Graphs

Definitions

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Graphs

- Definition
  - $G = (V, E)$
    - V = Finite and non empty set of vertices (simple or complex data)
    - E = Finite set of edges, that define a binary relation on V
- Directed/Undirected graphs
  - Directed
    - Edge = sorted pair of vertices $(u, v) \in E$ and $u, v \in V$
  - Undirected
    - Edge = unsorted pair of vertices $(u, v) \in E$ and $u, v \in V$

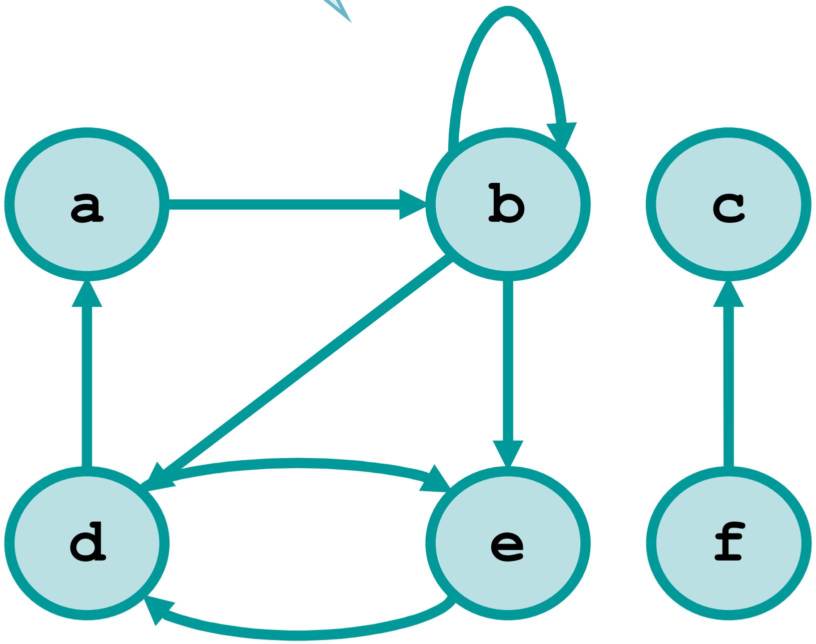
Applications

Domain	Vertex	Edge
communications	phone, computer	fiber optic, cable
circuits	gate, register, processor	wire
mechanics	joint	spring
finance	stocks, currencies	transactions
transports	airoport, station	air corridor, railway line
games	position on board	legal move
social networks	person	friendship
neural networks	neuron	synapsis
chemical compounds	molecules	link

Example: Directed graph

```
G = {V, E}
V = {a, b, c, d, e, f}
E = {(a,b),(b,b),(b,d),(b,e),
     (d,a),(d,e),(e,d),(f,c)}
```

Self-loop

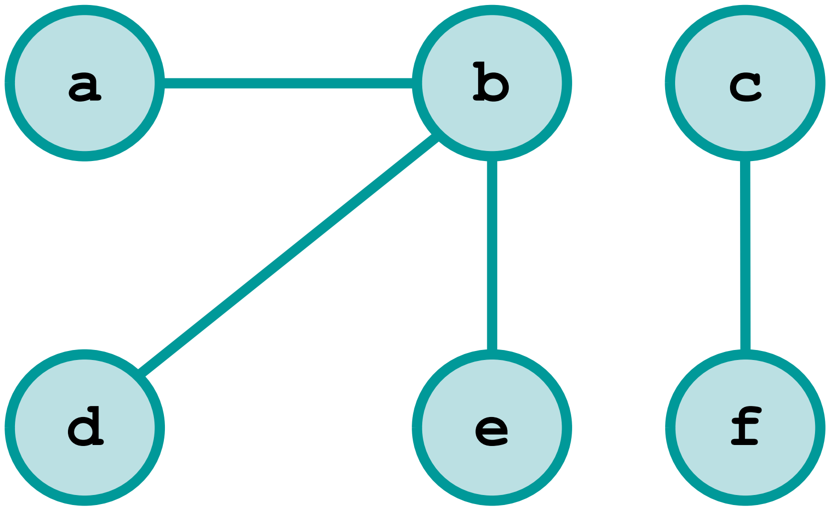


In some contexts self-loops may be forbidden.
If the context allows loops, but the graph is self-loop-free, it is called **simple**

Example: Undirected graph

```
G = {V, E}
V = {a, b, c, d, e, f}
E = {(a,b),(b,d),(b,e),(c,f)}
```

Self-loop



In some contexts self-loops may be forbidden.
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Edges

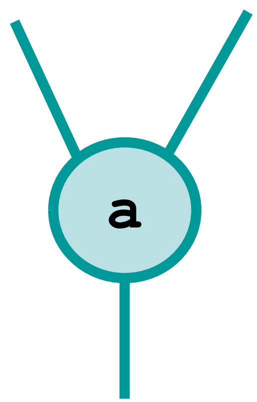
- Edges
  - An edge (a, b) can be
    - Incident from vertex a
    - Incident in vertex b
    - Incident on vertices a and b

  - Vertices a and b are adjacent
    - $a \rightarrow b \Leftrightarrow (a, b) \in E$

Edges

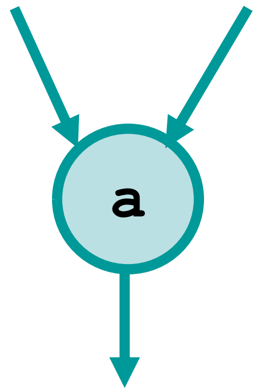
- Undirected graph
  - Degree (a) = number of incident edges

Degree (a) = 3



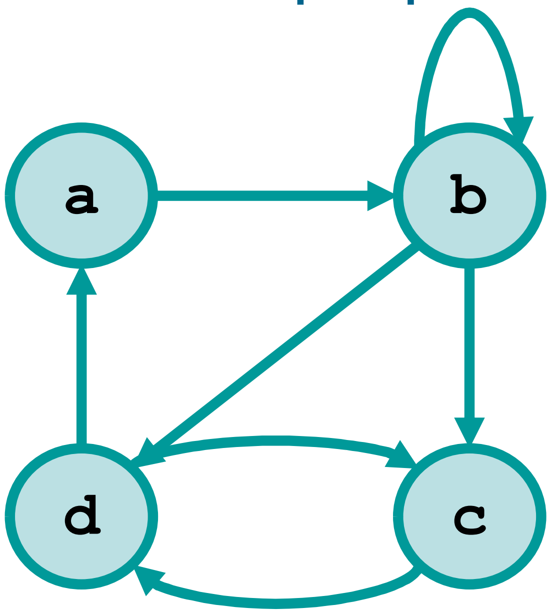
- Directed graph
  - In-degree (a) = number of incoming edges
  - Out-degree (a) = number of outgoing edges
  - Degree (a) = in-degree(a) + out-degree(a)

In-degree (a) = 2
Out-degree (a) = 1
Degree (a) = 3



Paths

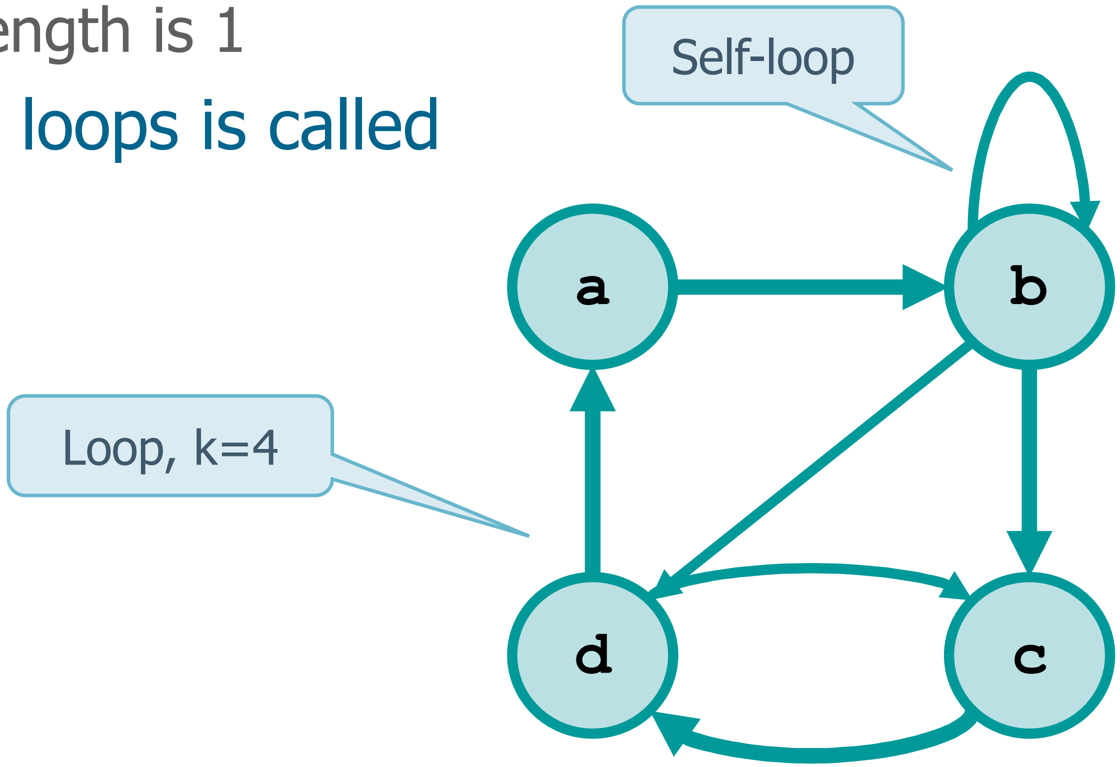
- Paths
  - A path p, $u \rightarrow_p u'$, is defined in G = (V, E) as
    - $\exists(v_0,v_1,v_2,\ldots,v_k) \mid u=v_0,\ u'=v_k,\ \forall i = 1,2,\ldots,k\ (v_{i-1},v_i) \in E$
  - k = length of the path
  - u' is reachable from u $\Leftrightarrow \exists$ p: $u \rightarrow_p u'$
  - Simple path p: distinct $(v_0,v_1,v_2,\ldots,v_k) \in p$



G = (V, E)
p: a $\rightarrow_p$ d : (a, b), (b, c), (c, d)
k = 3
d is reachable from a
p is a simple path

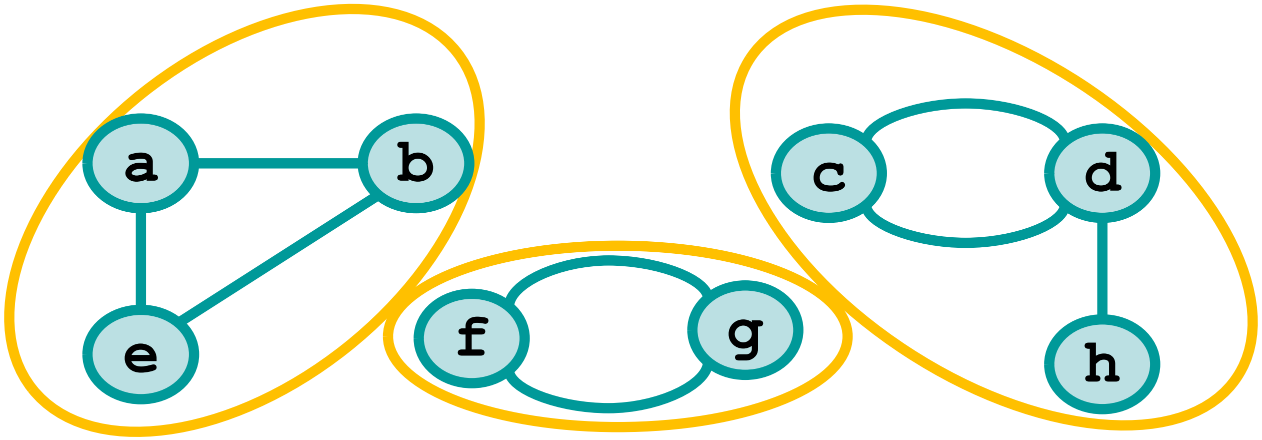
Loops

- Loops
  - A loop is defined as a path where
    - $v_0=v_k$, the starting and arrival vertices do coincide
  - Self-loop
    - Loops whose length is 1
  - A graphs without loops is called **acyclic**



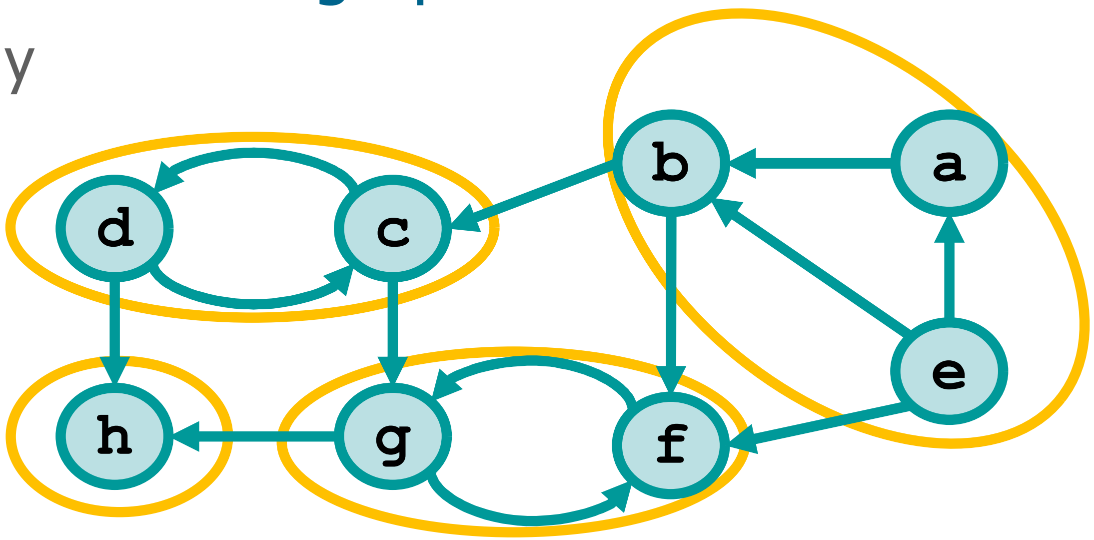
Connection in undirected graphs

- An undirected graph is said to be connected iff
  - $\forall v_i, v_j \in V$ there exists a path p such that $v_i \rightarrow_p v_j$
- In an undirected graph
  - Connected component
    - Maximal connected subgraph, that is, there is no superset including it which is connected
  - Connected undirected graph
    - Only one connected component



Connection in directed graphs

- A directed graph is said to be strongly connected iff
  - $\forall v_i, v_j \in V$ there exists two paths p, p′ such that $v_i \rightarrow_p v_j$ and $v_j \rightarrow_{p'} v_i$
- In a directed graph
  - Strongly connected component
    - Maximal strongly connected subgraph
  - Strongly connected directed graph
    - Only one strongly connected component



Dense/sparse graphs

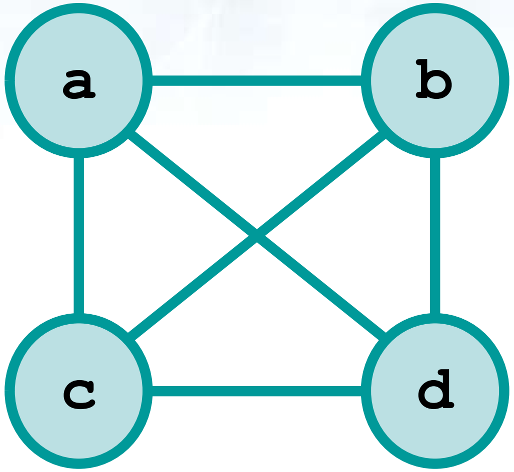
- Given a graph
  - $G = (V, E)$

  with
  - $|V|$ = cardinality of set V
  - $|E|$ = cardinality of set E
- We define
  - Dense graph
    - $|E| \cong |V|^2$

      A lot of edges
  - Sparse graph
    - $|E| << |V|^2$

      Few edges

Complete graph



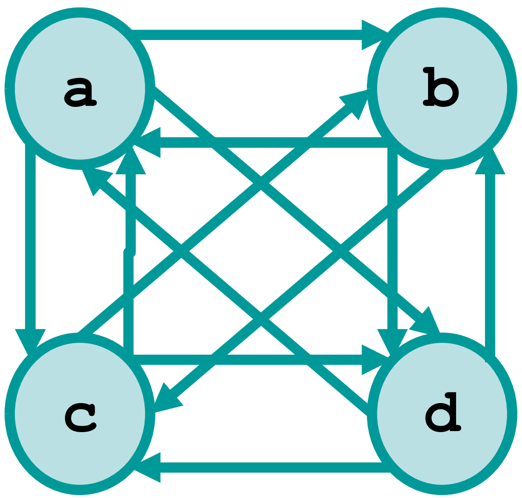
- Definition
  - $\forall v_i, v_j \in V \quad \exists (v_i, v_j) \in E$
- How many edges there are in a complete undirected graph?
  - |E| is given by the number of **combinations** of |V| elements taken 2 by 2
    - $|E| = \frac{|V|!}{(|V|-2)! \cdot 2!} = \frac{|V| \cdot (|V|-1) \cdot (|V|-2)!}{(|V|-2)! \cdot 2!} = \frac{|V| \cdot (|V|-1)}{2}$

Combinations: Order does not matter

Complete graph

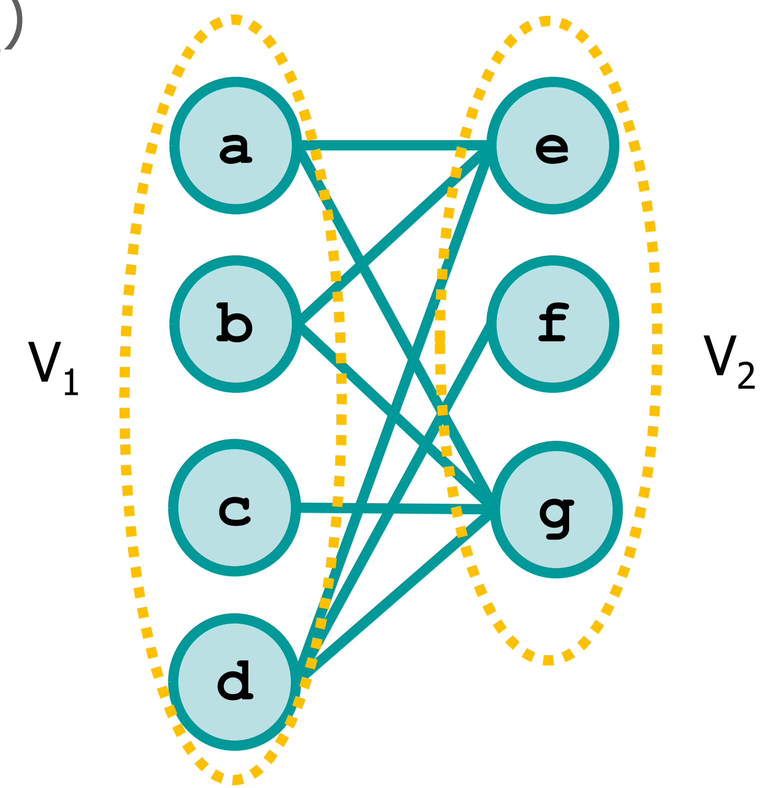
- How many edges there are in a complete directed graph?
  - |E| is the number of **dispositions** of |V| elements taken 2 by 2
    - $|E| = \frac{|V|!}{(|V|-2)!} = \frac{|V|\cdot(|V|-1)\cdot(|V|-2)!}{(|V|-2)!} = |V| \cdot (|V|-1)$

Dispositions:
Order matters



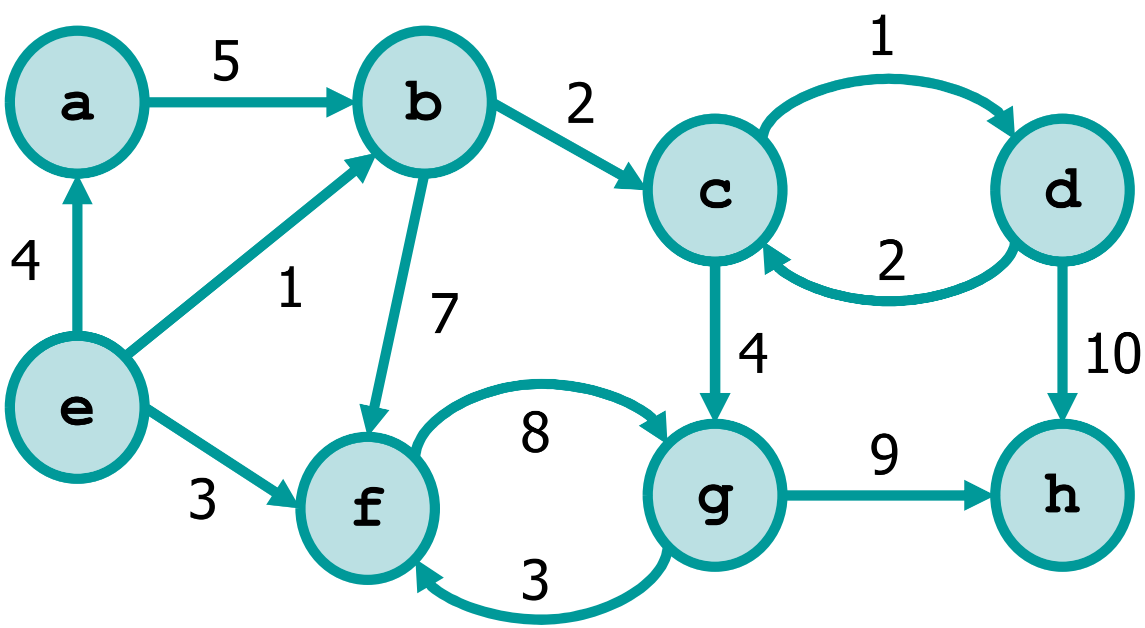
Bipartite graph

- Definition
  - Undirected graph where the V set may be partitioned in 2 subsets V_1 and V_2, such that
    - $\forall(v_i, v_j) \in E$ and ($v_i \in V_1$ and $v_j \in V_2$) or ($v_j \in V_1$ and $v_i \in V_2$)



Weighted graph

- A weighted graph is a graph whose edges have a weight, i.e.,
  - $\exists$ w: E $\rightarrow$ R | w(u,v) = weight of edge (u, v)
  - In practice, weights may be integers, reals, positive or negative values, etc.



Types of Graphs