1 Problem

$$t^2y'' + 3ty' - 3y = 0 \tag{1}$$

1. Check that $y_1(t) = t$ is a solution
2. Find the general solution

2 Solution

For part one, we just compute

$$y(t) = t \quad y'(t) = 1 \quad y''(t) = 0$$

and plug in to 1 to get

$$t^2(0) + 3t(1) - 3(t) = 0$$

For part 2, guess a solution of the form

$$y_2 = v(t)y_1 = tv(t) \quad y_2' = tv' + v \quad y_2'' = tv'' + 2v'$$

Plugging in to 1 we get

$$\begin{aligned} t^2(tv'' + 2v') + 3t(tv' + v) - 3(tv) &= t^3v'' + 2t^2v' + 3t^2v' + 3tv - 3tv \\ &= t^3v'' + 5t^2v' \end{aligned}$$

If we let $u = v'$ we have

$$0 = t^3u' + 5t^2u$$

or

$$-\frac{5}{t}dt = \frac{1}{u}du$$

integrating

$$-5\ln t = \ln u + C$$

$$u = Ct^{-5}$$

But, since we put $u = v'$

$$v = C\int t^{-5} = C\left(-\frac{1}{4}t^{-4} + D\right) \sim C(t^{-4} + D)$$

and

$$y_2(t) = C(t^{-4} + D) \cdot t = Ct^{-3} + Dt$$

solves 1 regardless of C, D. Since there are two degrees of freedom in this solution, it is in fact the general solution. Note that it includes y_1 as $(C = 0, D = 1)$.