Math 33B: Differential Equations

Review 6: Autonomous equations & stability

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1 Definitions

First order autonomous differential equations take the form

$$\frac{dy}{dt} = f(y), \tag{1}$$

where the function on the right-hand-side of (1), $f(y)$ does not depend on the independent variable (here, this is denoted by t). Autonomous equations of the form (1) are of course *separable*

$$\int \frac{1}{f(y)}\, dy = t + k. \tag{2}$$

However, it may be difficult to integrate $\displaystyle\int \frac{1}{f(y)}\, dy$ for an explicit solution in terms of y or it may not even be possible (in which case, one resorts to numerical methods). In such cases, we want to have a nice, simple way of obtaining *qualitative* information on the behavior of solutions without actually solving the ODEs. This is especially important for autonomous ODEs as they are used to model many physical processes in the real world.

Let us consider a value of y, say $y = y_0$, that renders the function $f(y)$ zero, i.e. $f(y_0) = 0$. Recall from our discussion on *direction fields* (see **Review 1**) that the arrows here have zero slope for all values of the independent variable, t. Therefore if we start the solution at a point $y = y_0$, then, following the zero-slope direction field arrows, it is obvious that the solution will stay at that value for all t. This means that $y(t) = y_0$ for all t is a *constant* solution to the ODE.

For example, in **Review 1**, we looked at a differential equation describing the velocity $v(t)$ of a falling object, given by $\dfrac{dv}{dt} = f(v) = 10 - 2v$ and we have shown that at $v = 5$, $f(v) = 0$ and hence $v(t) = 5$ is a constant solution to the ODE.

Let us now go back to using the notation $y(t)$ where y is the dependent variable and t is the independent variable.

The points $y = y_0$ that make $f(y)$ zero are called critical points (or equilibrium points).

The corresponding *constant* solutions given by $y(t) = y_0$ for all values of t are known as equilibrium solutions.

2 Qualitative solutions

Consider the autonomous ODE given by,

$$\frac{dy}{dt} = f(y) = y - 3. \tag{3}$$

If we were to use the method of isoclines to plot the direction field for (3), then, we'd see that all isoclines are horizontal and independent of t; in fact the equation of the isoclines is $y = 3 + c$ where c gives the slope of the arrows on the isoclines. See Fig. 1 for the direction field of (3) showing also several colorful, horizontal isoclines.

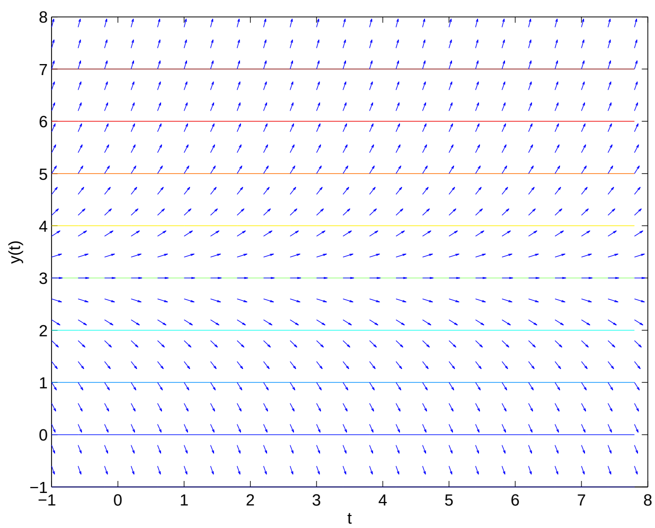

Figure 1: Isoclines and direction field for Eq. (3).

At $y = 3$, the function $f(y)$ in Eq. (3) is equal to zero which implies that $y = 3$ is a *critical point*. This corresponds to an equilibrium solution given by $y(t) = 3$ which remains constant for all values of t - this is represented by the integral curve, shown in black in Fig. 2. Four additional integral curves are shown in Fig. 2; the two solutions shown below $y = 0$ (green and red curves) are essentially the same curve but the green is *translated* to the right. Similarly, the purple curve is essentially the same as the blue one but it is translated to the right. This behavior is typical for ODEs whose isoclines are horizontal.

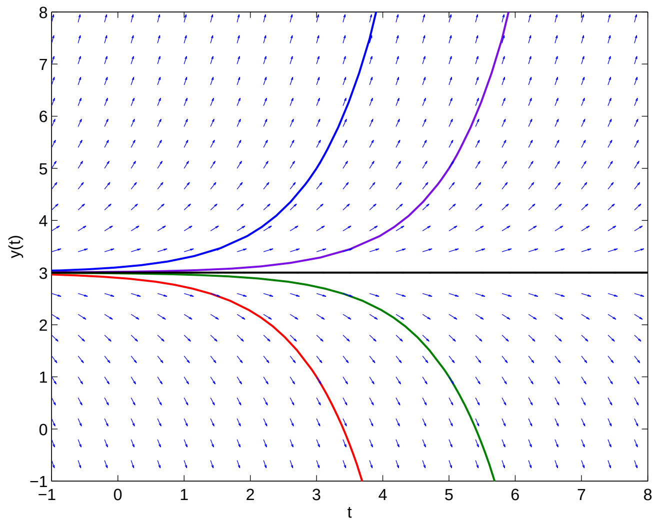

Figure 2: Equilibrium solution (black curve) and several non-equilibrium solutions (colored curves) for ODE (3).

What kind of information are we interested in?

- Given an autonomous ODE in the form of (1), we want to find all the critical points [these correspond to equilibrium solutions (i.e. constant solutions)]

  $\Longrightarrow$ Find all values of y for which $f(y) = 0$

- What happens above, below and in between equilibrium solutions?

  To determine the behavior of non-equilibrium solutions, we are interested in knowing where $f(y)$ is negative and where it is positive.

  $\Longrightarrow$ If $f(y) < 0$, then $\dfrac{dy}{dt} < 0$ and hence solutions are *decreasing*.

  $\Longrightarrow$ If $f(y) > 0$, then $\dfrac{dy}{dt} > 0$ and hence solutions are *increasing*.

Simple example: falling object

Again, let us look at the falling object example we covered in Lecture 2 (Review 1 notes).

$$\frac{dv}{dt} = f(v) = 10 - 2v. \tag{4}$$

Step 1: To obtain qualitative solutions to (4), we first look for critical points. These occur at $f(v) = 0$ hence,

$$f(v) = 10 - 2v = 0. \tag{5}$$

The only critical point is $v = 5$ and the equilibrium solution is $v(t) = 5$. This is shown in Fig. 3.

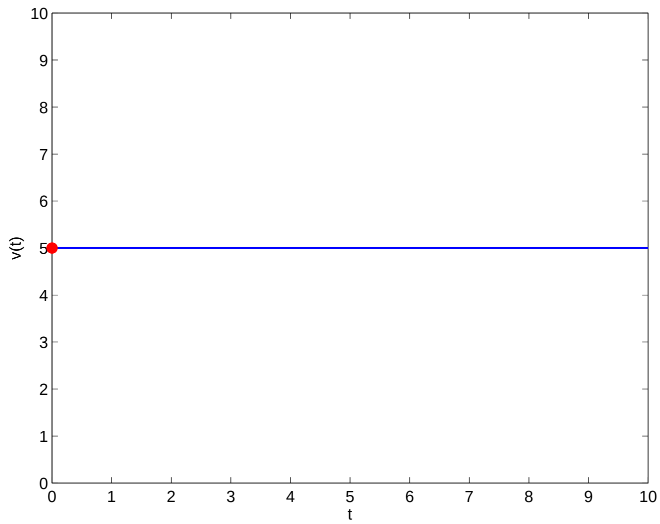

Figure 3: Equilibrium solution at $v(t) = 5$ for Eq. (4). The critical point is shown with a red, circle marker for $t = 0$.

Step 2: Find where $f(v) < 0$ and $f(v) > 0$.

Next, we plot $f(v)$ against v (see Fig. 4) to see for what values of v is $f(v)$ negative/positive. This shows that for $v > 5$, $f(v) < 0$. This implies that the gradient which is given by $\frac{dv}{dt} = f(v)$ is negative and hence solutions decrease. For $v < 5$, $f(v) > 0$ and hence solutions increase. The direction the solutions move toward is shown by arrows on the horizontal axis in Fig. 4. The arrows show that the solutions move toward the equilibrium solution, shown by a red circle marker in the same figure.

Now, if we were to start plotting a solution at a value *below* $v = 5$, the solution increases and approaches $v = 5$. If we start a solution *above* $v = 5$, the solution decreases and approaches $v = 5$. This behavior is shown in Fig. 5.

An equilibrium solution which is approached by all other solutions as t increases, is known to be *asymptotically stable.*

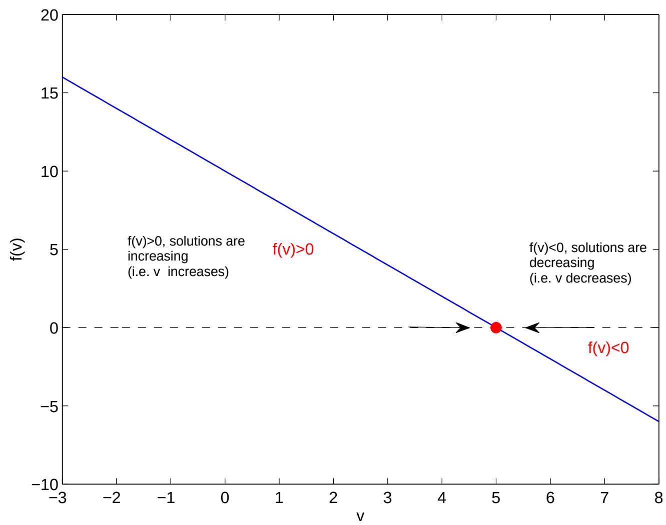

Figure 4: Plot of $f(v) = 10 - 2v$ against v. The function $f(v)$ is negative for $v > 5$ and positive for $v < 5$. The arrows show the direction the solutions move toward. In this case, all solutions move toward the critical point, $v = 5$.

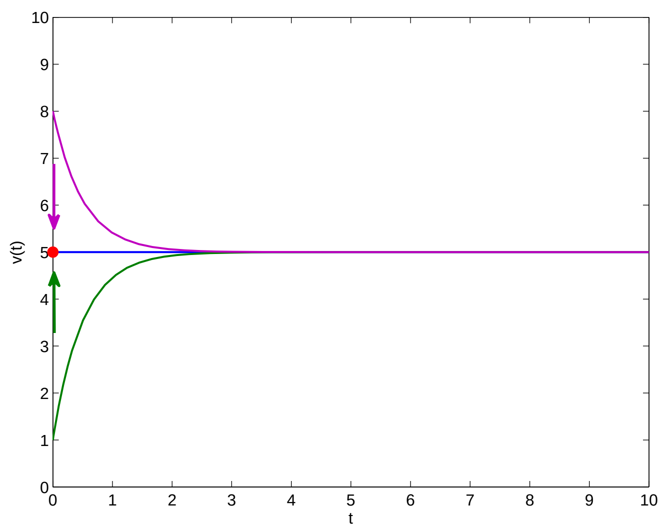

Figure 5: Equilibrium solution (blue line) and two non-equilibrium solutions (purple and green curves). Solutions that start above and below $v = 5$ approach $v = 5$ as t increases.

3 Logistic growth and classification of stability

Autonomous ODEs are often used to describe how 'population' increases. The term 'population' may refer to number of people, animals, bacteria as well as the *spread* of disease, a rumor etc.

A *simple* population growth model is given by,

$$\frac{dy}{dt} = ky \tag{6}$$

where k is a constant growth rate and $y(t)$ denotes the population of a given species at time t. This implies that the population keeps increasing indefinitely.

A *logistic growth* model gives a more realistic representation of how the 'population' increases. The growth rate that has been denoted by k in Eq. (6), is replaced by a linear relationship, $k = a - by$, where a and b are positive constants. This implies that as the population increases, the growth rate declines. Equation (6) is then replaced by

$$\frac{dy}{dt} = (a - by)y. \tag{7}$$

We wish to obtain qualitative information on the behavior of the solutions to Eq. (7) without solving the ODE. The ideas described in Section 2 apply here as well.

Typically we want to:

- Find critical points and equilibrium solutions;
- Determine where solutions are decreasing or increasing (this is equivalent to determining the *stability* of critical points).

Classification of critical points

In Section 2, we showed that the critical point was asymptotically stable because all (non-equilibrium) solutions approach it.

There are three types of stability:

1. If the arrows (corresponding to the direction the solutions move), move away from the equilibrium solution, i.e. $\leftarrow \bullet \rightarrow$ then the critical point is asymptotically unstable.
2. As we've seen in Section 2, if the arrows move toward the equilibrium solution $\rightarrow \bullet \leftarrow$, then the critical point is asymptotically stable.

3. Finally, if non-equilibrium solutions approach the critical point from one side but they move away from it from the other, i.e. $\rightarrow \bullet \rightarrow$, then the critical point is semistable.

Note: A worked-out example on *logistic growth with harvesting* covered in Lecture 9 will be posted on CCLE.