MATH 33B: DIFFERENTIAL EQUATIONS May 2^{nd}, 2013

Example: Solving nonhomogeneous ODEs with constant coefficients with the method of undetermined coefficients.

Find the G.S to:

$$y'' - 3y' + 2y = 2x^2 + 3e^{2x} \quad (1)$$

We know that the G.S. to (1) is:

$$y(x) = y_h + y_p. \quad (2).$$

We need to separately find y_h (the soln to the homog. eqn) and y_p (a particular soln to (1)).

Step 1: Find y_h.

The homog. ODE is $y'' - 3y' + 2y = 0$. (3)

This is a constant coef., homog. ODE so we use a characteristic eqn to find the gen soln to (3):

The char. eqn is: $m^2 - 3m + 2 = 0$.

$$(m-2)(m-1) = 0$$

$\Rightarrow$ The two roots are real and equal to $m_1 = 2$ and $m_2 = 1$

$$\Rightarrow \boxed{y_h(x) = c_1 e^{2x} + c_2 e^{x}}.$$

Step 2: Find y_p

We need to guess (intelligently!) a form for y_p.

We first look at $f(x)$ which is $f(x) = 2x^2 + 3e^{2x}$.

We have a polynomial of order 2 and an exponential and so y_p should have terms that match $f(x)$

Let's start with the polynomial

we should have Ax^2 (this matches the first term in $f(x)$)

and we should also include all its linearly independent derivs.

$\therefore$ $Ax^2 + Bx + C$ (4).

Then, we have the exponential term.

The form that matches $f(x)$ is De^{2x}. But, since the homog. soln already contains this term (through c_1e^{2x}) we need to multiply our guess by x $\Rightarrow$ a better guess is Dxe^{2x}. (5)

Putting (4) & (5) together:

$$\boxed{y_p = Ax^2 + Bx + C + Dxe^{2x}} \quad (6)$$

Next, we need to determine A, B, C & D.

From 6, $y_p' = 2Ax + B + 2Dxe^{2x} + De^{2x}$. (7)

$$y_p'' = 2A + 4Dxe^{2x} + 4De^{2x} \quad (8)$$

Sub. y_p, y_p', y_p'' in ①

$$(2A + 4Dxe^{2x} + 4De^{2x}) - 3(2Ax + B + 2Dxe^{2x} + De^{2x}) + 2(Ax^2 + Bx + C + Dxe^{2x}) = 2x^2 + 3e^{2x}.$$

Collect like terms together.

$$x^2(2A) + x(-6A + 2B) + xe^{2x}\cancel{(\overset{4D}{-6D} + 2D)}^{\,0} + e^{2x}(4D - 3D) + (2A - 3B + 2C) = 2x^2 + 3e^{2x}.$$

$$2Ax^2 + (2B - 6A)x + De^{2x} + (2A - 3B + 2C) = 2x^2 + 3e^{2x}.$$

Compare coef. of x^2, x, e^{2x}, constant on both sides.

x^2: $\quad 2A = 2 \quad \Rightarrow \quad A = 1$

x: $\quad 2B - 6A = 0 \quad \Rightarrow \quad 2B - 6 = 0 \quad \Rightarrow \quad B = 3$

e^{2x}: $\quad D = 3$

constant: $\quad 2A - 3B + 2C = 0 \quad \Rightarrow \quad 2 - 9 + 2C = 0 \quad \Rightarrow \quad C = 7/2$

$$\Rightarrow \boxed{y_p = x^2 + 3x + \frac{7}{2} + 3xe^{2x}.}$$

The G.S to ① is $y(x) = y_h + y_p$

$$\Rightarrow \boxed{y(x) = c_1 e^{2x} + c_2 e^{x} + x^2 + 3x + \frac{7}{2} + 3xe^{2x}}$$