TOPICS COVERED

1. SPLINE INTERPOLATION
2. INTRO to NUMERICAL SOLUTIONS of PDEs

SPLINE INTERPOLATION

splines considered to be continuous

$$S(x) = \{ p^{(k)}(x) \mid x_k < x < x_{k+1} \}$$

$$k = 0, 1, \dots, n-1$$

$$p^{(k)}(x) = \frac{M_k}{6}\left[\frac{(x_{k+1}-x)^3}{h} - h(x_{k+1}-x)\right] + \frac{M_{k+1}}{6}\left[\frac{(x-x_k)^3}{h} - h(x-x_k)\right] + \frac{y_k}{h}(x_{k+1}-x) + \frac{y_{k+1}}{h}(x-x_k)$$

The unknown M_k's satisfy the tri-diagonal system (*).

These are $(n-1)$ equations involving $M_0, \dots, M_n$, i.e. $(n+1)$ variables.

We need to prescribe suitable end conditions (for M_0 and M_n) to get a unique solution.

The values can be prescribed as follows-

(1) FREE BOUNDARY CONDITIONS

(aka natural spline)

$M_0 = 0 = M_n$,

at the two ends, the curve approaches a straight line.

(2) M_0, M_n are prescribed $M_0 = S_0, M_n = S_n$

(3) $S''(x)$ approaches a constant value at x_0, x_1.

$\therefore M_0 = M_1, \quad M_n = M_{n-1}$

H.T.

Q. $f(x) = \sqrt{x}$, find a spline in $[0, 10]$

Find the error table.

Q. Derive the spline equation.

Q.

x	1	2	3	4
y	1.5	2.2	3.1	4.3

Use cubic spline interpolation to compute

$y(1.2)$ and $y'(1)$.

Also find the spline.

Use natural spline

Spline interpolation gives a much better approximation for the derivative.

APPLICATION to BVP

$$y'' + A(x)y' + B(x)y = C(x)$$

$$y(a) = A$$

$$y(B) = B$$

HT. Solve $y'' - y = 0$

$y(0) = 0 = y(1) = 1$

$h = \frac{1}{2} \quad \therefore n = 2$

Now, if we have $A(x) \neq 0$

$$y'' + A(x)\,y' + B(x)y = C(x)$$

$$y(a) = \bar{A}, \quad y(b) = \bar{B}$$

At $x = x_k$, $A_k = 0$

$$A_k\, y_k' = -M_k - B_k y_k + C_k$$

$$y_k' = p_k'(x_k) = p_{k-1}'(x_k)$$

we need to solve both y_k & M_k,

We get a system of $2n$ eq^{n}s involving $(n-1) + (n+1) = 2n$ unknowns

Q. $y''' + 2y' + y = 30x$

$y(0) = 0$, $y(1) = 0$

$h = 0.5$

$\therefore n = 2$

$\therefore$ 4 eq^{n}s to be solved.

New topic : PDE

Basic idea : $u(x,y)$

PDE involves $f(u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}) = 0$

We will concern ourselves w/ 2nd order PDEs.

$f(u, u_x, u_y, u_{xx}, u_{yy},$

Principle part: part of eq^n involving 2nd order derivatives.

Linear Principle Part

$$A(x,y)\, u_{xx} + 2B(x,y)\, u_{xy} + C(x,y)\, u_{yy} + f(u, u_x, u_y, x, y) = 0$$

$$(x,y) \mapsto (\xi, \eta)$$

CLASSIFICATION	CONDITION	
1. HYPERBOLIC PDE Example: Wave eq. $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$	$B^2 - AC > 0$	$\exists$ transformation $(x,y) \mapsto (\xi,\eta)$ $u_{\xi\eta} + G(u, u_\xi, u_\eta, \xi, \eta) = 0$
2. PARABOLIC PDE eg. HEAT EQUATION $\frac{\partial u}{\partial t} = c \frac{\partial^2 u}{\partial x^2}$ Imp when process involves diffusion.	$B^2 = AC$	$u_{\xi\xi} = F(\cdots)$
3. Elliptic PDE $u_{xx} + u_{yy} = f(u, u_x, u_y, x, y)$ $\nabla^2 u = 0$: LAPLACE $\nabla^2 u = g(x,y)$: POISSON		

eg. $\vec{E} = \left(-\frac{\partial \varphi}{\partial x}, -\frac{\partial \varphi}{\partial y}\right)$

$\varphi \rightarrow$ potential f^n

PARABOLIC PDE

$$\frac{\partial u}{\partial t} = c\frac{\partial^2 u}{\partial x^2}$$

- x : space variable (spatial domain)
- t : time

In all decay processes $u \rightarrow 0$ but never "reaches" it,

$\therefore$ t is semi-infinite

$\therefore$ we define $t > 0$,

$0 < x < a$

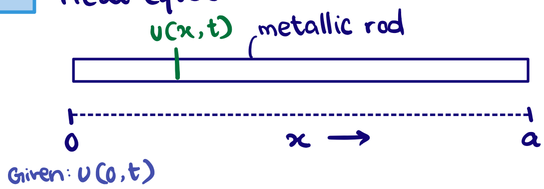
& prescribe B.C.s : $u(0,t) = U_0(t)$, $t > 0$

$u(a,t) = U_a(t)$

& I.C. (at $t=0$) : $u(x,0) = f(x)$, $\underbrace{0 < x < a}_{\text{spatial domain}}$

a may also be a function of time.

EXAMPLE Heat equation



In some cases the rod may be insulated, then we prescribe "heat flux" at the ends:

$$\frac{\partial u}{\partial x}(0,t) = \alpha = 0$$

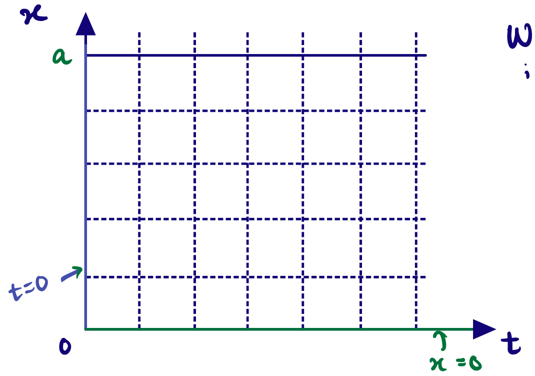
$$\frac{\partial u}{\partial x}(a,t) = \beta = 0$$

Let's start with the simplest case first,

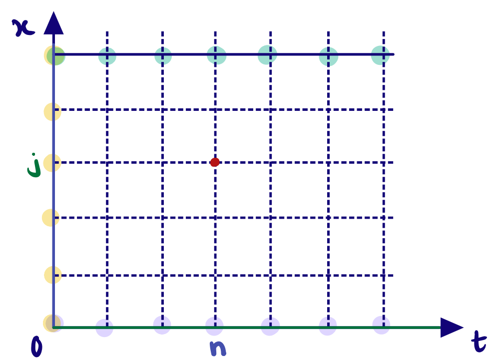
$$\frac{\partial u}{\partial x}(0,t) = \alpha = 0$$

$$\frac{\partial u}{\partial x}(a,t) = \beta = 0$$

Domain : $\begin{cases} x \in [0,a] \\ t > 0 \end{cases}$



We divide the domain into grid points.



time level

$$u_j^n = u(x_j, t_n)$$

spatial location

$t_n = n\delta t$, $n = 0, 1, \ldots$ (can go on to ∞)

$x_j = j\delta x$, $j = 0, 1, \cdots N$

Need to find u_j^n.

IC → (1) $u(x,0) = f(x) \Rightarrow u_j^0 = f_j$

BC → (2) $u(0,t) = u_0 \Rightarrow u_0^n = u_0$

$u(a,t) = u_a \Rightarrow u_N^n = u_a$

To find u_j^n , $j = 1, \ldots, N-1$
when $n > 0$

We adopt a "forward marching in time procedure".

At any stage, U_j^n is known, our task is to find U_j^{n+1}, $j=1,2,\dots,N-1$, $n \geq 0$.

$$\underset{(\checkmark)}{U_j^n} \longrightarrow \underset{(?)}{U_j^{n+1}}$$

Satisfy the equation at (x_j, t_n).

$$\left.\frac{\partial U}{\partial t}\right|_j^n = c \left.\frac{\partial^2 U}{\partial x^2}\right|_j^n$$

FORWARD TIME CENTRAL SPACE

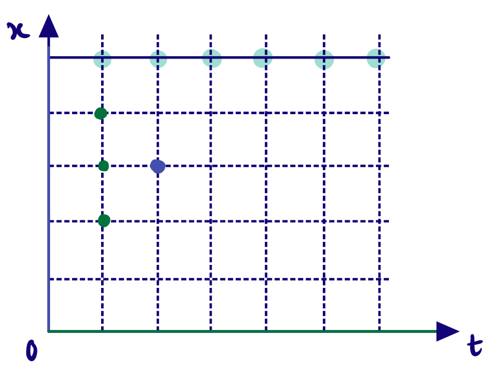
$$- 3U_j^0 - 4$$

$$\frac{U_j^{n+1} - U_j^n}{\delta t} = c\,\frac{U_{j+1}^n - 2U_j^n + U_{j-1}^n}{(\delta x)^2}$$

The unknown U_j^{n+1} can be expressed in terms of U_j^n s.

$$U_j^{n+1} = r\,U_{j+1}^n + (1-2r)U_j^n + r\,U_{j-1}^n, \quad j=1,2,\dots,N-1$$

where $r = \dfrac{c\,\delta t}{(\delta x)^2}$



T.E. $= O(\delta t, (\delta x)^2)$

↳ The residue by which the exact solution fails to satisfy the difference eq^n.

Derive error term.