Graphs

Single Source Shortest Paths for DAGs

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Shortest paths on weighted DAGs

- For a DAG the SSSPs problem can be solved with a simplified algorithm
- Shortest paths are always well defined even if there are negative-weight edges
  - This is because, obviously, negative-weight cycles cannot exist in a DAG

Shortest paths on weighted DAGs

- As there are no cycles it is enough to
  - Topologically sort the DAG
    - Impose a linear order on the vertices

Perform a DFS computing end-processing times
Order vertices using the end-processing times

  - Relax all vertices following the sorted order given by the topological sort
    - In other words, it suffices to make just one pass over the vertices in the topological sorted order
    - As we process a vertex, we relax each edge that leaves the vertex

SSSP for DAGs

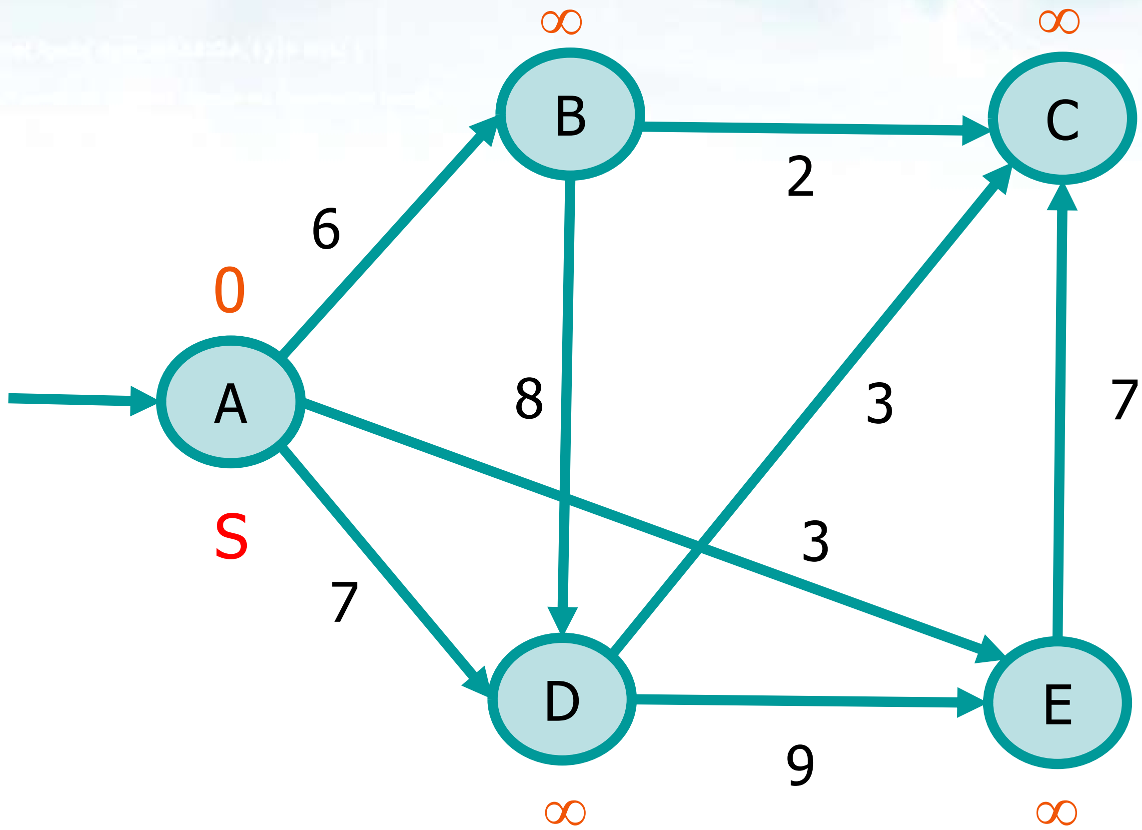
Pseudo-code

```
sssp_for_DAGs (G, w, s)
  topological sort the vertices of G
  initialize_single_source (G, s)

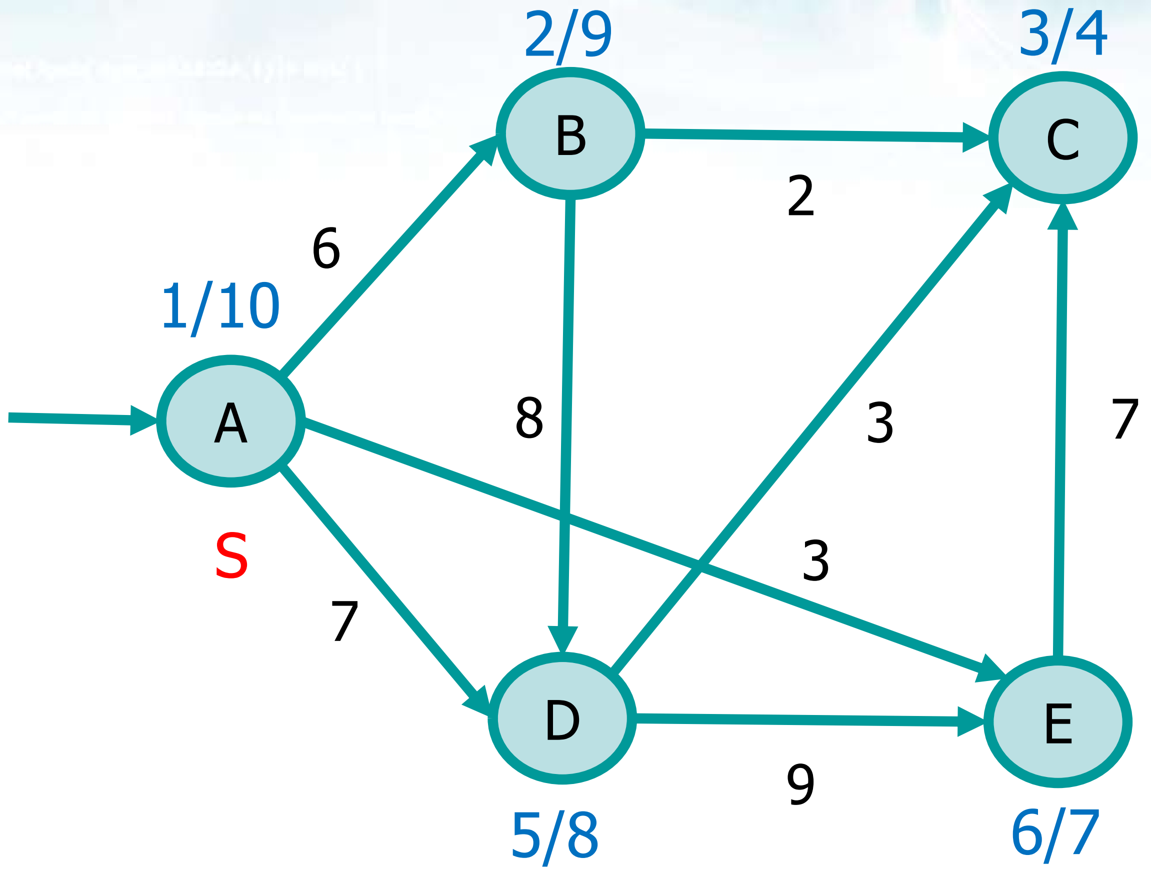
  for each vertex u ∈ V
    for each vertex v ∈ adjacency list of u
      relax (u, v, w)
```

Taken in topologically sorted order

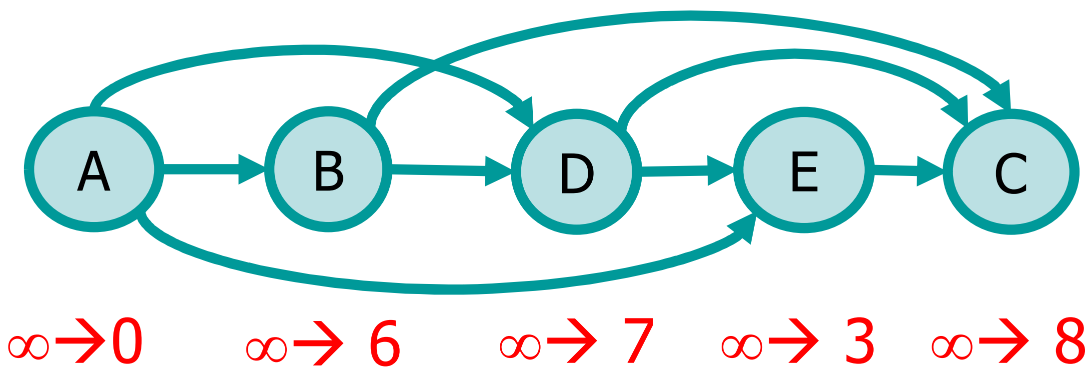
Example



Solution



Relaxation order
(A, B)
(A, D)
(A, E)
(B, D)
(B, C)
(D, E)
(D, C)
(E, C)



Complexity

Pseudo-code

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sssp_for_DAGs (G, w, s)
  topological sort the vertices of G
  initialize_single_source (G, s)

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    for each vertex v ∈ adjacency list of u
      relax (u, v, w)
```

$\Theta(|V|+|E|)$

$\Theta(|V|)$

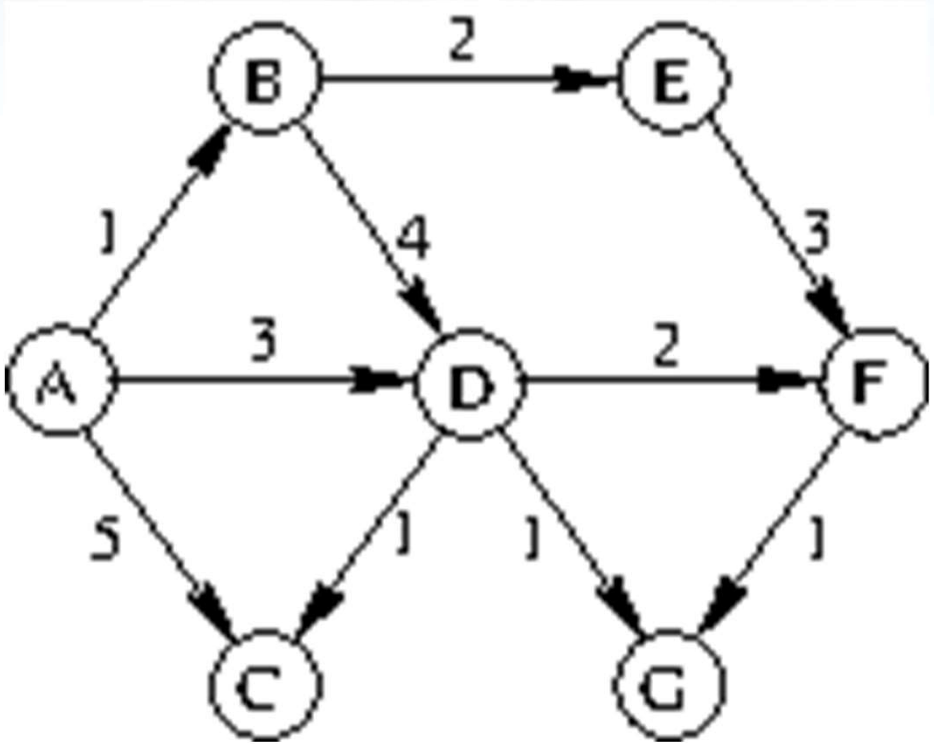
Executed E times alltogheter

$\Theta(1) \rightarrow \Theta(|E|)$

Taken in topological sorted order

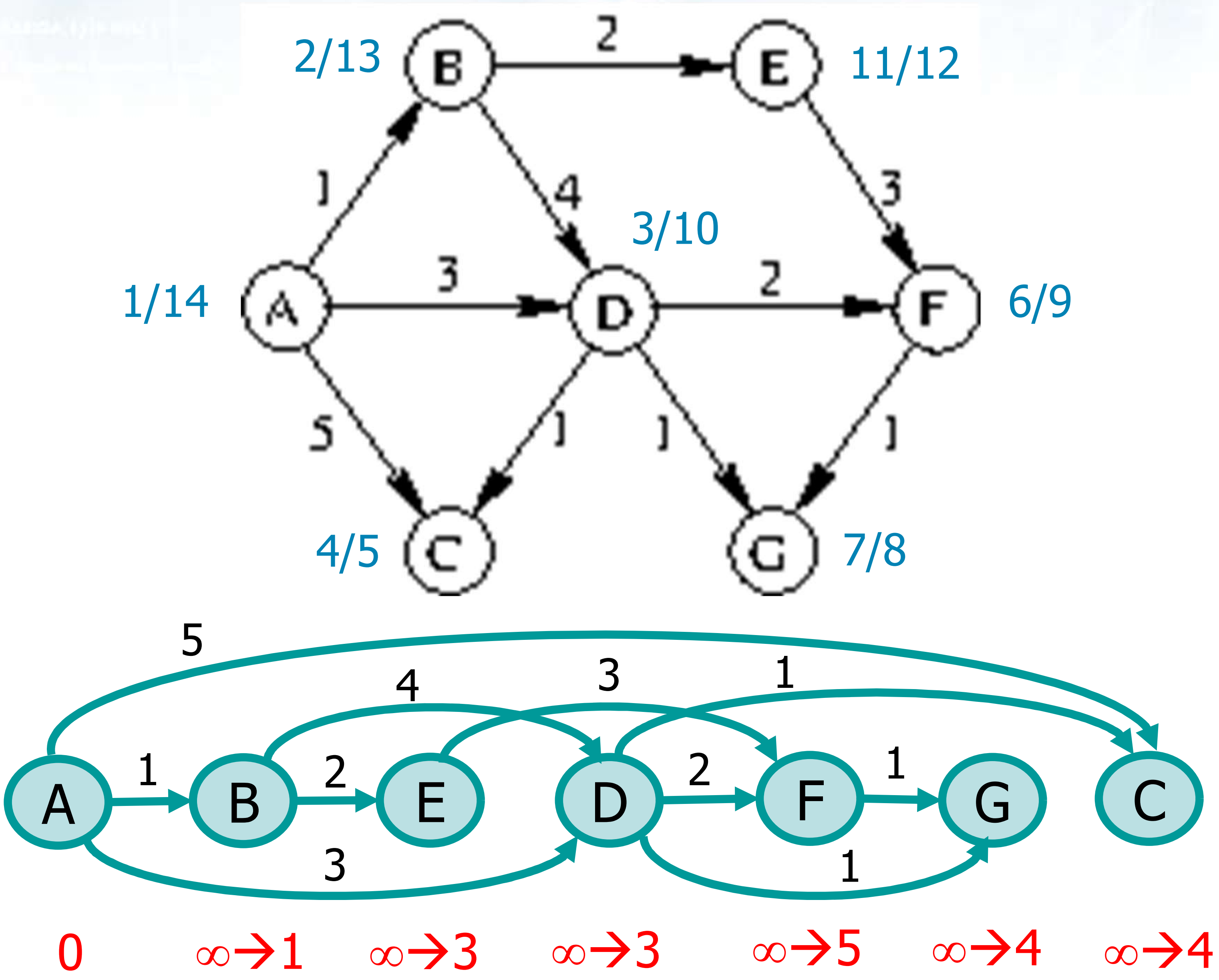
Overall running time complexity
$T(n) = \Theta(|V| + |E|)$

Exercise



- Given the following graph find all shortest paths starting from vertex A

Solution


2/13
B
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F
6/9
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1
1
1
4/5
C
G
7/8
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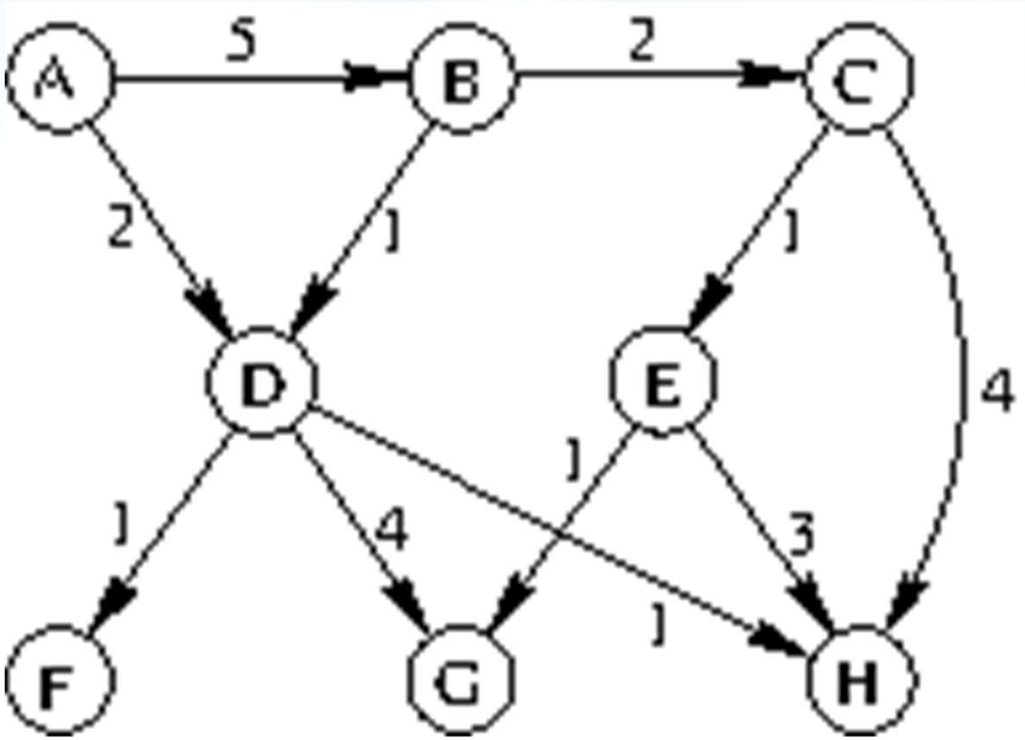
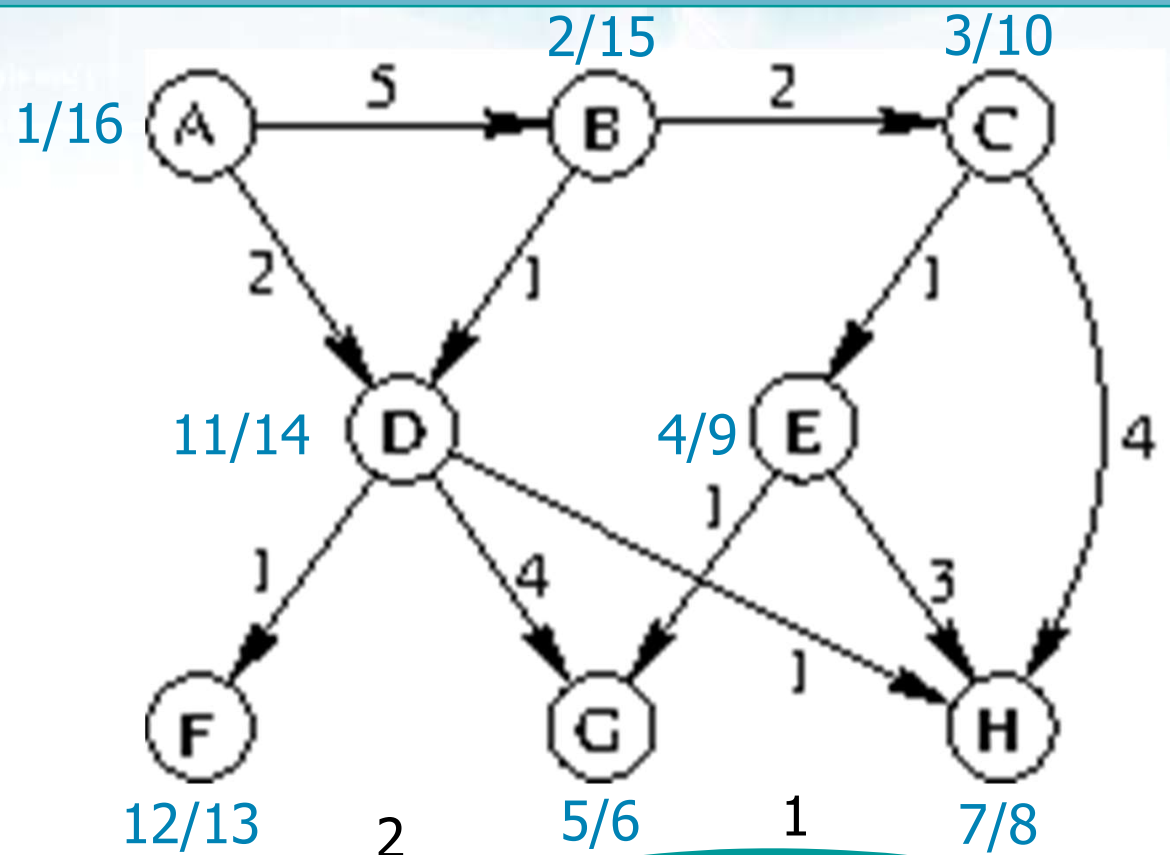
1
2
2
1
A
B
E
D
F
G
C
3
1
0
∞→1
∞→3
∞→3
∞→5
∞→4
∞→4

Exercise



- Given the following graph find all shortest paths starting from vertex A

Solution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G
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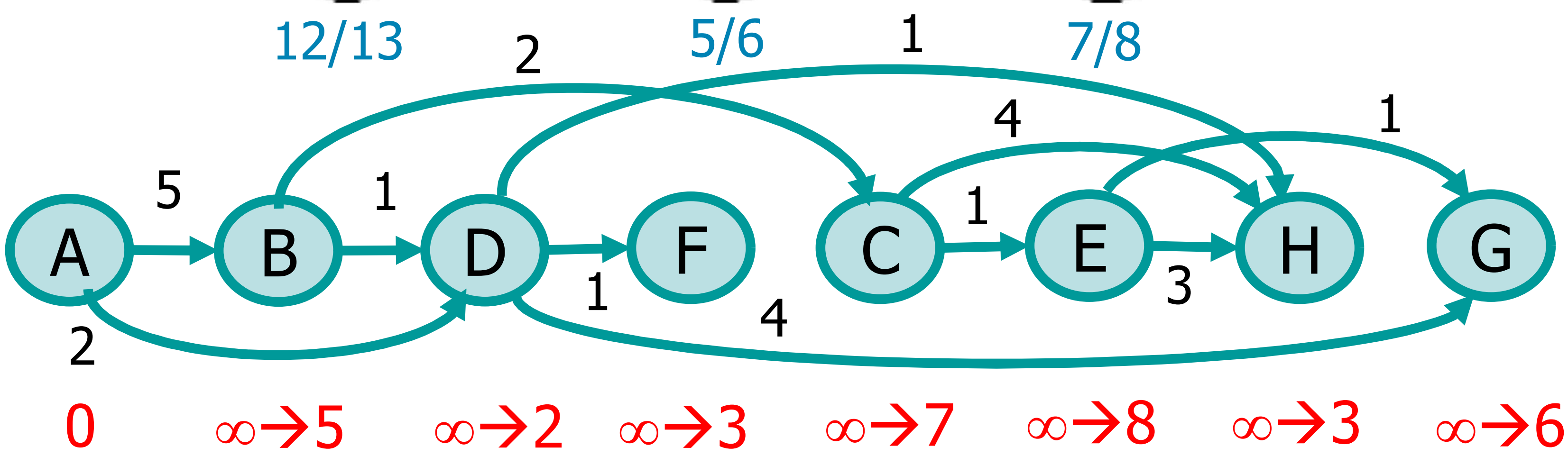
12/13
5/6
7/8


2
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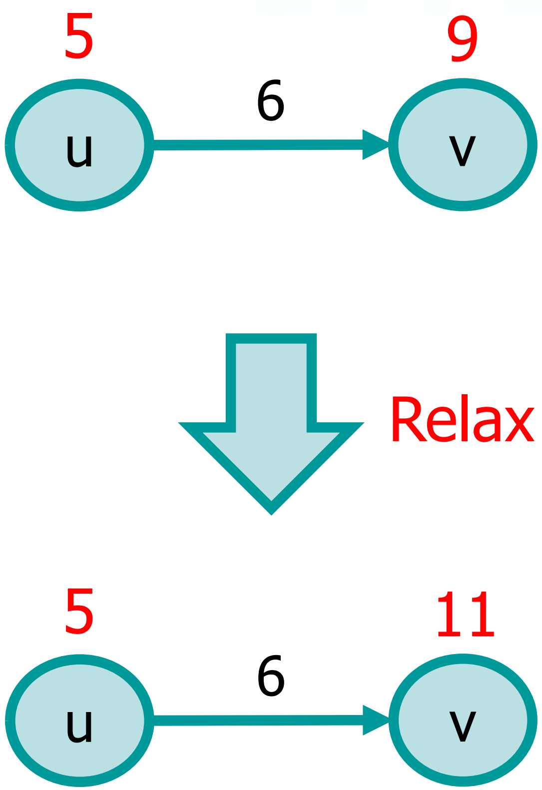
4
0
∞→5
∞→2
∞→3
∞→7
∞→8
∞→3
∞→6

Longest path on weighted DAG

- Problem intractable on generic weighted graph
- As on a DAG there are no cycles, the problem become computationally feasible
  - Topologically sort the DAG
  - For all ordered vertices
    - Apply the "inverse" relaxation rule starting from that vertex

```
inverse_relax (u, v, w) {
  if (v.dist < u.dist + w(v,u)) {
    v.dist = u.dist + w(v,u)
    v.pred = u
  }
}
```

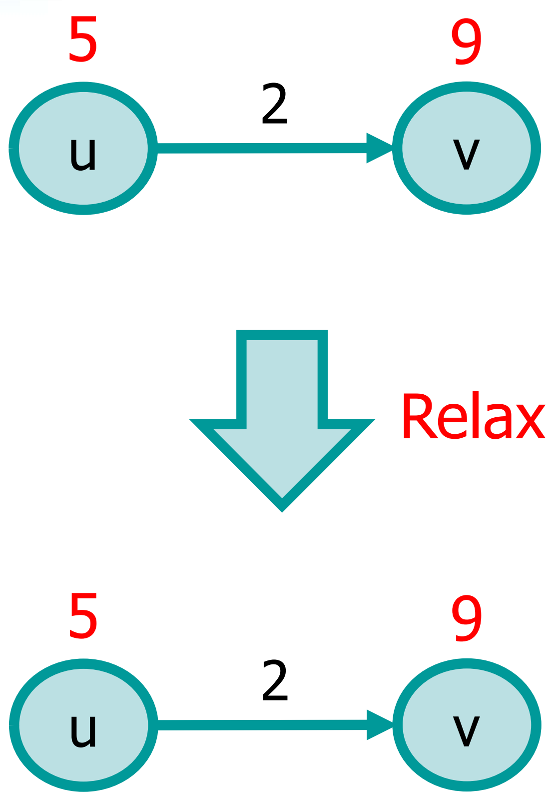
Example



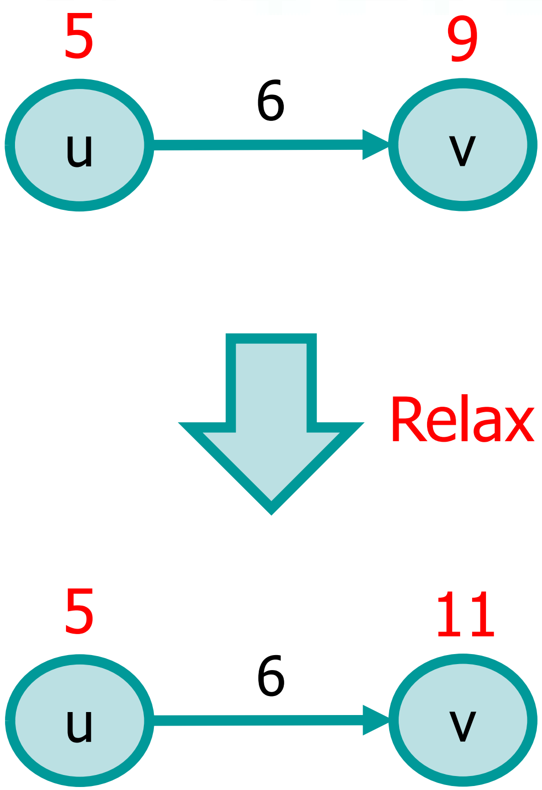

```
v.dist = 9
u.dist = 5
w(u,v) = 2
v.dist <
  u.dist + w(u,v)
```

Longest path from s to v = longtest path from s to u + edge (u,v)

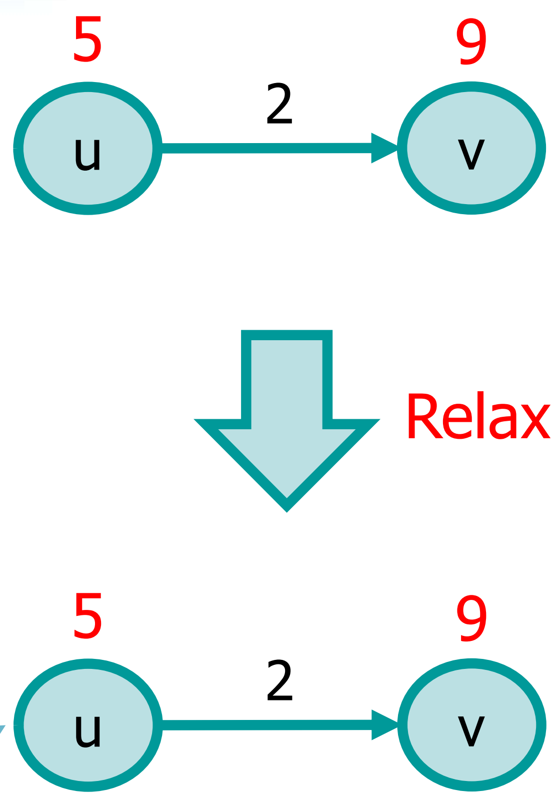
```
v.dist =
     = u.dist + w(u,v)
     = 5 + 2 = 7
v.pred = u
```



Example



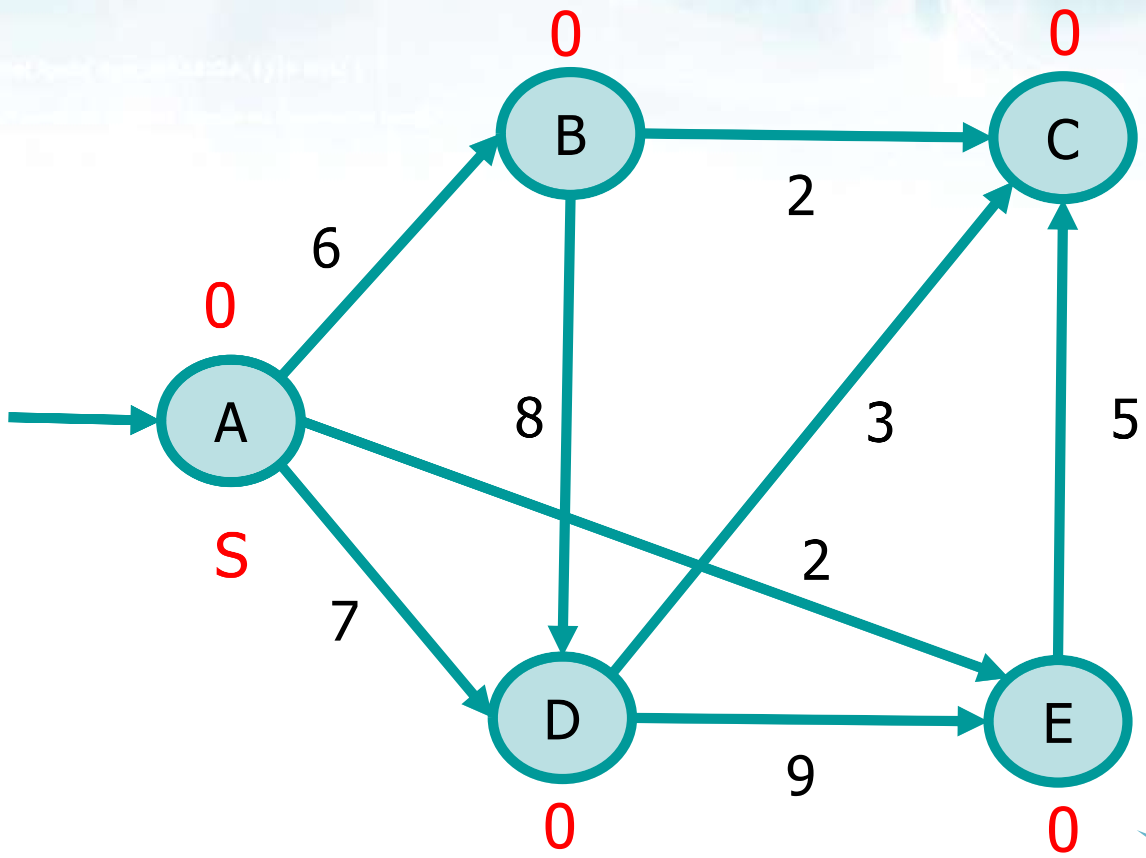

```
v.dist = 9
u.dist = 5
w(u,v) = 2
v.dist >
  u.dist + w(u,v)
```



Relaxation has no effect

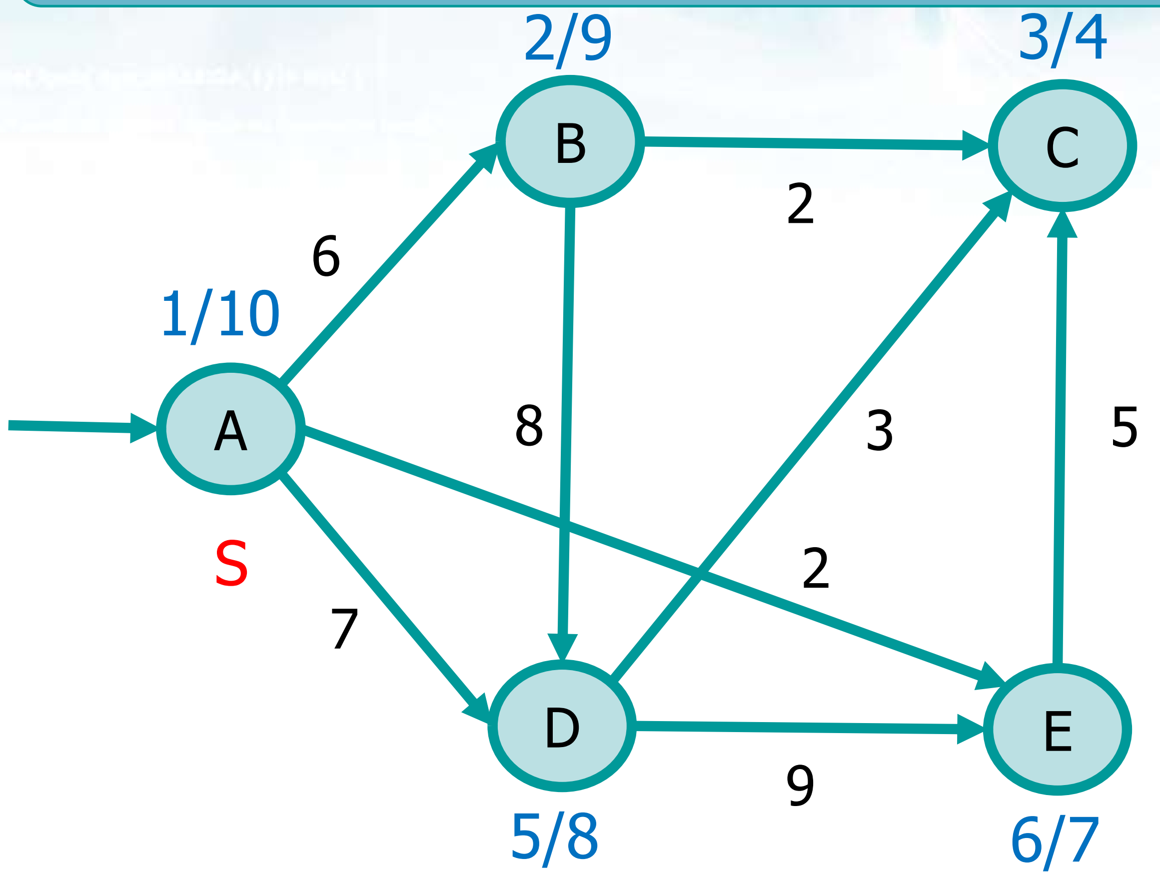
```
v.dist = unchanged
       = 6
v.pred = unchanged
```

Example



The initial estimate is equal to zero for all vertices

Solution



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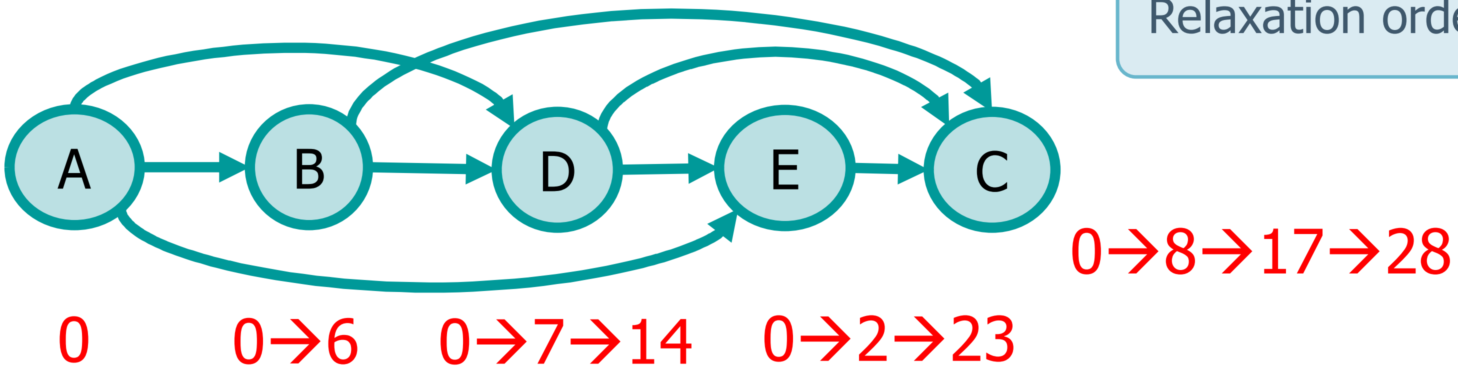
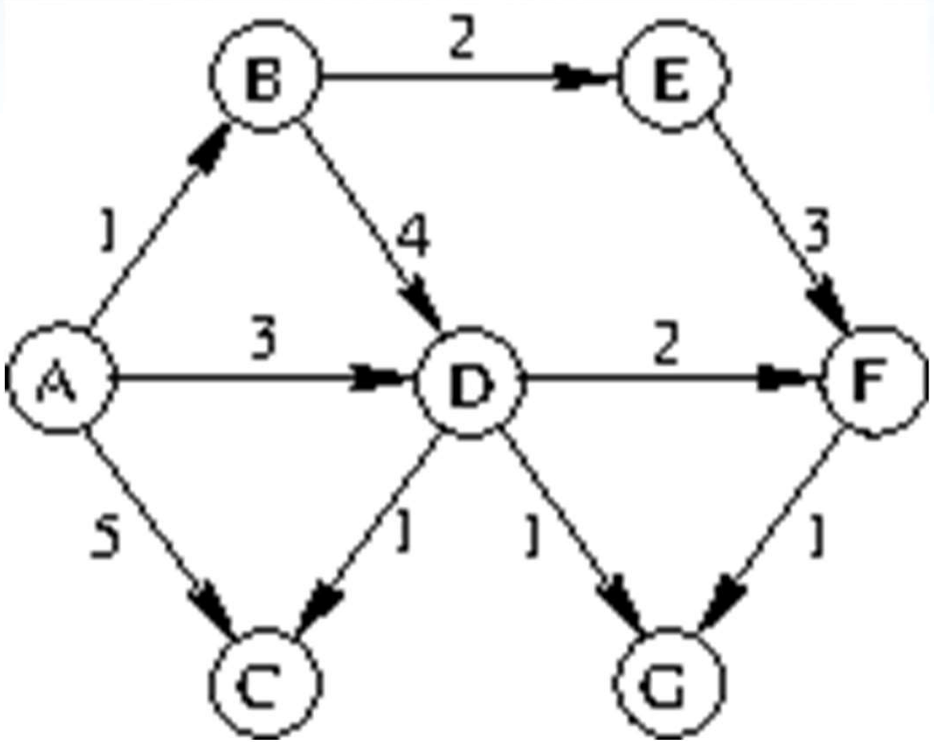


Complexity

- As the algorithm for the shortest paths for DAGs

Exercise



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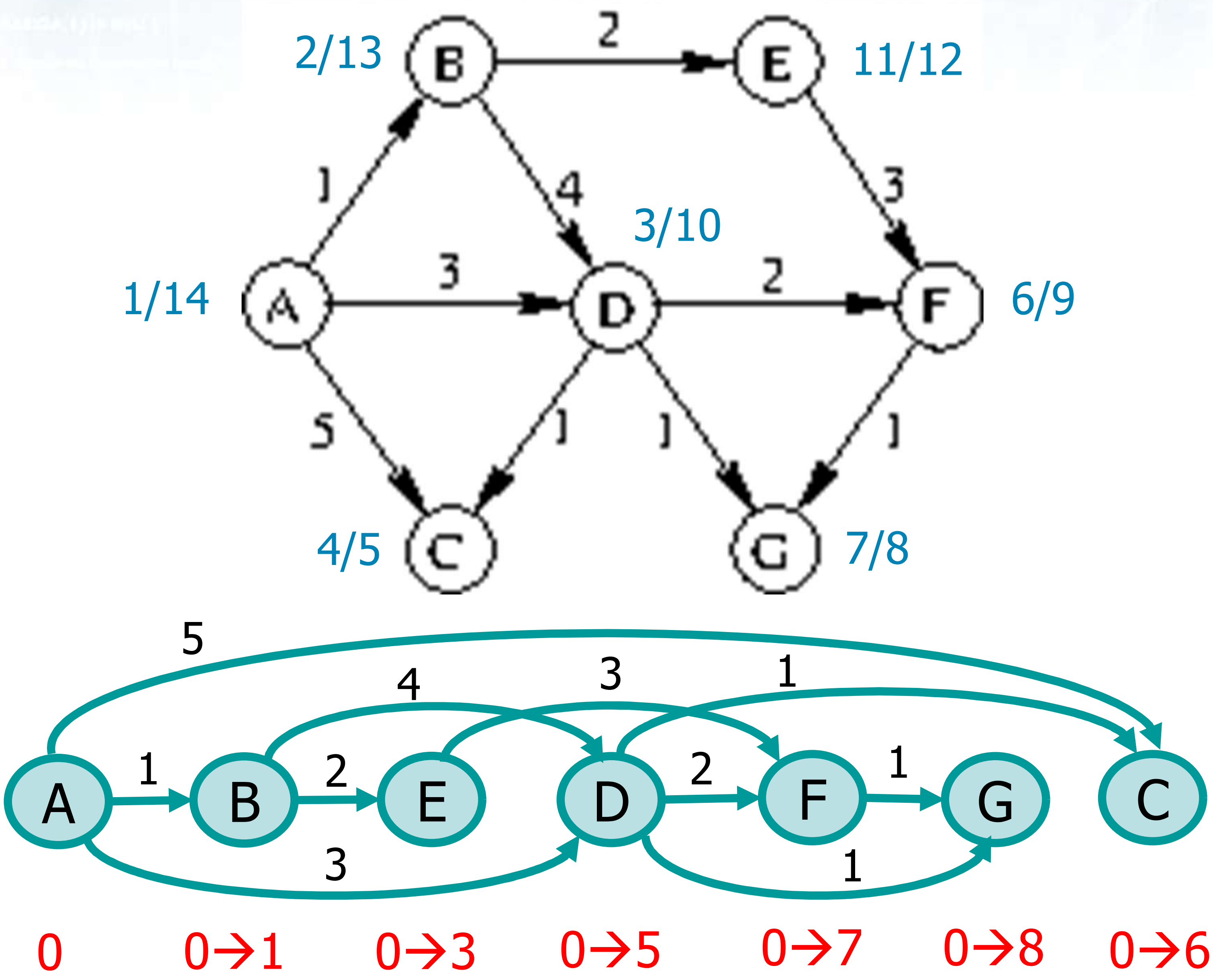
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0→1
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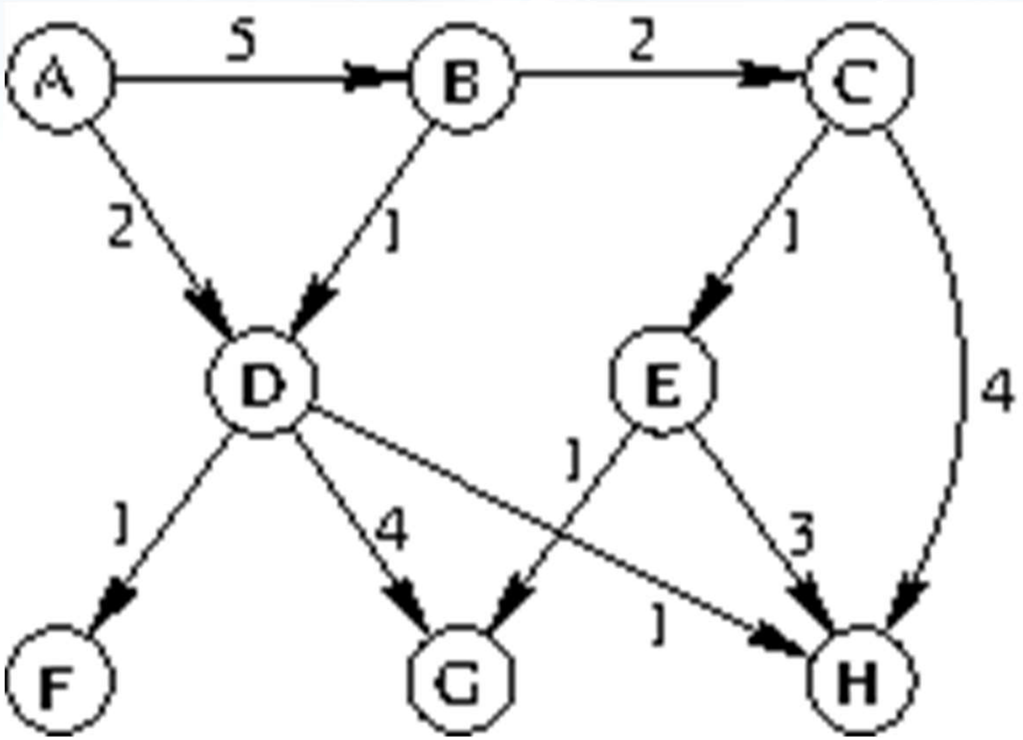
0→6

Exercise



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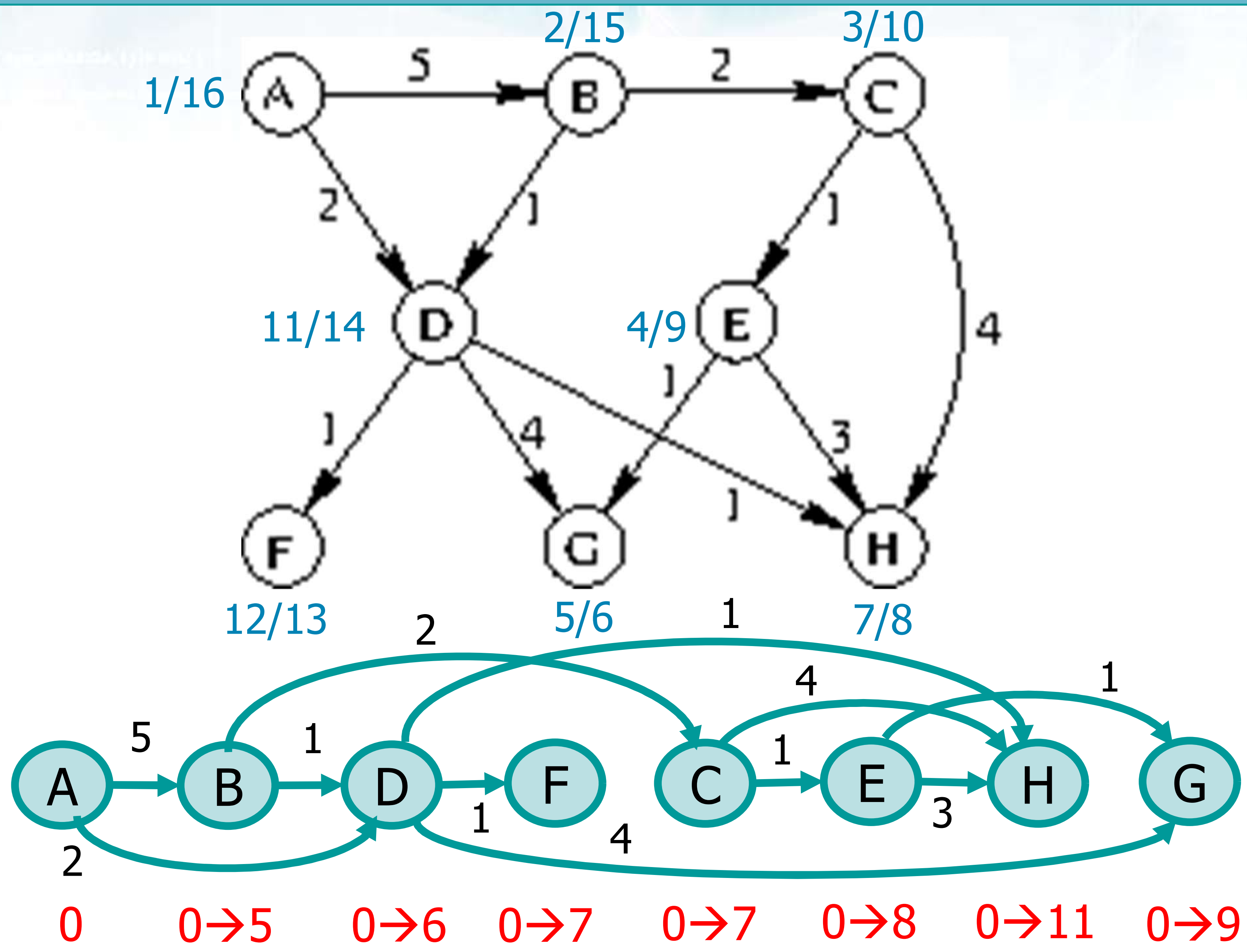

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0→9