TOPICS COVERED :

- more on Burger's equation.
- ELLIPTIC PDE
- ADIS SCHEME

H.T. + L.T.

Q. $u_t + c u_x = \nu u_{xx}$, $\nu > 0$, $-\frac{L}{2} < x < \frac{L}{2}$

(I) I.C. $u(x,0) = \begin{cases} 1 & , x < 0 \\ 0 & , x \geq 0 \end{cases}$

$u(-L/2, t) = 1$, $u(L/2, t) = 0$

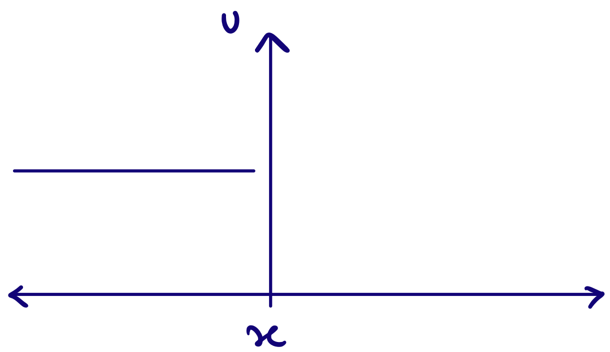
Take $L = 10$, use Crank-Nicolson scheme to discretise.

(II) I.C. $u(x,0) = \sin x$, $0 < x < 1$

B.C. $u(0,t) = u(1,t) = 0$, $t > 0$

(III) I.C. $u(x,0) = 4x(1-x)$, $0 < x < 1$

B.C. $u(0,t) = u(1,t) = 0$, $t > 0$



$t = 0$

$u = f(x+ct)$, $t > 0$

$u(x,t) = f(x+ct)$

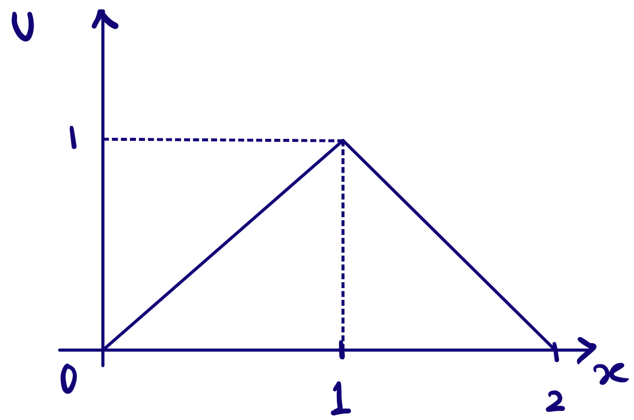
$u(x,0) = f(x) = \begin{cases} 1 & , x < 0 \\ 0 & , x > 0 \end{cases}$

For a hyperbolic PDE, we can have a solution in spite of a singularity.

Things get a bit tricky on adding the νU_{xx} term.

Let us consider

$$u(x,0) = \begin{cases} x, & 0 < x < 1 \\ 2-x, & 1 < x < 2 \end{cases}$$



The diffusion term ν has a "flattening" / "smoothening" effect, which increases as ν increases.

Q. Nonlinear Burger's Equation.

$$U_t + UU_x = \nu U_{xx}, \quad \nu > 1.$$

$u(0,t) = U_0$, $U(L,t) = U$,

$u(x,0) = f(x)$.

Discretise the above equation by Crank-Nicolson scheme and solve it iteratively through Newton's linearisation technique. Determine the tri-diagonal system at each iteration.

Explicit Scheme for advection - diffusion equation
(linear Burgers equation)

$$U_t + CU_x = \nu U_{xx}$$

F.T.C.S.

$$\frac{U_j^{n+1} - U_j^n}{\delta t} + C\frac{U_{j+1}^n - U_{j-1}^n}{2\delta x} = \nu \frac{U_{j+1}^n - 2U_j^n + U_{j-1}^n}{\delta x^2}$$

If $\nu << 1$

$$\frac{U_j^{n+1} - U_j^n}{\delta t} + C\frac{U_{j+1}^n - U_{j-1}^n}{2\delta x} = 0$$

$$\varepsilon_j^{n+1} - \varepsilon_j^n + r\left(\varepsilon_{j+1}^n - \varepsilon_{j-1}^n\right) = 0$$

$$\xi - 1 + r\left(e^{i\theta} - e^{-i\theta}\right) = 0$$

$$\xi = 1 - r\,(i\sin\theta) \qquad , \quad r = \frac{c\,\delta t}{\delta x}$$

$$|\xi| = \sqrt{1 + r^2\sin^2\theta} > 1 \quad \forall\, \theta$$

Unconditionally unstable.

H.T.

If $\nu \sim O(1)$

The modified equation (i.e. the FDE when we expand all variables about (x_j, t_n) and remove the 2^{nd} and higher order derivatives wrt t by x using the series expansion).

Qs. $$\frac{\partial u}{\partial t} + \overbrace{u\frac{\partial u}{\partial x}}^{\text{advection term}} - \overbrace{\nu \frac{\partial^2 u}{\partial x^2}}^{\text{diffusion term}} = 0$$

I.C. $u(x,0) = \sin(\pi x)$, $0 < x < 1$

B.C. $u(0,t) = u(1,t) = 0$, $t > 0$

$\delta x = 0.05$, $r = \delta t / \delta x^2 = 1$

$\nu = 1$

H.T. Get the ensuing tri-diagonal system at every iteration.

Use Crank-Nicolson scheme for discretisation.

When ν is small $(\to 0)$, the PDE reduces to a hyperbolic equation (called the wave equation).

Elliptic PDE (next class)

$$\partial u \;-\; c\,\overbrace{\nabla}^{\text{Laplacian}}{}^2 u = c\left[\partial^2 u \;+\; \partial^2 u\right]$$

$$\frac{\quad}{\partial t} = \ldots \left[\frac{\quad}{\partial x^2} + \frac{\quad}{\partial y^2} \right]$$

$$R : 0 < x < a, \quad 0 < y < b$$

or $$R : x^2 + y^2 \leq a^2$$

∂R is the boundary of R.

Conditions: I.C. $u(x, y, 0) = f(x, y)$ in R

B.C. $u(x, y, t)$ is prescribed on ∂R.

Grid points are (x_i, y_j, t_n), $u(x_i, y_i, t_n) = U_{ij}^n$

$$t_n \longrightarrow t_{n+1}$$

Knowing U_{ij}^n, obtain U_{ij}^{n+1}

H.T. The drawback of using C-N scheme (or any implicit scheme).

(Try discretising and see).

$$U_t = c\nabla^2 U, \quad U_{ij}^0 = f_{ij}$$

$$U_{ij}^{n+1} \quad \forall \; i = 0, N, \; j = 0, M$$

Last class :

$u(x,y,t)$

$u_t = \nu \nabla^2 u$

IBVP in which x, y are the space variables, $x, y \in R$.

∂R : boundary of R

Conditions : I.C. $u(x,y,0) = f(x,y)$ in R

B.C. $u(x,y,t)$ is prescribed on ∂R .

Here too we adopt the "forward marching in time" procedure .

$$\underbrace{U^n_{i,j}}_{\text{known}} \longrightarrow \underbrace{U^{n+1}_{i,j}}_{\text{to find}}$$

If we use the Crank-Nicolson scheme,

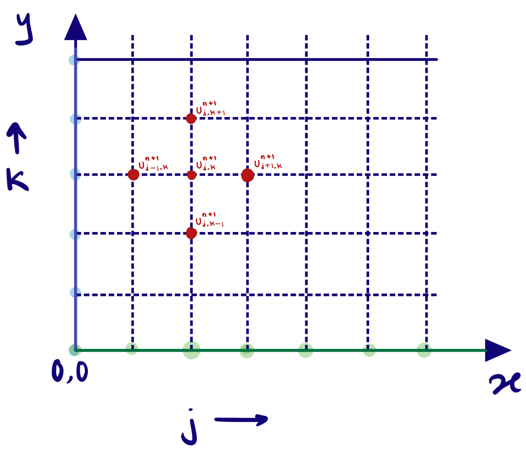
$$U^{n+1}_{j,k} - U^n_{j,k} = \frac{r_x}{2}\left[U^{n+1}_{j+1} - 2U^{n+1}_{j} + U^{n+1}_{j-1} + U^n_{j+1} - 2U^n_j + U^n_{j-1}\right]_{y=y_k}$$

$$+ \frac{r_y}{2}\left[U^{n+1}_{k+1} - 2U^{n+1}_{k} + U^{n+1}_{k-1} + U^n_{k+1} - 2U^n_k + U^n_{k-1}\right]_{x=x_j} + O(\delta t^2, \delta x^2, \delta y^2)$$

$$\left.\begin{array}{l} j = 1, \dots, N-1 \\ k = 1, \dots, M-1 \end{array}\right\} (N-1)\times(M-1) \text{ number of equations involving } (N-1)\times(M-1) \text{ variables}$$

The system is compact, but we have a *pentadiagonal* system .

Pictorially, this looks like



At $t = t_n$

To counter this problem we use a fractional time step.

$$t_n \xrightarrow{\text{step I}} t_{n+1/2} \xrightarrow{\text{step II}} t_{n+1}$$

Alternating Direction Implicit Scheme

To obtain the solution at time-step t_{n+1} using the solution at time step t_n, we go via two steps:

Step I: Advancing from t_n to $t_{n+1/2}$ is done using implicit/explicit discretisation of derivatives w.r.t x or w.r.t y (either/or) and explicit/implicit discretisation of derivatives wrt y or wrt x.
Through this we obtain $U_{ij}^{n+1/2}$ $\forall\, i, j$.

Step II. The solution is advanced from $t_{n+1/2}$ to t_{n+1} by reverse procedure, i.e.
explicit/implicit discretisation of derivatives w.r.t x or w.r.t y (either/or) and implicit/explicit discretisation of derivatives wrt y or wrt x.
Through this we obtain U_{ij}^{n+1} $\forall\, i, j$.

For the equation

$$U_t = \nu (U_{xx} + U_{yy})$$

Step 1 :

$$\frac{U^{n+1/2}_{i,j} - U^{n}_{i,j}}{\delta t/2} = \nu \left[\underbrace{\frac{U^{n+1/2}_{i+1,j} - 2U^{n+1/2}_{i,j} + U^{n+1/2}_{i-1,j}}{(\delta x)^2}}_{x \text{ is implicit}} + \underbrace{\frac{U^{n}_{i,j+1} - 2^{n}_{i,j} + U^{n}_{i,j-1}}{(\delta y)^2}}_{y \text{ is explicit}} \right]$$

We can form a system of equations of the form

$$a_i U^{n+1/2}_{i-1,j} + b_i U^{n+1/2}_{i,j} + c_i U^{n+1/2}_{i+1,j} = d_i$$

where

$$a_i = \frac{\nu}{2(\delta x)^2}, \quad b_i = -\frac{1}{\delta t} - \frac{\nu}{(\delta x)^2}, \quad c_i = \frac{\nu}{2(\delta x)^2}$$

$$d_i = -\frac{\nu}{2}\left[\frac{U^{n}_{i,j+1} - 2^{n}_{i,j} + U^{n}_{i,j-1}}{(\delta y)^2}\right] - \frac{U^{n}_{i,j}}{\delta t}$$

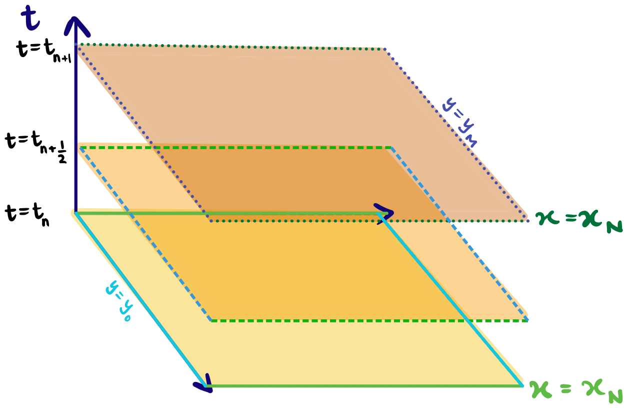
for $i = 1, \dots, N-1$.

We see that we can solve a tri-diagonal system

$$A^j U^{n+1/2}_j = D^j$$

where $U^{n+1/2}_j = \begin{bmatrix} U_{1,j} \\ U_{2,j} \\ \vdots \\ U_{N-1,j} \end{bmatrix}$

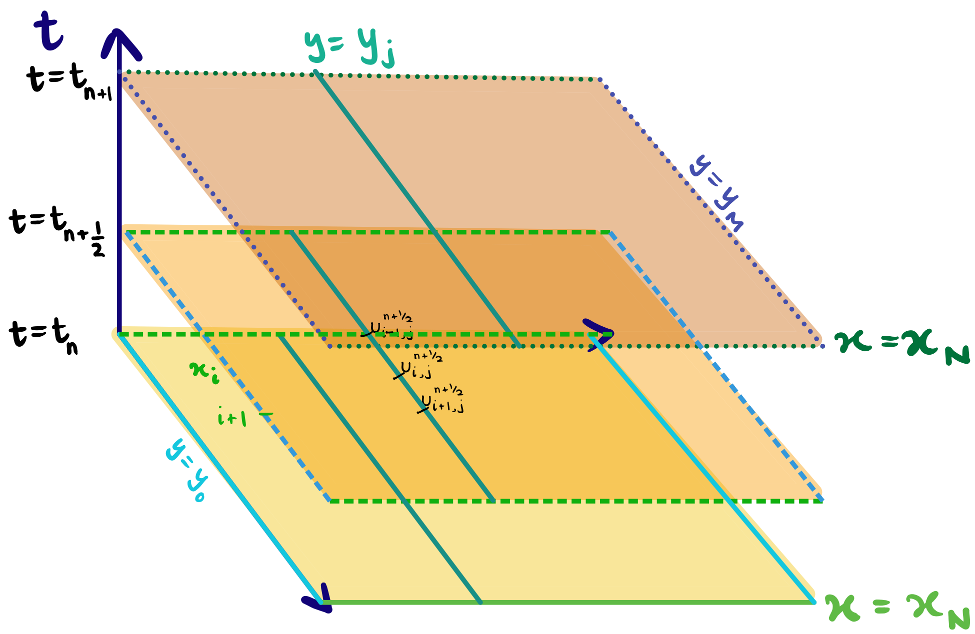
Here, j is fixed and we find $U^{n+1/2}_{i-1,j}$, $U^{n+1/2}_{i,j}$, $U^{n+1/2}_{i+1,j}$



for $j = 1, \dots, M-1$

$$A^j U^{n+1/2}_j = D^j$$

We obtain $U^{n+1/2}_{i,j}$ $\forall i,j$



<u>Step 2</u> :

$$\frac{U_{i,j}^{n+1} - U_{i,j}^{n+1/2}}{\delta t/2} = \nu \left[\underbrace{\frac{U_{i+1,j}^{n+1/2} - 2U_{i,j}^{n+1/2} + U_{i-1,j}^{n+1/2}}{(\delta x)^2}}_{x \text{ is explicit}} + \underbrace{\frac{U_{i,j+1}^{n+1} - 2U_{i,j}^{n+1} + U_{i,j-1}^{n+1}}{(\delta y)^2}}_{y \text{ is implicit}} \right]$$

At a fixed i,

$$A^i U_i^{n+1/2} = D^i$$

where $$U_i^{n+1/2} = \begin{bmatrix} U_{i,1} \\ U_{i,2} \\ \vdots \\ U_{i,M-1} \end{bmatrix}$$

A^i is an $(M-1) \times (M-1)$ tri-diagonal system.

This method is computationally easier to implement.

Use of fractional time step:

the greater the number of variables, the smaller the fraction and the greater the number of the fractional steps.

Question.

$$\frac{\partial u}{\partial t} = \nabla^2 u \quad , \quad \begin{matrix} -1 < x < 1 \\ -1 < y < 1 \end{matrix} \quad , \; t > 0$$

$$\text{I.C.} : \; u(x, y, 0) = \cos \frac{\pi x}{2} \cos \frac{\pi y}{2}$$

$$u = 0 \text{ on } x = \pm 1 \, , \; y = \pm 1$$

$$\delta x = \delta y = \tfrac{1}{2} \; , \; r = \tfrac{1}{6}$$

$$r = \frac{\nu \delta t}{2 \delta x^2} = \frac{\nu \delta t}{2 \delta y^2}$$

→ ADI scheme: perform analysis

H.T. Get the tri-diagonal system for all time steps.

Solve above problem with smaller step sizes.

In Steady state, $\frac{\partial u}{\partial t} \to 0$

$$\nabla^2 u = 0 \quad \to \text{laplace equation}$$

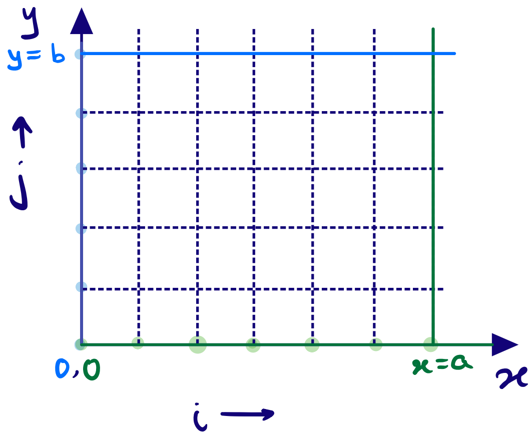
else $\nabla^2 u = f(x, y) \to$ Poisson equation.

Elliptic PDE, a BVP.

$x^2 + y^2 + z^2 \leq a^2$ ... v. imp. equation in modelling.

Given : $U_{xx} + U_{yy} = f(x,y)$, $\begin{cases} 0<x<a \\ 0<y<b \end{cases}$

U is prescribed at $x=0, a$ and $y=0, b$.



Central Difference Scheme

$$\frac{U_{i+1,j} - 2U_{i,j} + U_{i-1,j}}{(\delta x)^2} + \frac{U_{i,j+1} - 2U_{i,j} + U_{i,j-1}}{(\delta y)^2} = f_{ij}$$

$i = 1, \ldots, N-1$

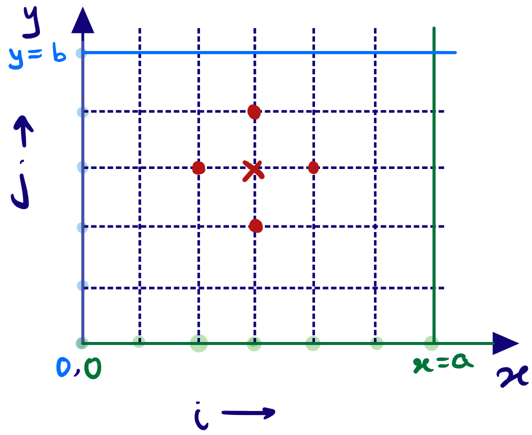
$j = 1, \ldots, M-1$

B.C. : $\begin{cases} U_{0,j} \ , \ U_{N,j} & \text{for } j = 1, 2, \ldots, M-1 \\ U_{i,0} \ , \ U_{i,M} & \text{for } i = 1, \ldots, N-1 \end{cases}$

Unknown U_{ij}

How do we solve this system?

Gauss Seidel Method



Guess the solution.

Modify the guess.

$$\frac{U_{i+1,j}^{(k)} - 2U_{ij}^{(k+1)} + U_{i-1,j}^{(k+1)}}{(\delta x)^2} + \frac{U_{i,j+1}^{(k)} - 2U_{i,j}^{(k+1)} + U_{i,j-1}^{(k+1)}}{(\delta y)^2} = f_{ij}$$

Let $\beta = \frac{\delta x}{\delta y}$

(check formulae)

To start the iteration, we guess $U_{ij}^{(0)}$ $\forall\, i,j$.

The iteration is said to converge if

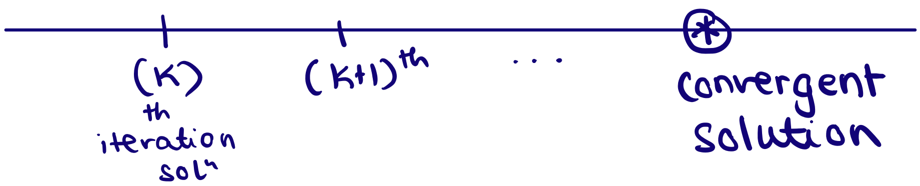
$$\max_{i,j} \left\{ \left| U_{i,j}^{(k+1)} - U_{i,j}^{(k)} \right| \right\} < \varepsilon \quad \forall\, k \geq K.$$

$$|a_{i,j}| \geqslant \sum_{\substack{j=1 \\ j \neq i}}^{M-1} |a_{ij}| \qquad \forall\, i$$

... sufficient condition for convergence.

Convergence may occur without being diagonally dominant.
However, the order of convergence is low (linear order)
i.e. the method is slow.

Next class: Successive Overrelaxation.



H.T.

Question. $\nabla^2 U = -10(x^2+y^2+10)$, $\begin{cases} 0 < x < 3 \\ 0 < y < 3 \end{cases}$

$U = 0$ on the boundary.

$\delta x = \delta y = 1$

Question. $-\nabla^2 U + 0.1\,U = 1$, $\begin{cases} 0 \leq x \leq 1 \\ 0 \leq y \leq 1 \end{cases}$

$U = 0$ on $x = 0$, $y = 0$

$\frac{\partial U}{\partial n} = 0$ on $x = 1$, $y = 1$

n is the unit normal.

(This condition is known as the no-flux situation).

$\delta x, \delta y = 0.5$
0.1
0.05

Question. $\nabla^2 U = 2\dfrac{\partial U}{\partial x} - 2$, $\begin{cases} 0 \leqslant x \leqslant 1 \\ 0 \leqslant y \leqslant 1 \end{cases}$

$U = 0$ on the boundary (homogeneous B.C.).

$\delta x = \delta y = 1/3$

$\delta x = \delta y = 0.1, 0.05, \ldots$