- Intro to SOR
- Hyperbolic PDEs

Given the Laplace Equation:

$$\nabla^2 U = 0 , \quad 0 \leq x, \ y \leq 1$$

$$\text{B.C.}: \begin{cases} U = 2x , & y = 0 \\ U = 2x-1 , & y = 1 \end{cases} , \quad 0 < x < 1$$

$$\begin{cases} U_x + U = 2-y , & x = 0 \\ U = 2-y , & x = 1 \end{cases} , \quad 0 \leq y \leq 1$$

$$h = k = 1/3$$

# Convergence Acceleration

$$U_{ij}^{(k+1)} = \frac{1}{2(1+\beta^2)} \left[ U_{i+1,j}^{(k)} + U_{i-1,j}^{(k+1)} + \beta^2 \left( U_{i,j+1}^{(k)} + U_{i,j-1}^{(k+1)} \right) \right]$$

$k \geq 0$

Check & confirm later

$$| \bar{U}_{ij} - U_{ij}^{(k)} | > \alpha \, | \bar{U}_{ij} - \bar{U}_{ij}^{(k+1)} |$$

$\alpha < 1$

# Successive Over-Relaxation Technique

Let $U_{ij}^{(k+1)}$ be the Gauss Seidel iteration at the $(k+1)^{th}$ iter.

To accelerate the convergence, consider

$$\bar{U}_{ij}^{(k+1)} = \underbrace{\bar{U}_{ij}^{(k)}}_{\text{modified value at } k\text{th iter.}} + \overbrace{\omega}^{\text{relaxation parameter}} \left( \bar{U}_{ij}^{(k+1)} - \bar{U}_{ij}^{(k)} \right)$$

Overrelaxation if $1 < \omega < 2$.

$$\bar{U}_{ij}^{(k+1)} = U_{ij}^{(k+1)} \quad \text{if } \omega = 1$$

Under relaxation if $0 < \omega < 1$

Q. $\nabla^2 T = 100$, $0 < x < 3$, $0 < y < 6$

B.C.: $\begin{cases} T = 0, & x = 0 \\ T = 200, & x = 3 \end{cases}$

$\begin{cases} \dfrac{\partial T}{\partial y} = 100, & y = 0 \\ \dfrac{\partial T}{\partial y} = -100, & y = 6 \end{cases}$

(multigrid, SSOR..)

Q. $-\nabla^2 U + 0.1 U = 1$, $0 \leq x, y \leq 1$

$U = 0$ on $x = 0$, $y = 0$

$\dfrac{\partial U}{\partial n} = 0$, $x = 1$, $y = 1$

$\delta x = \delta y = 0.5$

Q. $\nabla^2 U = 2\dfrac{\partial U}{\partial x} - 2$, $0 \leq x, y \leq 1$

B.C.: $U = 0$ on boundary

R: $\begin{cases} 0 \leq x \leq 1, \quad 0 \leq y \leq 1 \\ \delta x = \delta y = 1/3 \end{cases}$

Tricky Problem

$$\frac{\partial T}{\partial t} - \alpha \frac{\partial^2 T}{\partial x^2} = 0$$

$$\frac{0.5\, T_j^{n-1} - 2T_j^n + 1.5\, T_j^{n+1}}{\delta t} - \alpha \left[ (1+d)\frac{(T_{j-1}^n - 2T_j^n + T_{j+1}^n)}{(\delta x)^2} - d\frac{(T_{j-1}^{n-1} - 2T_j^{n-1} + T_{j+1}^{n-1})}{(\delta x)^2} \right] = 0$$

Find T.E. for an arbitrary value of d. Check for consistency.

Q. $\dfrac{\partial \theta}{\partial t} = \alpha \nabla^2 \theta + \dfrac{q}{\rho C_p}$ , $\theta(r, \phi)$

$$\frac{\partial \theta}{\partial t} = \alpha \left( \frac{\partial^2 \theta}{\partial r^2} + \frac{1}{r}\frac{\partial \theta}{\partial r} + \frac{1}{r}\frac{\partial^2 \theta}{\partial \phi^2} \right) + \frac{q}{\rho C_p}$$

$\theta$ is prescribed at $r = 1$.

$\theta(r, 0) = \theta(r, 2\pi)$

$\theta(0, r, \phi) = f(r, \phi)$

We have seen how to handle IBVP : parabolic

PBVP : elliptical

pure BVP

# Hyperbolic PDE

Wave equation $u_{tt} = a^2 u_{xx}$

$$(x, t) \longmapsto (\xi, \eta)$$

where $\xi = x - at$

$\eta = x + at$

$$U_t = U_\xi \cdot \frac{\partial \xi}{\partial t} + U_\eta \cdot \frac{\partial \eta}{\partial t}$$

The reduced equation is

$$\frac{\partial^2 U}{\partial \xi \partial \eta} = 0$$

$$\Rightarrow \frac{\partial}{\partial \xi}\left(\frac{\partial U}{\partial \eta}\right) = 0 \Rightarrow \frac{\partial U}{\partial \eta} = g(\eta)$$

$$\Rightarrow U = f(\xi) + g(\eta) = f(x+at) + g(x-at)$$

where $f, g$ are arbitrary functions.

If $U(x,0) = f(x) + g(x) = F(x)$ (say)

The problem is an IBVP

IC : $U(x,0) = F(x)$ , $U_t(x,0) = G(x)$

$$U(x,t) = f(x+at) + g(x-at)$$

$$\frac{\partial U}{\partial t} = a \cdot f'(x+at) - ag'(x-at)$$

$$\frac{\partial U}{\partial t}(x,0) = af'(x) - ag'(x) = G(x)$$

$$\Rightarrow f'(x) - g'(x) = \frac{1}{a} G(x)$$

$$\Rightarrow f(x) - g(x) = \frac{1}{a}\int G(x)\,dx + c_1$$

$$\int (f' - g')\,dx = \frac{1}{a}\int_0^x G(\tau)\,d\tau + c_1$$

$$f + g = F(x)$$

$$f - g = \frac{1}{a}\int_0^x G(\tau)\,d\tau + c_1$$

$$f = \frac{1}{2}\left[F(x) + \int_0^x \frac{1}{a} G(\tau)\,d\tau + c_1\right]$$

$$g = \frac{1}{2}\left[F(x) - \int_0^x G(\tau)\,d\tau - c_1\right]$$

$$f(x+at) = \frac{1}{2}\left[F(x+at) + \int_0^{x+at} \frac{1}{a} G(\tau)\,d\tau + c_1\right]$$

$$g(x-at) = \frac{1}{2}\left[F(x-at) - \int_0^{x-at} G(\tau)\,d\tau - c_1\right]$$

$$u(x,t) = \frac{1}{2}\left[F(x+at) + F(x-at) + \int_{x-at}^{x+at} G(\tau)\,d\tau\right]$$

D'Alembert's Equation (domain of dependence)

→ Solution at any time is determined by the initial condition.

→ External points (outside domain of dependence) does not influence.

- Courant Condition

Q. $u_{tt} = c^2 u_{xx}$

$u(0,t) = u(1,t) = 0$

$u(x,0) = \sin(\pi x)$, $\frac{\partial u}{\partial t}(x,0) = 0$, $0 \leq x \leq 1$

Take $c = 1$, $\delta x = 1/5$, $\nu = \delta t/\delta x = 0.5$ $\Rightarrow$ $\delta t = 0.1$

Q.

$$U_{tt} = c^2 U_{xx}$$

$$U(0,t) = U(300,t) = 0$$

$$U(x,0) = \begin{cases} 0 & , 0 \leq x \leq 100 \\ 100 \sin\left[\frac{\pi(x-100)}{120}\right] & , 100 < x \leq 220 \\ 0 & , 220 \leq x \leq 300 \end{cases}$$

$$\frac{\partial U}{\partial t}(x,0) = 0$$

$$\nu = \frac{a\,\delta t}{\delta x} = 1$$

(example: simple harmonic wave)

scale the space variable down.

$$\frac{\partial U}{\partial t} + c\frac{\partial U}{\partial x} = 0$$

$$U_{tt} - c^2 U_{xx} = 0$$

$$\left(\frac{\partial}{\partial t} - \frac{c\,\partial}{\partial x}\right)\left(\frac{\partial U}{\partial t} + c\frac{\partial U}{\partial x}\right) = 0$$

$$\frac{dt}{1} = \frac{dx}{c} = \frac{dU}{0}$$

$U = f(x-ct)$ , $f$ is arbitrary

$$U(x,t) = f(x-ct)$$

$$U_t + c\,U_x = 0.$$

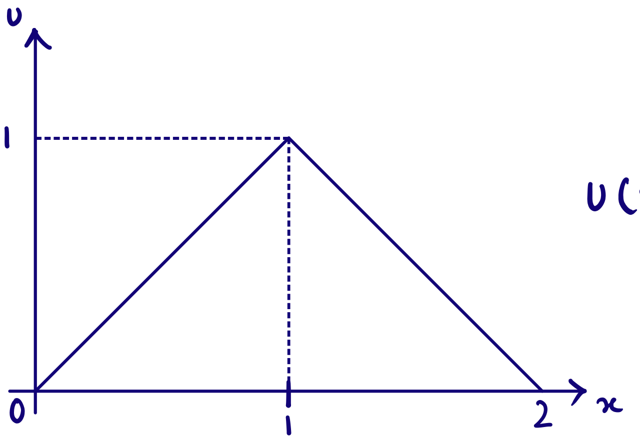
$$U(x,0) = g(x)$$

$$u(x,0) = g(x)$$

$$u(0,t) = U_0 \ , \quad t \geq 0$$

eg.

$$u(x,0) = \begin{cases} 0 \ , & x < 0 \\ x \ , & 0 < x < 1 \\ 2-x \ , & 1 \leq x \leq 2 \\ 0 \ , & x > 2 \end{cases}$$




$$u(x,t) = f(x-ct) = \begin{cases} 0 \ , & x-ct < 0 \\ x-ct \ , & ct \leq x \leq 1+ct \\ 2-x+ct \ , & 1+ct \leq x \leq 2+ct \\ 0 \ , & x > 2+ct \end{cases}$$

(shift)

Stability analysis —

- $U_t + cU_x = 0$

  $u(x,0) = f(x) \ , \quad x > 0$

  $u(0,t) = U_0 \ , \quad t > 0$

  FTCS.