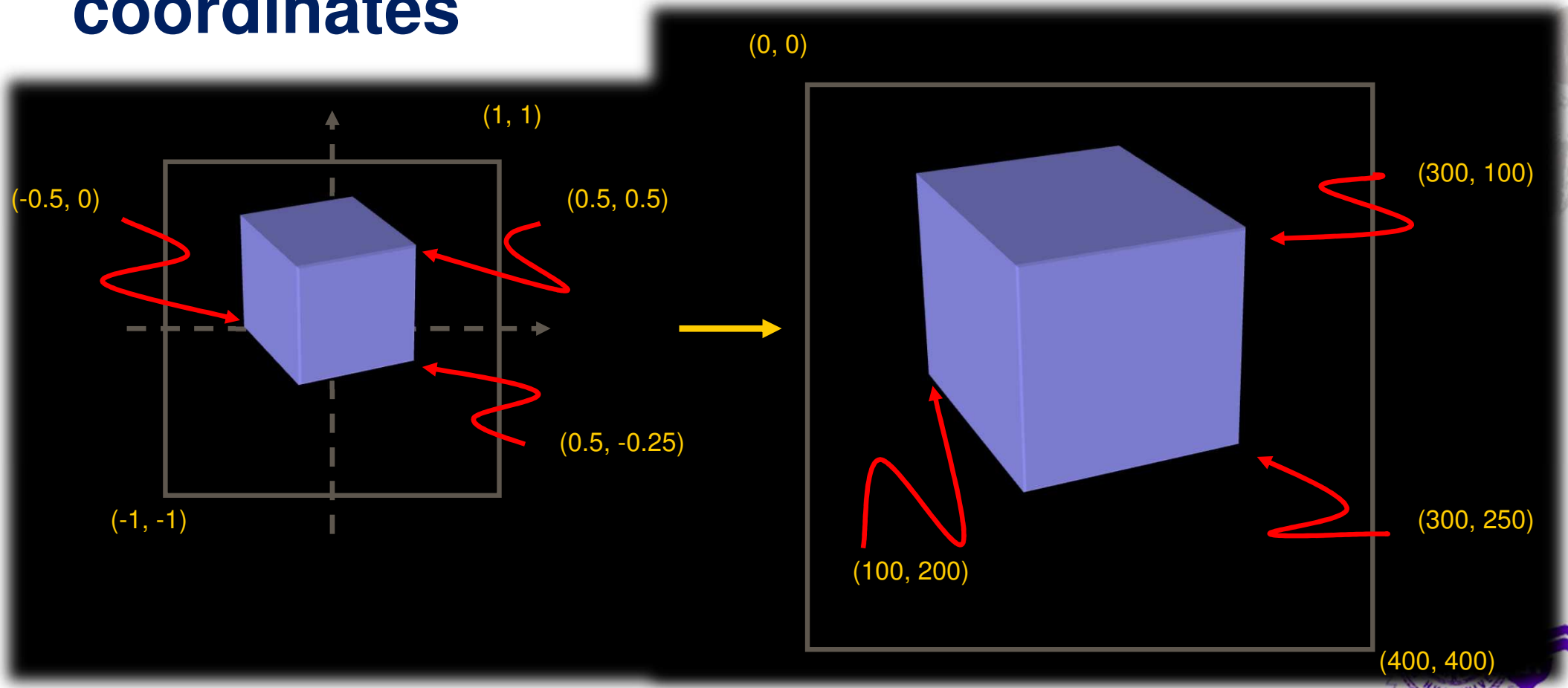
Review of 3D Viewing Process

- From Normalized Space to Screen Space



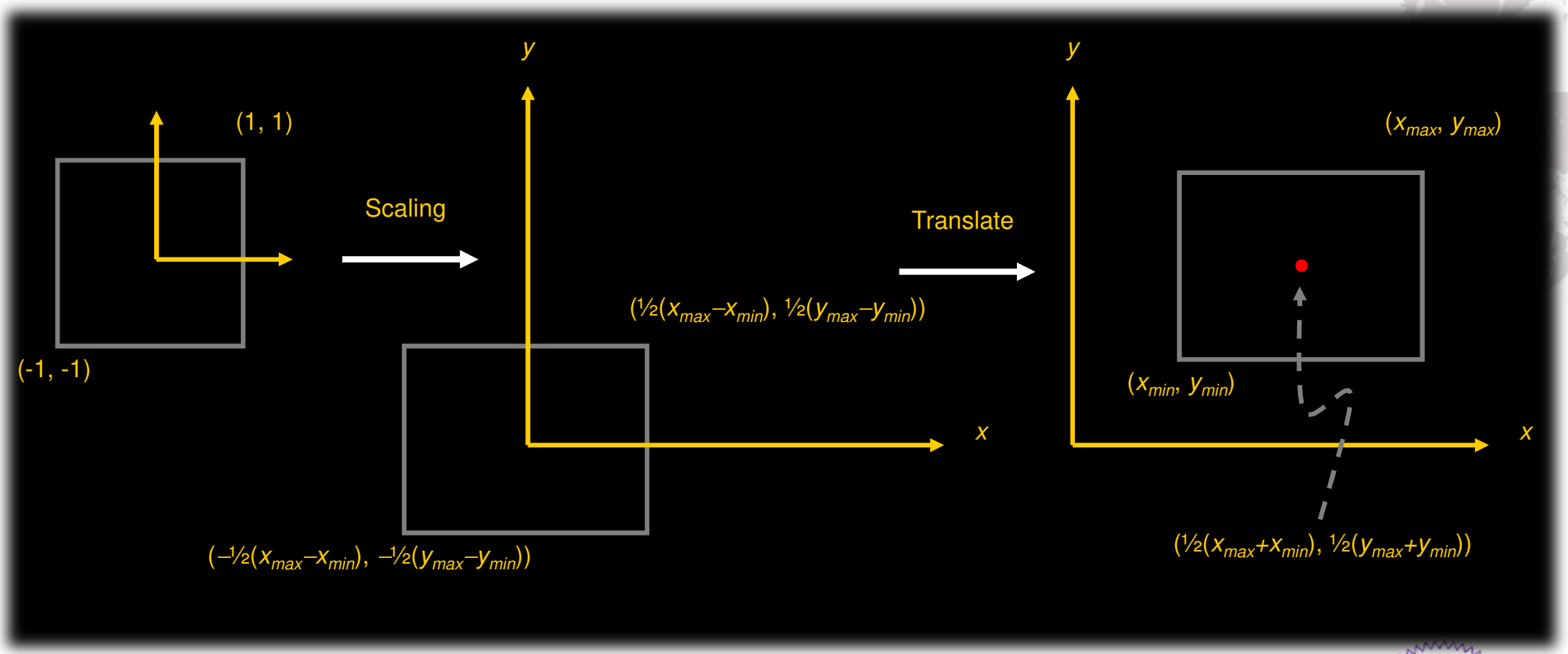
Viewport Transformation

- Transform from normalized clipping coordinates to 2D display device coordinates



Viewport Transformation

- Composition of viewport transformation



Viewport Transformation

- Matrix representation of viewport transformation

$$M_{viewport} = T\left(\frac{x_{max}+x_{min}}{2}, \frac{y_{max}+y_{min}}{2}, 0\right) \cdot S\left(\frac{x_{max}-x_{min}}{2}, \frac{y_{max}-y_{min}}{2}, 1\right)$$

$$= \begin{bmatrix} \frac{x_{max}-x_{min}}{2} & 0 & 0 & \frac{x_{max}+x_{min}}{2} \\ 0 & \frac{y_{max}-y_{min}}{2} & 0 & \frac{y_{max}+y_{min}}{2} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Viewport Transformation

- **Depth Range Adjustment**
  - **E.g., set the depth range within [0, 1]**

$$M_{viewport} = T(\frac{x_{\max} + x_{\min}}{2}, \frac{y_{\max} + y_{\min}}{2}, 0.5) \cdot S(\frac{x_{\max} - x_{\min}}{2}, \frac{y_{\max} - y_{\min}}{2}, \frac{1}{2})$$

Translate the range from [-0.5, 0.5] to [0, 1]

Scale the range from [-1, 1] to [-0.5, 0.5]

OpenGL Transformations

glOrtho

glFrustum

gluPerspective

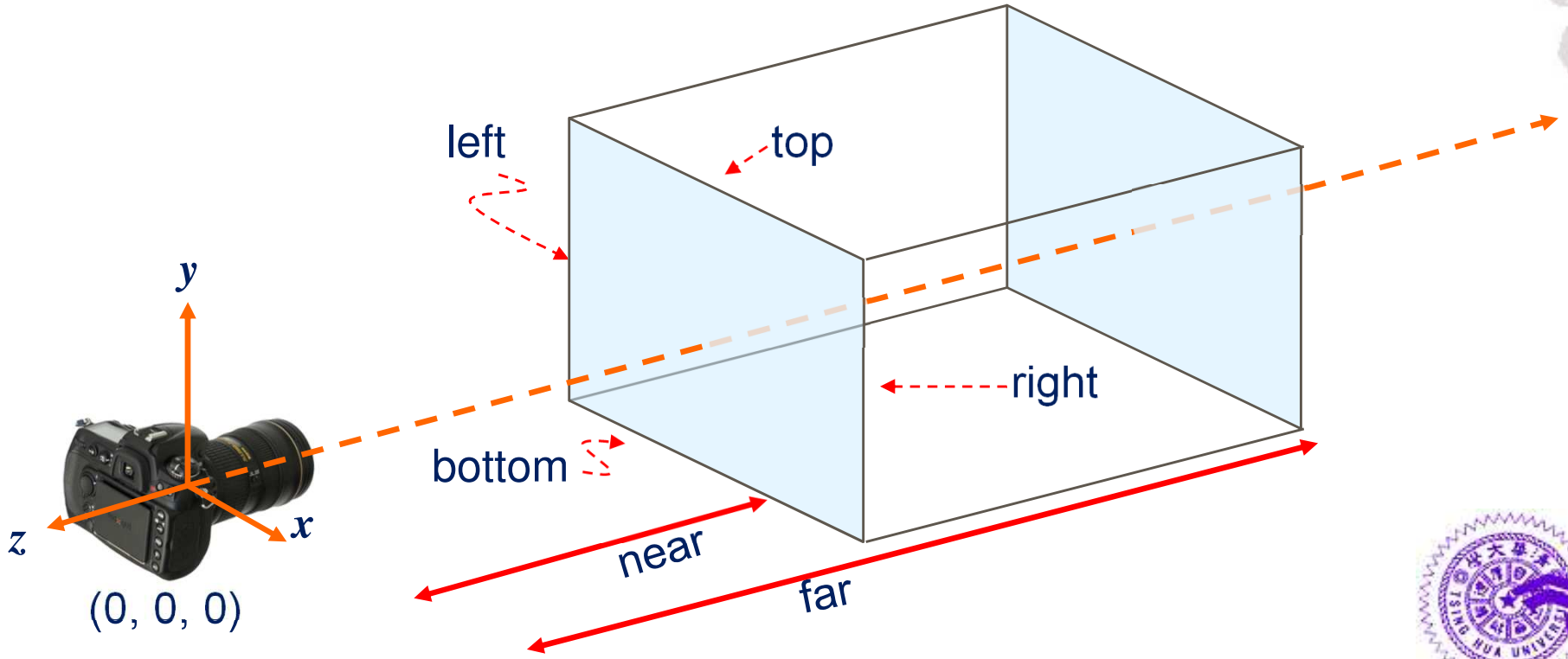
gluLookAt

OpenGL Conventions

- In order to derive the similar equations for OpenGL glOrtho, glFrustum, and gluPerspective, the following terms should be aware of
  - $x_{max} = Right; x_{min} = Left$
  - $y_{max} = Top; y_{min} = Bottom$
  - $z_{near} = -Near; z_{far} = -Far$
- It is because the definition of Near and Far in the APIs are the distances to the viewpoint instead of the Z coordinates

glOrtho

- **OpenGL Orthographic Projection**
  - **glOrtho(*left*, *right*, *bottom*, *top*, *near*, *far*)**
    - ***left*, *right*, *bottom*, *top*: coordinates**
    - ***near*, *far*: distances to viewpoint.**



glOrtho

- OpenGL Orthographic Transformation Matrix
  - Orthographic (parallel) projection and orthographic normalization

$$\begin{pmatrix} \frac{2}{Right-Left} & 0 & 0 & t_x \\ 0 & \frac{2}{Top-Bottom} & 0 & t_y \\ 0 & 0 & \frac{-2}{Far-Near} & t_z \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

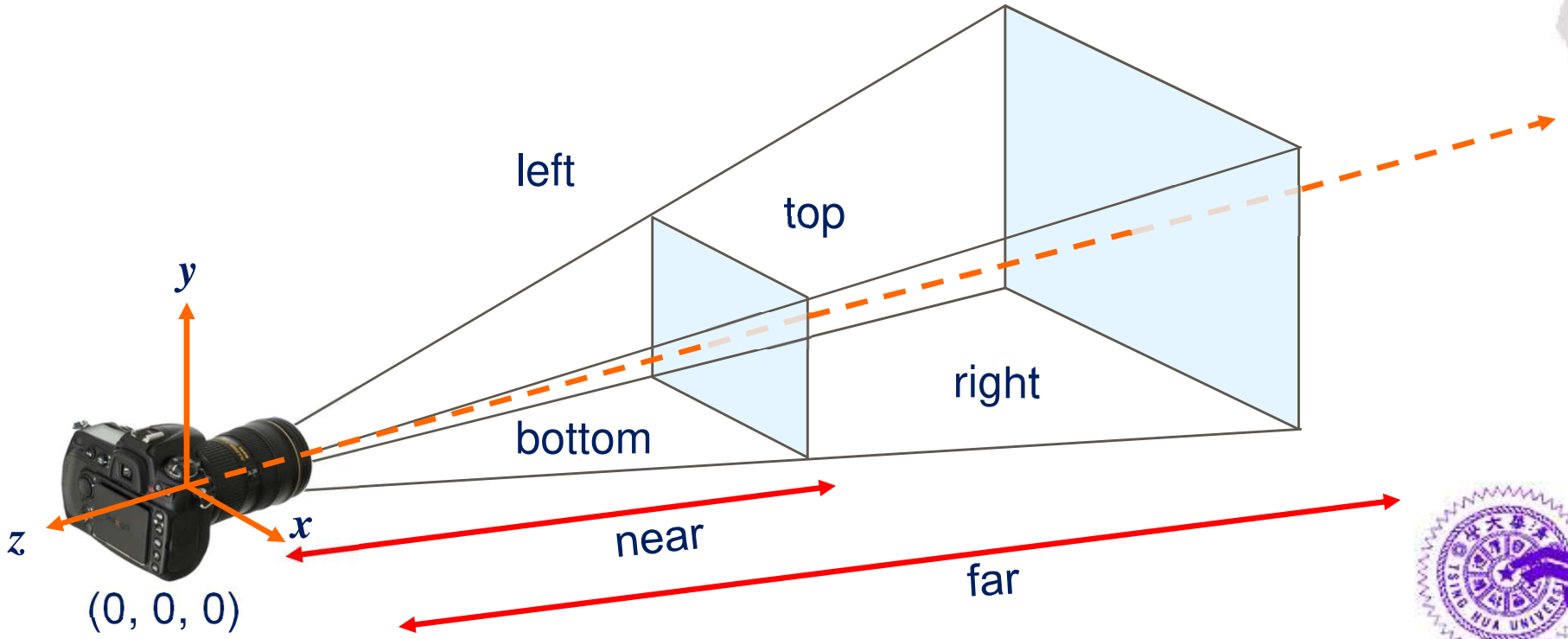
$$t_x = -\frac{Right+Left}{Right-Left}$$

$$t_y = -\frac{Top+Bottom}{Top-Bottom}$$

$$t_z = -\frac{Far+Near}{Far-Near}$$

glFrustum

- OpenGL Perspective Projection
  - glFrustum(*left*, *right*, *bottom*, *top*, *near*, *far*)
    - *left*, *right*, *bottom*, *top*: coordinates
    - *near*, *far*: distances to viewpoint



glFrustum

- OpenGL Perspective Transformation Matrix
  - Perspective projection and perspective normalization

$$\begin{pmatrix} \frac{2\,Near}{Right-Left} & 0 & A & 0 \\ 0 & \frac{2\,Near}{Top-Bottom} & B & 0 \\ 0 & 0 & C & D \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

$$A = \frac{Right + Left}{Right - Left}$$

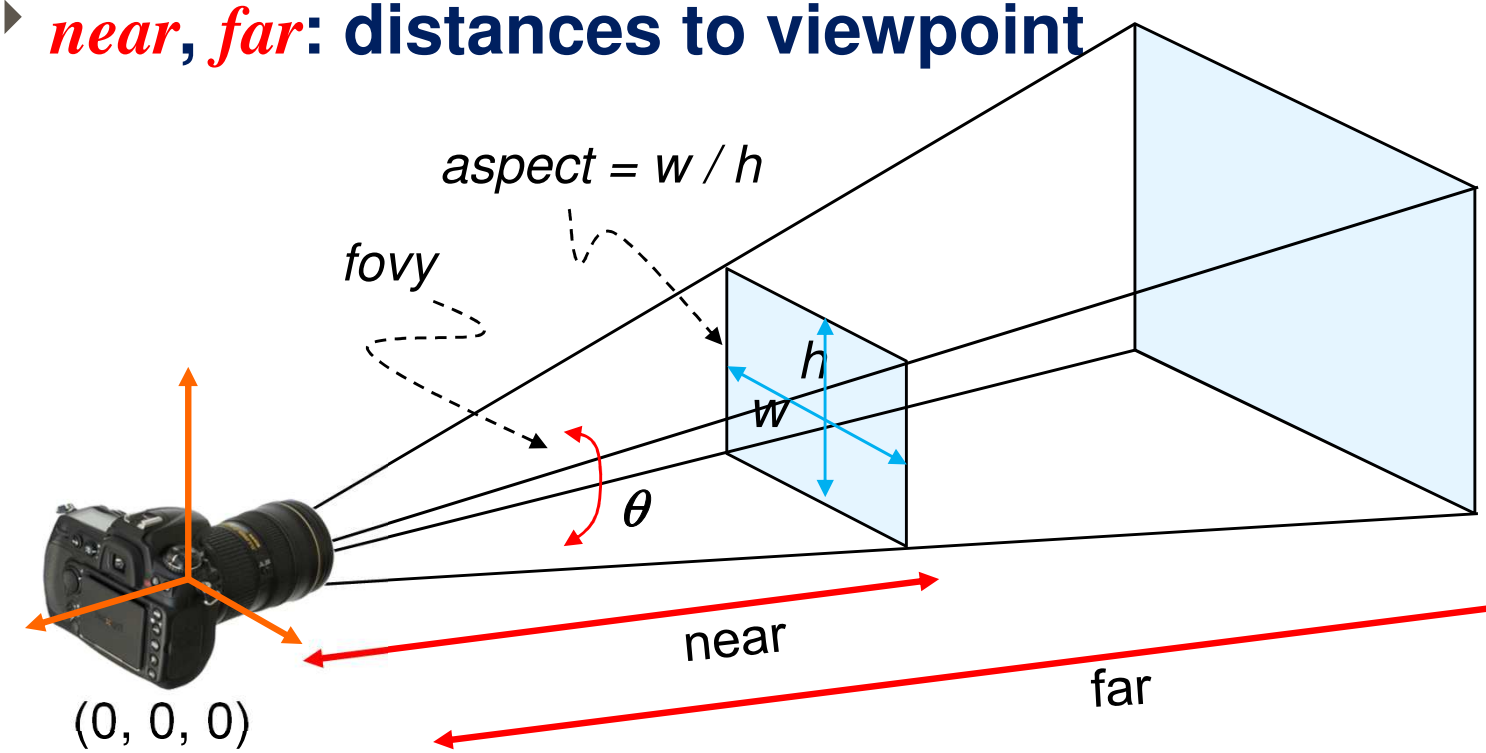
$$B = \frac{Top + Bottom}{Top - Bottom}$$

$$C = -\frac{Far + Near}{Far - Near}$$

$$D = -\frac{2Far\,Near}{Far - Near}$$

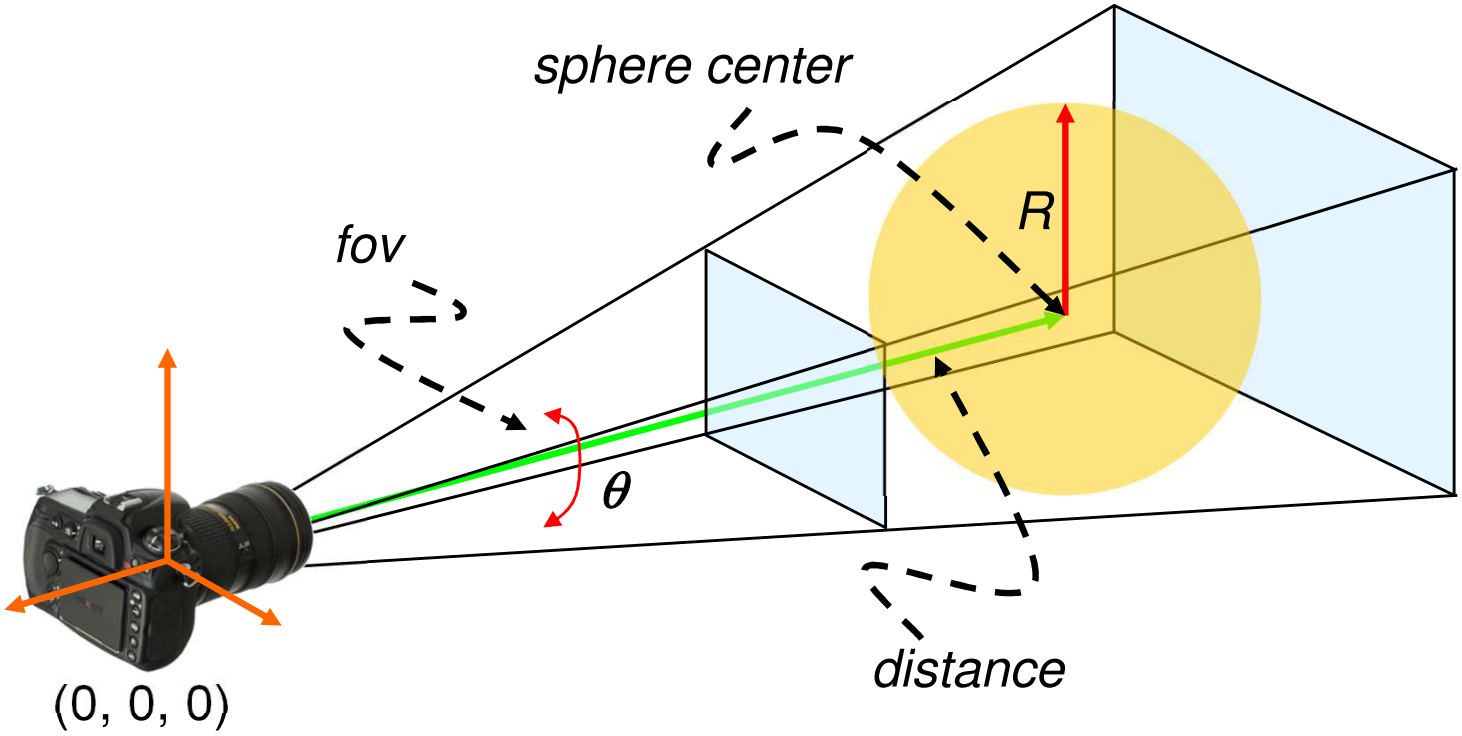
gluPerspective

- OpenGL Perspective Projection
  - gluPerspective(*fovy*, *aspect*, *near*, *far*)
    - *fovy*: field of view angle
    - *aspect*: aspect ratio of width to height
    - *near*, *far*: distances to viewpoint



gluPerspective

- How to derive a good field of view angle
  - Determine the bounding sphere of the scene
  - $fov = 2 \times \tan^{-1}(R/distance)$
    - *R*: sphere radius; *distance*: camera to sphere center



gluPerspective

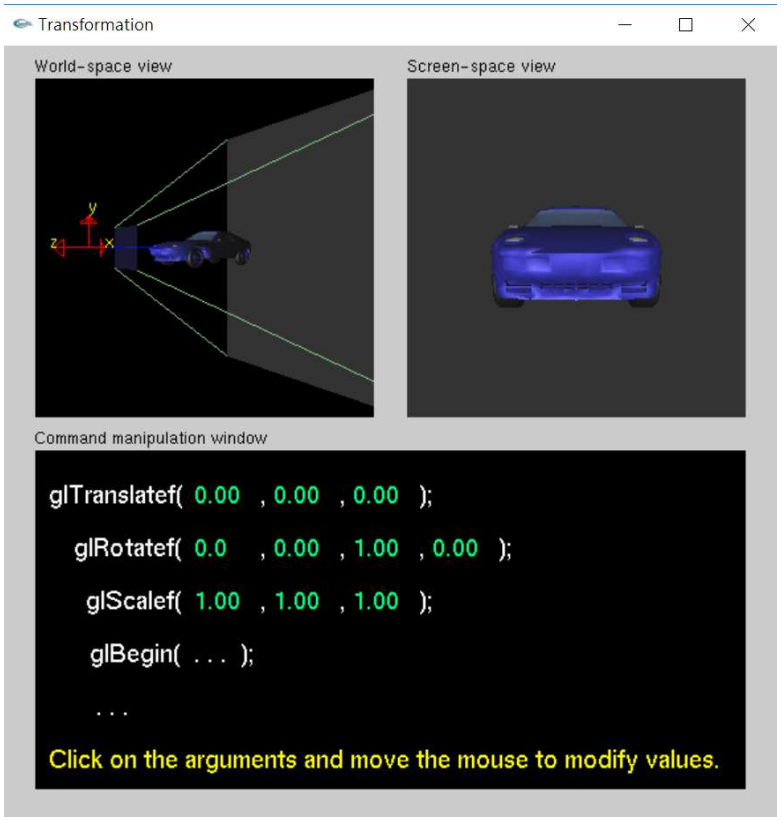
- **OpenGL Perspective Transformation Matrix**
  - **Perspective projection and perspective normalization**

$$\begin{pmatrix} \frac{f}{aspect} & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & \frac{far + near}{near - far} & \frac{2 \cdot far \cdot near}{near - far} \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

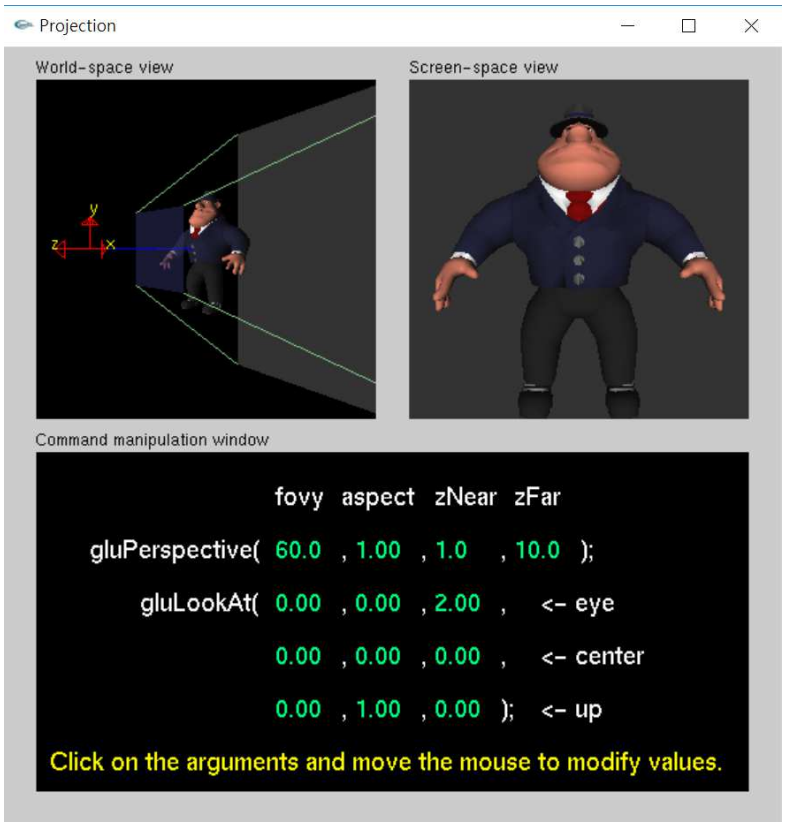
$$f = \cot\left(\frac{fovy}{2}\right)$$

OpenGL Transformation Demo

- **Transformation tutorial for OpenGL by Nate Robins (https://user.xmission.com/~nate/tutors.html)**



Geometrical Transformation



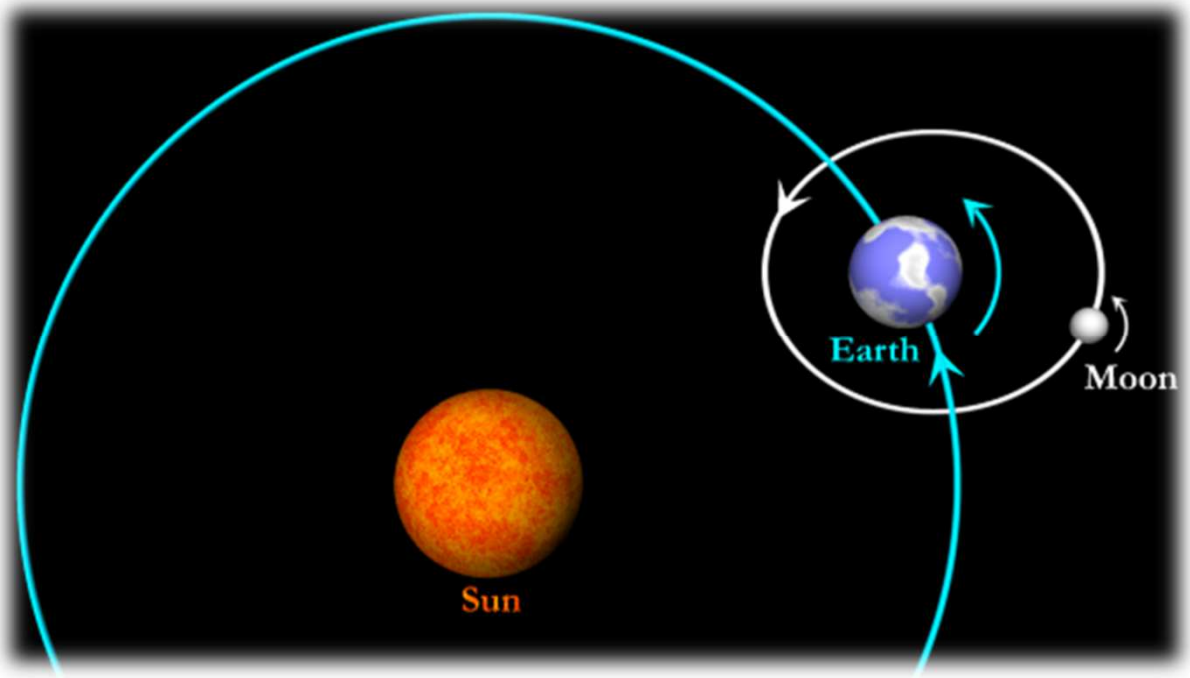
Viewing and Projection Transformation

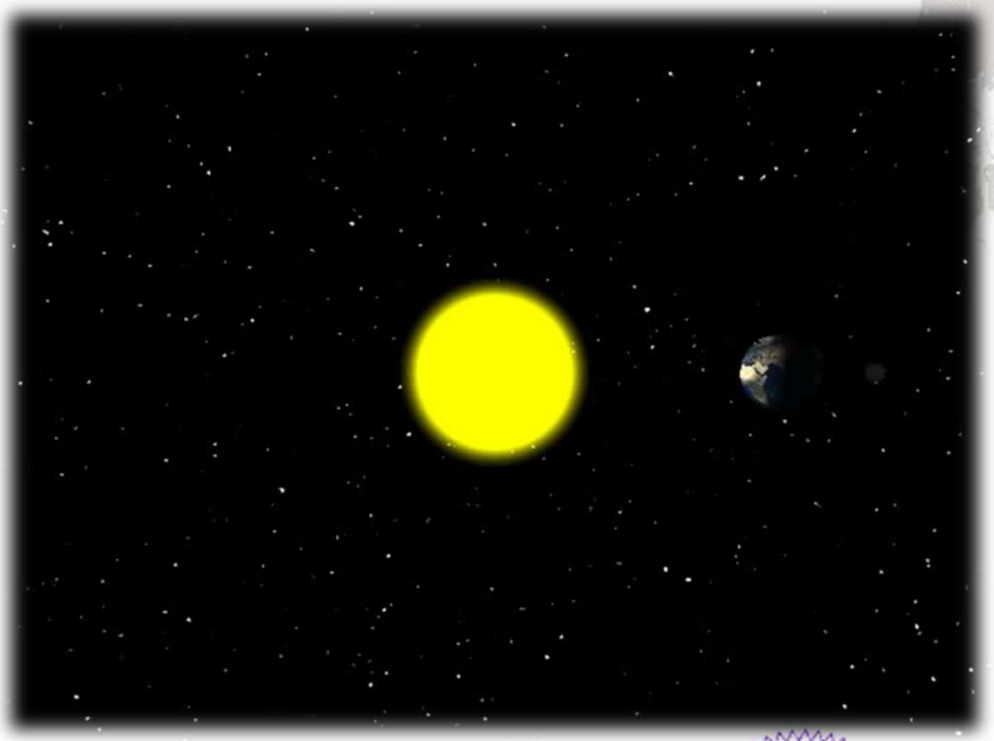
Managing Transformation Matrices

Matrix Stack

Why need to manage the matrices?

- **Consider a simplified solar system with sun, earth, and moon**



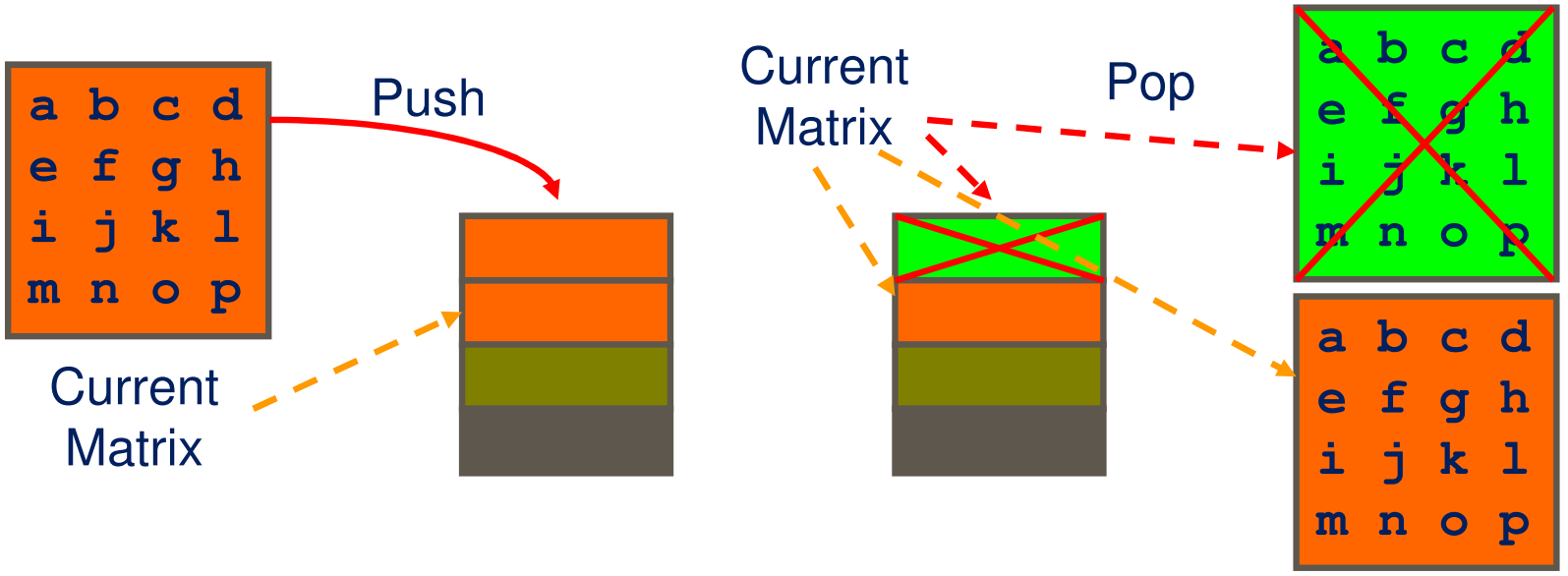


Matrix Stack

- Matrix stack is useful to remember some transformation matrix at certain point and restore it back when necessary
- OpenGL used to manage the matrix stack for user but is now deprecated
- You can implemented a similar mechanism by your own or use other way to manage your matrices

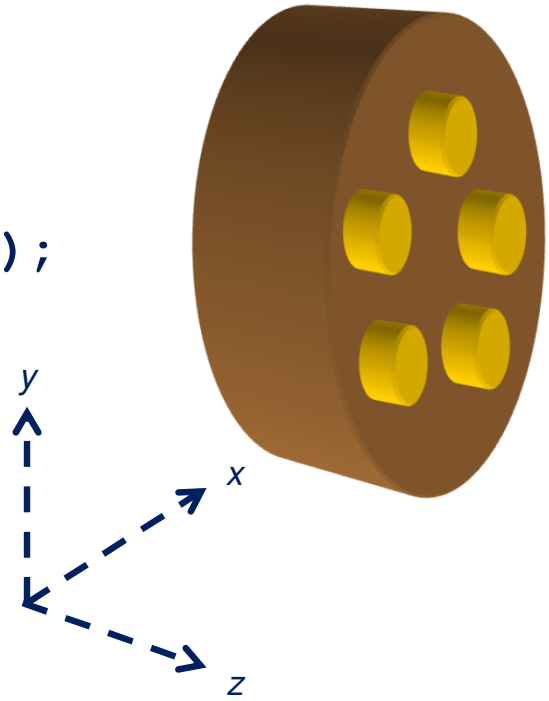
Matrix Stack in OpenGL

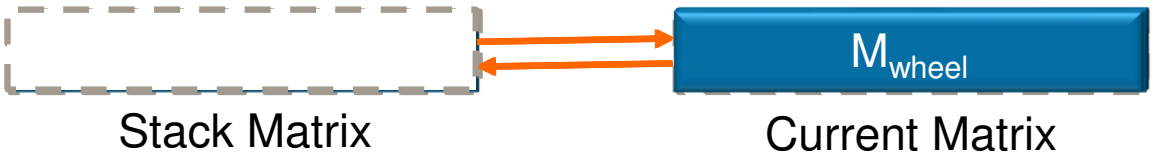
- Top of the matrix stack is the current matrix
- Any transformation is deal with the current matrix, i.e. the top of matrix stack
- Use *PushMatrix* or *PopMatrix* to push into or pop out of the matrix stack



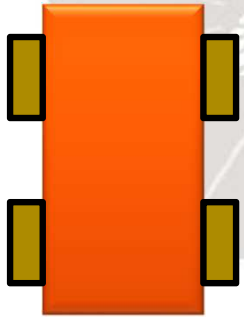
Example of Using Matrix Stack

```
draw_wheel_and_bolts()
{
    int i;
    draw_wheel();
    for(i=0;i<5;i++){
        glPushMatrix();
            glRotatef(72.0*i, 0.0, 0.0, 1.0);
            glTranslatef(3.0, 0.0, 0.0);
            draw_bolt();
        glPopMatrix();
    }
}
```

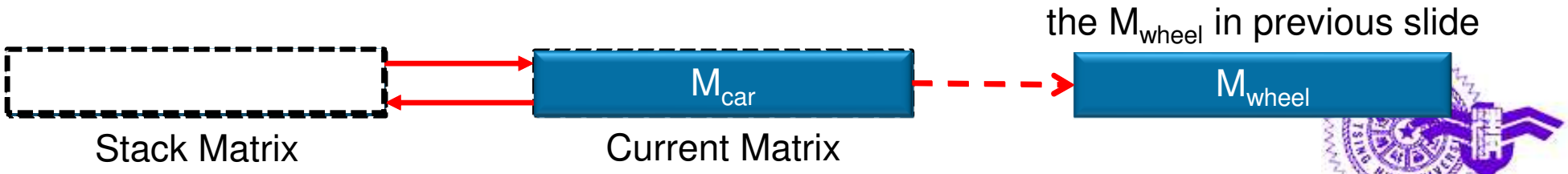




Example of Using Matrix Stack



```
draw_body_and_wheel_and_bolts()
{
    draw_car_body();
    glPushMatrix();
       glTranslatef(40, 0, 30);   /*move to first wheel position*/
       draw_wheel_and_bolts();
    glPopMatrix();
    glPushMatrix();
       glTranslatef(40, 0, -30);    /*move to 2nd wheel position*/
       draw_wheel_and_bolts();
    glPopMatrix();
    ...                          /*draw last two wheels similarly*/
}
```



Composition of 3D Transformations

- Example I: Building a Solar System

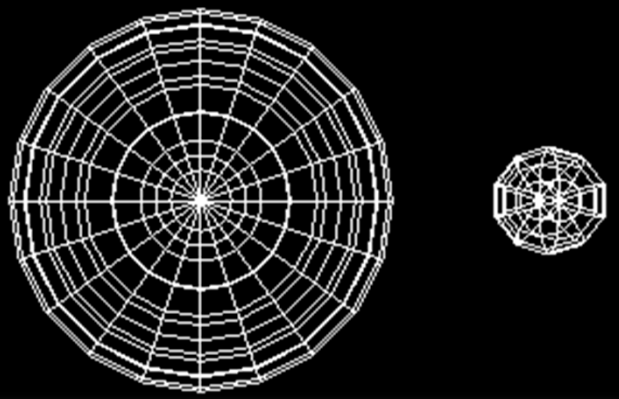
```
void display(void)
{
   glClear(GL_COLOR_BUFFER_BIT);
   glColor3f(1.0, 1.0, 1.0);
   glPushMatrix();
   glutWireSphere(1.0, 20, 16);    /* draw sun */
   glRotatef((GLfloat) year, 0.0, 1.0, 0.0);
   glTranslatef(2.0, 0.0, 0.0);
   glRotatef((GLfloat) day, 0.0, 1.0, 0.0);
   glutWireSphere(0.2, 10, 8);    /* draw smaller planet */
   glPopMatrix();
   glutSwapBuffers();
}
```

Store the sun position

Eg. 360°×(date/365)

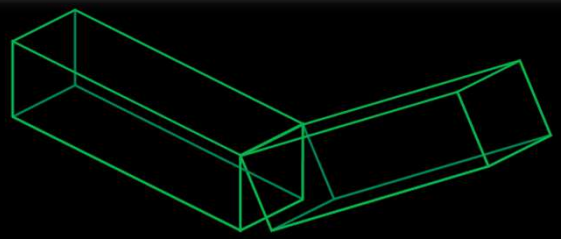
Eg. 360°×(hour/24)

Restore the sun position

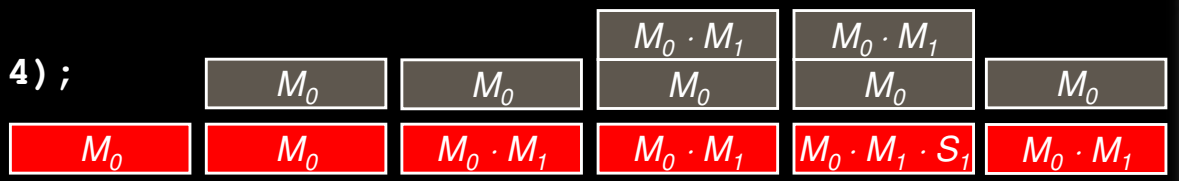


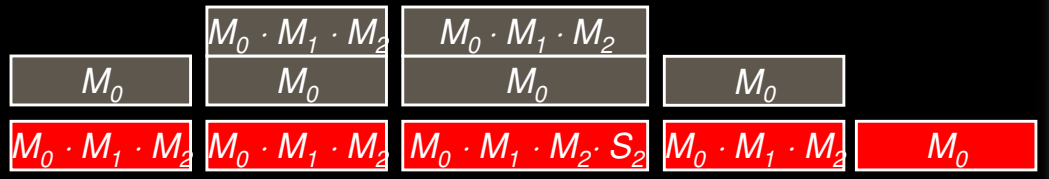
Composition of 3D Transformations

- Example II: Articulated Robot Arm



```
void display(void)
{
    glClear(GL_COLOR_BUFFER_BIT);
M0  glPushMatrix();
    glTranslatef(-1.0, 0.0, 0.0);
M1  glRotatef((GLfloat) shoulder, 0.0, 0.0, 1.0);
    glTranslatef(1.0, 0.0, 0.0);
M0·M1  glPushMatrix();
M0·M1·S1  glScalef(2.0, 0.4, 0.4);
    glutWireCube(1.0);
M0·M1  glPopMatrix();
    glTranslatef(1.0, 0.0, 0.0);
M2  glRotatef((GLfloat) elbow, 0.0, 0.0, 1.0);
    glTranslatef(1.0, 0.0, 0.0);
M0·M1·M2  glPushMatrix();
M0·M1·M2·S2  glScalef(2.0, 0.4, 0.4);
    glutWireCube(1.0);
M0·M1·M2  glPopMatrix();
M0  glPopMatrix();
    glutSwapBuffers();
}
```





Q&A