Automatic Control: Course Review

Define Laplace Variable

```
s = tf('s');
```

Define System Given Matrices

```
A=[-3 2;-2 -3];
B=[1;0];
C=[0 1];
D = 0;

sys = ss(A,B,C,D);
```

Find Laplace Transform of x(t)

```
U = 1/s;
tol = 1e3;
x0 = [1;1];
X = zpk(minreal((s*eye(length(A))-A)\(x0 + B * U),tol));
```

Find Laplace Transform of y(t)

```
Y  = zpk(minreal(C*X + D*U, tol));
```

Compute PFE (Partial Fraction Expansion)

To compute y(t) we must remember that if the PFE has complex conjugate poles $Y(s) = \frac{R}{s-\sigma_0-\omega_0 j} + \frac{R^*}{s-\sigma_0+\omega_0 j}$ (note the residue R is the one associated with the complex root having positive imaginary part) then the antitransform is given by $y(t) = 2|R|e^{\sigma_0 t}\cos(\omega_0 t + arg(R))\epsilon(t)$ (keep in mind that the sign of σ_0 and ω_0 is the **opposite** of the one you find in the PFE $Y(s) = \frac{R}{s-\sigma_0-\omega_0 j}$)

```
[num_X1, den_X1] = tfdata(X(1), 'v');
[num_X2, den_X2] = tfdata(X(2), 'v');

[r1, p1] = residue(num_X1, den_X1);
[r2, p2] = residue(num_X2, den_X2);

[num_Y, den_Y] = tfdata(Y, 'v');
[r_Y, p_Y] = residue(num_Y, den_Y);
```

Define Transfer Function

```
s=tf('s');
H=(s+5)/(s^2+3*s+2);
```

Find system poles and zeros

```
zeros_H = zero(H);
poles_H = pole(H);
```

System in ZPK form

A system is in Zero Pole Gain Form if $H(s) = K_\infty \frac{(s-z_1)(s-z_2)\ldots(s-z_m)}{(s-p_1)(s-p_1)\ldots(s-p_n)}$

```
s=tf('s');
H=4*(2*s+6)/(s^2+3*s+2);
zpk(H)
```

```
ans =

    8 (s+3)
  -----------
  (s+2) (s+1)

Continuous-time zero/pole/gain model.
```

System in DC Gain Form

A system is in DC Gain Form if $H(s) = K \frac{(1-\frac{s}{z_1})(1-\frac{s}{z_2})\ldots(1-\frac{s}{z_m})}{s^r(1-\frac{s}{p_1})(1-\frac{s}{p_2})\ldots(1-\frac{s}{p_{n-r}})}$

```
% No MATLAB Statement :(
```

From State-Space Representation to Transfer Function

In the presence of **zero initial conditions** ($x(0) = 0$) we can compute H(s) with $H(s) = C(sI - A)^{-1}B + D$. The solution is **unique**

```
H = C*inv(s*eye(length(A))-A)*B+D;

% Another way of doing it

sys = ss(A,B,C,D);
H = tf(sys);
```

From Transfer Function to State-Space Representation

The solution is **not unique**. If the Transfer Function is not Strictly Proper (m = n proper, m < n strictly proper) we need to divide Numerator by Denominator. If H(s) is **strictly proper** no preliminary manipulations are needed. We would then need to choose from **Controller Canonical Form** or **Observer Canonical Form**, meaning that we have two ways to populate the A,B,C,D matrices with the H(s) coefficients. With matlab it all reduces to using two statements

```
H=(s^2+3*s+1)/(s^3+s^2+s+1);
sys=ss(H);
```

Internal Stability of a System: System Natural Modes

The system natural modes can be derived from the eigenvalues of the A matrix. In the case of eigenvalues with algebraic multiplicity $\mu_i > 1$ we need to study the **minimal polynomial**

- Considering the i-th **distinct** eigenvalue $\lambda_i (i = 1, \dots, r)$ with minimal polynomial multiplicity $\mu_i^{'}$ the functions $m_{i,1}(t) = e^{\lambda_i t}, m_{i,2}(t) = te^{\lambda_i t}, \dots, m_{i,\mu_i^{'}}(t) = \frac{t^{\mu_i^{'}-1}}{(\mu_i^{'}-1)!}e^{\lambda_i t}$ are the **natural modes** associated with the eigenvalue λ_i
- For each couple of complex conjugate eigenvalues $\lambda = \sigma_0 \pm j\omega_0$ with minimal polynomial multiplicity $\mu_i^{'}$ the natural modes are: $m_1(t) = e^{\sigma_0 t}\cos(\omega_0 t + \phi), m_2(t) = te^{\sigma_0 t}\cos(\omega_0 t + \phi), \dots, m_{\mu_i^{'}}(t) = \frac{t^{\mu_i^{'}-1}}{(\mu_i^{'}-1)!}e^{\sigma_0 t}\cos(\omega_0 t + \phi)$

Internal Stability of a System: Resume

An LTI system is:

- **Internally Stable** if and only if $\Re(\lambda_i(A)) \leq 0$ and $\mu^{'}(\lambda_i(A)) = 1$ for all the eigenvalues such that $\Re(\lambda_i(A)) = 0$
- **Asymptotically Stable** if and only if $\Re(\lambda_i(A)) < 0$

```
A=[0 0 0;0 0 1;0 0 -5];
eig(A);
% Since we have an eigenvalue with polinomial multiplicity of 2, we need to
% study $(sI-A)^{-1}$ to find the minimal multiplicity
A1=minreal(zpk(inv(s*eye(3)-A)));
```

Bibo Stability of a System

An LTI system is **BIBO stable** if and only if **all** the poles of the transfer function H(s) have **strictly negative** real part. We call **system poles** of an LTI system the roots of the denominator $D_H(s)$ of the transfer function $H(s)$ obtained **after** the simplification of all the common factors between the numerator and denominator (**zero-pole cancellation**). A system without any zero-pole cancellation in the computation of the transfer function is called **minimal** or in **minimal form**

Steady-State Response: Step Input

For a step input signal $u(t) = \bar{u}\epsilon(t) \rightarrow U(s) = \frac{\bar{u}}{s}$ the steady state response is $y_{ss}(t) = \bar{y}\epsilon(t) \rightarrow \bar{y} = \lim s \rightarrow 0sY(s) = \bar{u} \cdot H(0)$ where $H(0)$ is the **generalized DC gain**

```
s=tf('s');
H=1/((s+2)*(s+10));
K = dcgain(H);

% It can also be used for transfer functions with poles in zero (we have to
% remember to multiply the transfer function by $s^r$
H=1/(s*(s+2)^2);
K = dcgain(s*H);
```

Steady-State Response: Sinusoidal Input

For a sinusoidal input signal $u(t) = \bar{u}\sin(\omega_0 t)$ the steady state response is $y_{ss}(t) = \bar{y}(\omega_0)\sin(\omega_0 t + \phi(\omega_0))$ with $\bar{y}(\omega_0) = \bar{u}|H(j\omega_0)|, \phi(\omega_0) = \angle(H(j\omega_0))$. We can use the command *bode* to get the values of $\bar{y}(\omega_0), \phi(\omega_0)$

```
s=tf('s');
H=1/((s+2)*(s+10));
w0=0.5;
[m,f]=bode(H,w0);   % m is in linear scale, f is in DEGREES
f_rad = f/180*pi;   % [Radiants]
```

Time Delay

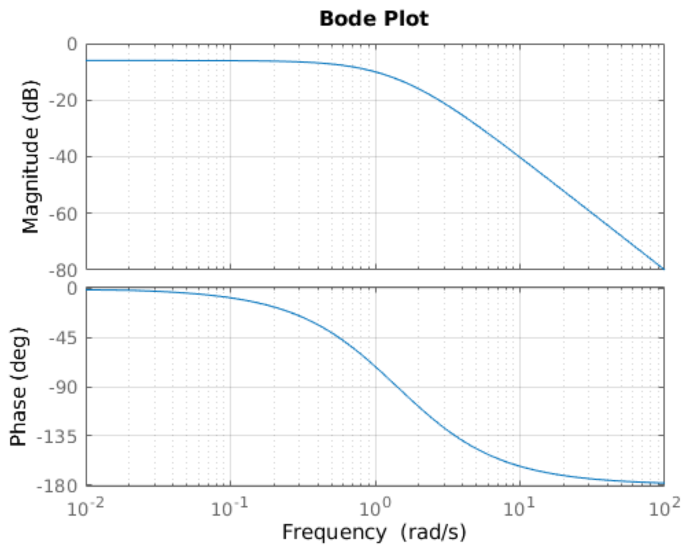
An LTI system whose dynamic behavior is described by a transfer function H(s) in the presence of a time delay can be represented as $H_{delay}(s) = H(s)e^{-\theta s}$. Since the time delay introduces a non-real rational component in the transfer function, we cannot compute the anti-transform as is. We must approximate the exponential function using the **Padè Approximation**:

- **First Order**: $e^{-\theta s} \approx \frac{1-\frac{\theta}{2}s}{1+\frac{\theta}{2}s}$
- **Second Order**: $e^{-\theta s} \approx \frac{1-\frac{\theta}{2}s+\frac{\theta^2 s^2}{12}}{1+\frac{\theta}{2}s+\frac{\theta^2 s^2}{12}}$

```
s=tf('s');
H=1/(s^2+s+1);
H.inputdelay=2;
pade_order = 1;
SYSX = zpk(pade(H,pade_order));
```

Bode Diagram

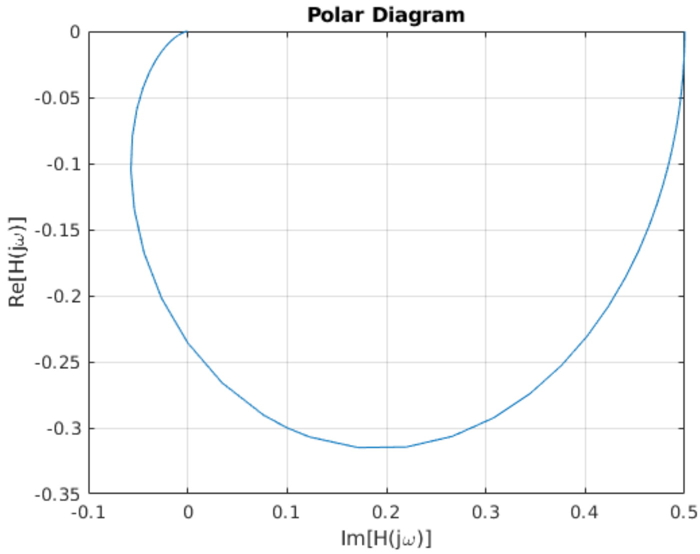
```
s=tf('s');
H=1/(s^2+3*s+2);
figure, bode(H);
title("Bode Plot");
grid on;
```



Polar Diagram

```
s=tf('s');
H=1/(s^2+3*s+2);
```

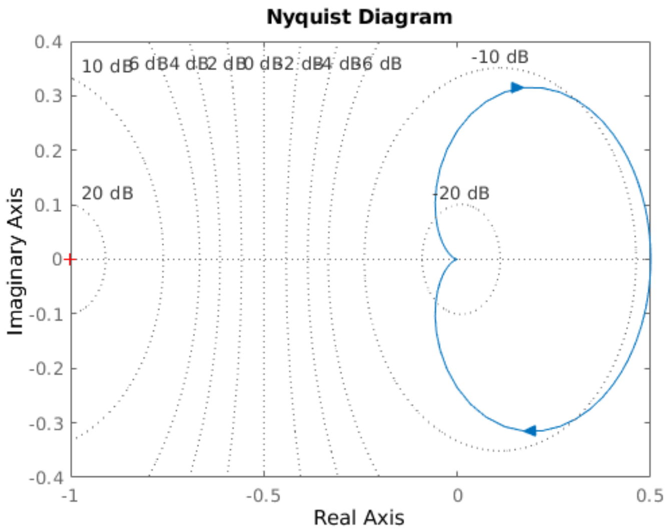
```
[re,im]=nyquist(H);
figure, plot(squeeze(re), squeeze(im));
title("Polar Diagram");
xlabel("Im[H(j\omega)]");
ylabel("Re[H(j\omega)]");
grid on;
```



Nyquist Diagram

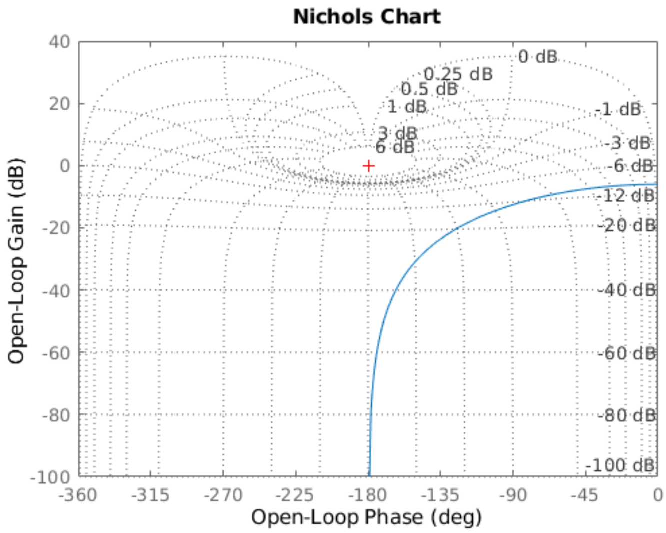
Keep in mind that Matalb does not plot the images of the semicircles $\rho \to 0$

```
s=tf('s');
H=1/(s^2+3*s+2);
figure,nyquist(H);
grid on;
```



Nichols Diagram

```
s=tf('s');
H=1/(s^2+3*s+2);
figure,nichols(H);
grid on;
```



Well-Posedness of a Feedback Control System

A feedback control system is said to be **well-posed** if:

- $G(s)$ is minimal and strictly proper
- $C(s)$ is minimal and proper

Stability of a Feedback Control System

1. Check for unstable zero-pole cancellations when $L(s) = C(s) \cdot G(s)$ is formed. If there are any, the system is **unstable**
2. Study the poles of $T(s) = \frac{L(s)}{1+L(s)}$: if all of them have **strictly negative real part** the system is **stable**

```
C=(s-1)/(s+1);
G=1/((s-1)*(s+2));
L=G*C;  % Loop Transfer Function before semplification
L=minreal(L);   % Loop Transfer Function after zero-pole cancellations
```

Nyquist Stability Criterion

The previous stability criteria do not give informations about how $L(s)$ affects stability or how **robust** the system is in the presence of perturbations. We then introduce the **Nyquist Stability Critetion**. Consider the Nyquist diagram of $L(s)$ and the complementary sensitivity function $T(s)$. Denote with:

- N the number of encirclements of the Nyquist diagram of $L(s)$ around the critical point $(-1, j0)$ ($N > 0 \rightarrow$ clockwise, $N < 0 \rightarrow$ counter-clockwise). N should be **well-defined** meaning that if the Nyquist diagram crosses the **critical point** $(-1, j0)$ the system is unstable
- Z the number of poles of $T(s)$ with **strictly positive real part**
- P the number of poles of $L(s)$ with **strictly positive real part**

Then we have $N = Z - P$. Since we know that a feedback control system is stable if and only if all the poles of $T(s)$ have **strictly negative real-part** $\rightarrow Z = 0 \rightarrow N = -P$. So the steps to determine if a feedback control system is stable are:

1. Determine the number P of the poles of $L(s)$ with **strictly positive real part**
2. Draw the Nyquist diagram of $L(s)$ and compute the number N of encirclements of the plot around the critical point $(-1, j0)$
3. Compute the number Z of poles of $T(s)$ with **strictly positive real part** as $Z = N + P$
4. Then the system is stable if and only if $Z = 0 \rightarrow N = -P$

Constant Magnitude Loci

Using the function T_Grid and S_Grid we can plot on the Nichols plane the constant magnitude loci that can help us to determine how to modify the Controller Transfer Function in order to satisfy stability margin requirements

```
s=tf('s');
L=1/(s*(s+2)*(s+4));
figure, nichols(L), hold on;
T_grid(0.4);
S_grid(2.6);
ngrid;
```