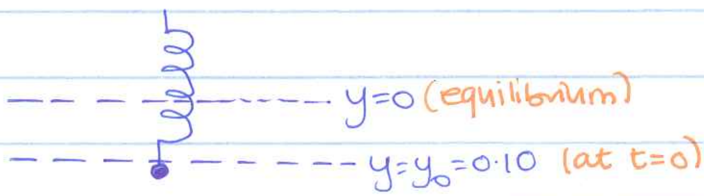
MATH 33B: DIFFERENTIAL EQUATIONS

May 2013.

Examples: HARMONIC MOTION

1. Undamped, unforced harmonic motion.

Consider a spring which has been stretched 0.10 m downwards from its equilibrium position:



The object attached to it has mass of 5 kg and the spring has a spring constant of 5 N/m.

Describe the displacement $y(t)$ of the mass if it is pushed upwards with a velocity of 1.44 m/s. (note: this is a very similar to the example covered in lecture 18 with the exception that we have an upwards velocity initially)

Since there's no damping, the eqn of motion is:

$$my'' + ky = 0 \quad \text{or} \quad y'' + w_0^2 y = 0 \quad \text{where } w_0 = \frac{k}{m}.$$

$$w_0 = \frac{k}{m} = \frac{5}{5} = 1 \quad \Rightarrow \quad y'' + y = 0 \quad ①$$

Using the char. eqn. $r^2 + 1 = 0$ (use r for the root since m represents mass here to avoid confusion) the root are: $\lambda = \pm i$.

and, hence, the G.S. is $\boxed{y(t) = c_1 \cos t + c_2 \sin t}$ (2)

Apply I.Cs.

Initially, the object is displaced 0.10 m downwards:

$$y(0) = c_1 = 0.10 \Rightarrow \underline{c_1 = 0.10}$$

The initial velocity of the object is 1.44 m/s upwards: $v(0) = y'(0) = -1.44$ (recall that in deriving the SHM eqn, we chose the downwards direction as +ve).

from (2), $y'(t) = -c_1 \sin t + c_2 \cos t$

$$y'(0) = c_2 = -1.44 \Rightarrow \underline{c_2 = -1.44}$$

Back in (2), $y(t) = 0.10 \cos t - 1.44 \sin t$. (3)

It's customary to combine a cosine - sine sdn like (3) into a single cosine term:

Eq. (3) is equivalent to: $y(t) = A\cos(\omega_0 t - \phi)$ where $\omega_0 = 1$ and A and ϕ are related to c_1 & c_2 as follows:

amplitude of motion

$$A = \sqrt{[c_1^2 + c_2^2]} = \sqrt{[0.10^2 + (-1.44)^2]} \approx 1.44.$$

$$\tan\phi = \frac{c_2}{c_1} = \frac{-1.44}{0.10} \Rightarrow \arctan\left(\frac{-1.44}{0.10}\right) = \arctan(-14.4)$$

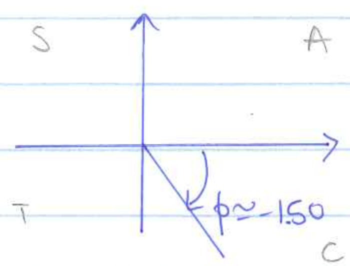
phase of motion

$$\phi \approx -1.50 \text{ rad}.$$

our calculation for ϕ should satisfy:

$$\cos\phi = \frac{\overset{+ve}{c_1}}{\underset{+ve}{A}} > 0 \quad \& \quad \sin\phi = \frac{\overset{-ve}{c_2}}{\underset{+ve}{A}} < 0$$

now, $\arctan(-14.4) \simeq -1.50$ is in the fourth quadrant:



where $\cos\phi$ is +ve and $\sin\phi$ is -ve hence the conditions above are satisfied and $\phi \simeq -1.50$ is the correct phase constant.

The displacement therefore is: $\boxed{y(t) = 1.44\cos(t + 1.50)}$

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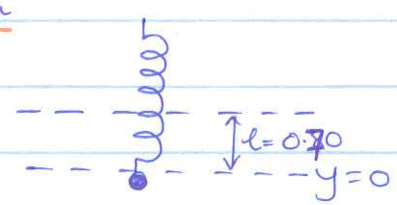
2. Damped, unforced harmonic motion.

A 5 kg object stretches a spring by 0.80 m:

at eqm



The mass has a damper hooked up $\therefore$ damping constant is equal to $\mu = 75$ kg/s. The mass is initially displaced 0.39 m downwards and then it is released.

(i) Does the system experience underdamping, critical damping or overdamping?

(ii) Determine $y(t)$ and plot the solution.

Now, the eqn of motion ~~also~~ needs to take into account the damping.

We have: $my'' + \mu y' + ky = 0$

or $y'' + 2cy' + \omega_0^2 y = 0$ ① where $2c = \frac{\mu}{m}$ & $\omega_0 = \frac{k}{m}$.

The spring constant, k, hasn't been given to us so we first need to determine it.

Find k.

At equilibrium, $mg = kl$ [∃ only 2 forces: gravity & force due to spring (i.e. Hooke's law)]

$$\therefore k = \frac{mg}{l} = \frac{5 \times 9.8}{0.7}$$

$$k = 70 \text{ N/m}.$$

① becomes $y'' + \frac{75}{5}y' + \frac{70}{5}y = 0$

the IVP is $\boxed{y'' + 15y' + 14y = 0 \text{ ② with } y(0) = 0.35\text{m} ; y'(0) = 0}$

(i) To determine what kind of damping the system experiences, we need to look at the roots of the characteristic eqn for ②: $\lambda^2 + 15\lambda + 14 = 0$. ③

Since the discriminant of ③ i.e. $\Delta = 15^2 - 4 \times 14 = 169 > 0$, then the system will experience overdamping.

(ii) To find $y(t)$, we proceed to solve the IVP.

Eq. ③ gives: $\lambda = \dfrac{-15 \pm \sqrt{169}}{2} \Rightarrow \lambda_1 = -1$ & $\lambda_2 = -14$

$\therefore$ the gen. soln is $y(t) = c_1 e^{-t} + c_2 e^{-14t}$. ④.

note: for systems experiencing unforced, harmonic motion, real roots of the char. eqn will always be negative.

Apply I.C.s to ④

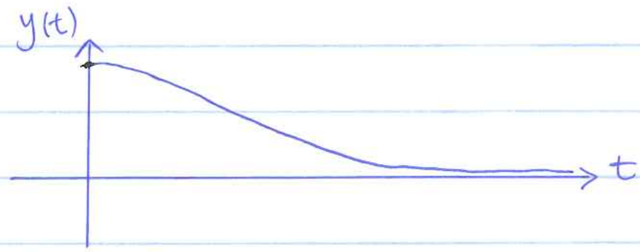
$y(0) = 0.35 \Rightarrow y(0) = c_1 + c_2 = 0.35$ ⑤

$y'(t) = -c_1 e^{-t} - 14c_2 e^{-14t} \Rightarrow y'(0) = -c_1 - 14c_2 = 0 \Rightarrow c_1 = -14c_2$. ⑥

In ⑤, $-14c_2 + c_2 = 0.35 \Rightarrow c_2 = \dfrac{0.35}{-13} = -0.03$.

In ⑥, $c_1 = -14 \cdot (-0.03) = 0.42$.

$\Rightarrow \boxed{y(t) = 0.42e^{-t} - 0.03e^{-14t}}$.



overdamping: no oscillations and $y \to 0$ as $t \to \infty$.