Math 33B: Differential Equations

Review 5: Existence & uniqueness of IVPs

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1 Objectives

Suppose we have a first order initial-value-problem (IVP):

$$\frac{dy}{dx} = f(x, y) \quad \text{with} \quad y(a) = b \tag{1}$$

What information do we want to know when given an IVP?

- Does a solution exist?
- If it does, is the solution unique?
- If a solution does exist, how do we find it?

The objectives of this course are mainly concerned with the last question. Typically we don't want to waste time trying to find a solution that doesn't exist. Also, if we are looking for a solution to a physical situation, we want to know that there is a *unique* solution corresponding to the physical process.

In this review, we *informally* state the theorems that guarantee that a solution exists and is unique and show how to apply them to IVPs. The theorem proofs are rather involved and beyond the scope of this course but they are covered in any advanced calculus textbook.

2 Existence & uniqueness theorems

In the following statements of the existence and uniqueness theorems, we refer back to the IVP given by Eq. (1).

Existence: **If** $f(x,y)$ is *continuous* 'near' a point (a,b), **then** at least one solution to (1) *exists*.

Uniqueness: Given that a solution exists 'near' (a,b) then, **if** the partial derivative of $f(x,y)$ wrt y, i.e. $\frac{\partial f}{\partial y}$, is *continuous* 'near' a point (a,b), **then** the solution is *unique*.

Figure 1 shows what we mean by 'near' (a,b) in the boxed definitions above; the black rectangle represents a region which contains the initial condition, (a,b). This rectangle represents the largest interval within which we can check whether $f(x.y)$ and $\frac{\partial f}{\partial y}$ are continuous (note: this region doesn't really have to be represented by a rectangle, it can be any shape).

An integral curve passing through (a,b) is also shown within the rectangle: it is guaranteed by the theorems that if f and $\frac{\partial f}{\partial y}$ are continuous, then the solution through (a,b) exists and is unique even if it is for a very short interval. Note that outside of the rectangle (where the hypotheses of the theorems aren't satisfied), we have no idea whether solutions exist/are unique.

The definitions in the boxes give **sufficient** conditions for existence and uniqueness. This means that if $f(x,y)$ is <u>*not*</u> continuous in the rectangle, then a solution *may or may not* exist. Similarly, if $\frac{\partial f}{\partial y}$ is <u>*not*</u> continuous in the rectangle, then a solution *may or may not* be unique.

3 Applying the theorems to IVPs

Consider the IVP,

$$x\frac{dy}{dx} = y \quad \text{with} \quad y(a) = b. \tag{2}$$

Before we apply the theorems, we need to put the ODE in the form given in (1). So, by rearranging Eq. (2), the ODE is expressed as,

$$\frac{dy}{dx} = \frac{y}{x}. \tag{3}$$

Obviously this can be very easily solved for $y(x)$ but our objective here is to apply the theorems to state whether a solution exists and whether it is unique before attempting to solve the IVP.

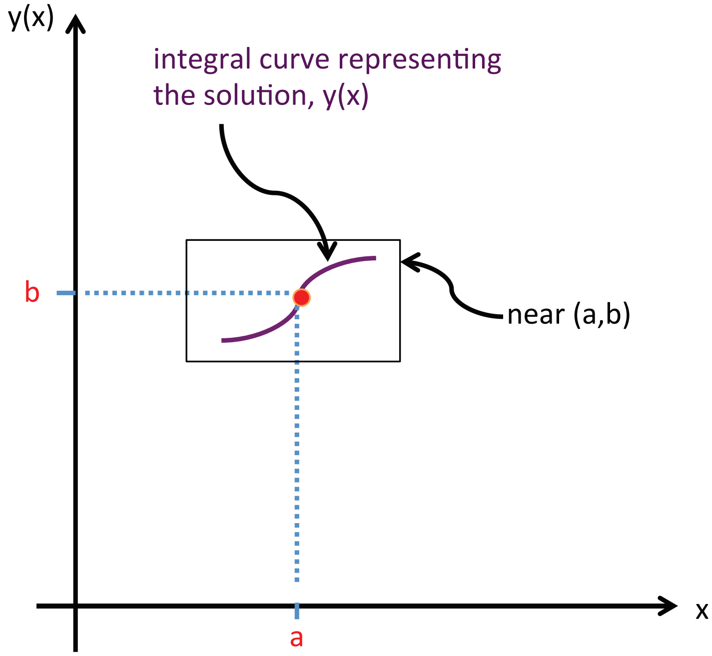

Figure 1: Plot showing the region defined as *'near' the point* (a, b) in the existence and uniqueness theorems. If f and $\frac{\partial f}{\partial y}$ are continuous there, then a solution exists and is unique.

Existence

In Eq. (3), $f(x, y) = \dfrac{y}{x}$.

It's easy to see that $\dfrac{y}{x}$ is continuous near *any* (a, b) where $a \neq 0$. This means that our rectangle within which $f(x, y)$ is continuous and hence the existence theorem may be applied, can be as big as we want as long as it doesn't contain the points $a = 0$ (these lie on the vertical axis).

Therefore, by the *existence* theorem, a solution exists near any (a, b) as long as $a \neq 0$.

Note that, at $a = 0$, $\dfrac{y}{x}$ is discontinuous and the existence theorem hypothesis is not satisfied. A solution may or may not exist but we can't conclude anything from the theorem itself.

Uniqueness

To apply the uniqueness theorem, we first need to find $\dfrac{\partial f}{\partial y}$. Differentiating $f(x, y)$ partially wrt y gives $\dfrac{\partial f}{\partial y} = \dfrac{1}{x}$.

Again, $\dfrac{1}{x}$ is continuous near *any* (a, b) where $a \neq 0$.

Therefore, by the *uniqueness* theorem, the solution is unique near any (a, b) as long as $a \neq 0$.

To reinforce the ideas of existence and uniqueness, let us proceed to find the general solution to the ODE. The ODE is separable so by separating variables and integrating, we obtain the following general solution

$$y = kx, \tag{4}$$

where k is a constant. Figure 2 shows plots of Eq. (4) for several values of k, including the one that passes through (a, b) given by $y = (b/a)x$.

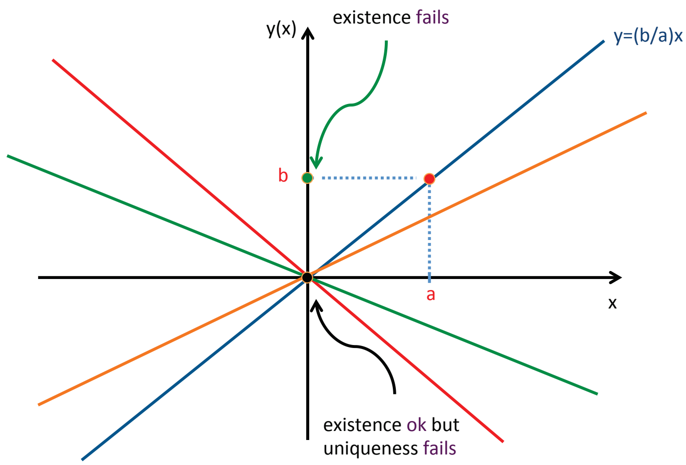

Figure 2: Plots of the general solution (4) for several values of k. Existence fails for the IVP with $y(0) = b$ where $b \neq 0$ and uniqueness fails for the IVP with $y(0) = 0$ as the initial condition.

We note the following:

1. As long as $a \neq 0$, we can solve the IVP with $y(a) = b$. Then the solution is unique: for example, in Fig. 2 only the blue line satisfies the ODE and passes through (a, b).

2. The y-axis represents all the points where $a = 0$. If we take a point $(0, b)$ where $b \neq 0$, then,

   (a) the hypotheses of the existence and uniqueness theorems aren't satisfied so we cannot conclude whether a unique (or any) solution exists just by using the theorems;

   (b) having solved the ODE, we know that for a solution to pass through $(0, b)$, this would have to be completely vertical. Such a solution cannot exist and therefore existence fails at $(0, b)$ where $b \neq 0$.

3. Let us now take the point $(0, 0)$ as our initial condition to the IVP. We note that here,

   (a) the hypotheses of the existence and uniqueness theorems aren't satisfied because f and $\frac{\partial f}{\partial y}$ are discontinuous. Just by using the theorems then we conclude that we don't know whether a unique (or any) solution exists;

   (b) having solved the ODE however, we know that $(0, 0)$ satisfies the general solution (4) for any value of k. This means that *all* solutions pass through the origin and satisfy the IVP with $y(0) = 0$, as shown in Fig. 2. A solution exists but it is not unique i.e. uniqueness fails.