Brief notes on Astrophysical Discs

Jessica Lok (jhyl3@cam.ac.uk)

Part III Dynamics of Astrophysical Discs (DAD). Lectured by Henrik Latter in Lent 2025. This is a set of revision notes which covers most things. For additional useful resources supplementing lecture content, see Henrik's course page, Gordon's course page from 2020 and Gordon's slides from 2013 (here and here). What is covered in this document:

1. Orbital dynamics: Newton's laws of motion, particle dynamics, angular momentum invariants, epicycles
2. Viscous accretion discs: governing equations, viscous stress tensor, self-gravitation, thermodynamics, 1D diffusion equation
3. Local disc models: shearing sheet, incompressible dynamics: hydrodynamic instability, 2D vortex patches, magnetorotational instability; compressible dynamics: 3D hydrodynamic instability, gravitational instability; satellite-disc interactions
4. Select exam questions, as far back as 2013. Will maybe be updated at some point.

If there are mistakes (particularly in exam question notes) please let me know!

1 Orbital dynamics

- Newton's laws

$$\ddot{\mathbf{r}} = -\nabla\Phi \qquad \begin{cases} \ddot{r} - r\dot{\phi}^2 = -\partial_r\Phi & (1.1) \\ r\ddot{\phi} + 2\dot{r}\dot{\phi} = 0 & (1.2) \\ \ddot{z} = \partial_z\Phi & (1.3) \end{cases}$$

- $\dot{\mathbf{r}} = \dot{r}\,\mathbf{e}_r + r\dot{\phi}\,\mathbf{e}_\phi$, $|\dot{\mathbf{r}}|^2 = \dot{r}^2 + (r\dot{\phi})^2 = \dot{r}^2 + h^2/r^2$
- $\Omega = \dot{\phi}$, $\Phi_{\text{eff}} = \Phi + \frac{h^2}{2r^2}$
- $h = r^2\dot{\phi} = r^2\Omega$
- $h^2 = r^4\dot{\phi}^2 = r^3\, r\dot{\phi}^2 = r^4\Omega^2$
- Circular orbits, $\dot{r} = \ddot{r} = 0$. Then $r\dot{\phi}^2 = \partial_r\Phi$, $h^2 = r^3\, r\dot{\phi} = r^3\partial_r\Phi$
- Horizontal epicycle frequency

$$\kappa^2 = \partial^2_{rr}\Phi_{\text{eff}}(r,0) = \partial^2_{rr}\Phi(r,0) + \frac{3h^2}{r^4} = \partial^2_{rr}\Phi(r,0) + 3\Omega^2 = \frac{1}{r^3}\frac{dh^2}{dr} = 4\Omega^2 + 2r\Omega\frac{d\Omega}{dr} \qquad (1.4)$$

  It also follows that $\partial^2_{rr}\Phi(r,0) = \Omega^2 + 2r\Omega\frac{d\Omega}{dr}$

- Perturbed state, basic + [perturbation]: $\mathbf{r} = r\,\mathbf{e}_r + [\delta r\,\mathbf{e}_r + \delta z\,\mathbf{e}_z]$. $\ddot{\delta r} + \kappa^2\delta r = 0$, $\ddot{\delta z} + \Omega_z^2\delta z = 0$. Perturbations undergo oscillations if $\kappa^2 > 0$, $\Omega_z^2 > 0$.
- If $\kappa, \Omega_z = \Omega$ (Keplerian), then orbits are closed. Otherwise orbits are not closed, and the periapsis precesses at the apsidal precession rate,

$$\dot{\phi}_{\text{peri}} = \frac{\Delta\phi_{\text{peri}}}{\Delta t} = \Omega - \frac{2\pi}{\Delta t} = \Omega - \kappa, \qquad (1.5)$$

  where Δt is the time taken for the particle to return to periapsis. The nodal precession rate is $\Omega - \Omega_z$.

2 Viscous accretion discs

- Mass conservation $D_t\rho = -\rho\nabla\cdot\mathbf{u}$
- Momentum equation $D_t\mathbf{u} = -\nabla\Phi - \frac{1}{\rho}\nabla p + \frac{1}{\rho}\nabla\cdot\mathbf{\Pi}$
- Energy equation $\rho T\, D_t s = \mathcal{H} - \mathcal{C}$. Perfect gas: $D_t p = -\gamma p\nabla\cdot\mathbf{u} + (\gamma-1)[\mathcal{H} - \mathcal{C}]$
- Gravitational potential $\Phi = \Phi_{\text{ext}} + \Phi_{\text{sg}}$, $\nabla^2\Phi_{\text{sg}} = 4\pi G\rho$
- Heat flux (Fickian) $\mathbf{F} = -16\sigma/(3\kappa\rho)\,\nabla T$, cooling rate $\mathcal{C} = \nabla\cdot\mathbf{F}$.

- Viscous stress tensor $\mathbf{\Pi} = 2\mu\mathbf{S} + \mu_b(\nabla\cdot\mathbf{u})\mathbf{I}$
- Symmetric strain tensor $\mathbf{S} = \frac{1}{2}(\nabla\mathbf{u} + (\nabla\mathbf{u})^T) - \frac{1}{3}(\nabla\cdot\mathbf{u})\mathbf{I}$, divergence $2\mu\,\nabla\cdot\mathbf{S} = \mu(\nabla^2\mathbf{u} + \frac{1}{3}\nabla(\nabla\cdot\mathbf{u}))$
- Heating rate $\mathcal{H} = \mathbf{\Pi} : \nabla\mathbf{u} = 2\mu\,\mathbf{S}^2 + \mu_b(\nabla\cdot\mathbf{u})^2$

Cylindrical polars, Newtonian framework

- ϕ-component of directional derivative $(\mathbf{u}\cdot\nabla\mathbf{u})_\phi = \frac{u_r}{r}\partial_r(ru_\phi) + \frac{u_\phi}{r}\partial_\phi u_\phi + u_z\partial_z u_\phi$
- ϕ-component of divergence of viscous stress tensor $(\nabla\cdot\mathbf{\Pi})_\phi = \frac{1}{r^2}\partial_r(r^2\Pi_{r\phi}) + \frac{1}{r}\partial_\phi\Pi_{\phi\phi} + \partial_z\Pi_{z\phi}$
- Assumptions:
  - (i) Evolution of disc is slower than orbital motion $\implies u_\phi \simeq r\Omega(r)$, thus $\partial_t u_\phi = \partial_z u_\phi = 0$
  - (ii) Axisymmetry $\implies \partial_\phi = 0$
  - (iii) Viscous torque vanishes on the surface $\implies \partial_z\Pi_{z\phi} = 0$
- ϕ-component of EoM applying assumptions $ru_r\rho\frac{dh}{dr} = -\partial_r(r^2\Pi_{r\phi})$
- Mass conservation applying assumptions $\partial_t\rho + \frac{1}{r}\partial_r(r\rho u_r) + \partial_z(\rho u_z) = 0$
- $r\phi$ component of the viscous stress tensor $\Pi_{r\phi} = \mu\left[\cancel{\frac{1}{r}\partial_\phi u_r} + r\partial_r\left(\frac{u_\phi}{r}\right)\right] = \mu r\frac{d\Omega}{dr}$
- Radial mass flux $\mathcal{F} = \int_{-\infty}^{\infty}\int_0^{2\pi} ru_r\rho\,d\phi\,dz$
- Viscous torque $\mathcal{G} = \int_{-\infty}^{\infty}\int_0^{2\pi} -r^2\Pi_{r\phi}\,d\phi\,dz = -\int_{-\infty}^{\infty}\int_0^{2\pi}\mu\,d\phi\,dz\,r^3\frac{d\Omega}{dr} = -2\pi\overline{\nu}\Sigma r^3\frac{d\Omega}{dr}$
- Surface density defined by $2\pi\Sigma = \int_{-\infty}^{\infty}\int_0^{2\pi}\rho\,d\phi\,dz$
- Mean radial velocity defined by $2\pi r\Sigma\overline{u_r} = \mathcal{F}$
- Mean kinematic viscosity defined by $2\pi\overline{\nu}\Sigma = \int_{-\infty}^{\infty}\int_0^{2\pi}\mu\,d\phi\,dz$
- Vertically-integrated ϕ-component of EoM $\mathcal{F}\frac{dh}{dr} = -\partial_r\mathcal{G}$
- Vertically-integrated mass conservation equation $\partial_t\Sigma + \frac{1}{2\pi r}\partial_r\mathcal{F} = 0$ assumes ρu_z vanishes at $\pm\infty$
- 1D diffusion equation for viscous evolution

$$\partial_t\Sigma + \frac{1}{r}\partial_r\left[\left(\frac{dh}{dr}\right)^{-1}\partial_r\left(\overline{\nu}\Sigma r^3\frac{d\Omega}{dr}\right)\right] = 0 \tag{2.1}$$

Keplerian

- $\Omega = (GM)^{1/2}r^{-3/2}$, $\frac{d\Omega}{dr} = -\frac{3}{2}(GM)^{1/2}r^{-5/2}$, $h = r^2\Omega = \sqrt{GMr}$
- $\mathcal{G} = 3\pi(GM)^{1/2}\overline{\nu}\Sigma r^{1/2}$, $dh/dr = \frac{1}{2}(GM)^{1/2}r^{-1/2}$
- 1D diffusion equation

$$\partial_t\Sigma = \frac{3}{r}\partial_r\left[r^{1/2}\partial_r\left(r^{1/2}\overline{\nu}\Sigma\right)\right] \tag{2.2}$$

Miscellaneous topics

- Luminosity generated by accretion: assume Keplerian, and do a rough scaling analysis. $L \sim r^2\mathcal{H}$, $\mathcal{H} \sim \mu\Omega^2$. From $\mathcal{F}(dh/dr) = -\partial_r\mathcal{G}$, get $\dot{M} \sim \nu\Sigma$, hence $L \sim r^2\dot{M}\Omega^2 \sim GM\dot{M}/r$. e.g. for BH, $r_s \sim GM/c^2 \implies L \sim \dot{M}c^2$.

3 Local disc models

3.1 Shearing sheet

- Point P following circular flow. Position vector $\mathbf{r}_0 = (r_0, \phi_0 + \Omega_0 t, 0)$. Reference frame corotating with P, local Cartesian coordinates centred on $\mathbf{r}_0$

$$\underset{\text{radial}}{x = r - r_0}, \qquad \underset{\text{azimuthal}}{y = r_0(\phi - \phi_0 - \Omega_0 t)}, \qquad \underset{\text{vertical}}{z = z}. \tag{3.1}$$

- Tidal potential $\Phi_t = \Phi - \frac{1}{2}\Omega_0^2 r^2$. In local coordinates $\Phi_t = \mathrm{cst.} - \Omega_0 S_0 x^2 + \frac{1}{2}\Omega_{z0}^2 z^2 + \mathrm{h.o.t.}$ Shearing rate $S = -r\, d\Omega/dr$. Keplerian shearing rate $S_0 = 3\Omega_0/2$. Horizontal epicyclic frequency $\kappa^2 = 2\Omega(2\Omega - S)$
- Coriolis force $2\Omega_0\, \mathbf{e}_z \times \mathbf{u} = 2\Omega_0(-\dot{y}, \dot{x}, 0)$
- Particle dynamics: steady-state solutions ($\ddot{x} = \dot{x} = 0$) $x = \mathrm{cst.}$, $y = y_0 - S_0 x t$, i.e. linear shear flow, particles trace out differential rotation
- Angular momentum / canonical y-momentum $p_y = \dot{y} + 2\Omega_0 x = \mathrm{cst.}$ (integral of y-component of EoM)

3.2 Incompressible dynamics

- Ignore z-structure (but consider a 3D shearing box). Tidal potential $\Phi_t = -\Omega_0 S_0 x^2$. Incompressibility $\nabla\cdot\mathbf{u} = 0$, $\rho = \mathrm{cst.}$
- Hydrodynamic momentum $\partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + 2\Omega_0\,\mathbf{e}_z \times \mathbf{u} = -\nabla\Phi_t - \frac{1}{\rho}\nabla p + \nu\nabla^2\mathbf{u}$
- Basic state $p = \mathrm{cst.}$, $\mathbf{u} = -Sx\,\mathbf{e}_y$ ‘circular differential rotation’

Shearing waves, hydrodynamic instability

- Perturbations $\mathbf{v}(x,y,z,t)$, $p'(x,y,z,t)$. Define $\psi \equiv p'/\rho$
- Governing equations (ignoring viscous terms which can be made to decay) NOT LINEARISED

$$[\partial_t - Sx\partial_y + \mathbf{v}\cdot\nabla]v_x - 2\Omega v_y = -\partial_x\psi \tag{3.2}$$
$$[\partial_t - Sx\partial_y + \mathbf{v}\cdot\nabla]v_y + (2\Omega - S)v_x = -\partial_y\psi \tag{3.3}$$
$$[\partial_t - Sx\partial_y + \mathbf{v}\cdot\nabla]v_z = -\partial_z\psi \tag{3.4}$$

- Ansatz for perturbations $\mathbf{v} = \tilde{\mathbf{v}}(t)\,e^{i\mathbf{k}\cdot\mathbf{x}}$, $\psi = \tilde{\psi}(t)\,e^{i\mathbf{k}\cdot\mathbf{x}}$, $\mathbf{k} = \mathbf{k}(t)$
- ‘Choose’ a wavenumber

$$(\partial_t - Sx\partial_y)\mathbf{v} = \left(\frac{d\tilde{\mathbf{v}}}{dt} + i\left(\frac{d\mathbf{k}}{dt}\cdot\mathbf{x}\right)\tilde{\mathbf{v}} - ik_y Sx\tilde{\mathbf{v}}\right)e^{i\mathbf{k}\cdot\mathbf{x}} = \frac{d\tilde{\mathbf{v}}}{dt}\,e^{i\mathbf{k}\cdot\mathbf{x}} \tag{3.5}$$

  if $d\mathbf{k}/dt = Sk_y\,\mathbf{e}_x$. $\implies$ $k_x = k_{x0} + Sk_y t$, $k_y = \mathrm{cst.}$, $k_z = \mathrm{cst.}$ Nonlinear term $\mathbf{v}\cdot\nabla\mathbf{v} = i(\mathbf{k}\cdot\mathbf{v})\mathbf{v} = 0$ because $\nabla\cdot\mathbf{v} = i\mathbf{k}\cdot\mathbf{v} = 0$.

$$\frac{d^2}{dt^2}(k^2\tilde{v_x}) + \kappa^2 k_z^2 \tilde{v_x} = 0 \tag{3.6}$$

- Axisymmetric ($k_y = 0$): $k_x = k_{x0}$. Dispersion relation $\omega^2 = \kappa^2 k_z^2/k^2$. If $\kappa^2 > 0$, harmonic oscillator - inertial waves (incompressible plane waves, $\mathbf{c}_p\cdot\mathbf{c}_g = 0$). Unstable if $\kappa^2 < 0$. Same as Rayleigh's criterion for centrifugal instability: $dh^2/dr < 0$
- Non-axisymmetric: leading waves for $k_x < 0$, trailing waves for $k_x > 0$. Asymptotic limit $t \to \infty$, $k^2 \to S^2 k_y^2 t^2$, and

$$\frac{d^2}{dt^2}(t^2\tilde{v_x}) + \frac{\kappa^2 k_z^2}{S^2 k_y^2}\tilde{v_x} = 0 \tag{3.7}$$

  admits algebraic solutions $\tilde{v_x} \propto t^\sigma$, $\sigma = -3/2 \pm [1/4 - \kappa^2 k_z^2/(S^2 k_y^2)]^{1/2}$

2D vortices

- Ignore z-component. Stream function χ, $\mathbf{v} = \nabla\times(\chi\,\mathbf{e}_z) \implies v_x = \partial_y\chi$, $v_y = -\partial_x\chi$. Vorticity in z-direction $\zeta = \nabla\times(\nabla\times\chi\,\mathbf{e}_z) = -\nabla^2\chi\,\mathbf{e}_z$. Poisson $\nabla^2\chi = -\zeta$
- Governing equations

$$[\partial_t - Sx\partial_y + \mathbf{v}\cdot\nabla]\mathbf{v} + \begin{pmatrix} -2\Omega v_y \\ (2\Omega - S)v_x \\ 0 \end{pmatrix} = -\nabla\psi \quad \xrightarrow{\nabla\times} \quad [\partial_t - Sx\partial_y]\zeta = \frac{\partial(\chi,\zeta)}{\partial(x,y)} = -\frac{\partial\chi}{\partial y}\frac{\partial\zeta}{\partial x} + \frac{\partial\chi}{\partial x}\frac{\partial\zeta}{\partial y} \tag{3.8}$$

- Elliptical vortex patch, constant vorticity ζ_0. Area of vortex patch conserved, $\pi ab = \mathrm{cst.}$ Evolution of shape depends on (treated separately, sum effects)
  (i) ζ_0, through $\nabla^2\chi = -\zeta_0$ (Kirchhoff). Patch rotates at $\dot{\theta} = ab\zeta_0/(a+b)^2$

(ii) Background shear (Kida) distorts, shears, and rotates the patch

$$\frac{\dot{a}}{a} = -\frac{\dot{b}}{b} = S\sin\theta\cos\theta, \qquad \dot{\theta} = \frac{S(b^2\cos^2\theta - a^2\sin^2\theta)}{a^2 - b^2} \tag{3.9}$$

Aspect ratio $r = a/b$. $\dot{r}/r = \dot{a}/a - \dot{b}/b = 2\dot{a}/a$.

$$\frac{\dot{r}}{r} = 2S\sin\theta\cos\theta, \qquad \dot{\theta} = \frac{S(\cos^2\theta - r^2\sin^2\theta)}{r^2 - 1} + \frac{r\zeta_0}{(r+1)^2} \tag{3.10}$$

- Equilibria $\dot{r} = 0 \implies \theta = 0, \pi/2$. If $\theta = 0$, $\dot{\theta} = 0 \implies \zeta_0/S = -(r+1)/[r(r-1)]$. Circular vortices ($r = 1$, $\zeta_0 \to \pm\infty$) don't exist. Cyclonic $r < 1 \implies \zeta_0 > 0$; anti-cyclonic $r > 1 \implies \zeta_0 < 0$.
- Stability of equilibria, perturbations $\dot{\delta r} = 2Sr\,\delta\theta$, $\dot{\delta\theta} = S\,\partial_r f\,\delta r$, $f = 1/(r^2-1) + r\zeta_0/[S(r+1)^2]$. Then $\ddot{\delta r} - 2S^2 r\,\partial_r f\,\delta r = 0$: marginal stability $r = \sqrt{2} - 1$ (cyclonic minimum), stability for $r > \sqrt{2} - 1$ i.e. all anticyclonic and some cyclonic vortices are stable.
- Kida vortex core (linear approximation) $\chi = x^2/b^2 + y^2/a^2$, velocity and pressure

$$\mathbf{u} = \frac{S}{r-1}\left(\frac{y}{r}\,\mathbf{e}_x - rx\,\mathbf{e}_y\right) \tag{3.11}$$

- Elliptical instability (3D)

$$\omega^2 = \Omega^2\left[2 - \frac{3}{2r(r-1)}\right]\left[2 - \frac{3r}{2(r-1)}\right] \tag{3.12}$$

  only grows when $3/2 < r < 4$

Magnetorotational instability

- Nonideal induction equation $\partial_t\mathbf{B} + \mathbf{u}\cdot\nabla\mathbf{B} = \mathbf{B}\cdot\nabla\mathbf{u} + \eta\nabla^2\mathbf{B}$, solenoid $\nabla\cdot\mathbf{B} = 0$, inviscid momentum + Lorentz force $\partial_t\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + 2\Omega\,\mathbf{e}_z\times\mathbf{u} = -\nabla\Phi_t - \nabla T/\rho_0 + \mathbf{B}\cdot\nabla\mathbf{B}/(\mu_0\rho_0)$, total pressure $T = p + B^2/(2\mu_0)$
- Basic state $P = \text{cst.}$, $\mathbf{B} = B_0\,\mathbf{e}_z$, $B_0 = \text{cst.}$, $T = \text{cst.}$, $\mathbf{u} = -Sx\,\mathbf{e}_y$
- Perturbations $\mathbf{u}'$, $\mathbf{T}'$, $\mathbf{B} = B_0(\mathbf{e}_z + \mathbf{b}')$ only depend on z, growth rate σ, i.e. $\propto \exp(ikz + \sigma t)$. Linearised perturbation equations

$$\partial_t\mathbf{u}' - Su_x'\,\mathbf{e}_y + 2\Omega\,\mathbf{e}_z\times\mathbf{u}' = -\frac{1}{\rho_0}\nabla T' + v_a^2\,\partial_z\mathbf{b}' \tag{3.13}$$

$$\nabla\cdot\mathbf{b}' = 0, \qquad \nabla\cdot\mathbf{u}' = 0, \qquad \partial_t\mathbf{b}' = \partial_z\mathbf{u}' - Sb_x'\,\mathbf{e}_y + \eta\partial_{zz}^2\mathbf{b}' \tag{3.14}$$

- $b_z' = 0$ (solenoid), $u_z' = 0$ (incompressible), $T' = 0$ (EoM z-component). Introduce $\overline{\sigma} = \sigma + \eta k^2$. Alfvén length $\ell_a \equiv v_a/\Omega$. Need $\ell_a \lesssim H \iff v_a \lesssim v_s \iff \beta \gtrsim 1$
- Dispersion relation $\sigma^2\overline{\sigma}^2 + \kappa^2\overline{\sigma}^2 + 2\sigma\overline{\sigma}v_a^2k^2 + v_a^2k^2(v_a^2k^2 - 2\Omega S) = 0$. Need $\sigma^2 > 0$ for instability. Combination of magnetic tension and inertial forces, no pressure ($T' = 0$)
- Ideal $\sigma^4 + (2v_a^2k^2 + \kappa^2)\sigma^2 + v_a^2k^2(v_a^2k^2 - 2\Omega S) = 0$. Look at sum and product of roots. If $v_a^2k^2 - 2\Omega S < 0 \implies k^2 < 2S/(\Omega\ell_a^2)$, $S > 0$, then wlog $\sigma_1^2 > 0$ while $\sigma_2^2 < 0$ - instability. Maximum growth rate $\sigma_{\max} = S/2$ at $\ell_a^2k^2 = 1 - [\kappa/(2\Omega)]^4$. (Solve quadratic) - zeros at $v_ak/\kappa = 0$, $(4\Omega^2/\kappa^2 - 1)^{1/2}$; beyond which the instability switches off (too much magnetic tension). Keplerian: $v_ak/\kappa \equiv k\ell_a$. Maximum growth rate $\sigma_{\max} = 3\Omega/4$ at $\ell_a^2k^2 = 15/16$ - explosive growth (growth on dynamical timescales). Wavenumber range $\ell_ak = 0, \sqrt{3}$
- Strong-field limit: if $\ell_a > H$ then MRI switches off. Then instability grows if $1/H \lesssim k < \sqrt{3}/\ell_a$
- Weak-field limit: MRI paradox (singular limit $\sigma_{\max} = S/2$ as $v_a \to 0 \implies \ell_a \to 0$) resolved by including magnetic diffusivity, relevant at small scales - recall $k\ell_a \sim \mathcal{O}(1)$. Diffusion length $\ell_\eta \equiv \eta/v_a$. Expanding nonideal DR out, need product of roots $\kappa^2k^4\eta^2 + v_a^2k^2(v_a^2k^2 - 2\Omega S) < 0 \implies k^2 < 3/(\ell_a^2 + \ell_\eta^2) \implies k^2 \lesssim 3/\ell_\eta^2 \to 0$ as $v_a \to 0$.
- General / intermediate: $H \gtrsim \lambda_{\text{MRI}} \gtrsim \max(\ell_a, \ell_\eta) \iff v_s \gtrsim v_a \gtrsim \eta\Omega/v_s \iff \mathrm{R}_\mathrm{m}^2 \gtrsim \beta \gtrsim 1$. Magnetic Reynolds number $\mathrm{R}_\mathrm{m} \equiv Hv_s/\eta$.

3.3 Compressible dynamics

3D hydrodynamic instability

- Vertically-stratified, isothermal shearing box; Keplerian, non-SG disc; tidal potential $\Phi_t = -\frac{3}{2}\Omega^2 x^2 + \frac{1}{2}\Omega^2 z^2$; basic state $\mathbf{u} = -\frac{3}{2}\Omega x\,\mathbf{e}_y$, $\rho_0 = \rho_{00}\exp[-z^2/(2H^2)]$
- Small axisymmetric perturbations ρ', $\mathbf{u}'$, $\rho' = \tilde{\rho}(z)e^{ikx-i\omega t}$, pseudo-enthalpy $\tilde{h} = c^2\tilde{\rho}/\rho_0$
- Quantisation: second-order ODE (Hermite's equation) for $\tilde{h}$ in z; ξ must be integer to admit polynomial solutions

$$\frac{d^2\tilde{h}}{dz^2} - \frac{z}{H^2}\frac{d\tilde{h}}{dz} + \frac{\xi}{H^2}\tilde{h} = 0, \qquad \xi = \frac{\omega^2}{\Omega^2} - H^2k^2\left(\frac{\omega^2}{\omega^2-\Omega^2}\right) \tag{3.15}$$

- Dispersion relation

$$\omega^4 - (\Omega^2 + c^2k^2 + \Omega^2 n)\omega^2 + \Omega^4 n = 0 \tag{3.16}$$

Recover 2D density waves (non-SG) when $n = 0$. Otherwise, two limits: $\omega/\Omega \gg 1 \implies \omega^2 = \Omega^2(1+n)+c^2k^2$, p-modes (pressure); $\omega/\Omega \ll 1 \implies \omega^2 = n\Omega^2/(1+n+H^2k^2)$, r-modes (Rossby)

Gravitational instability

- Self-gravitating 2D disc (e.g. razor thin), $\Sigma = \int \rho\,dz$, $\rho = \Sigma\,\delta(z)$, mass conservation $\partial_t\Sigma + \nabla\cdot(\Sigma\mathbf{u}) = 0$, momentum equation $\partial_t\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + 2\Omega\,\mathbf{e}_z\times\mathbf{u} = -\nabla\Phi_t - \nabla P/\Sigma - \nabla\Phi_{d,m}$, $\Phi_t = -\Omega S x^2$, Poisson $\nabla^2\Phi_d = 4\pi G\Sigma\,\delta(z)$, $P = \int p\,dz$, EoS $P = P(\Sigma)$
- SG potential $\tilde{\Phi}_d = -2\pi G/|k|\,\exp(-k|z|)\tilde{\Sigma}$, $k \neq 0$. Deal with Poisson by FT in x and y, e.g.

$$\tilde{\Sigma} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Sigma(x,y,t)\exp(-ik_x x - ik_y y)\,dx\,dy. \tag{3.17}$$

FT Poisson $(-k^2 + \partial_{zz}^2)\tilde{\Phi}_d = 4\pi G\tilde{\Sigma}\,\delta(z)$, Laplace for $z<0$, $z>0$ admits $\tilde{\Phi}_d = A_\pm e^{\mp kz}$ and JC at $z=0$: $[\tilde{\Phi}_d]_-^+ = 0$, $[\partial_z\tilde{\Phi}_d]_-^+ = 4\pi G\tilde{\Sigma}$

- Basic state $\Sigma = $ cst., $\mathbf{u} = -Sx\,\mathbf{e}_y$. Axisymmetric disturbances $\mathbf{v}$, P', Φ'_d, e.g. $\Sigma' = \tilde{\Sigma}'\exp(ik_x x - i\omega t)$. LINEARISED perturbation equations

$$\begin{aligned} \partial_t\Sigma' + \Sigma(\nabla\cdot\mathbf{v}) &= 0 \\ \partial_t\mathbf{v} - Sv_x\,\mathbf{e}_y + 2\Omega\,\mathbf{e}_z\times\mathbf{v} &= -\nabla\Phi'_{d,m} - \frac{1}{\Sigma}\nabla P' \\ P' &= v_s^2\Sigma' \\ \nabla^2\Phi'_d &= 4\pi G\Sigma'\,\delta(z) \end{aligned}$$

$$\begin{aligned} -i\omega\tilde{\Sigma}' + ik_x\Sigma\tilde{v_x} &= 0 \\ -i\omega\tilde{v_x} - 2\Omega\tilde{v_y} &= -ik_x\left[\tilde{\Phi}_{d,m} + v_s^2\frac{\tilde{\Sigma}'}{\Sigma}\right] \\ -i\omega\tilde{v_y} + (2\Omega - S)\tilde{v_x} &= 0 \\ \tilde{\Phi}_{d,m} &= -\frac{2\pi G}{|k_x|}\tilde{\Sigma}' \end{aligned}$$

- Dispersion relation $\omega^2 = \kappa^2 - 2\pi G\Sigma|k_x| + v_s^2k_x^2$. Density waves. Inertial (restoring), self-gravity (destabilising), acoustic (restoring). Toomre stability parameter $Q \equiv v_s\kappa/(\pi G\Sigma)$. Jeans length $k_J = \pi G\Sigma/v_s^2 (\equiv 1/(QH)$ if Keplerian) is where ω is minimised. Unstable wavelengths exist if $\omega^2(k_J) < 0 \iff Q < 1$ and the range of unstable wavenumbers is bounded by $k_J(1\pm\sqrt{1-Q^2})$.
- If $1 < Q \lesssim 1.5$ the disc is nonlinearly unstable to large amplitude perturbations. Thermostatic regulation if T varies, e.g. heating by gravitoturbulence is stabilising, cooling is destabilising

3.4 Satellite-disc interactions

Particle dynamics with an embedded satellite

- Inviscid, non-SG 2D shearing sheet, embedded satellite at $\mathbf{r}_0$ (i.e. middle of shearing sheet). Previously ψ was a pseudo-enthalpy. Now ψ is the satellite potential, $\psi = -GM_s/\sqrt{x^2+y^2}$
- Governing equations

$$\ddot{x} - 2\Omega\dot{y} = 2\Omega S x - \partial_x\psi, \qquad \ddot{y} + 2\Omega\dot{x} = -\partial_y\psi \tag{3.18}$$

If $\psi = 0$, then $\ddot{x} + \kappa^2 x = 0 \implies x = x_0 + \mathrm{Re}\left[Ae^{-i\kappa t}\right]$, $y = (y_0 - Sx_0 t) - \frac{2\Omega}{\kappa}\,\mathrm{Re}\left[iAe^{-i\kappa t}\right]$ where the steady-state solution is $x = x_0$, $y = y_0 - Sx_0 t$; wlog set $y_0 = 0$. Amplitude A is a complex CONSTANT. $p_y = \kappa^2 x_0/(2\Omega)$ and $\varepsilon = \frac{1}{2}\kappa^2|A|^2 - \Omega S p_y^2/\kappa^2$ are conserved

- If $\psi \neq 0$, then do linear perturbation theory. Also note $\dot{p}_y = -\partial_y \psi$ - canonical y-momentum not conserved in encounter. $A = A(t)$ still complex. Perturbations $x_1(t) \equiv \mathrm{Re}\left[Ae^{-i\kappa t}\right]$ and $y_1(t)$... Linearise gradient of satellite potential $\partial_x \psi = (GM_s/x_0^2)\,(1+S^2t^2)^{-3/2}$, $\partial_y \psi = -(GM_s/x_0^2)\,St\,(1+S^2t^2)^{-1/2}$, neglecting terms $\sim\mathcal{O}(x_1, y_1)$ then governing equation is

$$\ddot{x}_1 + \kappa^2 x_1 = -\frac{GM_s}{x_0^2 S} f(t), \qquad f(t) = \frac{S}{(1+S^2t^2)^{3/2}} + \frac{2\Omega}{(1+S^2t^2)^{1/2}} \tag{3.19}$$

with integration constant set to 0. $f(t)$ is symmetric about $t=0$ and peaks at $t=0$.

- Solve using Green's function H satisfying $\ddot{H} + \kappa^2 H = \delta(t-\xi)$. For $\xi < t$, $H = 0$. For $\xi > t$, $H = C_1 \cos\left[\kappa(t-\xi)\right] + C_2 \sin\left[\kappa(t-\xi)\right]$ subject to $[H]_-^+ = 0$, $[H_t]_-^+ = 1$ at $\xi = t$. Then $H = \frac{1}{\kappa}\sin\left[\kappa(t-\xi)\right]$ for $\xi > t$ and

$$x_1 = -\frac{GM_s}{x_0^2 S\kappa}\int_{-\infty}^{t} \sin\left[\kappa(t-\xi)\right] f(\xi)\, d\xi = \mathrm{Re}\left[-\frac{GM_s}{x_0^2 S\kappa}\int_{-\infty}^{t} ie^{-i\kappa(t-\xi)} f(\xi)\, d\xi\right] \tag{3.20}$$

$$\implies A = -i\frac{GM_s}{x_0^2 S\kappa}\int_{-\infty}^{t} e^{i\kappa\xi} f(\xi)\, d\xi = -i\frac{GM_s}{x_0^2 S\kappa} C(t), \tag{3.21}$$

i.e. epicyclic oscillations. In limit $t \to \infty$, make variable change $\zeta = S\xi$, write in terms of Bessel functions

$$C_\infty = \int_{-\infty}^{\infty} e^{i(\kappa/S)\zeta}(1+\zeta^2)^{-3/2}\, d\zeta + \frac{2\Omega}{S}\int_{-\infty}^{\infty} e^{i(\kappa/S)\zeta}(1+\zeta^2)^{-1/2}\, d\zeta = \frac{2\kappa}{S}K_1(\kappa/S) + \frac{4\Omega}{S}K_0(\kappa/S) \tag{3.22}$$

is just a constant - $C_\infty \simeq 3.4$ (Keplerian). $A_{\max} = -iGM_s C_\infty/(x_0^2 S\kappa)$

- $f(t)$ peaks at $t = 0$. This motivates the impulse approximation, which supposes that in limit $S \gg \Omega$, $f \simeq 2\delta(t)$. Then

$$\ddot{x}_1 + \kappa^2 x_1 = -\frac{2GM_s}{x_0^2 S}\delta(t) \implies x_1 = \begin{cases} 0, & t < 0; \\ -\frac{2GM}{x_0^2 S\kappa}\sin(\kappa t), & t \geq 0. \end{cases} \tag{3.23}$$

Equivalent $C_\infty = 2$.

- Criterion for linear theory to be valid: need $|A/x_0| \ll 1$. Keplerian - $\kappa \sim \Omega \sim S \implies x_0/r_0 \gg (M_s/M)^{1/3}$ - Hill radius of the embedded satellite. Outside Hill radius: epicycles occur. Inside: particles get accreted.

- Specific energy $\varepsilon + \psi$ conserved in encounter. $\Delta\psi = 0$ because $\psi = 0$ long before and after encounter. Thus $\Delta\varepsilon = 0$ implies

$$0 = \frac{1}{2}\kappa^2 |A_{\max}|^2 - \frac{\Omega S}{\kappa^2}(2p_y \Delta p_y) \implies \Delta p_y = \frac{(GM_s C_\infty)^2}{2S^3 x_0^5}, \tag{3.24}$$

so particles at $x_0 > 0$ gain p_y and move out and those that start at $x_0 < 0$ lose p_y and move in.

- y-force on disc per unit x at location x is $F_y = \Delta p_y m_p \times$ encounter rate pul. Encounter rate pul given by number of particles $(\Sigma/m_p) \times$ y-velocity at x ($|Sx|$) thus

$$F_y = \frac{(GM_s C_\infty)^2 \Sigma}{2S^2 x^4}\,\mathrm{sgn}(x) \tag{3.25}$$

- Criterion for gap opening: if gravitational torque $\sim r_0 \int_{r_h}^{\infty} F_y\, dx >$ viscous torque $r_0^2 \Pi_{xy}$:

$$\bar{\nu} \lesssim \left(\frac{M_s}{M}\right) r_0^2 \Omega \implies \frac{M_s}{M} \gtrsim \alpha \left(\frac{H}{r_0}\right)^2 \tag{3.26}$$

4 Notes on exam questions

Notes I wrote after attempting exam questions. I did do most of the questions but didn't write notes for all of them. This will make much more sense if you try the questions first. All Henrik years except 2019, 2023 (Gordon) and 2013 (Sijme-Jan Paardekooper). As you will find with AFD the style of questions differs quite drastically between lecturers.

2013 Q2 - GI with softening

In razor-thin discs, the effects of self-gravity are exaggerated compared to a real disc with finite thickness. Introducing a softening length ε smooths the SG potential over a distance ε. Then the potential is Plummer with $\varepsilon \sim H$. Dispersion relation modified by SG term having damping factor of $\exp(-k\varepsilon)$. Without softening the wavenumber of maximum growth is the Jeans wavenumber $k_J = 1/(QH)$. ND $s = k/k_J$. With ND softening δ, s_* (maximum growth) satisfies a non-algebraic equation, with condition $s_* < 1/(1+\delta)$. Softening ($\delta > 0$) increases the wavelength of maximum growth as well as the critical Toomre parameter Q_c beneath which instability occurs.

2014 Q1 - Internal photoevaporation

If interested see Clarke et al. (2001)

2014 Q2 - GI in a layer of dust

TLDR: dust with Stokes drag. In uncoupled limit recover dispersion relation for GI in gas. In weakly-coupled limit there are two damped density waves and a secular mode which is always unstable at sufficiently small k.

2D compressible shearing sheet. Tidal potential only in radial. Governing equations for dust — EoM includes Stokes drag, parametrised by inverse Stokes number ε. Basic state: dust velocity = gas velocity. Given $P = c^2\sigma$. Axisymmetric perturbations (only x-dependence) and growth rate s. Defined $\overline{\omega}^2$ to be the normal dispersion relation for GI in gas (in uncoupled limit, $\varepsilon = 0$, recover $\overline{\omega}^2 < 0$ as instability criterion). Get $Q < 1$ from explicitly evaluating the (wavenumber) roots of $\overline{\omega}^2 = 0$ and stipulating the k roots have to be real — look at the determinant. Then weakly coupled case ($0 < \varepsilon \ll 1$): expand s to first order in ε. We are told the third mode is $s_0 = 0$. The other two modes have real $s_1 < 0$ (i.e. decay, evaluate first order) and $s_0 = \pm i\overline{\omega}$. Hence these are damped density waves, and not overstability. The third mode is secular and the sign of s_1 depends on the sign of $\overline{\omega}^2 - \Omega^2$, with instability ($s_1 > 0$) if $\overline{\omega}^2 < \Omega^2$. If k is sufficiently small the dust is always unstable. Rewrite the criterion $\overline{\omega}^2 - \Omega^2 < 0$ in terms of Q, and find that $Q < \Omega/(ck)$. If dust has radial extent L, then this sets an upper bound to $1/k$. Thus the 'rough' instability criterion is $Q \lesssim L/H$.

2014 Q3 - MRI with vertical stratification

TLDR: the vertically stratified system admits solutions where the vertical wavenumber k is quantised. The MRI is entirely stabilised when the lowest mode ($n = 1$) is in the stable regime.

Compressible, vertically stratified shearing sheet (tidal potential is radial + vertical). EoM with Lorentz force and IE. Basic state the usual except density has z-dependence via $\rho = \rho_0 h$ with h a function of the ND parameter z/H. Perturbations are in the form $e^{st}F$ where $F = F(z)$. Linearised pert equations give dispersion relation which becomes the typical MRI dispersion relation provided $F'' + k^2 hF = 0$ holds, where $'$ denotes derivatives w.r.t. z. Admits Legendre polynomials in $\tanh(z/H)$, with $k_n = \sqrt{n(n+1)}/H$, i.e. the vertical wavenumbers are quantised. Instability if product of roots is negative, i.e. $v_a^2k^2 - 3\Omega^2 < 0$. Disc is MRI-stable if the lowest mode $k_1 = \sqrt{2}/H$ is stable, which is equivalent to $\beta < 4/3$. Last part asks for n of fastest growing mode when $\beta = 24$. Quoted Keplerian result of $\ell_a^2k^2 = 15/16$ but this does not give integer n; closest is $n = 3$.

2015 Q1 - Steady-state disc structure, disc emission (also ES1 Q4)

2015 Q2 - Compressible dynamics and GI

Keplerian, 2D shearing sheet. Total energy is conserved. Scale Toomre parameter to obtain $Q \sim (M_*/M_d)(H/R)$. FT and solve delta forcing at $z = 0$ to solve for SG potential. In a 3D disc, SG force in the momentum equation can be written as the divergence of a gravitational stress tensor. Some fuckery with vector calculus — or take integral over volume and integrate by parts, that also works too... There are two terms in $\mathbf{T}$ — the 'pressure' (which is negative), and $\mathbf{gg}$, which has both isotropic and deviatoric parts; recall it is $r^2T_{r\phi}$ which contributes to angular momentum transport.

2015 Q3 - MRI with radial thermal stratification

TLDR: the thermal structure modifies the MRI dispersion relation by adding $N^2\sigma^2 + N^2k^2v_a^2$. This has undetermined implications on the maximum growth rate and the corresponding k.

Have extra 'potential temperature' equations (this came up in COS), only advection. Tidal potential radial only. Basic state has $\theta = 0$. Without making any assumptions about the nature of the perturbations, NINE governing

LINEARISED equations total (3 from EoM, 3 from IE, PT, incompressibility, solenoidal). Considered plane-wave perturbations with z-dependence only and growth rate s, hence $u'_z = b'_z = \psi' = 0$. Get modified MRI dispersion relation – addition of $N^2\sigma^2 + N^2k^2v_a^2$. If $B_0 = 0$ (hydrodynamic limit) then recover Hoiland-Solberg instability criterion: $N^2 + \Omega^2 < 0$. Unsure about the order-of-magnitude estimate for N^2 and commenting on hydrodynamical stability as the question seems to suggest that $N^2 < 0$ because $\partial_r S < 0$, although it is a well-established result that discs (according to the viscous alpha model) have $\partial_r S > 0$ and thus are hydrodynamically stable but MRI-unstable. If $B_0 \neq 0$ then maximum growth occurs when $\ell_a^2 k^2 = 1 - (N^2+\Omega^2)^2/(16\Omega^4)$. If $N^2 < 0$ then this maximum growth wavenumber is pushed up, i.e. max growth occurs at shorter wavelengths.

2016 Q1 - Diffusion equation, self-similar treatment, vertical structure, viscous instability

Radiation pressure dominant, inviscid alpha disc. Vertical scalings give $\overline{\nu}\Sigma \sim \Sigma^{-1}$ — satisfies the viscous instability criterion ($d(\overline{\nu}\Sigma)/d\Sigma < 0$). This can be interpreted by considering the radial mass flux in the region around an overdensity localised at $r = r_0$. In a Keplerian disc, $\mathcal{F} \sim -r^{1/2}\partial_r(r^{1/2}\overline{\nu}\Sigma)$. Expand to find $\mathcal{F} = -\overline{\nu}\Sigma - r\,[d(\overline{\nu}\Sigma)/d\Sigma]\,\partial_r\Sigma$. For $r > r_0$, $\partial_r\Sigma < 0$, hence if $d(\overline{\nu}\Sigma)/d\Sigma < 0$ then there is more flux heading in negative r; the converse is true. The quicker way given $\overline{\nu}\Sigma \sim \Sigma^{-1}$ is to note that then $\delta\mathcal{F} \sim -\partial_r(\overline{\nu}\Sigma) \sim -\partial_r(1/\Sigma) \sim \partial_r\Sigma$ and sketch the schematic of an overdensity. Conversely in 2022 Q1 (Vertical structure of SG discs), in a marginally-stable SG alpha disc we find that $\overline{\nu}\Sigma \sim \Sigma^7$ and thus such a disc is viscously stable.

2016 Q2 - Waves and 'planets' in the shearing sheet

3D, Keplerian, no vertical stratification, non-SG, barotropic. Consider small, 3D axisymmetric perturbations $\propto e^{ik_xx+ik_zz-i\omega t}$. The dispersion relation is $\omega^4 - (\Omega^2 + c^2k^2)\omega^2 + \Omega^2c^2k_z^2 = 0$. Inertial waves ($\omega^2 = \Omega^2k_z^2/k^2$) in the limit $\omega^2 \ll \Omega^2 \ll c^2$. For a polytropic and adiabatic fluid, can rewrite equations in terms of the enthalpy (which is a potential). Then, given a velocity field (e.g. of a planet), can solve for Q. Because barotropic, $\nabla Q \parallel \nabla P \parallel \nabla\rho$ hence $Q = \mathrm{cst.}$ defines the shape/surface of the planet.

2016 Q3 - Maxwell's Saturn's rings

Raf probably did this one.

2017 Q1 - Viscous evolution and vertical structure

Keplerian, steady-state SD profile, localised mass supply (Dirac), vertical structure. Dwarf nova disc in partially-ionized regime, $\kappa \propto T^{10}$. In the first case, question says $F = 0$, so the energy equation $dT/dz \propto -F$ implies that $dT/dz = 0$ i.e. vertical isothermality, but the vertical density suggests otherwise... why?

2017 Q2 - Vertical shear instability

If interested see Nelson et al. (2013)

Shearing sheet model with vertical shear, also in $\mathbf{e}_x$, parametrised by some parameter q. Incompressible. Energy equation: energy is not conserved, shear provides a source of free energy (note that background shear from circular rotation can be written as a flux, because it derives from a tidal potential). Disc is Keplerian (with associated background shear rate; the vertical shear is in addition to this). Equilibrium does not seek plane wave solutions explicitly but rather some undefined function $f(\xi)$ with $\xi = k_xx + k_zz$. For $\mathbf{u}' \propto f(\xi)$ and $P' \propto g(\xi)$, find that need $f \propto g'$. Asked to show that under this ansatz the nonlinear $\mathbf{u}' \cdot \nabla\mathbf{u}'$ vanishes — see e.g. hydrodynamical instability. Note that $k^2 = k_x^2 + k_z^2$. Dispersion rate and maximum growth rate in limit of small shear $q \ll 1$ — can make arguments to get there, otherwise differentiate w.r.t. k_x/k_z. Maximum growth rate is Ωq. This is not exactly explosive growth, because although $\Omega = 1/t_{\mathrm{dyn}}$, growth rate is scaled by small q...

2017 Q3 - Density waves and dust

2018 Q1 - Accretion onto a neutron star

Magnetic truncation radius defined by where the magnetic pressure exceeds the ram pressure of the accretion flow from infinity. This gives one scaling. The other scaling comes from accretion flow from infinity — balancing the specific KE with the potential at r_m. The dipole rotates at rate Ω_m, and the fluid flow matches this rotation rate near r_m, and can be flung from the system if (ignoring magnetic energy) KE exceeds potential at r_m — this is propeller flow. The NS accretes mass throughout its lifetime and if it exceeds some Chandrasekhar-like limit then will collapse into BH. Disc evolution, Keplerian, with nonzero magnetic torque at $r = r_{\mathrm{in}}$ (not necessarily r_m). Integrate to find that the typical steady-state SD profile is modified by the magnetic torque,

$$\overline{\nu}\Sigma = \frac{\dot{M}}{3\pi}\left[1 - \sqrt{\frac{r_{\mathrm{in}}}{r}}(1-\lambda)\right], \qquad \lambda = \frac{T_m}{h(r_{\mathrm{in}})\dot{M}}.$$

If $\lambda \ll 1$, then approach typical hydrodynamic limit. If $\lambda \gg 1$, then magnetic torque dominates and $\overline{\nu}\Sigma$ approaches far-field $\dot{M}/3\pi$ limit from above, rather than below. T_m is the azimuthally and vertically averaged Maxwell stress, $T_m \simeq r^2B^2H/\mu_0$.

2018 Q2 - Vertical structure of a slowly cooling disc

REALLY struggled with this one. Can't comment because I don't know how to do it

2018 Q3 - Oscillatory convection

If interested see Latter (2016)

2021 Q1 - Hyper-accreting black holes

2021 Q2 - Gravitoturbulent protoplanetary discs

Keplerian, SG, razor-thin, axisymmetric density waves. If no SG (i.e. just acoustic-inertial waves) $c_p c_g = c_s^2$. With SG, $c_p c_g = 0$ at the Jeans wavenumber; $c_p c_g > 0$ for $k > k_J$ etc. If $Q > 1$ (stability) then yes these are oscillations that are extremely dispersive (i.e. both normal and anomalous dispersion); a localised wavepacket spreads out very quickly. In general, it is useful to note that $c_p c_g = (d\omega^2/dk)/(2k)$. An example of a turbulent viscosity prescription is to take $\overline{\nu} \sim k_1^{-2}\Omega$, where k_1 is the wavenumber (short) of the longest unstable mode, i.e. consider the upper limit on eddy size to be $\sim k_1^{-1}$ and eddy velocity on dynamical timescales.

2021 Q3 - Elliptical instability in a Kida vortex core (also ES2 Q3)

2022 Q1 - Vertical structure of SG discs

2022 Q2 - Spreading of a narrow planetary ring

Razor-thin, Keplerian, only dynamic viscosity. If Σ, u_x, P are all functions of x, t, and $u_y = -(3/2)\Omega x$, then can reduce system (via mass conservation and y-component of EoM) to a single diffusion equation, $\partial_t \Sigma = 3\partial_x^2(\nu\Sigma)$. Parabolic system does not admit 'wave'/oscillating solutions. Hence look for perturbations of the form $\exp(i\mathbf{k}\cdot\mathbf{x}+st)$, and given $\nu = \nu(\Sigma) \implies \nu' = (d\nu/d\Sigma)\,\Sigma'$ and find $s > 0$ if $d(\nu\Sigma)/d\Sigma < 0$, the usual viscous instability criterion. Given prescription $\nu = A\Sigma^2$, and $\Sigma(x,t)$ is some symmetric function of x within $|x| < w$, i.e. there is a ring of material centred at $x = 0$ which spreads out radially. The speed at which the ring spreads is u_x at $|x| = w$ — this gives the first result $\dot{w}w = 9A\sigma^2$. Substitute back into diffusion equation to get $\dot{\sigma}/\sigma = -\dot{w}/w \implies (\sigma w)_t = 0$. Mass in ring conserved. Lastly find that $\sigma^4 = \sigma_0^4 + Bt$ where B is some constant — used $\dot{w}w = 9A\sigma^2$ result to solve this.

2022 Q3 - Stratified incompressible shearing sheet

2023 Q1 - GI in viscous discs

2023 Q2 - Satellite interaction

2023 Q3 - Oscillations of the scale height

If interested see Dubrulle et al. (1995)

2024 Q1 - Viscous and magnetic accretion

2024 Q2 - Radial drift of dust and the streaming instability

TLDR: in a global disc model, when radial pressure effects included, the gas rotates sub-Keplerian with relative difference $\sim (H/r)^2$; this changes the background gas velocity. In the local shearing sheet, due to dust-gas coupling, the equilibrium state of the dust is altered — the dust drifts radially inward. Radial drift is maximised when $\Omega_K\tau = 1$.

Non-turbulent gas, thin disc. Scaling for H given by z-component of EoM (in hydrostatic balance). Perfect gas with $s = \ln(P\rho^{-\gamma})$ and $s = s(r)$ results in vertical density structure $\rho(z) \propto (1 - z^2/H^2)^{1/(\gamma-1)}$, where $H \sim c_{s,0}/\Omega_K$. Gas does not rotate Keplerian — consider the radial component of the EoM. If $d\rho_0/dr < 0$ (which is generally the case) then $\Omega < \Omega_K$ (sub-Keplerian). In the shearing sheet, gas equilibrium velocity $v_y = V - (3/2)\Omega x$ where $V < 0$ for sub-Keplerian rotation. (Easier to sketch streamlines.) Effect on dust equilibrium — find that $\Delta\mathbf{u} \propto V$ and Δu_x (radial) is maximised when $\Omega_K\tau = 1$, where τ is the stopping time (hence $\Omega_K\tau$ is the dimensionless stopping time $\sim$ Stokes number). Last part on axisymmetric (but has z-component) gas wave disturbance which perturbs the gas field $\propto \exp(i\mathbf{k}\cdot\mathbf{x} - i\omega t)$ giving rise to perturbations $\propto \exp(i\mathbf{k}\cdot\mathbf{x})$ in the SD and velocity field of the dust. Showed that $\mathcal{D}_\tau(\mathcal{D}_\tau^2 + \Omega_K^2)\mathcal{D}\sigma' = Fe^{-i\omega t}$ where F is some constant, and $\mathcal{D} = \partial_t + i(\mathbf{k}\cdot\Delta\mathbf{u})$, $\mathcal{D}_\tau = \mathcal{D} + 1/\tau$. Last part on showing algebraic growth in σ' subject to some condition — unsure how to proceed.

2024 Q3 - Vortensity in the shearing sheet

TLDR: vortensity is conserved. Instead of using mass conservation to link SD perturbation to velocity perturbation, use definition of vortensity instead. This is helpful when considering non-axisymmetric perturbations. Dispersion relation from eigenvalues of matrix; recover usual non-SG result if $k_y = 0$.

Non-SG, locally isothermal ($P = c_s^2\Sigma$), razor-thin, Keplerian $\Phi_t = -(3/2)\Omega^2 x^2$. Vortensity (otw known as potential vorticity) $q = \Sigma^{-1}\left[2\Omega + (\nabla\times\mathbf{u})\cdot\mathbf{e}_z\right]$ is conserved, $D_t q = 0$: easiest to start with EoM and note that because we are confined to 2D, the term $(2\Omega\,\mathbf{e}_z + \nabla \times \mathbf{u}) \cdot \nabla\mathbf{u} = 0$. Now move to shearing sheet in axisymmetric equilibrium. Typical basic state but now suppose the SD varies radially by $\Sigma = \Sigma_0 \exp[\sigma(x)]$. Goal is to solve for $\sigma(x)$, given vortensity $q(x)$. Two equations: vortensity definition, and one relating u_y to σ — combining gives 2nd order ODE for σ. Linearise in small σ (this affects the exponential term only). Given step function for vortensity with regime switch at some value d; BCs are $\sigma \to 0$ as $|x| \to \infty$, continuity of σ and $\partial_x\sigma$ at d, and $\partial_x\sigma = 0$ at $x = 0$ (symmetry). Last part on time-dependent non-axisymmetric perturbations in $\mathbf{u}$ and Σ, where e.g. $\mathbf{u}' = \tilde{\mathbf{u}}(t)\exp[ik_x(t)x + ik_y y]$. In order for linearised perturbation equations to be 'consistent', need terms involving x to vanish — gives time evolution of k_x — this is the same result and condition in the notes (3D shearing box, non-linearised, but crucially incompressible. The incompressibility condition actually causes the nonlinear terms to vanish). To show vortensity perturbation $\tilde{q} = \mathrm{cst.}$, use vortensity conservation. Obtain form for $\tilde{\Sigma}/\Sigma_0$ in terms of components of $\tilde{\mathbf{u}}$ from linearising definition of vortensity, setting $\tilde{q} = 0$. Substituting back into perturbed EoM gives $d\tilde{\mathbf{u}}/dt = \mathbf{A}(t)\tilde{\mathbf{u}}$ where $\mathbf{A}$ is a 2x2 matrix. When $k_y = 0$, recover dispersion relation for non-SG acoustic-inertial (density) waves from eigenvalues of $\mathbf{A}$.