Lecture summary

1. 2-Point BVP with mixed B.C.
2. 2nd order forward difference.
3. Control volume method for piecewise continuous (ODE)
4. Higher (3rd) order will substitution and reduction of order.

Two-point linear BVP:

$$y'' + A(x)y' + B(x)y = C(x), \quad 0 < x < a$$

TYPES OF BOUNDARY CONDITIONS

Given B.C.	Name	method by which to solve
1. $y_0 = y(x_0)$ $y_n = y(x_n)$	DIRICHLET	Thomas algo, tri-diag
2. $y'_0 = y'(x_0)$ $y'_n = y'(x_n)$	NEUMANN	
3. $\alpha_0 y_0 + \alpha_1 y'_0 = \gamma_1$ $\beta_0 y_n + \beta_1 y'_n = \gamma_2$	ROBIN [MIXED: $\alpha_1 = \beta_0 = 0$]	

In (2) and (3) we are given the derivatives are the boundary points, where we cannot apply the central the central difference method.

However using forward and backward diff. methods at x_0 and x_n respectively yields us an error of order $O(h)$:

$$y'_0 = \frac{y_1 - y_0}{h} + O(h) \; ; \; y'_n = \frac{y_n - y_{n-1}}{h} + O(h)$$

How to get consistency in error?

* INTRODUCE FICTITIOUS POINTS y_{-1} and y_{n+1}

$$y_{-1} = y_0 - h$$
$$y_{n+1} = y_n + h$$

"have no physical meaning(?), aid in approximation and getting consistent order of error!!"

the B.C. can be discretised by taking-

$$y'_0 = \frac{y_1 - y_{-1}}{2h} + O(h^2)$$

&

$$y'_n = \frac{y_{n+1} - y_{n-1}}{2h} + O(h^2)$$

At $i = 0, 1, \dots, n, n+1$ $\}$ $(n+1)+2 = (n+3)$ points

- The ODE can be discretised at $i = 0, 1, \dots, n$, giving $(n+1)$ equations.
- 2 BCs

$$\alpha_0 y_0 + \alpha_1 \frac{y_1 - y_{-1}}{2h} = \gamma_1 \quad \dots (i)$$

$$\beta_0 y_n + \beta_1 \frac{y_{n+1} - y_{n-1}}{2h} = \gamma_2 \quad \dots (ii)$$

$(n+1)$
$+$
2

$n+3$

system is consistent

(i) $$y_{-1} = y_1 - \left(\frac{\gamma_1 - \alpha_0 y_0}{\alpha_1}\right)(2h)$$

(ii) $$y_{n+1} = y_{n-1} + \left(\frac{\gamma_2 - \beta_0 y_n}{\beta_1}\right)(2h)$$

#1

(HT) DISCRETISE the BVP

$$y'' + Ay' + By = C$$

with B.C. $\begin{cases} \alpha_0 y_0 + \alpha_1 y_0' = \gamma_1 \\ \beta_0 y_n + \beta_1 y_n' = \gamma_2 \end{cases}$

and find the reduced tri-diagonal $(n+1) \times (n+1)$ system $AX = b$,

$$X^T = [x_0, x_1, \dots, x_n]$$

#2

(HT) <u>EXERCISE</u> : $y'' - 2y = 0$,

$\left.\begin{array}{l} y(0) = 1 \\ y'(1) = 0 \end{array}\right\}$ BOUNDARY CONDITIONS

$x_i = ih$, $h = 0.2$

Unknowns : $y_1, y_2, \dots, y_n$

$\to n$ unknowns

Discretise the ODE for $i = 1, \dots, n$. [$\because y_0$ is known $\therefore i=0$ can be removed]

Introduce a fictitious point y_{n+1} :

$$y_n' = 0 \Rightarrow y_{n+1} = y_{n-1}$$

#3

(HT) Show that truncation error for BVP using central difference method is $O(h^2)$.

TYPES OF ERRORS :

ERROR = $Y_i - y_i$; Y_i = Exact solution, y_i = Numerical sol^n

But this def^n isn't much use ∵ we don't know the exact sol^n in the first place, ∴ we try to find the diff. components of the error, where they come from and try to estimate atleast some upper bound for them.

1. TRUNCATION ERROR

The residue by which the exact solution fails to satisfy the difference eq^n.

$T.E. = L(Y_i)$ at x_i,

where Y_i is the exact solution & $L[y_i]=0$ is the discretised eq^n.

2. ROUND-OFF ERROR – unavoidable

If y_i is the numerical sol^n, and $\bar{y}_i$ is the value of y_i after rounding off to some decimal place,

$$y_i = \bar{y}_i + \varepsilon_i \Rightarrow \varepsilon_i = y_i - \bar{y}_i \quad \Big\} \varepsilon_i = \text{roundoff error}$$

3. STABILITY

A check for stability is if $|\varepsilon_i| < M \ \forall i$.

CONSISTENCY OF SCHEME

T.E. $\rightarrow 0$ as $h \rightarrow 0$

■ 2nd ORDER FORWARD DIFFERENCE

To write y_i' as:

$$y_i' = Ay_i + By_{i+1} + Cy_{i+2} + O(h^2)$$

From Taylor Series, we have:

$$y_{i+1} = y_i + hy_i' + \frac{h^2}{2!}y_i'' + \frac{h^3}{3!}y_i''' + \cdots \quad \ldots(i)$$

$$y_{i+2} = y_i + 2hy_i' + \frac{(2h)^2}{2!}y_i'' + \frac{(2h)^3}{3!}y_i''' + \cdots \quad \ldots(ii)$$

$4\times(i)$

$$\Rightarrow \quad 4y_{i+1} = 4y_i + 4hy_i' + \frac{(2h)^2}{2!}y_i'' + \frac{4h^3}{3!}y_i''' + \cdots \quad \ldots(iii)$$

$(ii)-(iii)$

$$\Rightarrow \quad y_{i+2} - 4y_{i+1} = -3y_i - 2hy_i' + 0 + \frac{4h^3}{3!}y''' + \cdots \quad \ldots(iv)$$

$$\Rightarrow \quad y_{i+2} - 4y_{i+1} + 3y_i = -2hy_i' + O(h^3)$$

$$\Rightarrow \quad y_i' = -\frac{(3y_i - 4y_{i+1} + y_{i+2})}{2h} + O(h^2)$$

$$\boxed{y_i' = -\left(\frac{3y_i - 4y_{i+1} + y_{i+2}}{2h}\right) + O(h^2)}$$

■ 2nd ORDER BACKWARD DIFFERENCE

To write y_i' as:

$$y_i' = \bar{A}y_i + \bar{B}y_{i-1} + \bar{C}y_{i-2} + O(h^2)$$

Using a similar procedure as above we get :

$$\boxed{y_i' = \frac{3y_i - 4y_{i-1} + y_{i-2}}{2h} + O(h^2)}$$

eg. Form a tridiagonal system for :

$$y'' - 2y = 0$$
$$y(0) = 1$$
$$y'(1) = 0$$

Discretise for $i = 1, 2, \dots, n-1$,

use the above 2nd order formulae.

Till now we have assumed that the concerned functions were continuous and infinitely differentiable. But it may not necessarily be so.

■ BVP with piece-wise continuous coefficients, which is expressed as

$$\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] + q(x)y = r(x)$$

$$a < x < b$$

$p(x)$, $q(x)$, $r(x)$ may be piecewise-continuous functions of x.

B.C. $\alpha_0 y(a) + \alpha_1 y'(a) = \gamma_1$

$\beta_0 y(b) + \beta_1 y'(b) = \gamma_2$

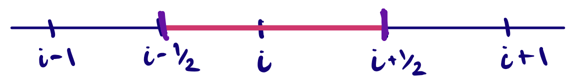
Here we use an approach called

(Read more)

- ## CONTROL VOLUME APPROACH

Each grid-point is enclosed in a control volume (denoted by ——)

— : cell face



within the control volume,

Integrate the eq^n b/w $x_{i-1/2}$ to $x_{i+1/2}$,
the control volume, where y is approx by y_i (central pt enclosed by control vol)

[this approach is usually considered for PDEs, hence we call the region volume, though for ODE it is an interval]

$$\int_{x_{i-1/2}}^{x_{i+1/2}} \frac{d}{dx}\left(p(x)\frac{dy}{dx}\right)dx + \int_{x_{i-1/2}}^{x_{i+1/2}} q(x)\,y\,dx = \int_{x_{i-1/2}}^{x_{i+1/2}} r(x)\,dx$$

$$\Rightarrow \left[p(x)\frac{dy}{dx}\right]_{x_{i-1/2}}^{x_{i+1/2}} + \left[q_{i-}\,y_i\frac{\delta x_{i-1}}{2} + q_{i+}\,y_i\frac{\delta x_i}{2}\right] = r_{i-1}\frac{\delta x_{i-1}}{2} + r_{i+}\frac{\delta x_i}{2}$$

where $\delta x_i = x_{i+1} - x_i$, $\delta x_{i-1} = x_i - x_{i-1}$

(Within the control volume, $y \approx y_i$)

In this approach we can also have non-uniform grid dist.

The final discretised equation is

$$\left[P_{i+\frac{1}{2}} \frac{y_{i+1}-y_i}{\delta x_i} + P_{i-\frac{1}{2}} \frac{y_i - y_{i-1}}{\delta x_{i-1}}\right] + y_i \left[q_{i-} \frac{\delta x_{i-1}}{2} + q_{i+} \frac{\delta x_i}{2} \right]$$

$$= r_{i-1} \frac{\delta x_{i-1}}{2} + r_{i+} \frac{\delta x_i}{2}$$

$i = 1, 2, \dots, n-1$

This will lead to a tri-diagonal system of order $(n-1)$.

Show that if p_i, q_i, r_i are continuous this will be same as the central difference method.

Solve:

Q. $y'' - 2xy' - 2y = -4x$
$y(0) - y'(0) = 0$
$2y(1) - y'(1) = 1$

LAB TASK
$h = 0.1, 0.05$ & so on..

Use 2nd order FORWARD & BACKWARD eq^n.

Q. $y'' + 2xy' + 2y = 4x$
$y(0) = 1, \quad y(0.5) = 1.279$
$h = 0.1$

How about HIGHER ORDER BVP?
(≥ 3)

$$y''' + A(x)y'' + B(x)y' + C(x)y = D(x)$$
$$y(0) = \alpha, \quad y'(0) = \varepsilon$$
$$y'(L) = \lambda$$
$$0 < x < L$$

Satisfy the eq at any grid point

$$y_i''' = (y_i'')' = \frac{y_{i+1}'' - y_{i-1}''}{2h} + O(h^2)$$

$$= \frac{1}{2h^3}\left(y_{i+2} - 2y_{i+1} + 2y_{i-1} - y_{i-2}\right) + O(h^2)$$

$i = 2, \dots, n-2 \rightarrow$ $n-3$ eq^n
+ 2 B.C.
$\overline{n-1}$ eq^n
NO unique sol^n.

Will not yield a tri-diagonal system.
Also, # eq $\neq$ # unknowns.

How else can we solve this?

Reduce 3rd order eq^n to 2nd order:

COUPLED EQ^{n}s,

note: z depends on y

$$\begin{cases} z = y' & \text{(i)} \\ z'' + A(x)z' + B(x)z + C(x)y = D(x) & \text{(ii)} \end{cases}$$
$$y(0) = \alpha$$
$$z(0) = \varepsilon$$
$$z(L) = \lambda$$

Integrate (i) between x_{i-1}, x_i using TRAPEZOIDAL RULE,

$$\frac{dy}{dx} = z$$

$$\Rightarrow \int_{x_{i-1}}^{x_i} dy = \int_{x_{i-1}}^{x_i} z\, dx$$

$$\Rightarrow y_i - y_{i-1} = \frac{\delta x_{i-1}}{2}(z_i + z_{i-1})$$

careful about derivations
check last step

Use central diff. to discretise (ii)

You will have $2(n-1)$ eq^n involving $y_1, y_2, \dots, y_{n-1}$

& $z_1, z_2, \dots, z_{n-1}$

$\rightarrow 2(n-1)$ var,

Get this discretised eq^n

Q. $y''' + 4y'' + y' - 6y = 1$

$y(0) = y'(0) = 0, \quad y(1) = 1$

$h = 0.25$