

TOPICS COVERED

1. Solving 3rd order BVP by BLOCK TRIDIAGONAL METHOD.
2. Solving special cases of 4th order BVPs
3. Introduction of non-linear BVP.
Recap of Newton Raphson method.

HIGHER ORDER BVP

$$y''' + A(x)y'' + B(x)y' + C(x)y = D(x)$$

$$y(0) = \alpha, \quad y'(0) = \varepsilon$$

$$y'(L) = \lambda$$

$$0 < x < L$$

Using central diff method, 2 issues:

L (1) Do not get tri-diagonal system.

(2) # unknowns > # equations

We reduce 3rd order eq["] to 2nd order:

COPLED EQ's,
note: z depends on y

$$\begin{cases} z = y' & (i) \\ z'' + A(x)z' + B(x)z + C(x)y = D(x) & (ii) \\ y(0) = \alpha \\ z(0) = \varepsilon \\ z(L) = \lambda \end{cases}$$

- Integrate (i) between x_{i-1}, x_i using TRAPEZOIDAL RULE ($\because z$ is an unknown function of y)

$$\frac{dy}{dx} = z$$

$$\Rightarrow \int_{x_{i-1}}^{x_i} dy = \int_{x_{i-1}}^{x_i} z dx$$

$$\Rightarrow y_i - y_{i-1} - \frac{h x_i}{2} (z_i + z_{i-1}) = 0$$

$$\Rightarrow y_i - y_{i-1} - \frac{h}{2} (z_i + z_{i-1}) = 0$$

- Use central diff. to discretise (ii)

You will have $2(n-1)$ eq^h involving y_1, y_2, \dots, y_{n-1}

& z_1, z_2, \dots, z_{n-1}

$\rightarrow 2(n-1)$ var.,

$$\frac{z_{i+1} - 2z_i + z_{i-1}}{h^2} + A_i \frac{z_{i+1} - z_{i-1}}{2h} + B_i z_{i-1} + C_i y_i = D_i$$

$$i = 1, \dots, n-1$$

Combine ① and ② into a matrix equation.

$$Z_i = \begin{pmatrix} y_i \\ z_i \end{pmatrix}$$

and express the paired equations in matrix form as follows:

$$\bar{A}_i \bar{Z}_{i-1} + \bar{B}_i \bar{Z}_i + \bar{C}_i \bar{Z}_{i+1} = \bar{D}_i$$

$$\Rightarrow \begin{pmatrix} -1 & -\frac{h}{2} \\ 0 & \frac{1}{h^2} - \frac{A_i}{2h} \end{pmatrix} \begin{pmatrix} y_{i-1} \\ z_{i-1} \end{pmatrix} + \begin{pmatrix} 1 & -\frac{h}{2} \\ C_i & -\frac{2}{h^2} + B_i \end{pmatrix} \begin{pmatrix} y_i \\ z_i \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{h^2} + \frac{A_i}{2h} \end{pmatrix} \begin{pmatrix} y_{i+1} \\ z_{i+1} \end{pmatrix} = \begin{pmatrix} 0 \\ D_i \end{pmatrix}$$

Here

$$\bar{A} = \begin{pmatrix} -1 & -\frac{h}{2} \\ 0 & \frac{1}{h^2} - \frac{A_i}{2h} \end{pmatrix}, \quad \bar{B} = \begin{pmatrix} 1 & -\frac{h}{2} \\ C_i & -\frac{2}{h^2} + B_i \end{pmatrix}$$

$$\bar{C} = \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{h^2} + \frac{A_i}{2h} \end{pmatrix}, \quad \bar{D} = \begin{pmatrix} 0 \\ D_i \end{pmatrix}$$

$$\Rightarrow \bar{A}_i \bar{Z}_{i-1} + \bar{B}_i \bar{Z}_i + \bar{C}_i \bar{Z}_{i+1} = \bar{D}_i$$

y_0, z_0 are known (given as B.C.s)
 and y_n does not enter this system,
 while z_n is known (B.C.),

we take $i = 1$ to $n-1$

At $i=1$

$$\bar{A}_1 \bar{Z}_0 + \bar{B}_1 \bar{Z}_1 + \bar{C}_1 \bar{Z}_2 = \bar{D}_1$$

\bar{A}_1 and \bar{Z}_0 are known
 $\Rightarrow \bar{B}_1 \bar{Z}_1 + \bar{C}_1 \bar{Z}_2 = \bar{D}_1 - \bar{A}_1 \bar{Z}_0$

for $i = 2$ to $n-2$

$$\bar{A}_i \bar{Z}_{i-1} + \bar{B}_i \bar{Z}_i + \bar{C}_i \bar{Z}_{i+1} = \bar{D}_i$$

At $i = n-1$

$$\bar{A}_{n-1} \bar{Z}_n + \bar{B}_{n-1} \bar{Z}_{n-1} + \bar{C}_{n-1} \bar{Z}_n = \bar{D}_{n-1}$$

Now here, in the product $\bar{C}_{n-1} \bar{Z}_n$, all values are known.
 $\therefore \bar{A}_{n-1} \bar{Z}_n + \bar{B}_{n-1} \bar{Z}_{n-1} = \bar{D}_{n-1} - \bar{C}_{n-1} \bar{Z}_n$

This can be expressed as a tri-diagonal system-

$$\begin{bmatrix} \bar{B}_1 & \bar{C}_1 & 0 & \cdots & 0 & 0 & 0 \\ \bar{A}_2 & \bar{B}_2 & \bar{C}_2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & \bar{A}_{n-1} & \bar{B}_{n-1} \end{bmatrix} \begin{bmatrix} Z_1 \\ Z_2 \\ \vdots \\ Z_{n-1} \end{bmatrix} = \begin{bmatrix} \bar{D}_1 - \bar{A}_1 \bar{Z}_0 \\ \bar{D}_2 \\ \vdots \\ \bar{D}_{n-1} - \bar{C}_{n-1} \bar{Z}_n \end{bmatrix}$$

$$\mathbf{A} \quad \mathbf{X} = \mathbf{B}$$

This kind of matrix is called a BLOCK-TRIDIAGONAL MATRIX.

which can be reduced to

$$\begin{bmatrix} I & M_1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & I & M_2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & I \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_{n-1} \end{bmatrix} = \begin{bmatrix} \bar{D}'_1 \\ \bar{D}'_2 \\ \vdots \\ \bar{D}'_{n-1} \end{bmatrix}$$

ALGORITHM for BLOCK ELIMINATION METHOD:

Here

$$M_1 = (\bar{B}_1)^{-1} \bar{C}_1 \quad \text{and} \quad \bar{D}'_1 = (\bar{B}_1^{-1})(\bar{D}_1 - \bar{A}_1 \bar{Z}_0)$$

and for $i=2, \dots, n-1$,

$$\bar{B}'_i = \bar{B}_{i+1} - \bar{A}_i M_{i-1}$$

$$M_i = (\bar{B}'_i)^{-1} \bar{C}_i, \quad \bar{D}'_i = (\bar{B}'_i)^{-1} (\bar{D}_i - \bar{A}_i \bar{D}'_{i-1})$$

$$\text{where } Z_{n-1} = \bar{D}_{n-1}$$

$$Z_i = \bar{D}'_i - M_i Z_{i+1} \quad i = n-2, n-3, \dots, 2, 1$$

This method does not give us $(\frac{y_n}{z_n})$ directly.

Will have to do a substitution to get the value.

$$Q. \quad y''' + 4y'' + y' - 6y = 1$$
$$y(0) = y'(0) = 0, \quad y(1) = 1$$
$$h = 0.25$$

NEXT CLASS: 4th order coupled system .

$$Q. \quad y''' + A(x)y' + B(x)y' + C(x)y = D(x)$$
$$y'(0) = \alpha$$
$$y(L) = \beta$$
$$y'(L) = \kappa$$

Block condition can also be acquired in other cases,

e.g.

$$y''' + 81y = 81x^2$$

$$y(0) = y(1) = y''(0) = y''(1) = 0$$

$$\text{Let } p = y'' \Rightarrow y'' - p = 0$$

$$p'' + 81y = 81x^2$$

$$y''_i - p_i = \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} - p_i = 0$$

$$\frac{p_{i+1} - 2p_i + p_{i-1}}{h^2} + 81y_i = 81x_i^2$$

$$\underbrace{\begin{pmatrix} \frac{1}{h^2} & 0 \\ 0 & \frac{1}{h^2} \end{pmatrix}}_{\bar{A}} \begin{pmatrix} y_{i-1} \\ p_{i-1} \end{pmatrix} + \underbrace{\begin{pmatrix} -\frac{2}{h^2} & -1 \\ 81 & -\frac{2}{h^2} \end{pmatrix}}_{\bar{B}} \begin{pmatrix} y_i \\ p_i \end{pmatrix} + \underbrace{\begin{pmatrix} \frac{1}{h^2} & 0 \\ 0 & \frac{1}{h^2} \end{pmatrix}}_{\bar{C}} \begin{pmatrix} y_{i+1} \\ p_{i+1} \end{pmatrix} = \begin{pmatrix} 0 \\ 81x_i^2 \end{pmatrix}$$

$$h = 1/3, n = 3, n-1 = 2$$

$$i = 1$$

$$\begin{pmatrix} 9 & 0 \\ 0 & 9 \end{pmatrix} \begin{pmatrix} y_0 \\ p_0 \end{pmatrix} + \begin{pmatrix} -18 & -1 \\ 81 & -18 \end{pmatrix} \begin{pmatrix} y_1 \\ p_1 \end{pmatrix} + \begin{pmatrix} 9 & 0 \\ 0 & 9 \end{pmatrix} \begin{pmatrix} y_2 \\ p_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 81(\frac{1}{9}) \end{pmatrix}$$

~~$$9y_0 - 18y_1 - p_1 + 9y_2 = 0$$~~

~~$$9p_0 + 81y_1 - 18p_1 + 9p_2 = 9$$~~

$$\begin{pmatrix} 9 & 0 \\ 0 & 9 \end{pmatrix} \begin{pmatrix} y_1 \\ p_1 \end{pmatrix} + \begin{pmatrix} -18 & -1 \\ 81 & -18 \end{pmatrix} \begin{pmatrix} y_2 \\ p_2 \end{pmatrix} + \begin{pmatrix} 9 & 0 \\ 0 & 9 \end{pmatrix} \begin{pmatrix} y_3 \\ p_3 \end{pmatrix} \\ = \begin{pmatrix} 0 \\ 81 \left(\frac{4}{9} \right) \end{pmatrix} = \begin{pmatrix} 0 \\ 36 \end{pmatrix}$$

$$9y_1 - 18y_2 - P_2 + \cancel{9y_3} = 0$$

$$9P_1 + 81y_2 - 18P_2 + \cancel{9P_3} = 36$$

o Straight-forward Method

$$\text{Q. } y_i^{(IV)} = y_i'' = \frac{y_{i+1}'' - 2y_i'' + y_{i-1}''}{h^2} \\ = \frac{1}{h^2} \left[\frac{1}{h^2} \left\{ y_{i+2} - 2y_{i+1} + y_i \right. \right. \\ \left. \left. - 2y_{i+1} + 4y_i - 2y_{i-1} \right. \right. \\ \left. \left. y_i - 2y_{i-1} + y_{i-2} \right\} \right] \\ = \frac{1}{h^4} \left[y_{i+2} - 4y_{i+1} + 6y_i - 4y_{i-1} + y_{i-2} \right]$$

Substituting this in

$$y^{(IV)} + 81y = 81x^2$$

$$y(0) = y(1) = y''(0) = y''(1) = 0$$

and multiplying by h^4 on both sides,
we get

$$y_{i+2} - 4y_{i+1} + 6y_i - 4y_{i-1} + y_{i-2} + 81h^4 y_i = 81h^4 x_i^2$$

$$y''_0 = y''_n = 0 \Rightarrow y_0 = y_n$$

By introducing y_{-1} and y_{n+1} ,

$$y''_0 = \frac{y_1 - 2y_0 + y_{-1}}{h^2} \Rightarrow y_{-1} = 2y_0 - y_1 \\ = -y_1$$

$$y''_n = \frac{y_{n+1} - 2y_n + y_{n-1}}{h^2} \Rightarrow y_{n+1} = -y_{n-1}$$

Now, taking $h = \frac{1}{3}$, write the equation for
 $i=1, 2$

$$\underline{i=1}$$

$$\cancel{y_3} - 4y_2 + 6y_1 - \cancel{4y_0} + \cancel{y_{-1}} + 81\left(\frac{1}{3}\right)^4 y_1 = 81\left(\frac{1}{3}\right)\left(\frac{1}{3}\right)^2$$

$$-4y_2 + 6y_1 = \frac{1}{9} \quad \dots (a)$$

$$\underline{i=2}$$

$$\cancel{-y_2} \cancel{y_4} - 4y_3 + 6y_2 - 4y_1 + \cancel{y_0} + 81\cancel{h^4} \cancel{y_2} = 81\left(\frac{1}{3}\right)\left(\frac{4}{9}\right)$$

$$6y_2 - 4y_1 = \frac{4}{9} \quad \dots (b)$$

$$6(a) + 4(b) \Rightarrow (36 - 16)y_1 = \frac{6 + 16}{9}$$

$$\Rightarrow \frac{10}{28} y_1 = \frac{11}{9}$$

$$\Rightarrow y_1 = \frac{11}{90}$$

$$\Rightarrow y_2 = \frac{1}{4} \left(\frac{66}{90} - \frac{10}{90} \right) = \frac{1}{4} \times \frac{56}{90} \text{ us}$$

$$\text{Ans. } y_1 = \frac{11}{90}, \quad y_2 = \frac{7}{45}$$

Q. $y^{(n)} + 81y = \phi(x)$
 $\phi(x)$ is given by the table:

Our goal is to
get a compact
system.

x	y_3	$2/3$	1
$\phi(x)$	81	162	243

$$y(0) = y'(0) = y''(1) = y'''(1) = 0$$

Take $h = y_3$

Ans. Introduce fictitious points y_{-1}, y_{n+1}
i.e. y_{-1} and y_n

$$y'_0 = 0 \Rightarrow y_{-1} = y_1$$

$$y''_3 = 0 \Rightarrow \frac{y_4 - 2y_3 + y_2}{h^2} = 0$$

$$\Rightarrow y_4 = 2y_3 - y_2$$

$$y'''_3 = 0 \Rightarrow (y''_3)' = \frac{y''_4 - y''_2}{2h}$$

$$= \frac{y_5 - 2y_4 + y_3}{2h} - \frac{y_3 - 2y_2 + y_1}{2h}$$

$$\Rightarrow y'''_3 = \frac{y_5 - 2y_4 + 2y_2 - y_1}{2h} = 0$$

$$\Rightarrow y_5 = 2y_4 - 2y_2 + y_1$$

Discretise for $i = 1, 2, 3$

$$y_{i+2} - 4y_{i+1} + 6y_i - 4y_{i-1} + y_{i-2} + 81h^4 y_i = h^4 \phi(x_i)$$

$i=1$

$$y_3 - 4y_2 + \underline{6y_1} - \cancel{4y_0} + \cancel{y_{-1}} + 81h^4 y_1 = \frac{1}{81} \cdot 81$$

$$\Rightarrow y_3 - 4y_2 + 8y_1 = 1$$

$i=2$

$$y_4 - 4y_3 + 6y_2 - 4y_1 - \cancel{y_0} + \cancel{81h^4 y_2} = \frac{1}{81} \cdot \cancel{\frac{162}{81}}$$

$$\Rightarrow 2y_3 - y_2 - 4y_3 + 7y_2 - 4y_1 = 2$$

$$\Rightarrow -2y_3 + 6y_2 - 4y_1 = 2$$

$i=3$

$$y_5 - 4y_4 + 6y_3 - 4y_2 + y_1 + 81h^4 y_3 = \frac{1}{81} \cancel{\frac{243}{81}}$$

$$\Rightarrow 2y_4 - 2y_3 + y_1 - 4y_4 + 6y_3 - 4y_2 + y_1 + y_3 = 3$$

$$\Rightarrow -2y_4 + 7y_3 - 6y_2 + 2y_1 = 3$$

$$\Rightarrow -4y_3 + 2y_2 + 7y_3 - 6y_2 + 2y_1 = 3$$

$$\Rightarrow 3y_3 - 4y_2 + 2y_1 = 3$$

which upon solving gives us

$$y_1 = \frac{8}{13}, \quad y_2 = \frac{22}{13}, \quad y_3 = \frac{37}{13}$$

$$Q. \quad y'' + A(x)y' + B(x)y + C(x)y = D(x)$$

$$y'(0) = \alpha$$

$$y(L) = \beta$$

$$y'(L) = \kappa$$

How to reduce the system?

$$y' - z = 0$$

TRICK:

Integrate between x_i and x_{i+1} instead.

$$\int_{x_i}^{x_{i+1}} dy - \int_{x_i}^{x_{i+1}} zdz = 0$$

$$y_{i+1} - y_i - \frac{h}{2}(z_i + z_{i+1}) = 0$$

$$i = 1, 2, \dots, (n-1)$$

$$Q. \quad y''' + 4y'' + y' - 6y = 1$$

$$y(0) = y'(0) = 0, \quad y'(1) = 1$$

$$h = 0.1, 0.05, 0.01$$

"optimal step size".

$$Q. \quad y'' + 2y = \frac{x^2}{9} + \frac{2}{3}x + 4$$

$$y(0) = y'(0) = y(3) = y'(3) = 0$$

$$h = 1.$$

$$Q. \quad y^{IV} - y''' + y = x^2$$
$$y(0) = y'(0) = 0, \quad y(1) = 2, \quad y'(1) = 0$$
$$h = y_3$$

NON-LINEAR BVP

$$G(y, y', y'') = 0$$

e.g. (arbitrary)

$$y'' + (y')^2 + y^2 = x \quad , \quad a < x < b$$

$$y(a) = y_a, \quad y(b) = y_b$$

$$x_i = a + i\Delta x = a + ih \quad , \quad i = 1, 2, \dots, N-1$$

$$y_0 = y_a \quad , \quad y_N = y_b$$

$$\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} + \frac{(y_{i+1} - y_{i-1})^2}{4h^2} + y_i^2 = x_i$$

$i = 1, 2, \dots, N-1$

forms a system of $(N-1)$ equations with $(N-1)$ variables.
 PRO \downarrow a compact system
 CON \downarrow cannot be expressed in matrix form due to non-linear nature.

NEWTON RAPHSON METHOD for solving nonlinear equations iteratively .

$$\phi(x) = 0 \quad , \quad x \text{ is the root.}$$

Let α_k be an approximation of α .

$$\alpha = \alpha_k + \text{Error} = \alpha_k + \Delta \alpha$$

$\Delta \alpha \rightarrow \text{unknown.}$

$$\phi(\alpha_k + \Delta\alpha) = 0 \quad \dots (i)$$

Expand (i) by Taylor Series -

$$\phi(\alpha_k) + \Delta\alpha \phi'(\alpha_k) + \frac{(\Delta\alpha)^2}{2!} \phi''(\alpha_k) + \dots = 0$$

The unknown $\Delta\alpha$ satisfies the infinite degree polynomial given above.

If we retain only the terms upto linear order of $\Delta\alpha$, we get

$$\phi(\alpha_k) = \Delta\alpha + \phi'(\alpha_k) \approx 0$$

$$\Delta\alpha = -\frac{\phi(\alpha_k)}{\phi'(\alpha_k)}$$

$$\alpha_{k+1} = \alpha_k + \Delta\alpha$$

$$\Rightarrow \alpha_{k+1} = \alpha_k - \frac{\phi(\alpha_k)}{\phi'(\alpha_k)}, \quad k=0, 1, \dots$$

$\alpha_{k+1} \rightarrow$ next approximation for α .

We repeat the above procedure till $|\phi(\alpha_k)| < \epsilon$.

To start the iteration we consider an initial approximation α_0 .

NR method has a quadratic order of convergence.

↳ Convergence depends on initial guess (α_0) .

CAUCHY SEQUENCE (recall - Real Analysis)

$$\{\alpha_k \mid k \geq 0\}, \quad \alpha_k \rightarrow \alpha \text{ as } k \rightarrow \infty$$

A sequence $X = (x_n) \in \mathbb{R}$ is said to be a Cauchy sequence if $\forall \epsilon > 0 \exists N(\epsilon) \in \mathbb{N}$ s.t. $\forall N \in \mathbb{N} \exists n, m \geq N$,
 $|x_n - x_m| < \epsilon$.

Similarly here, for convergence we can check

$$|\alpha_{k+1} - \alpha_k| < \epsilon$$

for a pre-assigned $\epsilon > 0$.