Math 33B: Differential Equations

Review 9: Method of undetermined coefficients

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1 Homogeneous & nonhomogeneous ODEs

So far, when it comes to second order ODEs, we have looked at *linear* differential equations of the form,

$$y'' + p(x)y' + q(x)y = f(x). \tag{1}$$

Further, we required for $p(x)$ and $q(x)$ to be constants **and** for $f(x) = 0$.

We called the resulting ODE a *homogeneous* ODE with constant coefficients:

$$y'' + py' + qy = 0. \tag{2}$$

In **Review 8** we discussed how to obtain solutions to the homogeneous equation given by (2). The results of the method we employed in **Review 8** are stated in the box below.

Given a second order linear, homogeneous equation with constant coefficients,

$$y'' + py' + qy = 0, \tag{3}$$

solve its characteristic equation,

$$m^2 + pm + q = 0. \tag{4}$$

The general solution depends on the roots to Eq. (4).

1. For real + distinct roots,
$$y_h(x) = c_1 e^{m_1 x} + c_2 e^{m_2 x}. \tag{5}$$
2. For complex + distinct roots, $m = a \pm ib$,
$$y_h(x) = e^{ax}[c_1 \cos(bx) + c_2 \sin(bx)]. \tag{6}$$
3. For a single, real root, $m = -p/2$,
$$y_h(x) = c_1 e^{-px/2} + c_2 x e^{-px/2}. \tag{7}$$

Note the subscript h in the general solutions in Eqs. (5)-(7). This denotes that the solutions are solutions to the *homogeneous* ODE (i.e. when $f(x) = 0$).

In this review, we are interested in finding solutions to the *nonhomogeneous* equation given by (1) when the coefficients $p(x)$ and $q(x)$ are, again, constants, i.e. we are looking for solutions to:

$$y'' + py' + qy = f(x). \tag{8}$$

Structure of the general solution to (8)

The general solution to the nonhomogeneous ODE is given as

$$y(x) = y_h(x) + y_p(x), \tag{9}$$

where y_h is the solution to the homogeneous ODE with constant coefficients [this can take any one of the three forms given in Eqs. (5)-(7)] and y_p is **a** particular solution to the nonhomogeneous ODE.

Proof

We are trying to show that $y = y_h + y_p$ is a solution to Eq. (8).

We start off by assuming that y is a solution to the nonhomogeneous ODE. This implies that it should satisfy Eq. (8); so we have the following equation:

$$y'' + py' + qy = f(x). \tag{10}$$

We also assume that whatever y_p is, it should also satisfy Eq. (8):

$$y_p'' + py_p' + qy_p = f(x). \tag{11}$$

Now, take Eq. (11) and subtract it from (10):

$$(y - y_p)'' + p(y - y_p)' + q(y - y_p) = 0. \tag{12}$$

Equation (12) looks like the homogeneous equation given by (2). This implies that $y - y_p$ must be a solution to the homogeneous ODE which, in turn, implies that $y - y_p$ must equal to y_h (the solution to the homogeneous equation):

$$y - y_p = y_h \tag{13}$$

which is,

$$y = y_h + y_p. \tag{14}$$

So, y is a solution to the nonhomogeneous equation and it is made up of y_h and y_p as shown in Eq. (14).

Our objective now is to find a particular solution y_p so we can put it together with y_h and form the general solution to the nonhomogeneous problem.

2 Method of undetermined coefficients

We are now ready to look for solutions to

$$y'' + py' + qy = f(x) \tag{15}$$

where $f(x) \neq 0$. Note that sometimes this $f(x)$ term is referred to as the forcing term.

When does the method work?

Before we introduce the method let's just state when it can be used. The M.U.C can only be used when $f(x)$ consists of a rather limited class of functions:

(i) exponentials;

(ii) sines & cosines;

(iii) polynomials.

Sums and products of these functions work as well. We talked about these types of functions being the only ones that will work with this method and the underlying reason has to do (yet again) with linear independence. What all of the above classes of functions have in common is that they have a finite number of linearly independent derivatives.

For example, take the function $f(x) = \sin x$ and differentiate wrt x a few times:

$$f' = \cos x, \;\; f'' = -\sin x, \text{ etc}$$

Only $\sin x$ and $\cos x$ are linearly independent. The addition of the next order derivative i.e $f'' = -\sin x$ is *linearly dependent* to the original function.

As an additional example let's take the polynomial $f(x) = x^3$; the first three derivatives are:

$$f' = 3x^2,\ f'' = 6x,\ f''' = 6.$$

These are all linearly independent to x^3; however the addition of the fourth derivative, which is equal to 0, renders the set *linearly dependent*.

To contrast the above, consider the functions $f(x) = \dfrac{1}{x}$ and $f(x) = \ln x$; these functions have an infinite number of linearly independent derivatives: as you differentiate them wrt x, you obtain higher and higher powers of x in the denominator.

The only functions that have the property of having a finite number of linearly independent derivatives are the three classes specified above.

In what follows, we will look at examples where the forcing term, i.e. $f(x)$, is made up of exponentials, sines and cosines or polynomials.

2.1 Exponentials

Consider the nonhomogeneous ODE:

$$y'' - 3y' - 4y = 3e^{2x}. \tag{16}$$

As we've shown earlier in this review, the general solution is made up of the solution to the homogeneous version of Eq. (16), i.e.

$$y'' - 3y' - 4y = 0 \tag{17}$$

and a particular solution to (16).

The solution to (17) is:

$$y_h(x) = c_1 e^{4x} + c_2 e^{-x}. \tag{18}$$

Note: check out the worked-out example on CCLE for more details on the solution to this problem.

Now, we focus on how to get a particular solution to the nonhomogeneous ODE by using the method of undetermined coefficients. The method really boils down to having an *educated guess* for y_p. We obtain a first guess by looking at the forcing term, $f(x)$ and choosing y_p to have a similar form. So, in this case, since $f(x) = 3e^{2x}$, our guess for a particular solution is

$$y_p(x) = Ae^{2x}, \tag{19}$$

where A is the undetermined coefficient that we need to determine. Notice that the exponent in y_p matches the one in the exponential function in $f(x)$. All we are left to do is determine

A and, if everything works out smoothly, Eq. (19) should give us a particular solution to (16).

We proceed as follows. We want to determine A in such a way that y_p satisfies (16). If y_p is to satisfy the nonhomogeneous equation, then we expect the following equation to hold true:

$$y_p'' - 3y_p' - 4y_p = 3e^{2x}. \tag{20}$$

From Eq. (19) we obtain y_p' and y_p'':

$$y_p' = 2Ae^{2x} \tag{21}$$
$$y_p'' = 4Ae^{2x} \tag{22}$$

Next, we substitute Eqs. (19), (21) and (22) in (20):

$$(4A - 6A - 4A)e^{2x} = 3e^{2x}. \tag{23}$$

Finally, comparing coefficients of e^{2x}, we get $A = -1/2$. Substituting $A = -1/2$ in Eq. (19) we have $y_p = -\frac{1}{2}e^{2x}$ as the desired particular solution.

Putting together the homogeneous solution and the particular solution we've just found, we obtain the general solution to (16) as:

$$y(x) = c_1e^{4x} + c_2e^{-x}. \tag{24}$$

Exceptional cases

The method of undetermined coefficients is a relatively simple 'guessing' method which seems to work well, albeit for a small class of functions. However, there are situations where the method will fail to allow us to determine the *undetermined* coefficient(s).

More specifically, this will happen when $f(x)$ contains a function which already appears in the homogeneous solution.

For example, if we were looking for a particular solution satisfying:

$$y'' - 3y' - 4y = 4e^{4x} \tag{25}$$

and started off with a guess $y_p = Ae^{4x}$ then, it follows (and it is easily verified) that if we were to substitute y_p in $y'' - 3y' - 4y$, we get zero; this isn't surprising since we already know that c_1e^{4x} is a fundamental solution satisfying the homogeneous equation. With the

LHS of (25) being zero though, the undetermined coefficient, A, vanishes and we are left with no way of determining A.

As a *general rule*, if $f(x)$ clashes with a fundamental solution in y_h, we want to multiply our initial guess of y_p with x (or higher powers of x if needed). For instance, in the ODE given by (25), our *first* initial guess of $y_p = Ae^{4x}$ won't work since this is the same solution as one of the fundamental solutions (ignoring constants). Our 'educated' guess therefore is to choose $y_p = Axe^{4x}$. If we now proceed to determine A as before, we find that $A = 4/5$ and our desired particular solution is $y_p = \dfrac{4}{5}xe^{4x}$. Note that a detailed working of this example is available on CCLE.

2.2 Sines & cosines

Now, consider the same ODE as before with the RHS being a sine term:

$$y'' - 3y' - 4y = 2\sin x. \tag{26}$$

The homogeneous solution is given by (18).

The RHS of (26) is $f(x) = 2\sin x$ and we note that $f(x)$ does not contain any terms that belong in the homogeneous solution. So, in this case we can go ahead and guess a form of y_p that is similar to $2\sin x$,

$$y_p = A\sin x. \tag{27}$$

So, we have assumed a form of y_p which we hope it works and again, all we are left to determine is A.

Upon differentiating $y_p = A\sin x$ and substituting in Eq. (26), we have:

$$-5A\sin x - 3A\cos x = 2\sin x + 0\cos x. \tag{28}$$

Note we have included the $+0\cos x$ term on the RHS of (28) to be able to compare coefficients of $\sin x$ and $\cos x$ on both sides.

We actually find that we cannot determine A since the two equations we are left with, i.e.

$$-5A = 2 \quad \text{and} \quad 3A = 0 \tag{29}$$

leave us with no choice of A that satisfies both of them.

We need to rethink our guess of y_p. To determine A correctly, we need to include *all* the linearly independent derivatives of $f(x)$. A better guess therefore would be to start off with a y_p that includes both sine and cosine terms:

$$y_p = A\sin x + B\cos x, \tag{30}$$

where A and B are the undetermined coefficients.

Differentiating (30) to get y_p', y_p'' and substituting back in (26) gives us:

$$(-5A + 3B)\sin x + (-5B - 3A)\cos x = 2\sin x \tag{31}$$

Comparing coefficients of $\sin x$ and $\cos x$, we have $A = -5/17$ and $B = 3/17$.

The desired particular solution is therefore,

$$y_p = -\frac{5}{17}\sin x + \frac{3}{17}\cos x, \tag{32}$$

and the general solution is,

$$y(x) = c_1 e^{4x} + c_2 e^{-x} - \frac{5}{17}\sin x + \frac{3}{17}\cos x. \tag{33}$$

2.3 Polynomials

For our final example we consider,

$$y'' - 3y' - 4y = x^2, \tag{34}$$

where $f(x) = x^2$.

Again, the homogeneous solution is given by (18). For the particular solution guess, though it would make sense to choose $y_p = Ax^2$ to match $f(x)$, for the same reasons discussed in Subsection 2.2, we choose a form for y_p that contains x^2 as well as *all* its linearly independent derivatives. Ignoring coefficients, these are the functions x and 1. So, our chosen form for y_p is:

$$y_p = Ax^2 + Bx + C \tag{35}$$

where A, B and C are to be determined.

Proceeding in the same way as for the previous examples, we take (35), differentiate wrt x for y_p' and y_p'' and substitute in (34). This gives:

$$-4Ax^2 + x(-6A - 4B) + (-4C + 2A - 3B) = x^2 \tag{36}$$

Comparing coefficients of x^2, x and the constants on both sides (note on the RHS the only term is x^2 which means that coefficients of x and the constants are 0):

$$-4A = 1, \quad 6A + 4B = 0 \quad \text{and} \quad -4C + 2A - 3B = 0. \tag{37}$$

From Eqs. (37), we have $A = -1/4$, $B = 3/8$ and $C = -13/32$, hence, the desired particular solution is:

$$y_p = -\frac{1}{4}x^2 + \frac{3}{8}x - \frac{13}{32}. \tag{38}$$

Putting y_h and y_p together, gives the general solution to (34) as:

$$y(x) = c_1 e^{4x} + c_2 e^{-x} - \frac{1}{4}x^2 + \frac{3}{8}x - \frac{13}{32}. \tag{39}$$

3 Summary

Given a second order linear, nonhomogeneous equation,

$$y'' + p(x)y' + q(x)y = f(x), \tag{40}$$

its general solution is given by

$$y(x) = y_h(x) + y_p(x), \tag{41}$$

where $y_h(x)$ is the solution to the homogeneous ODE (when $f(x) = 0$) and $y_p(x)$ is a particular solution to the nonhomogeneous equation. We use the *method of undetermined coefficients* to obtain a particular solution to (40) when p and q are constants and when $f(x)$ takes an exponential function, sines and/or cosines, polynomials or any sums and products of the aforementioned classes of functions.

The table below summarizes the forms $f(x)$ can take and the corresponding guess for $y_p(x)$.

Table 1: ‘Simple’ forms of $f(x)$

	$f(x)$	Guess for $y_p(x)$
1.	ae^{bx}	Ae^{bx}
2.	$a\cos\beta x$ <u>or</u>, $a\sin\beta x$ <u>or</u>, $a\cos\beta x + b\sin\beta x$	$A\cos\beta x + B\sin\beta x$
3.	n^{th} degree polynomial	$A_n x^n + A_{n-1}x^{n-1} + ... + A_0$

Finally, note that if $f(x)$ contains a term that belongs to the homogeneous solution, then the guess for $y_p(x)$ included in Table 1, needs to be multiplied by x (or, if needed, higher powers of x).