MATH 33B: DIFFERENTIAL EQUATIONS — April 2013

Example: solving linear, first order ODEs.

$$\frac{dy}{dx} \cdot \tan x + y = 1 \qquad (1)$$

Find the general soln.

Solution process

1) Express the ODE given by (1) in standard form:

divide by $\tan x$,

$$\frac{dy}{dx} + \frac{y}{\tan x} = \frac{1}{\tan x}. \qquad (2)$$

where $p(x) = \frac{1}{\tan x}$ and $f(x) = \frac{1}{\tan x}$.

2) Find the integrating factor (I.F) using $\mu(x) = e^{\int p(x)\,dx}$.

$$\int p(x)\,dx = \int \frac{1}{\tan x}\,dx = \int \frac{\cos x}{\sin x}\,dx \qquad \left(\tan x = \frac{\sin x}{\cos x}\right)$$

$$\int p(x)\,dx = \ln \sin x$$

Note: Here, we are looking for <u>any</u> function whose derivative is $p(x)$ so any constant of integration would do (even $k=0$). For simplicity, we choose to set $k=0$ when looking for the integrating factor.

$\therefore$ the I.F is:

$$\mu(x) = e^{\int p(x)\,dx} = e^{\ln \sin x} = \sin x$$

3) Multiply both sides of ② by the I.F.

$$\sin x \frac{dy}{dx} + \sin x \frac{y}{\tan x} = \sin x \frac{1}{\tan x}.$$

Simplifying,

$$\sin x \frac{dy}{dx} + \cos x \cdot y = \cos x \quad ③$$

4) Reduce the LHS of ③ in the form $\frac{d}{dx}[\mu(x) \cdot y(x)]$

$$\frac{d}{dx}\Big[\underbrace{\boxed{\sin x}}_{\text{I.F.}} \cdot \underbrace{\boxed{y}}_{\text{solution}}\Big] = \sin x \frac{dy}{dx} + y \cdot \cos x^{*}. \quad ④$$

* make sure you check that the reduced form $\frac{d}{dx}[\mu(x) \cdot y(x)]$ is equivalent to the LHS of ③.

5) Integrate both sides of ④ wrt x:

$$\sin x \cdot y = \int \cos x\, dx = \sin x + k$$

$$\therefore \boxed{y(x) = 1 + \frac{k}{\sin x}.}$$

general solution