**Math 33b, Winter 2013, Tonći Antunović - Homework 6**

From the textbook solve the problems:
Section 9.1: 4, 6, 8, 10, 12, 16, 18, 20, 22, 24

And also the problems below:

**Problem 1.** Find the eigenvalue and the eigenvectors for the matrix

$$A = \begin{pmatrix} 9 & 6 \\ -5 & -4 \end{pmatrix}$$

**Solution:** The characteristic polynomial is

$$\det(\lambda I - A) = \det \begin{pmatrix} -9+\lambda & -6 \\ 5 & 4+\lambda \end{pmatrix} = (-9+\lambda)(4+\lambda)+30 = \lambda^2 - 5\lambda - 6 = (\lambda-6)(\lambda+1).$$

Therefore, the eigenvalues are 6 and $-1$.

The eigenspace for the eigenvalue $-1$ is the nullspace of

$$A + I = \begin{pmatrix} 10 & 6 \\ -5 & -3 \end{pmatrix}$$

and is obviously generated by the vector $(3, -5)^T$ (so all eigenvectors are $(3x, -5x)^T$)

The eigenspace for the eigenvalue 6 is the nullspace of

$$A - 6I = \begin{pmatrix} 3 & 6 \\ -5 & -10 \end{pmatrix}$$

and so is generated by the vector $(2, -1)^T$ (so all eigenvectors are $(2x, -x)^T$).

**Problem 2.** Find the fundamental set of solutions for the system $\mathbf{y}' = A\mathbf{y}$ where

$$A = \begin{pmatrix} 2 & -2 \\ 1 & -1 \end{pmatrix}$$

**Solution:** The characteristic polynomial of $A$ is

$$\det(\lambda I - A) = \det \begin{pmatrix} \lambda - 2 & 2 \\ -1 & \lambda + 1 \end{pmatrix} = (\lambda - 2)(\lambda + 1) + 2 = \lambda^2 - \lambda,$$

so the eigenvalues are 0 and 1.

For the eigenvalue 0 the eigenspace is the nullspace of

$$A = \begin{pmatrix} 2 & -2 \\ 1 & -1 \end{pmatrix}$$

which is spanned by the vector $(1, 1)^T$.

For the eigenvalue 1 the eigenspace is the nullspace of

$$A - I = \begin{pmatrix} 1 & -2 \\ 1 & -2 \end{pmatrix}$$

which is spanned by the vector $(2, 1)^T$.

Therefore, we have the solutions

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad e^t \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

and since they are independent they form the fundamental set of solutions and the general solution is

$$C_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 e^t \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

**Problem 3.** Find the fundamental set of solutions for the system $\mathbf{y}' = A\mathbf{y}$ where

$$A = \begin{pmatrix} 1 & 0 & \sqrt{2} \\ 1 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

**Solution:** The characteristic polynomial is

$$\det(\lambda I - A) = \det \begin{pmatrix} \lambda - 1 & 0 & -\sqrt{2} \\ -1 & \lambda & 0 \\ 0 & 0 & \lambda - 2 \end{pmatrix} = \lambda(\lambda - 1)(\lambda - 2).$$

Therefore eigenvalues are 0, 1 and 2.

For the eigenvalue 0 the eigenspace is the nullspace of

$$A = \begin{pmatrix} 1 & 0 & \sqrt{2} \\ 1 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

which is spanned by $(0,1,0)^T$.

For the eigenvalue 1 the eigenspace is the nullspace of

$$A = \begin{pmatrix} 0 & 0 & \sqrt{2} \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

which is spanned by $(1,1,0)^T$.

For the eigenvalue 2 the eigenspace is the nullspace of

$$A = \begin{pmatrix} -1 & 0 & \sqrt{2} \\ 1 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

which is spanned by $(2,1,\sqrt{2})^T$.

So we found the solutions

$$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad e^t \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad e^{2t} \begin{pmatrix} 2 \\ 1 \\ \sqrt{2} \end{pmatrix}.$$

Since there are three of them and they are linearly independent they form a fundamental set of solutions and the general solutions is given by

$$C_1 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + C_2 e^t \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + C_3 e^{2t} \begin{pmatrix} 2 \\ 1 \\ \sqrt{2} \end{pmatrix}.$$