Math 33B: Differential Equations

Review 4: Exact ODEs

Apr. 15, 2013 *Instructor: Aliki M.*

1 Introduction to exact ODEs

So far we have looked at two types of first order ODEs:

1. Separable [Review 2]
2. linear [Review 3]

We have also seen how to transform a nonlinear, nonseparable ODE into a separable one [see *Homework 1, Problem 5(ii)*] thus putting the differential equation in a form which we know how to solve.

This review is concerned with a class of differential equations known as *exact* for which we can also identify a method of solution. We will demonstrate the meaning of *exactness* through an example before going in to the details of the solution method.

Consider the following ODE

$$4xy + 1 + (2x^2 + \cos y)\frac{dy}{dx} = 0; \tag{1}$$

where Eq. (1) is neither separable nor linear (here, the term which dictates that ODE (1) is nonlinear is $\cos y$).

Now, consider a function $f(x, y)$ given as,

$$f(x, y) = 2x^2y + x + \sin y. \tag{2}$$

Note that the function $f(x, y)$ depends on both variables, x and y, where $y = y(x)$. Let us now determine the *total derivative* of $f(x, y)$ given by $\dfrac{d}{dx}\big[f(x, y)\big]$.

Differentiating (2) *implicitly* gives,

$$\frac{d}{dx}\big[f(x, y)\big] = 4xy + 2x^2\frac{dy}{dx} + 1 + \cos y\frac{dy}{dx}; \tag{3}$$

note that the first two terms in Eq. (3) are a result of the product rule when $2x^2y$ is differentiated with respect to x.

Rearranging (3) yields

$$\frac{d}{dx}\big[f(x,y)\big] = 4xy + 1 + (2x^2 + \cos y)\frac{dy}{dx}, \tag{4}$$

which is equivalent to the LHS of Eq. (1). Using Eq. (4), the ODE (1) may be expressed as,

$$\frac{d}{dx}\big[f(x,y)\big] = 0, \tag{5}$$

where $f(x,y)$ is given by (2). Equation (5) states that the **total** derivative of $f(x,y)$ wrt x is zero. This means upon integrating both sides wrt x, we have

$$f(x,y) = c,$$

which is equal to

$$2x^2y + x + \sin y = c; \tag{6}$$

where c is a constant. Equation (6) is the implicit form of the general solution to ODE (1). As usual, the particular solution is obtained by solving for the constant c through application of an initial condition.

Using this 'special' function $f(x,y)$ we were able to reduce an ODE which looked pretty difficult to solve at first and determine the general solution. This approach is undoubtedly quite simple and works for all first order **exact** ODEs but there are two main concerns:

1. How can we tell an ODE is exact?
2. How can we determine the appropriate $f(x,y)$ for every exact ODE we encounter?

2 Multivariable calculus definitions

At this point, let us recall the definition of *partial derivatives* and *total derivative* of a multivariable function.

Partial Derivatives

Consider a function f of two variables such that $f = f(x,y)$. The *partial derivatives* of f are defined as the derivative with respect to one of the variables with all other variables held constant. The partial derivative of f wrt x is denoted by $\frac{\partial f}{\partial x}$ and it represents the slope along the x-direction while the partial derivative of f wrt y is denoted by $\frac{\partial f}{\partial y}$ and it represents the slope along the y-direction.

Also, note that an alternative way of denoting $\frac{\partial f}{\partial x}$ is f_x (where the subscript denotes partial differentiation wrt x). Similarly $\frac{\partial f}{\partial y}$ may be denoted by f_y (where the subscript denotes partial differentiation wrt y).

Total Derivative

The total derivative is a little more complex and it is really only relevant when the variables that make up f (here, x and y) are not *truly* independent. For example, y is not independent of x because $y = y(x)$. If we were asked to find the **total** rate of change of f wrt x, the partial derivative $\frac{\partial f}{\partial x}$ does not suffice because it doesn't give any information on how varying x also changes y. In such cases, to find the **true** slope of f along the x-direction we need to define the *total derivative* of the function $f(x, y)$ which is, in turn, expressed through the partial derivatives via the chain rule:

$$\frac{d}{dx}\Big[f(x,y)\Big] = \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx};$$

which simplifies to

$$\frac{d}{dx}\Big[f(x,y)\Big] = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx}. \tag{7}$$

3 Condition for exactness

We will consider the example we used in Section 1 given by ODE (1). We have also assumed the existence of a function $f(x, y)$ defined by Eq. (2). Now, let us take Eq. (2) and differentiate *partially* wrt x and wrt y. The partial derivative of $f(x, y)$ wrt x is given by,

$$\begin{aligned} \frac{\partial f}{\partial x} &= \frac{\partial}{\partial x}[2x^2y + x + \sin y]; \\ \frac{\partial f}{\partial x} &= 4xy + 1. \end{aligned} \tag{8}$$

Similarly, the partial derivative of $f(x, y)$ wrt y is given by,

$$\begin{aligned} \frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}[2x^2y + x + \sin y]; \\ \frac{\partial f}{\partial y} &= 2x^2 + \cos y. \end{aligned} \tag{9}$$

Observe that Eq. (8) gives the first two terms of Eq. (1) and Eq. (9) is the coefficient of

$\frac{dy}{dx}$ in Eq. (1). This implies that Eq. (1) may be expressed as,

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} = 0. \tag{10}$$

Note: any homogeneous* first order ODE that is *exact* may be put in the above form..

We recognize the LHS of (10) as the total derivative of $f(x, y)$ as defined by (7). This implies that we can rewrite (10) as

$$\frac{d}{dx}[f(x, y)] = 0. \tag{11}$$

Now let us consider a generalized form of ODE (1) given by

$$P(x, y) + Q(x, y)\frac{dy}{dx} = 0, \tag{12}$$

where $P(x, y)$ and $Q(x, y)$ are continuous.

According to Eq. (10) and the statement in blue directly below it, if there exists a function $f(x, y)$ such that,

$$P(x, y) = \frac{\partial f}{\partial x} \quad \text{and} \quad Q(x, y) = \frac{\partial f}{\partial y}, \tag{13}$$

then Eq. (12) is known to be **exact**.

Now if we partially differentiate P wrt y, we have

$$\frac{\partial P}{\partial y} = \frac{\partial}{\partial y}\Big(\frac{\partial f}{\partial x}\Big) = \frac{\partial^2 f}{\partial y \partial x}. \tag{14}$$

Similarly if we partially differentiate Q wrt x,

$$\frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}\Big(\frac{\partial f}{\partial y}\Big) = \frac{\partial^2 f}{\partial x \partial y}. \tag{15}$$

If P and Q *exist* and are *continuous* over some region $\mathcal{R}$, and all the partial derivatives of P and Q (i.e. $\frac{\partial P}{\partial x}, \frac{\partial P}{\partial y}, \frac{\partial Q}{\partial x}$ and $\frac{\partial Q}{\partial y}$) also exist and are continuous there, then, the order in which we take the first and second derivatives doesn't matter. In other words, if the continuity of the above-mentioned quantities is guaranteed then,

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}. \tag{16}$$

*Note that, here, the term *homogeneous* refers to the RHS of Eq. (10) being equal to zero.

Equation (16) is equivalent to,

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}. \tag{17}$$

Equation (17) gives a condition that needs to be satisfied for an ODE to be exact.

An ODE of the form $P(x, y) + Q(x, y)\dfrac{dy}{dx} = 0$ is exact **if and only if** $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$

Example: test for exactness

Suppose we wanted to show that ODE (1) is exact. Comparing (1) to (12), it's easy to see that,

$$P(x, y) = 4xy + 1 \quad \text{and} \quad Q(x, y) = 2x^2 + \cos y.$$

To show that an ODE of the form (12) is exact, we need to show that (17) holds true. Therefore, differentiating P partially wrt y (keeping x constant) gives,

$$\frac{\partial P}{\partial y} = 4x.$$

Similarly, differentiating Q partially wrt x (keeping y constant) gives,

$$\frac{\partial Q}{\partial x} = 4x.$$

So, $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x} = 4x$ and therefore ODE (1) is exact.

4 Solving exact ODEs

Having a simple way to check whether an ODE is exact doesn't really help us solve for the solution to the differential equation. To solve an exact ODE, we need to find this special function $f(x, y)$ that we mentioned in Section 1.

Once we know that the ODE we are trying to solve is exact, then we also know that there exists a function $f(x, y)$ which satisfies,

$$\frac{\partial f}{\partial x} = P(x, y) \tag{18}$$

and

$$\frac{\partial f}{\partial y} = Q(x, y). \tag{19}$$

Note that we can integrate Eq. (18) wrt x and Eq. (19) wrt y to obtain $f(x, y)$. Let us start from (18); integrating wrt x gives,

$$f(x, y) = \int P(x, y)\, dx + k_1(y). \tag{20}$$

Note that because two variables are involved in f, we do not get a constant of integration but an arbitrary, pure function of y, given by $k_1(y)$ (to see this, imagine partially differentiating with respect to x again; its not only constants that go to zero). Similarly, we integrate (19) wrt y

$$f(x, y) = \int Q(x, y)\, dy + k_2(x), \tag{21}$$

where $k_2(x)$ is an arbitrary, pure function of x.

A differential equation of the form $\mathbf{P(x,y) + Q(x,y)\frac{dy}{dx} = 0}$ is **exact** if there exists a function $f(x, y)$ such that $\mathbf{f_x = P(x,y)}$ and $\mathbf{f_y = Q(x,y)}$.

The ODE is then expressed as $\mathbf{\frac{d}{dx}\Big[f(x,y)\Big] = 0}$ and the general solution to the exact differential equation is given by $\mathbf{f(x,y) = c}$.

Example: solving exact ODEs

Let us now return to ODE (1). We have already shown in the previous section that the ODE is exact.

Using Eqs. (18) and (19), we can write down the following two equations,

$$\frac{\partial f}{\partial x} = 4xy + 1 \tag{22}$$

and

$$\frac{\partial f}{\partial y} = 2x^2 + \cos y. \tag{23}$$

Integrating (22) wrt x gives,

$$f(x, y) = \int 4xy + 1\, dx = 2x^2 y + x + k_1(y). \tag{24}$$

Integrating (23) wrt y gives,

$$f(x, y) = \int 2x^2 + \cos y\, dy = 2x^2 y + \sin y + k_2(x). \tag{25}$$

Both Eqs. (24) and (25) define $f(x, y)$ but in (24) there's an unknown function of y while in (25), there's an unknown function of x. However, given that we know that (24) and (25) should be equal, comparing the two equations gives,

$$k_1(y) = \sin y \quad \text{and} \quad k_2(x) = x. \tag{26}$$

Using $k_1(y) = \sin y$ in (24), $f(x, y)$ is given by

$$f(x, y) = 2x^2 y + x + \sin y, \tag{27}$$

which is the function we defined in Eq. (2) at the beginning of this review. Now, according to the boxed definition on page 6, the exact ODE (1) may be expressed as,

$$\frac{d}{dx}\Big[2x^2 y + x + \sin y\Big] = 0. \tag{28}$$

Integrating (28) wrt x gives a 1-parameter* family of solutions of the exact differential equation given as follows

$$2x^2 y + x + \sin y = c, \tag{29}$$

where c is the constant of integration.

*Here, the *parameter* refers to the constant c in Eq. (29); a general solution to *any first order ODE* will be represented by a 1-parameter family of solutions. This parameter is determined using a given initial condition.