

TOPICS COVERED

1. Solution of Non-linear BVP
2. Newton's method for linearisation
3. Quasi-linear form of BVP

NON-LINEAR BVP

$$y'' = f(x, y, y')$$

$$y(a) = y_a, \quad y(b) = b$$

$$x_i = a + i \Delta x, \quad a < x < b$$

$$\frac{y_{i+1} - 2y_i + y_{i-1}}{\Delta x^2} - f\left(x_i, y_i, \frac{y_{i+1} - y_{i-1}}{2\Delta x}\right) = 0$$

$$\Rightarrow F_i(y_{i-1}, y_i, y_{i+1}) = 0$$

TASK: To solve $(N-1)$ non linear algebraic equations

involving $(N-1)$ variables y_1, y_2, \dots, y_{N-1} .

We solve the n /linear system (*) iteratively using the NEWTON-RAPHSON METHOD.

At any k th iteration, let $y_i^{(k)}$ be the approximation and Δy_i is the error.

$$y_i = \begin{array}{c} y_i^{(k)} \\ | \\ \text{exact value} \end{array} + \begin{array}{c} \Delta y_i \\ | \\ \text{approximate value} \end{array} \quad \begin{array}{c} \text{error} \\ | \end{array}$$

which satisfy (*), i.e.

$$F_i(y_{i-1}^{(k)} + \Delta y_{i-1}, y_i^{(k)} + \Delta y_i, y_{i+1}^{(k)} + \Delta y_{i+1}) = 0$$

We expand using the Taylor Series (for 3 variables)

$$F_i(y_{i-1}^{(k)}, y_i^{(k)}, y_{i+1}^{(k)}) + \frac{\partial F_i}{\partial y_{i-1}} \left| \begin{array}{c} (k) \\ \Delta y_{i-1} + \frac{\partial F_i}{\partial y_i} \end{array} \right| \Delta y_{i-1} + \frac{\partial F_i}{\partial y_{i+1}} \left| \begin{array}{c} (k) \\ \Delta y_{i+1} \end{array} \right|$$

+ ...

$$y_i = y_i^{(k)} + \Delta y_i$$

$\Delta y_0 = 0 = \Delta y_N \quad \because y_0 \text{ and } y_N \text{ are prescribed.}$

NEGLECTING higher order (> 2) derivative terms, we get

$$F_i(y_{i-1}, y_i, y_{i+1}) + \frac{\partial F_i}{\partial y_{i-1}} \left| \begin{array}{c} (k) \\ \Delta y_{i-1} + \frac{\partial F_i}{\partial y_i} \end{array} \right| \Delta y_i + \frac{\partial F_i}{\partial y_{i+1}} \left| \begin{array}{c} (k) \\ \Delta y_{i+1} \end{array} \right|$$

which involves unknowns as $\Delta y_1, \Delta y_2, \dots, \Delta y_{N-1}$
appearing linearly in the system of $(N-1)$ eqns.

which forms an $(N-1) \times (N-1)$ tri-diagonal system,

$$AX = b, X^T = [\Delta y_1 \ \Delta y_2 \ \dots \ \Delta y_{N-1}]$$

Solving this gives us an approximate value of the error term
We use these to get the approximation for the $(k+1)^{th}$ iteration-

$$y_i^{(k+1)} = y_i^{(k)} + \Delta y_i \quad i=1, 2, \dots, N-1$$

which generates a sequence of iterates

$$y_i^{(k)}, \quad k=0, 1, \dots$$

To start the method, $y_i^{(0)}$ needs to be prescribed.
 ↴ initial assumption

The procedure is repeated till the convergence criterie is achieved.

$$|\Delta y_i| < \varepsilon_i, |y_i^{(k+1)} - y_i^{(k)}| < \varepsilon$$

$$\forall i, k \geq K$$

$$\Rightarrow |\Delta y_i| < \varepsilon \quad \forall i, k \geq K$$

↑
a finite number

$$\max_{1 \leq i \leq N-1} |\Delta y_i| < \varepsilon, \quad \varepsilon \text{ is an arbitrarily small, positive number.}$$

STOP

PROS of this method

* Quadratic rate of convergence.

(in practice we see that convergence occurs in 4-5 iterations of N-R
If it seems to be taking longe
change $y_i^{(0)}$ guess)

Exercise 1. $y'' + 1 + (y')^2 = 0$

$$y(0) = 1, \quad y(1) = 2$$

Write as $a_i \Delta y_{i-1} + b_i \Delta y_i + c_i \Delta y_{i+1} = d_i$
 $i = 1, 2, \dots, N-1$

At i , $y_i y''_i + 1 + (y'_i)^2 = 0$

$$y_i \left(\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} \right) + 1 + \left(\frac{y_{i+1} - y_{i-1}}{2h} \right)^2 = 0$$

$$\underbrace{\frac{y_i y_{i+1} - 2y_i^2 + y_{i-1}}{h^2} + 1 + \frac{y_{i+1}^2 - 2y_{i+1} y_{i-1} + y_{i-1}^2}{4h^2}}_{f_i} = 0$$

$$\frac{\partial F}{\partial y_{i-1}} = \frac{y_i}{h^2} - \frac{2y_{i+1}}{24h^2} + \frac{2y_{i-1}}{24h^2} = \frac{y_i}{h^2}$$

$$F_i(y_{i-1}, y_i, y_{i+1}) + \frac{\partial F_i}{\partial y_{i-1}} \Big|_{\Delta y_{i-1}}^{(k)} + \frac{\partial F_i}{\partial y_i} \Big|_{\Delta y_i}^{(k)} + \frac{\partial F_i}{\partial y_{i+1}} \Big|_{\Delta y_{i+1}}^{(k)}$$

$$\frac{y_{i+1}}{h^2} - \frac{4y_i}{h^2} + \frac{y_{i-1}}{h^2}$$

Solve for $h = 1/4 = 0.25$

Consider the problem:

$$\cdot \quad y'' - (y')^2 - y^2 + y + 1 = 0$$

$$\text{B.C.} \begin{cases} y(0) = 0.5 \\ y(\pi) = -0.5 \end{cases}$$

Solve iteratively by linearisation technique and obtain the ensuing tri-diagonal system at each iteration.

$$\left(\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} \right) - \left(\frac{y_{i+1} - y_{i-1}}{2h} \right)^2 - y_i^2 + y_i + 1 = 0$$

$$\left(\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} \right) - \frac{y_{i+1}^2 - 2y_{i+1}y_{i-1} + y_{i-1}^2}{4h^2} - y_i^2 + y_i + 1 = 0$$

$$\frac{\partial F}{\partial y_{i-1}} = \frac{1}{h^2} - \frac{y_{i-1} - y_{i+1}}{2h^2} \quad ; \quad \frac{\partial F}{\partial y_i} = -\frac{2}{h^2} - 2y_i + 1$$

$$\frac{\partial F}{\partial y_{i+1}} = \frac{1}{h^2} - \frac{y_{i+1} - y_{i-1}}{2h^2}$$

$$\underbrace{\left(\frac{1}{h^2} - \frac{y_{i-1} - y_{i+1}}{2h^2} \right)}_{a_i} \Delta y_{i-1} + \underbrace{\left(-\frac{2}{h^2} - 2y_i + 1 \right)}_{b_i} \Delta y_i + \underbrace{\left(\frac{1}{h^2} - \frac{y_{i+1} - y_{i-1}}{2h^2} \right)}_{c_i} \Delta y_{i+1} = - \underbrace{\left(\left(\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} \right) - \frac{y_{i+1}^2 - 2y_{i+1}y_{i-1} + y_{i-1}^2}{4h^2} - y_i^2 + y_i + 1 \right)}_{d_i}$$

$$y_i^{(k+1)} = y_i^{(k)} + \Delta y_i$$

$$\max_{1 \leq i \leq N-1} |\Delta y_i| < \epsilon \ll 1$$

[Take up this problem in the lab class,
be careful about storage]

Alternative (illustrated by example) (HT)

$$Q. \quad 3yy'' + (y')^2 = 0$$

$$\text{B.C.} \begin{cases} (y')^2 = 0 \\ y(0) = 0, \quad y(1) = 2 \end{cases}$$

Ans. Discretising -

$$3y_i \left(\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} \right) + \left(\frac{y_{i+1} - y_{i-1}}{2h} \right)^2 = 0$$

$$y_i = y_i^{(k)} + \Delta y_i$$

Replace all y_j with $y_j^{(k)} + \Delta y_j$ and expand.

Dropping the square and higher orders terms of Δy_{ij} , we get

$$\Delta y_{i-1} [\dots] + \Delta y_i [\dots] + \Delta y_{i+1} [\dots] = d_i \quad i = 1, 2, \dots, N-1$$

$Ax = d$ system.

$$Q. \quad y'' + 2yy' = 4 + 4x^3$$

$$\text{B.C.s} \quad \begin{cases} y(1) = 2 \\ y(2) = 4.5 \end{cases}$$

Ans.

$$\left(\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} \right) + \frac{\partial F}{\partial y_i} \left(\frac{y_{i+1} - y_{i-1}}{2h} \right) - 4x_i^3 - 4 = 0$$

$$a_i = \frac{\partial F}{\partial y_{i-1}} = \frac{1}{h^2} - \frac{y_i^{(k)}}{h}$$

$$b_i = \frac{\partial F}{\partial y_i} = \frac{y_{i+1}^{(k)} - y_{i-1}^{(k)}}{h} - \frac{2}{h^2}$$

$$c_i = \frac{\partial F}{\partial y_{i+1}} = \frac{1}{h^2} + \frac{y_i^{(k)}}{h}$$

$$d_i = 4 + 4x_i - y_i \left(\frac{y_{i+1} - y_{i-1}}{h} \right) - \left(\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} \right)$$

Fair initial approximation :

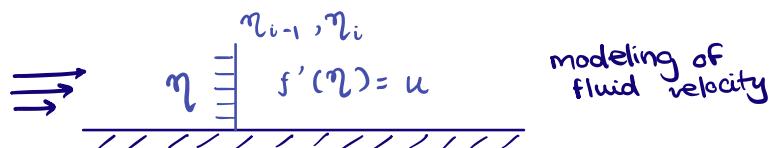
$$y-2 = \frac{2.5}{1} (x-1)$$

$$y^{(0)}(x) = 2.5x - 0.5$$

→ Ultimate solution should not depend on initial approx.

$$Q. \quad f''' + ff'' + 1 - (f')^2 = 0$$

$$f(0) = 0, \quad f'(0) = 0, \quad f'(10) = 1$$



$$F = f'$$

$$F'' + FF' + 1 - F^2 = 0$$

$$f(0) = 0, \quad F(0) = 0, \quad F(10) = 1$$

Steps:

1. Discretise eq"

2. Solve discretised equation

$$f_i^{(k+1)} = f_i^{(k)} + \Delta f_i, \quad f_i^{(k+1)} = F_i^{(k)} + \Delta F_i$$

$$X_i = \begin{pmatrix} \Delta f_i \\ \Delta F_i \end{pmatrix} \rightarrow k \geq 0$$

Get the block tri-diagonal equation

$$\bar{A}_i X_{i-1} +$$

Ans. Given:

$$F = f'$$

$$F = \frac{df}{d\eta} \Rightarrow F d\eta = df$$

$$\Rightarrow \int_{\eta_{i-1}}^{\eta_i} F d\eta = \int_{\eta_{i-1}}^{\eta_i} df$$

$$\Rightarrow \frac{\delta \eta_i (F_i + F_{i-1})}{2} = f_i - f_{i-1}$$

$$F_i - f_{i-1} - \frac{\delta \eta}{2} (F_i + F_{i-1}) = f_i - \frac{h}{2} (F_i + F_{i-1}) - f_{i-1} = 0$$

$$\Delta F_i - \frac{1}{h} (\Delta f_i + \Delta f_{i-1}) - \Delta f_{i-1} = 0$$

$$F'' + FF' + 1 - F^2 = 0$$

$$f(0) = 0, \quad F(0) = 0, \quad F(10) = 1$$

$$\frac{F_{i+1} - 2F_i + F_{i-1}}{h^2} + f_i \frac{F_{i+1} - F_{i-1}}{2h} - F_i^2 + 1 = 0$$

$$a_i = \frac{1}{h^2} - \frac{f_i}{2h} ; \quad b_i = -\frac{2}{h^2} - 2f_i$$

$$c_i = \frac{1}{h^2} + \frac{f_i}{2h}$$

$$d_i = \dots$$

$$A_i = \begin{pmatrix} -1 & -\frac{1}{h} \\ 0 & \frac{1}{h^2} - \frac{f_i^{(u)}}{2h} \end{pmatrix}$$

$$B_i = \begin{pmatrix} 1 & -h/2 \\ \frac{F_i^{(k)} - F_{i-1}^{(k)}}{2h} & -\frac{2}{h^2} - 2F_i^{(k)} \end{pmatrix}$$

$$C_i = \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{h^2} + \frac{f_i^{(u)}}{2h} \end{pmatrix}$$

(H1)

$$Q. \quad F'' + (2F + 4)F' = 0$$

$$B.C. \quad \begin{cases} F(0) = 0, \quad F''(0) = -0.1, \quad F'(0.087) = 0 \end{cases}$$

Another imp. BVP - modelling flow through orifice

$$\xrightarrow{\hspace{1cm}} f'$$

NEW TECHNIQUE

currently — Newton Linearisation technique

Solving algebraic system iteratively

↳ Not always good.



QUASI-LINEARISATION TECHNIQUE [Original paper: 1957, Bellman & Kalaba]

Consider a BVP

$$F(y'', y', y) = 0, \quad a < x < b$$

$$y(a) = y_a, \quad y(b) = y_b$$

→ x is the variable

→ F is a f^n of f^m .

Let $y^{(k)}$, $y'^{(k)}$, $y''^{(k)}$ be the approximate form of y, y', y'' respectively at any iteration k .

We expand F by Taylor Series about the approximate form $y^{(k)}, y'^{(k)}, y''^{(k)}$ by treating F as a function of y, y', y'' , we get

$$F(y^{(k)}, y'^{(k)}, y''^{(k)}) + (y - y^{(k)}) \left. \frac{\partial F}{\partial y} \right|^{(k)} \\ + (y' - y'^{(k)}) \left. \frac{\partial F}{\partial y'} \right|^{(k)} \cdot (y'' - y''^{(k)}) \left. \frac{\partial F}{\partial y''} \right|^{(k)} \dots$$

$$\left[\frac{\partial F}{\partial y} \right] + \left[\frac{\partial F}{\partial y''} \right]$$

Neglecting the square and higher order terms, we get the modified form of y, y', y'' denoted by

$y^{(k+1)}, y'^{(k+1)}, y''^{(k+1)}$, satisfying -

$$(y^{(k+1)} - y^{(k)}) \left. \frac{\partial F}{\partial y} \right|^{(k)} + (y'^{(k+1)} - y'^{(k)}) \left. \frac{\partial F}{\partial y'} \right|^{(k)} \\ + (y''^{(k+1)} - y''^{(k)}) \left. \frac{\partial F}{\partial y''} \right|^{(k)} = -F(y^{(k)}, y'^{(k)}, y''^{(k)})$$

... QUASI LINEAR FORM

We have reduced the given non-linear equation to a linear equation of $y^{(k+1)}$, which is referred to as the quasi-linear form of the given BVP.

ADVANTAGES : Reduction of non-linear system to linear system.

But it does not end here, we also have to solve this obtained linear system.

Q. find the QUASILINEAR FORM of

$$3yy'' + (y')^2 = 0$$

$$y(0) = 0$$

$$y(1) = 1$$

$$(y^{(k+1)} - y^{(k)}) \left(3y^{(k)} \right) + (y^{(k+1)} - y^{(k)}) \left(2y' \right) \\ + (y''^{(k+1)} - y''^{(k)}) \left(3y^{(k)} \right) = 0$$

$$3y^{(k)} y''^{(k+1)} + 2y^{(k)} y'^{(k+1)} + 3y''^{(k)} y^{(k+1)} \\ = 3y^{(k)} y''^{(k)} + 2(y')^2 + 3y^{(k)} y''^{(k)} \\ (\text{check})$$

(Try all given eqn's w/ both linearisation techniques)