CENG 384 - Signals and Systems for Computer Engineers
Spring 2020
Written Assignment 2

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1. (a) $y[n] = \sum_{k=1}^{\infty} x[n-k]$

   i. *memory*: The system is not memoryless because the output $y[n]$ depends on past values of $x[n]$.

   ii. *stability*: The system is not stable because when $x[n] = u[n]$, as n goes to infinity, $y[n] = \sum_{k=1}^{\infty} x[n-k] = n$ also goes to infinity

   iii. *causality*: The system is causal because the output depends only on current and previous values of the input.

   iv. *linearity*: Suppose $y_1 = \sum_{k=1}^{\infty} x_1[n-k]$, $y_2 = \sum_{k=1}^{\infty} x_2[n-k]$ and a, b are constants.

$$\begin{aligned}
\sum_{k=1}^{\infty}(a \cdot x_1[n-k] + b \cdot x_2[n-k]) &= \sum_{k=1}^{\infty}(a \cdot x_1[n-k]) + \sum_{k=1}^{\infty}(b \cdot x_2[n-k]) \\
&= a \cdot \sum_{k=1}^{\infty} x_1[n-k] + b \cdot \sum_{k=1}^{\infty} x_2[n-k] \\
&= a \cdot y_1[n] + b \cdot y_2[n]
\end{aligned}$$

   Hence the system is linear.

   v. *invertibility*:

$$\begin{aligned}
y[n+1] - y[n] &= \sum_{k=1}^{\infty} x[n+1-k] - \sum_{k=1}^{\infty} x[n-k] \\
&= \sum_{k=1}^{\infty} x[n-(k-1)] - \sum_{k=1}^{\infty} x[n-k] \\
&= \sum_{k=0}^{\infty} x[n-k] - \sum_{k=1}^{\infty} x[n-k] \\
&= x[n]
\end{aligned}$$

   $x[n] = y[n+1] - y[n]$ is the inverse of the system and therefore system is invertible.

   vi. *time-invariance*: By replacing $x[n]$ with $x[n-n_0]$, we get

$$y[n-n_0] = \sum_{k=1}^{\infty} x[n-n_0-k]$$

   Thus the system is time-invariant.

   (b) $y(t) = tx(2t+3)$

   i. *memory*: The system is not memoryless because the output $y(t)$ depends on both past and future values of $x(t)$.

   ii. *stability*: The system is not stable because when $x(t) = u(t)$, as t goes to infinity, $y(n) = tx(2t+3)$ also goes to infinity.

   iii. *causality*: The system is not causal because the output $y(t)$ depends on future values of $x(t)$.

iv. *linearity*: Suppose $y_1(t) = tx_1(2t+3)$, $y_2(t) = tx_2(2t+3)$ and a, b are constants.

$$a \cdot tx_1(2t+3) + b \cdot tx_2(2t+3) = a \cdot y_1(t) + b \cdot y_2(t)$$

Hence the system is linear.

v. *invertibility*: The system is not invertible because $y(0) = 0$ for every input $x(t)$, i.e to find the inverse system, we need to divide y with t, however this division can not be done when $t = 0$.

vi. *time-invariance*: By replacing $x(t)$ with $x(t - t_0)$, we get

$$y(t - t_0) = (t - t_0)x(2(t - t_0) + 3) \neq tx(2(t - t_0) + 3)$$

Thus the system is not time-invariant.

2. (a)

$$\int x(t) - 5y(t)dt = y(t)$$

(b) By differentiating and rearranging terms,

$$y'(t) + 5y(t) = x(t)$$

Solving characteristic equation

$$\alpha + 5 = 0 \quad \longrightarrow \quad \alpha = -5$$

Therefore homogeneous solution is $y_H = Ke^{-5t}$.

$$x(t) = (e^{-t} + e^{-3t})u(t) = e^{-t}u(t) + e^{-3t}u(t)$$

Since the system is linear, we can find particular solutions for $x_1(t) = e^{-t}u(t)$, $x_2(t) = e^{-3t}u(t)$ and add them. $x_1(t) = e^{-t}u(t)$, so $x_1(t) = 0$ for $t < 0$. For $t > 0$, the input is e^{-t}. The transfer function for this equation is $H(\lambda) = \dfrac{1}{\lambda + 5}$. $\lambda = -1$ and $H(-1) = \dfrac{1}{4}$. Therefore particular solution of $x_1(t)$ is $\dfrac{1}{4}e^{-t}u(t)$.
Similarly, particular solution of $x_2(t) = e^{-3t}u(t)$ is $H(-3)e^{-3t}u(t) = \dfrac{1}{2}e^{-3t}u(t)$.

$$y(t) = y_H(t) + y_P(t) = Ke^{-5t} + \frac{1}{4}e^{-t}u(t) + \frac{1}{2}e^{-3t}u(t)$$

To find the final solution, we need to determine K, using the fact that system is initially at rest.

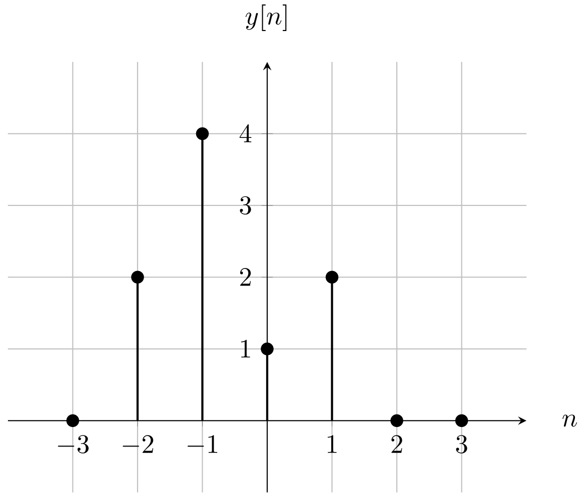
$$y(0) = K + \frac{1}{4} + \frac{1}{2} = 0 \quad \longrightarrow \quad K = -\frac{3}{4}$$

Hence,

$$y(t) = -\frac{3}{4}e^{-5t} + \frac{1}{4}e^{-t}u(t) + \frac{1}{2}e^{-3t}u(t)$$

3. (a)

$$\begin{aligned}
y[n] &= x[n] * h[n] \\
&= \sum_{k=-\infty}^{\infty} x[k] \cdot h[n-k] \\
&= \sum_{k=-\infty}^{\infty} (2\delta[k] \cdot \delta[n-k-1] + 2\delta[k] \cdot 2\delta[n-k+1] + \delta[k+1] \cdot \delta[n-k-1] + \delta[k+1] \cdot 2\delta[n-k+1]) \\
&= 2\sum_{k=-\infty}^{\infty} \delta[k] \cdot \delta[n-k-1] + 4\sum_{k=-\infty}^{\infty} \delta[k] \cdot \delta[n-k+1] + \sum_{k=-\infty}^{\infty} \delta[k+1] \cdot \delta[n-k-1] \\
&\qquad + 2\sum_{k=-\infty}^{\infty} \delta[k+1] \cdot \delta[n-k+1] \\
&= 2 \cdot \delta[n-1] + 4 \cdot \delta[n+1] + \delta[n] + 2 \cdot \delta[n+2] \\
y[n] &= 2 \cdot \delta[n+2] + 4 \cdot \delta[n+1] + \delta[n] + 2 \cdot \delta[n-1]
\end{aligned}$$



(b)

$$\frac{dx(t)}{dt} = \delta(t-1) + \delta(t+1)$$

Since convolution is distributive,

$$\begin{aligned} y(t) &= \frac{dx(t)}{dt} * h(t) \\ &= (\delta(t-1) * h(t)) + (\delta(t+1) * h(t)) \\ &= h(t-1) + h(t+1) \\ y(t) &= e^{-(t-1)} sin(t-1)u(t-1) + e^{-(t+1)} sin(t+1)u(t+1) \end{aligned}$$

4. (a)

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} e^{-\tau}u(\tau)e^{-2(t-\tau)}u(t-\tau)d\tau$$

For $t < 0$, $u(\tau)$ and $u(t-\tau)$ do not overlap. Thus $y(t) = 0$.
For $t \geq 0$, $u(\tau)$ and $u(t-\tau)$ overlap between 0 and t. So we can change the limits of the integral.

$$y(t) = \int_0^t e^{-\tau}e^{-2(t-\tau)}d\tau = e^{-2t}\int_0^t e^{\tau}d\tau = e^{-2t}(e^t - 1) = e^{-t} - e^{-2t}$$

(b)

$$y(t) = x(t) * h(t) = (u(t) - u(t-1)) * h(t) = \delta(t-1) * h(t) = h(t-1)$$
$$y(t) = e^{3(t-1)}u(t-1)$$

5. (a) Characteristic equation of the recurrence relation is

$$2\alpha^2 - 3\alpha + 1 = (2\alpha - 1)(\alpha - 1) = 0 \quad \longrightarrow \quad \alpha_1 = \frac{1}{2},\ \alpha_2 = 1$$

Therefore

$$y[n] = K_1\frac{1}{2^n} + K_2$$

Using initial conditions,

$$y[0] = K_1 + K_2 = 1$$
$$y[1] = \frac{K_1}{2} + K_2 = 0$$
$$\frac{K_1}{2} = 1 \quad \rightarrow \quad K_1 = 2 \quad \rightarrow \quad K_2 = -1$$

Hence

$$y[n] = \frac{1}{2^{n-1}} - 1$$

(b) Characteristic equation is

$$\alpha^3 - 3\alpha^2 + 4\alpha - 2 = (\alpha - 1)(\alpha - (1+j))(\alpha - (1-j)) = 0$$

Therefore

$$y(t) = K_1 e^t + K_2 e^{(1+j)t} + K_3 e^{(1-j)t}$$

Using initial conditions,

$$\begin{aligned}
y(0) &= K_1 + K_2 + K_3 && = 3 \\
y'(0) &= K_1 + K_2(1+j) + K_3(1-j) \\
&= (K_1 + K_2 + K_3) + j(K_2 - K_3) && = 1 \\
y''(0) &= K_1 + K_2(1+j)^2 + K_3(1-j)^2 \\
&= K_1 + 2j(K_2 - K_3) && = 2
\end{aligned}$$

$$j(K_2 - K_3) = -2 \quad \rightarrow \quad K_1 - 4 = 2 \quad \rightarrow \quad K_1 = 6$$
$$K_2 + K_3 = -3, \quad K_2 - K_3 = 2j \quad \rightarrow \quad K_2 = -\frac{3}{2} + j, \quad K_3 = -\frac{3}{2} - j$$

Hence,

$$\begin{aligned}
y(t) &= 6e^t + (-\frac{3}{2} + j)e^{(1+j)t} + (-\frac{3}{2} - j)e^{(1-j)t} \\
&= 6e^t + (-\frac{3}{2} + j)(cos(t) + jsin(t))e^t + (-\frac{3}{2} - j)(cos(t) - jsin(t))e^t \\
&= e^t(6 + (-\frac{3}{2}cos(t) - \frac{3}{2}jsin(t) + jcos(t) - sin(t)) + (-\frac{3}{2}cos(t) + \frac{3}{2}jsin(t) - jcos(t) - sin(t))) \\
y(t) &= e^t(6 - 3cos(t) - 2sin(t))
\end{aligned}$$

6. (a)

$$h_0[n] - \frac{1}{2}h_0[n-1] = \delta[n] \quad \longrightarrow \quad h_0[n] = \frac{1}{2}h_0[n-1] + \delta[n]$$

System is initially at rest, so $h_0[n] = 0$ for $n < 0$.

$$h_0[0] = 0 + 1 = 0$$

$\delta[n] = 0$ for $n > 0$. Thus for $n > 0$

$$h_0[n] = \frac{1}{2^n} + 0 = \frac{1}{2^n}$$

Hence,

$$h_0[n] = \frac{1}{2^n}u[n]$$

(b)

$$h[n] = h_0[n] * h_0[n] = \sum_{k=-\infty}^{\infty} \frac{1}{2^k}u[k] \cdot \frac{1}{2^{n-k}}u[n-k]$$

When $n < 0$, $u[k]$ and $u[n-k]$ do not overlap, therefore $h[n] = 0$.
When $n \geq 0$, $u[k]$ and $u[n-k]$ between 0 and n. So we can change the limits of the sum.

$$h[n] = \sum_{k=0}^{n} \frac{1}{2^k} \cdot \frac{1}{2^{n-k}} = \sum_{k=0}^{n} \frac{1}{2^n} = \frac{n}{2^n}$$

Hence

$$h[n] = \frac{n}{2^n}u[n]$$

(c)

$$y[n] - \frac{1}{2}y[n-1] = w[n]$$

Since the system is LTI

$$\frac{1}{2}y[n-1] - \frac{1}{4}y[n-2] = \frac{1}{2}w[n-1]$$

Subtracting these equations

$$y[n] - y[n-1] + \frac{1}{4}y[n-2] = w[n] - \frac{1}{2}w[n-1] = x[n]$$

The difference equation is

$$y[n] - y[n-1] + \frac{1}{4}y[n-2] = x[n]$$