Math 33b, Winter 2013, Tonći Antunović - Homework 9

From the textbook solve the problems:
Section 9.3: 10, 12, 16 and 18 (don't use numerical solver)
Section 9.4: 4, 6, 10, 20
Section 9.5: 12, 18, 22, 26, 28, 32.

And also the problems below:

Problem 1. Both the trace and the determinant of a 2x2 matrix A are equal to 2. Classify the equilibrium point and compute the characteristic polynomial. Then give one example of such a matrix A.

Solution: Characteristic polynomial is

$$p(\lambda) = \lambda^2 - T\lambda + D = \lambda^2 - 2\lambda + 2,$$

and since $T^2 - 4D = -4$ and $T > 0$ the equilibrium point is a spiral source. One example is

$$A = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}.$$

Problem 2. Any solution to the equation $y' = Ay$ is periodic with the period $3\pi/2$ (A is a 2x2 matrix). Classify the equilibrium point and give one example of such a matrix A.

Solution: The only case when all solutions are periodic is the center case, that is when we have two complex conjugate roots of the characteristic polynomial and when their real part is zero. Then roots are βi and $-\beta i$ and looking at the general form of the solution in this case we have that any solution is $2\pi/\beta$ periodic. So we need $2\pi/\beta = 3\pi/2$, so $\beta = 4/3$ and then the characteristic polynomial is

$$p(\lambda) = (\lambda - 4i/3)(\lambda + 4i/3) = \lambda^2 + 16/9.$$

An example of such a matrix is

$$A = \begin{pmatrix} 0 & -4/3 \\ 4/3 & 0 \end{pmatrix}.$$

Problem 3. Find the solution of the initial value problem $y' = Ay$, $y(0) = y_0$ where

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & -1 \end{pmatrix}, \qquad y_0 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

Solution: The characteristic polynomial is

$$\begin{aligned} p(\lambda) = \det(\lambda I - A) &= \det \begin{pmatrix} \lambda & -1 & -1 \\ -1 & \lambda & -1 \\ 0 & 0 & \lambda + 1 \end{pmatrix} \\ &= (\lambda+1) \det \begin{pmatrix} \lambda & -1 \\ -1 & \lambda \end{pmatrix} = (\lambda+1)(\lambda^2 - 1) = (\lambda-1)(\lambda+1)^2. \end{aligned}$$

So there are two eigenvalues $\lambda_1 = 1$ and $\lambda_2 = -1$. The eigenvectors for the eigenvalue $\lambda_1 = 1$ are the nullvectors of

$$A - I = \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & -2 \end{pmatrix}$$

which is easily seen to be the set of all multiples of the vector $(1, 1, 0)^T$.

The eigenvectors for the eigenvalue $\lambda_2 = -1$ are the nullvectors of

$$A + I = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

which are all vectors $(x_1, x_2, x_3)^T$ with the property $x_1 + x_2 + x_3 = 0$. We can find two independent such vectors say $(1, -1, 0)^T$ and $(1, 0, -1)^T$. Then the general solution is

$$y = C_1 e^t \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + C_2 e^{-t} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + C_3 e^{-t} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}.$$

Then

$$y = \begin{pmatrix} C_1 + C_2 + C_3 \\ C_1 - C_2 \\ -C_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

so we need $C_1+C_2+C_3=1$, $C_1-C_2=2$ and $-C_3=3$ which gives $C_1=3$, $C_2=1$, $C_3=-3$ and the solution is

$$y=3e^t\begin{pmatrix}1\\1\\0\end{pmatrix}+e^{-t}\begin{pmatrix}1\\-1\\0\end{pmatrix}-3e^{-t}\begin{pmatrix}1\\0\\-1\end{pmatrix}=e^t\begin{pmatrix}3\\3\\0\end{pmatrix}+e^{-t}\begin{pmatrix}-2\\-1\\3\end{pmatrix}.$$

Problem 4. Find the general solution of the equation $y'=Ay$ where

$$A=\begin{pmatrix}1&0&0\\0&0&2\\0&-2&0\end{pmatrix}.$$

Solution: The characteristic polynomial is

$$p(\lambda)=\det(\lambda I-A)=\det\begin{pmatrix}\lambda-1&0&0\\0&\lambda&-2\\0&2&\lambda\end{pmatrix}=(\lambda-1)\det\begin{pmatrix}\lambda&-2\\2&\lambda\end{pmatrix}=(\lambda-1)(\lambda^2+4),$$

whose zeros are $\lambda_1=1$ and $\lambda_2=2i$, $\lambda_3=-2i$. The eigenvectors for $\lambda_1=1$ are vectors in the nullspace of

$$A-I=\begin{pmatrix}0&0&0\\0&-1&2\\0&-2&-1\end{pmatrix}$$

which are multiples of the vector $(1,0,0)^T$. Complex eigenvectors for $\lambda_2=2i$ are the nullvectors of the matrix

$$A-2iI=\begin{pmatrix}1-2i&0&0\\0&-2i&2\\0&-2&-2i\end{pmatrix}$$

which are multiple vectors of $(0,1,i)^T$. Then the complex solution is

$$y=e^{2it}\begin{pmatrix}0\\1\\i\end{pmatrix}=(\cos(2t)+i\sin(2t))\left(\begin{pmatrix}0\\1\\0\end{pmatrix}+i\begin{pmatrix}0\\0\\1\end{pmatrix}\right)$$
$$=\left(\cos(2t)\begin{pmatrix}0\\1\\0\end{pmatrix}-\sin(2t)\begin{pmatrix}0\\0\\1\end{pmatrix}\right)+i\left(\cos(2t)\begin{pmatrix}0\\0\\1\end{pmatrix}+\sin(2t)\begin{pmatrix}0\\1\\0\end{pmatrix}\right)$$

and we obtain two independent solution as real and imaginary parts of this solution

$$\cos(2t)\begin{pmatrix}0\\1\\0\end{pmatrix}-\sin(2t)\begin{pmatrix}0\\0\\1\end{pmatrix},\qquad \cos(2t)\begin{pmatrix}0\\0\\1\end{pmatrix}+\sin(2t)\begin{pmatrix}0\\1\\0\end{pmatrix}.$$

Then general solution is

$$\begin{aligned}
y = C_1e^t\begin{pmatrix}1\\0\\0\end{pmatrix}+C_2\left(\cos(2t)\begin{pmatrix}0\\1\\0\end{pmatrix}-\sin(2t)\begin{pmatrix}0\\0\\1\end{pmatrix}\right)+&\\
C_3\left(\cos(2t)\begin{pmatrix}0\\0\\1\end{pmatrix}+\sin(2t)\begin{pmatrix}0\\1\\0\end{pmatrix}\right)&\\
= e^t\begin{pmatrix}C_1\\0\\0\end{pmatrix}+\cos(2t)\begin{pmatrix}0\\C_2\\C_3\end{pmatrix}+\sin(2t)\begin{pmatrix}0\\C_3\\-C_2\end{pmatrix}&.
\end{aligned}$$