Math 33B: Differential Equations

Review 10: Variation of parameters

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1 Introduction

The *method of variation of parameters* will be the last technique we look at on how to solve second order linear differential equations.

Again, we consider the following ODE

$$y'' + p(x)y' + q(x)y = f(x), \tag{1}$$

which is given here in standard form.

The method we are going to study in this review, is the most general one we've seen so far; there are no restrictions on $p(x)$ and $q(x)$ [i.e. they can be constant or nonconstant functions] and $f(x)$ doesn't have to be 'simple' for the method to work (like, for example, in the *method of undetermined coefficients* discussed in **Review 9**).

We will start with the homogeneous version of (1), i.e. when $f(x) = 0$; the method can be used to obtain the general solution to a differential equation when $p(x)$ and $q(x)$ are nonconstants which is a case we haven't dealt with until now.

2 Homogeneous differential equation

For this section, we consider

$$y'' + p(x)y' + q(x)y = 0, \tag{2}$$

where $p(x)$ and $q(x)$ can be nonconstant functions of x.

Since (2) is a *linear* ODE, we know that the general solution to (2) is:

$$y_h(x) = c_1y_1 + c_2y_2, \tag{3}$$

where, again, y_1 and y_2 are a set of fundamental solutions (i.e. they are linearly independent).

Now, the technique that follows requires that one of the fundamental solutions, say y_1, is known. **If y_1 is known** (either by inspection or because it is given in the problem statement) then, we have a method of getting the second solution, y_2 to the homogeneous ODE given by (2).

The idea is that we assume that the second solution is represented by the following equation

$$y_2(x) = v(x)y_1(x), \tag{4}$$

where $v(x)$ is an **unknown**,**nonconstant** function of x and $y_1(x)$ is the **known** solution.

The motivation for Eq. (4), comes from the fact that we need y_1 and y_2 to be linearly independent if they are to constitute a *fundamental set* of solutions to the homogeneous differential equation. It follows that, if they are linearly independent, they cannot be a constant multiple of each other; hence, why we demand that $v(x)$ is a nonconstant function of x. You will notice that this technique is not entirely new: we have used it to solve for constant coefficient, homogeneous ODEs when the characteristic equation gave us a single, repeated root. Much like what we are trying to do now, in that case, we only had one solution (from the characteristic equation) and we needed a method to find the second one (see **Case 3: Repeated root** in **Review 8** for more details).

All we know about y_2 at this point is that it is a solution to (2) and it is given by (4) with $v(x)$ still being unknown. So, all we need to do is determine $v(x)$ and we are done.

How to determine $\boldsymbol{v}(x)$

Since y_2 is a solution to the homogeneous ODE (2), we expect it to satisfy the following differential equation:

$$y_2'' + p(x)y_2' + q(x)y_2 = 0. \tag{5}$$

Our first step is to take y_2 given by Eq. (4) and differentiate wrt x to obtain y_2' and y_2'':

$$\begin{aligned} y_2' &= vy_1' + v'y_1 && (6)\\ y_2'' &= vy_1'' + 2v'y_1' + v''y_1 && (7) \end{aligned}$$

Now, substitute the expressions for y_2, y_2' and y_2'' in Eq. (5):

$$vy_1'' + 2v'y_1' + v''y_1 + p(x)(vy_1' + v'y_1) + q(x)vy_1 = 0. \tag{8}$$

Some of the terms in (8) cancel out which simplifies the equation. The easiest way to see this is to factorize v, v' and v''; so, Eq. (8) becomes:

$$v\Big(y_1'' + p(x)y_1' + q(x)y_1\Big) + v'\Big(p(x)y_1 + 2y_1'\Big) + v''y_1 = 0 \tag{9}$$

Firstly, we notice that the coefficient of v is equal to 0 since y_1 is a solution to (2) and hence:

$$y_1'' + p(x)y_1' + q(x)y_1 = 0. \tag{10}$$

Equation (9) therefore reduces to:

$$v'\Big(p(x)y_1 + 2y_1'\Big) + v''y_1 = 0, \tag{11}$$

which, when rearranged, gives:

$$\frac{v''}{v'} = -2\frac{y_1'}{y_1} - p(x). \tag{12}$$

Note that (12) is a *second* order ODE which can be solved (easily, using separation of variables) to determine $v(x)$. This is no different to the method of separation of variables used in first order ODEs (see **Review 2**); we just need to do this twice here.

Integrating (12) wrt x once, we have

$$\ln v' = -2\ln y_1 - \int p(x)\,dx. \tag{13}$$

Before we proceed to integrate once more to obtain $v(x)$, we first exponentiate both sides of (13):

$$v' = \frac{1}{y_1^2}e^{-\int p(x)\,dx}. \tag{14}$$

Integrating (14) wrt x gives us $v(x)$:

$$v(x) = \int \frac{1}{y_1^2}\,e^{-\int p(x)\,dx}\,dx. \tag{15}$$

Equation (15) gives a formula from which we can obtain $v(x)$ which can then be used in (4) to get y_2 given that y_1 is known. This formula can be used for *any* second order, linear, homogeneous ODE. Note that $p(x)$ in formula (15) corresponds to the coefficient of y' when the ODE is in standard form.

The main result from this section is summarized in the box below.

Given a second order, linear, homogeneous ODE of the form,

$$y'' + p(x)y' + q(x)y = 0, \tag{16}$$

the general solution is

$$y_h(x) = c_1 y_1 + c_2 y_2. \tag{17}$$

For the general case, where $p(x)$ and $q(x)$ in Eq. (16) are nonconstants, we can solve (16) if one of the fundamental solutions is already known. Therefore, assuming that $y_1(x)$ is known, the second solution is given by

$$y_2(x) = v(x)y_1(x), \tag{18}$$

where $v(x)$ may be determined using the following equation:

$$v(x) = \int \frac{1}{y_1^2}\, e^{-\int p(x)\, dx}\, dx. \tag{19}$$

3 Nonhomogeneous differential equation

The method of variation of parameters is most useful when applied to nonhomogeneous differential equations as most physical applications require the forcing term [i.e. $f(x)$] to be nonzero.

We are now looking for solutions to Eq. (1) where $p(x)$, $q(x)$ and $f(x)$ do not have to satisfy any prior conditions before the method can be used (this wasn't the case in the *method of undetermined coefficients*). This is a much more general method for solving linear ODEs. Note that this method will work on any nonhomogeneous linear ODE (of *any* order) but for the purposes of **33B** we will only look at second order differential equations.

Firstly, let's recall here that for a nonhomogeneous, linear ODE, the general solution is the linear combination of the solution to the homogeneous equation, y_h [this is given by (3)] and a particular solution, y_p, that satisfies the nonhomogeneous equation:

$$y(x) = y_h(x) + y_p(x). \tag{20}$$

Therefore, to construct the general solution to Eq. (1), we need to find a particular solution that satisfies the ODE and then add this to the homogeneous solution.

The method

The method may be used *only if* the homogeneous solution is known.

Step 1: Find homogeneous solution

Let us assume that we know the two fundamental solutions y_1 and y_2 that make up the homogeneous solution and we therefore have

$$y_h(x) = c_1 y_1(x) + c_2 y_2(x). \tag{21}$$

Step 2: General form of *a* particular solution to (1)

We form the desired particular solution, y_p by taking Eq. (21) and replacing the constants c_1 and c_2 with unknown functions of x. Naturally, we replace c_1 with $v_1(x)$ and c_2 with $v_2(x)$:

$$y_p(x) = v_1(x)y_1(x) + v_2(x)y_2(x). \tag{22}$$

Step 3: Find two independent equations to solve for v_1' and v_2'

Since y_p solves the nonhomogeneous ODE, we expect it to satisfy the differential equation such that:

$$y_p'' + p(x)y_p' + q(x)y_p = f(x). \tag{23}$$

We therefore need to substitute y_p and its derivatives in Eq. (23) and hope that we'll be able to solve for the unknown functions $v_1(x)$ and $v_2(x)$.

Let us start by differentiating y_p once. From Eq. (22), we have:

$$y_p' = v_1 y_1' + v_1' y_1 + v_2 y_2' + v_2' y_2. \tag{24}$$

Now, if we differentiate a second time to get y_p'', then that will become too messy. Further, by differentiating again, we'll pick up second derivatives of y_1, y_2 as well as v_1 and v_2.

We are ultimately trying to solve for v_1 and v_2. We will therefore need two independent equations to solve for the two unknowns. We can perhaps foresee that if we were to substitute y_p and its derivatives in (23), we would end up with just one equation and two unknowns and hence, we would have an under-determined problem which is obviously good for nothing.

At this point let's state that whatever form the functions v_1 and v_2 take, we demand that they satisfy the following equation:

$$v_1' y_1 + v_2' y_2 = 0; \tag{25}$$

note that the terms that appear in Eq. (25) are the terms that are shown in blue in Eq. (24); they are the ones that contain first derivatives of v_1 and v_2. Note that this isn't the last we'll see of this equation, we will come back it in Step 4.

Taking into account Eq. (25) in Eq. (24), y_p' reduces to the following equation:

$$y_p' = v_1 y_1' + v_2 y_2'; \tag{26}$$

and it follows that the second derivative, y_p'' is given by:

$$y_p'' = v_1 y_1'' + v_1' y_1' + v_2 y_2'' + v_2' y_2'. \tag{27}$$

We are now ready to substitute y_p [Eq. (22)], y_p' [Eq. (26)] and y_p'' [Eq. (27)] in the nonhomogeneous ODE (23).

Upon substituting in the LHS of Eq. (23), we have:

$$\begin{aligned} y_p'' + p(x) y_p' + q(x) y_p &= v_1 y_1'' + v_1' y_1' + v_2 y_2'' + v_2' y_2' \\ &+ p(x)\Big(v_1 y_1' + v_2 y_2'\Big) \\ &+ q(x)\Big(v_1 y_1 + v_2 y_2\Big) \end{aligned} \tag{28}$$

and we know that this should be equal to the forcing term on the RHS, $f(x)$.

By factorizing v_1 and v_2 in the LHS of (28) and equating the resulting equation to $f(x)$, we get:

$$v_1\Big(y_1'' + p(x) y_1' + q(x) y_1\Big) + v_2\Big(y_2'' + p(x) y_2' + q(x) y_2\Big) + v_1' y_1' + v_2' y_2' = f(x). \tag{29}$$

Now, Eq. (29) may be simplified considerably by noticing that, the coefficients of v_1 and v_2 are actually equal to zero; this is because y_1 and y_2 are solutions to the homogeneous equation. It follows that Eq. (29) reduces to:

$$v_1' y_1' + v_2' y_2' = f(x). \tag{30}$$

Step 4: Solve for v_1 and v_2

At this point, we note that we are left with <u>one</u> equation for v_1' and v_2' [this is Eq. (30)]. Recall that y_1 and y_2 are known fundamental solutions and of course $f(x)$ is a known forcing term so the only unknown terms in (30) are v_1' and v_2'. As previously mentioned, we need two independent equations to solve for the two unknown functions. In Step 3, we demanded that whatever v_1 and v_2 may be, they need to satisfy a certain condition. This condition will serve as our second equation [note that it is independent to (30)] which will help us solve for v_1' and v_2', and, ultimately, for v_1 and v_2.

Let's state here the two independent equations:

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\begin{aligned}
v_1' y_1 + v_2' y_2 &= 0 &(31)\\
v_1' y_1' + v_2' y_2' &= f(x) &(32)
\end{aligned}
$$

It is easy to see from Eq. (31), that v_1' is expressed in terms of v_2' as:

$$v_1' = -\frac{v_2' y_2}{y_1}. \tag{33}$$

Substituting (33) in (32) gives us an equation for v_2':

$$\Big(-\frac{v_2' y_2}{y_1} \Big) y_1' + v_2' y_2' = f(x). \tag{34}$$

Simplifying Eq. (34) gives:

$$v_2' = \frac{y_1 f(x)}{y_1 y_2' - y_2 y_1'} \tag{35}$$

and, upon substituting (35) in (33) gives an equation for v_1':

$$v_1' = \frac{-y_2 f(x)}{y_1 y_2' - y_2 y_1'}. \tag{36}$$

Upon integrating both sides of Eqs. (35) and (36) wrt x we obtain expressions for v_2 and v_1, respectively. Before we do that though, let's just note that the *denominator* in Eqs. (35) and (36) is the same and equal to the Wronskian of y_1 and y_2:

$$W(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1'. \tag{37}$$

We can therefore re-write Eqs. (36) and (35) as:

$$v_1' = \frac{-y_2 f(x)}{W(x)} \quad \text{and} \quad v_2' = \frac{y_1 f(x)}{W(x)}. \tag{38}$$

And, finally, upon integrating wrt x,

$$v_1 = \int \frac{-y_2 f(x)}{W(x)}\, dx \quad \text{and} \quad v_2 = \int \frac{y_1 f(x)}{W(x)}\, dx. \tag{39}$$

So, the desired particular solution takes the general form:

$$y_p(x) = \Big(\int \frac{-y_2 f(x)}{W(x)}\Big) y_1 + \Big(\int \frac{y_1 f(x)}{W(x)}\, dx\Big) y_2; \qquad (40)$$

where $f(x)$ is a known function and it is assumed that we either have y_1 and y_2 (from problem statement) *or* we can find them by solving the homogeneous equation [and, hence we also know $W(x)$].

Note also that, since y_1 and y_2 are *linearly independent* solutions to the linear, homogeneous equation, then, as discussed in **Review 7**, $W(x)$ is never equal to zero, hence the expressions for v_1 and v_2 are defined.

General notes

- You may memorize the forms that v_1 and v_2 take [given by Eqs. (39)] and use them in Eq. (40) to obtain the desired particular solution.
- A fully worked-out example on solving nonhomogeneous, second order linear ODEs is available on CCLE (see also Homework 5, Problems 4 & 5).
- This is a more general method than the *method of undetermined coefficients* and it can be used to solve any nonhomogeneous, linear ODE. However, note that while this method requires **integration** (to obtain v_1 and v_2), the method of undetermined coefficients is **purely algebraic** which, for some people, is considered to be an easier task.
- Finally, note that while it may seem that by using this method we can obtain the general solution to *any* nonhomogeneous, linear ODE, it does not mean that we will be able to integrate every time. So, to conclude, we can write down an expression for the desired particular solution [given by (40)] but we may not be able to carry out the integration to find analytic functions for v_1 and v_2. Having said that, for the purposes of 33B we'll be able to integrate just fine!

The important equations to remember for the *method of variation of parameters* for *nonhomogeneous* equations are given below.

Given a second order, linear, nonhomogeneous ODE of the form,

$$y'' + p(x)y' + q(x)y = f(x), \tag{41}$$

the general solution is

$$y(x) = y_h(x) + y_p(x). \tag{42}$$

For the general case, where $p(x)$ and $q(x)$ in Eq. (41) are nonconstants, we can solve (41) if the homogeneous solution is already known. Therefore, assuming that we have $y_h(x)$, the desired particular solution is given by

$$y_p(x) = v_1(x)y_1(x) + v_2(x)y_2(x) \tag{43}$$

where v_1 and v_2 are:

$$v_1(x) = \int \frac{-y_2 f(x)}{W(x)}\, dx \quad \text{and} \quad v_2(x) = \int \frac{y_1 f(x)}{W(x)}\, dx. \tag{44}$$

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