TOPICS COVERED

- More examples of discretisation of non-linear BVPs by Newton's method & quasilinear method.
- Intro to Spline interpolation

EXAMPLE (Recap of linearisation techniques)

$$y'' = 2 + y^2$$

$$y(0) = y(1) = 0$$

$$y'' - y^2 - 2 = 0$$

Discretise by Newton's linearisation technique:

$$\underbrace{\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} - y_i^2 - 2}_{F(y_{i+1},\, y_i,\, y_{i-1})} = 0$$

At any iteration, $y_i^{(k+1)} = y_i^{(k)} + \Delta y_i$

$$\frac{1}{h^2}\left[\left(y_{i+1}^{(k)} + \Delta y_{i+1}\right) - 2\left(y_i + \Delta y_i\right) + \left(y_{i-1} + \Delta y_{i-1}\right)\right]$$

$$- \left(y_i^{(k)} + \Delta y_i^{(k)}\right)^2 - 2 = 0 \quad , i = 1, 2, \dots, N$$

Unknowns are Δy_i's, expand and retain the terms only up to linear order of Δy_i's.

$$a_i \Delta y_{i-1} + b_i \Delta y_i + c_i \Delta y_{i+1} = d_i \, ,$$
$$i = 1, 2, \dots, N-1$$

Note: $y_i^{(k)}$ are known

$$\underbrace{\frac{1}{h^2}}_{a_i} \Delta y_{i-1} \underbrace{- \left(-\frac{2}{h^2} + 2y_i^{(k)}\right)}_{b_i} \Delta y_i + \underbrace{\frac{1}{h^2}}_{c_i} \Delta y_{i+1}$$

$$= \underbrace{2 - \frac{y_{i+1}^{(k)} - 2y_i^{(k)} + y_{i-1}^{(k)}}{h^2} + \left(y_i^{(k)}\right)^2}_{d_i}$$

*we do not need to store all $y_i^{(k)}$ $\forall k$, just last 2 are needed.

EXERCISE

$$f''' + ff'' + \beta(1 - f'^2) = 0$$

$$f(0) = f'(0) = 0, \quad f'(10) = 1$$

β is a real number.

This eq^n is used to model flow of fluids.

Ans. $F = f'$

$$F'' + fF' + 1 - F^2 = 0$$

Step 1: Discretise

$$\frac{h}{2}(F_i + F_{i-1}) = f_i - f_{i-1}$$

$$\frac{F_{i+1} - 2F_i + F_{i-1}}{h^2} + f_i \frac{F_{i+1} - F_{i-1}}{2h} + \beta(1 - F_i^2) = 0$$

(1)

$$(f_i + \Delta f_i) - (f_{i-1} + \Delta f_{i-1}) - \frac{h}{2}\left((F_i + \Delta F_i) + (F_{i-1} + \Delta F_{i-1})\right) = 0$$

(2)

$$\frac{1}{h^2}\left[(F_{i+1} + \Delta F_{i+1}) - 2(F_i + \Delta F_i) + (F_{i-1} + \Delta F_{i-1})\right]$$

$$+ \frac{(f_i + \Delta f_i)}{2h}\left[(F_{i+1} + \Delta F_{i+1}) - (F_{i-1} + \Delta F_{i-1})\right]$$

$$+ \beta(1 - F_i^2 - 2\Delta F_i F_i - \Delta F_i^2) = 0$$

$$\begin{pmatrix} -1 & -h/2 \\ 0 & \frac{1}{h^2} - \frac{f_i^{(k)}}{2h} \end{pmatrix} = A_i \qquad \begin{pmatrix} 1 & -h/2 \\ \frac{F_{i+1}^{(k)} - F_{i-1}^{(k)}}{2h} & -\frac{2}{h^2} - 2\beta F_i \end{pmatrix} = B_i \qquad \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{h^2} + \frac{f_i^{(k)}}{2h} \end{pmatrix} = C_i$$

$$\begin{pmatrix} -f_i^{(k)} + f_{i-1}^{(k)} + \frac{h}{2}(F_i^{(k)} + F_{i-1}^{(k)}) \\ -\beta + \beta(f_i^{(k)})^2 - \frac{F_{i+1}^{(k)} - 2F_i^{(k)} + F_{i-1}^{(k)}}{h^2} \end{pmatrix} = D_i$$

$$\max \{ |\Delta f_i|, |\Delta F_i| \}$$

Possible initial guess-

$$f^{(0)} = \frac{x^2}{20}$$

EXERCISE

$$f''' + ff'' + \beta(1 - f'^2) = 0$$

$$f(0) = f'(0) = 0, \quad f'(10) = 1$$

β is a real number.

At every iteration obtain the quasi-linear form.

Ans.

$$F(f''', f'', f', f) = 0$$

Let $f^{(k)}, f'^{(k)}, f''^{(k)}, f'''^{(k)}$ be the approximate form of f, f', f'', f''' respectively at any iteration k.

$$F(f^{(k)}, f'^{(k)}, f''^{(k)}, f'''^{(k)}) + (f - f^{(k)}) \left.\frac{\partial F}{\partial f}\right|^{(k)} + (f' - f'^{(k)}) \left.\frac{\partial F}{\partial f'}\right|^{(k)} + (f'' - f''^{(k)}) \left.\frac{\partial F}{\partial f''}\right|^{(k)} + (f''' - f'''^{(k)}) \frac{\partial F}{\partial f'''} + \cdots$$

Neglecting the square and higher order terms, we get the modified form

$$(f - f^{(k)}) \left.\frac{\partial F}{\partial f}\right|^{(k)} + (f' - f'^{(k)}) \left.\frac{\partial F}{\partial f'}\right|^{(k)} + (f'' - f''^{(k)}) \left.\frac{\partial F}{\partial f''}\right|^{(k)} + (f''' - f'''^{(k)}) \frac{\partial F}{\partial f'''} = F(f^{(k)}, f'^{(k)}, f''^{(k)}, f'''^{(k)})$$

In $F = f''' + f f'' + \beta(1 - f'^2)$

$$\frac{\partial F}{\partial f} = f''^{(k)}$$

$$\frac{\partial F}{\partial f'} = -2\beta f'^{(k)}$$

$$\frac{\partial F}{\partial f''} = f^{(k)}$$

$$\frac{\partial F}{\partial f'''} = 1$$

$$(f - f^{(k)})\left(f''^{(k)}\right) + \left(f' - f'^{(k)}\right)\left(-2\beta f'^{(k)}\right)$$

$$+ \left(f'' - f''^{(k)}\right)\left(f^{(k)}\right) + \left(f''' - f'''^{(k)}\right)$$

$$= F\left(f^{(k)}, f'^{(k)}, f''^{(k)}, f'''^{(k)}\right)$$

$$\Rightarrow \quad f'''^{(k+1)} + f^{(k)} f''^{(k+1)} + f''^{(k)} f^{(k+1)} - 2\beta f'^{(k)} f'^{(k+1)}$$
$$= f^{(k)} f''^{(k)} - \beta\left(f'^{(k)}\right)^2 - \beta$$

... REDUCED LINEAR BVP.

Now use central Block tri-diagonal system.

Let $g^{(k+1)} = f'^{(k+1)}$ ...(i)

$\therefore$ The equation becomes

$$g''^{(k+1)} + f^{(k)} g'^{(k+1)} - 2\beta g^{(k)} g^{(k+1)} + g'^{(k)} f^{(k+1)} = f^{(k)} g'^{(k)} - \beta (g^{(k)})^2 - \beta \quad \text{...(ii)}$$

where $f^{(k+1)}(0) = g^{(k+1)}(0) = 0, \quad g^{(k+1)}(10) = 1$

Eq. (i) using trapezoidal rule & simplifying gives us:

$$-\frac{h}{2}\left(g_i^{(k+1)} + g_{i-1}^{(k+1)}\right) + \left(f_i^{(k+1)} - f_{i-1}^{(k+1)}\right) = 0 \quad \text{... (i)a}$$

Eq. (ii) using central difference method, gives us:

$$\frac{1}{h^2}\left(g_{i+1}^{(k+1)} - 2g_i^{(k+1)} + g_{i-1}^{(k+1)}\right) + \frac{f_i^{(k)}}{2h}\left(g_{i+1}^{(k+1)} - g_{i-1}^{(k+1)}\right)$$

$$-2\beta g_i^{(k)} g_i^{(k+1)} + g_i'^{(k)} f_i^{(k+1)} = f_i^{(k)} g_i'^{(k)} - \beta (g_i^{(k)})^2 - \beta$$

$$\Rightarrow g_{i-1}^{(k+1)}\left[\frac{1}{h^2} - \frac{f_i^{(k)}}{2h}\right] + g_i^{(k+1)}\left[-\frac{2}{h^2} - 2\beta g_i^{(k)}\right]$$

$$+ g_{i+1}^{(k+1)}\left[\frac{1}{h^2} + \frac{f_i^{(k)}}{2h}\right] + g_i'^{(k)} f_i^{(k+1)}$$

$$A_i = \begin{pmatrix} -1 & -h/2 \\ 0 & \frac{1}{h^2} - \frac{f_i^{(k)}}{2h} \end{pmatrix} \qquad C_i = \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{h^2} + \frac{f_i^{(k)}}{2h} \end{pmatrix}$$

$$B_i = \begin{pmatrix} 1 & -h/2 \\ g_i'^{(k)} & -\frac{2}{h^2} - 2\beta g_i^{(k)} \end{pmatrix} \qquad D_i = \begin{pmatrix} 0 \\ \sim - \end{pmatrix}$$

NEW TOPIC:

Spline Interpolation

Drawbacks of current system:

- Require problem to form a tri-diagonal system

Pros:

- 2nd order accuracy
  - But maybe we can do better, & that too w/o much hassle.

"compact difference scheme" → HIGHER ORDERS

- f^n & its derivative are both considered to be unknown,
- → Hermite interpolation technique.

x	y
x_0	y_0
x_1	y_1
$\vdots$	$\vdots$
x_n	y_n

$P_n(x) = y$ s.t. $y_i = P_n(x_i)$

$i = 0, 1, \dots, n$.

x_i → node/grid points

usually associated w/ (P) Diff. eqs.

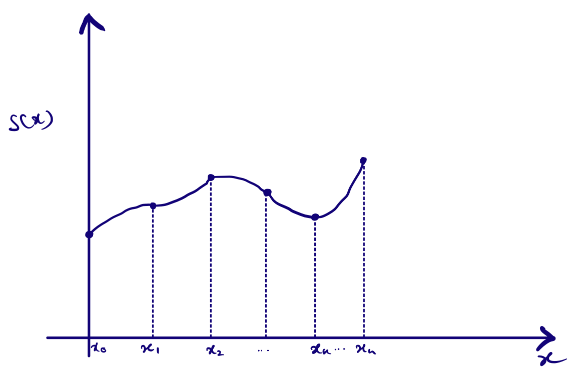
- Consider piece wise - interpol.
  - instead of taking whole interval (x_0, x_n), we take $p^{(k)}(x)$ as a polynomial of any degree (not necessarily k), which interpolates $y(x)$ in $[x_k, x_{k+1}]$

  for $k = 0, 1, \dots, n-1$.

→ $S(x)$ is a piece-wise interpolation polynomial.

$\rightarrow$ $S(x) = \{ p^{(k)}(x) \mid x_k \leq x \leq x_{k+1} , k = 0, 1, \dots, n-1 \}$

Recall : $p^{(k)}(x)$ is a cubic spline in $[x_k, x_{k+1}]$ if $p^{(k)}(x)$ is a cubic polynomial and has the same slope and curvature at e/
— (knot point $x_1, x_2, \dots, x_{n-1}$ where $p^{(k)}(x)$) are joined.



- $p^{(k)}(x)$ is a cubic polynomial
- $p^{(k)}(x_k) = y_k$, $p^{(k)}(x_{k+1}) = y_{k+1}$ $\forall k \in \{0, 1, \dots, n-1\}$

Now, how do we impose the co

- $p_k(x)$ in $[x_k, x_{k+1}]$
- $p_{k-1}(x)$ in $[x_{k-1}, x_k]$

$S(x) = \{ p_k(x) \mid x_k \leq x \leq x_{k+1} , k = 0, 1, \dots, n-1 \}$

$p^{(k)}(x_k) = p^{(k-1)}(x_k) = y_k$

$p^{(k)\prime}(x_k) = p^{(k-1)\prime}(x_k)$

$p^{(k)\prime\prime}(x_k) = p^{(k-1)\prime\prime}(x_k)$

Now, $S(x)$ is piecewise cubic in $[x_0, x_n]$, so

$S''(x)$ is piecewise linear in $[x_0, x_n]$.

$S''(x) = p^{(k)''}(x)$ in $[x_k, x_{k+1}]$

is a linear polynomial.

Let $M_k = S''(x_k)$, $M_{k+1} = S''(x_{k+1})$

Then

$$p^{(k)''}(x) = M_k \frac{(x - x_{k+1})}{(x_k - x_{k+1})} + M_{k+1} \frac{(x - x_k)}{(x_{k+1} - x_k)}$$

Note: M_k, M_{k+1} are given.

Let $h = x_{k+1} - x_k$ be constant for equi-spaced points $x_0, x_1, \dots, x_n$.

So $$p^{(k)''}(x) = \frac{M_k}{h}(x_{k+1} - x) + \frac{M_{k+1}}{h}(x - x_k)$$

$$\Rightarrow p^{(k)}(x) = \frac{M_k}{h}\frac{(x_{k+1} - x)^3}{6} + \frac{M_{k+1}}{h}\frac{(x - x_k)^3}{6}$$

$$C_k (x - x_k) + D_k (x_{k+1} - x)$$

$$C_k = \frac{1}{h}\left(y_{k+1} - M_{k+1}\frac{h^2}{6}\right)$$

$$D_k = \frac{1}{h}\left(y_k - M_k\frac{h^2}{6}\right)$$

$$p_k(x) = \frac{M_k}{6}\left[\frac{(x_{k+1}-x)^3}{h} - h(x_{k+1}-x)\right]$$

$$+ \frac{y_k}{h}(x_{k+1}-x) + \frac{M_{k+1}}{6}\left[\frac{(x-x_k)^3}{h} - h(x-x_k)\right]$$

$$+ \frac{y_{k+1}}{h}(x-x_k) \qquad , \quad k=0,1,\cdots,n-1$$

We have to determine M_k s , i.e. $M_0, M_1, \dots, M_n \rightarrow (n+1)$ unknowns

We impose the condition

$$p_k'(x_k) = p_{k-1}'(x_k) \quad \text{at } k=1,2,\dots,n-1.$$

$$p_k'(x) = \frac{M_k}{6}\left[\frac{-3(x_{k+1}-x)^2}{h} + h\right] - \frac{y_k}{h}$$

$$+ \frac{M_{k+1}}{6}\left[\frac{3(x-x_k)^2}{h} - h\right] + \frac{y_{k+1}}{h}$$

$$\Rightarrow \quad p_k'(x_k) = \frac{M_k}{6}\left[-2h\right] + \frac{M_{k+1}}{6}\left[-h\right] + \frac{\Delta y_k}{h}$$

$$= -\frac{h}{3}M_k - \frac{h}{6}M_{k+1} + \frac{\Delta y_{k+1}}{h}$$

& Solve for

$p_k'(x_{k+1})$, derive the eq^n to get it in terms of M_k & Δy_k (derive again, from start).

You will obtain a tri-diagonal system.