HW3 - 1

Homework 3: Worked-out solutions   April 2013

PROBLEM 1

(i) $\dfrac{dy}{dx} = 1 + y^2$ with $y(0) = 1$

$f(y) = 1 + y^2$: continuous everywhere hence at least one soln exists for the IVP.

$\dfrac{\partial f}{\partial y} = 2y$: continuous everywhere, therefore the solution is guaranteed to be unique.

(ii) $\dfrac{dy}{dx} = \sqrt{y}$ with $y(2) = 0$.

$f(y) = \sqrt{y}$: the function is defined for $y \geq 0$.

However, $f(0)$ is not continuous.

Let's review the definition of continuity at this point

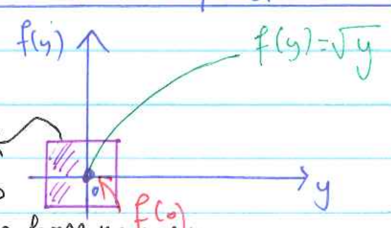
For a function, say $f(y)$, to be continuous at a point, say $y = a$, then:

(i) $f(a)$ must exist

(ii) the limit of $f(y)$ as y approaches $y = a$ from either side; i.e. $a - \varepsilon$ and $a + \varepsilon$ (where $\varepsilon > 0$) must exist and it should be equal to $f(a)$.

For example,

$\sqrt{y}$ isn't defined here so we can't approach $y=0$ from the left.



$f(0)$ exists but as the pt $y=0$ is approached from the left, the limit of $f(y)$ doesn't exist

HW3 - 2

<u>Back to problem 1 (ii),</u>

the fct $f(y)$ is continuous for $y>0$ so a soln that passes through $(2,0)$ is not guaranteed by the hypotheses of the thm.

Also, $\frac{\partial f}{\partial y} = \frac{1}{2} y^{-1/2} = \frac{1}{2\sqrt{y}}$ $\therefore \frac{\partial f}{\partial y}$ is continuous for all values y that lie in $y>0$.

$\Rightarrow$ uniqueness hypotheses are not satisfied for the I.C. $(2,0)$

<u>PROBLEM 2</u>

$$\frac{dy}{dx} = 3y^{2/3} \quad (1) \text{ with } y(0)=0.$$

<u>Show $y(x)=0$ & $y(x)=x^3$ are solutions to the IVP</u>

If they are solns to the IVP they need to satisfy the ODE and the I.C.

<u>$y(x)=0$.</u> Diff. wrt x: $\frac{d}{dx}(0)=0$

Sub. in RHS of (1), $3y^{2/3} = 3(0)^{2/3} = 0$

$\Rightarrow$ (1) is satisfied. Also, $y(x)=0$ passes through $(0,0)$ so the I.C is also satisfied.

<u>$y(x)=x^3$</u> Diff wrt x: $\frac{d}{dx}(x^3) = 3x^2$

Sub in RHS of (1), $3y^{2/3} = 3(x^3)^{2/3} = 3x^2$

(1) is satisfied.

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Also, at $x=0$, $y=x^3=0^3=0 \Rightarrow y=x^3$ passes through $(0,0)$.

Applying E & U thms:

existence $f(y)=3y^{2/3}$. : continuous everywhere. The thm guarantees that at least one solution exists.

uniqueness $\frac{\partial f}{\partial y} = 2y^{-1/3} = \frac{2}{y^{1/3}}$: continuous everywhere except $y=0$.

$\Rightarrow$ A unique solution around an interval that contains $y=0$ is not guaranteed by the uniqueness theorem which doesn't contradict the fact that there exist 2 solns passing through $(0,0)$.

PROBLEM 3

$\frac{dy}{dx} = \sqrt{y-x}$ with $y(a)=b$

$f(x,y)=\sqrt{y-x}$. : continuous everywhere except where $y \le x$ (note that, like Problem 1 (ii) we need to include all pts where $y=x$)

$\Rightarrow$ By the existence thm, at least one solution is guaranteed to exist for the IVP as long as $b>a$

$\frac{\partial f}{\partial y} = \frac{1}{2}(y-x)^{-1/2}$: continuous everywhere except where $y \le x$.

$\Rightarrow$ A unique soln is guaranteed for the IVP with $y(a)=b$ as the I.C. as long as $b>a$

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PROBLEM 4

$$x\frac{dy}{dx} - y = x^2\cos x.$$

divide by x to put in S.F.

$$\frac{dy}{dx} - \frac{1}{x}y = x\cos x \quad ①$$

where $p(x) = -1/x$

$f(x) = x\cos x.$

Find I.F.

$$\mu(x) = e^{\int p(x)dx} = \frac{1}{x}.$$

Multiply ① by $\mu(x)$

$$\frac{1}{x}\frac{dy}{dx} - \frac{1}{x^2}y = \cos x. \quad ②$$

② may be reduced as:

$$\frac{d}{dx}\left[\frac{1}{x}\cdot y\right] = \cos x. \quad ③$$

Integ. ③ wrt x.

$$\frac{y}{x} = \sin x + k \Rightarrow \boxed{y(x) = x\sin x + kx.}$$

Sketch of solns

This will depend on the value of k. Choosing $k=0$, $y = x\sin x$

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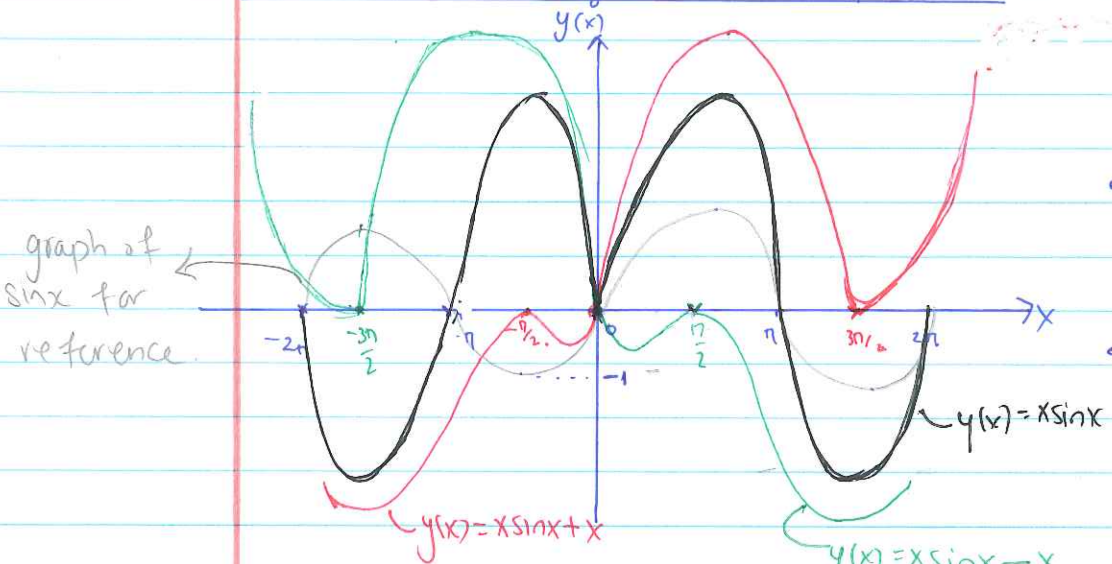
Choose x-range as $-2\pi < x < 2\pi$.



$k=0$, $y(x) = x\sin x$ (*)

- Roots of (*) are at $x = \pm\pi$, $x = \pm 2\pi$, $x = 0$.
- $y(x) > 0$ in $-\pi < x < \pi$
  $y(x) < 0$ in $\pi < x < 2\pi$
  $-2\pi < x < -\pi$.
- Note that a soln to the ODE needs to be continuous (or be comprised of continuous parts) so given the roots of $y(x)$, where $y(x)>0$, $y(x)<0$, we can plot a sketch of the soln pretty accurately.

For $k=1$, $y(x) = x\sin x + x = x(\sin x + 1)$
- Roots are at $x=0$, $x = -\pi/2$, $x = 3\pi/2$
- $y(x) > 0$ in $0 < x < 2\pi$
  $y(x) < 0$ in $-2\pi < x < 0$

Finally, choosing a negative value for k, say $k=-1$

$k=-1$, $y(x) = x\sin x - x = x(\sin x - 1)$
- Roots are at $x=0$, $x = \pi/2$, $x = -3\pi/2$
- $y(x) > 0$ in $-2\pi < x < 0$
  $y(x) < 0$ in $0 < x < 2\pi$.

Alternatively you could see that $y(-x)$ would be a reflection of $y(x)$ about the y-axis:

$y(x) = x\sin x + x \Rightarrow$ red curve

$y(-x) = x\sin x - x \Rightarrow$ green curve [note: $\sin(-x) = -\sin x$]

(ii) Back to the gen. soln: $y(x) = x\sin x + kx$

Applying I.C. $y(0) = 0 + 0 \neq -3$. $\nexists$ value that k can take

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that satisfies the I.C.

Applying the theorems.

From the ODE, $f(x,y) = \frac{y}{x} + x\cos x$.

$$\frac{\partial f}{\partial y} = \frac{1}{x}.$$

Both f and $\frac{\partial f}{\partial y}$ are discontinuous at $x=0$ which implies that the hypotheses of the theorems are not contradicted, as they do not guarantee that a soln exists in any interval that contains $x=0$.

PROBLEM 5

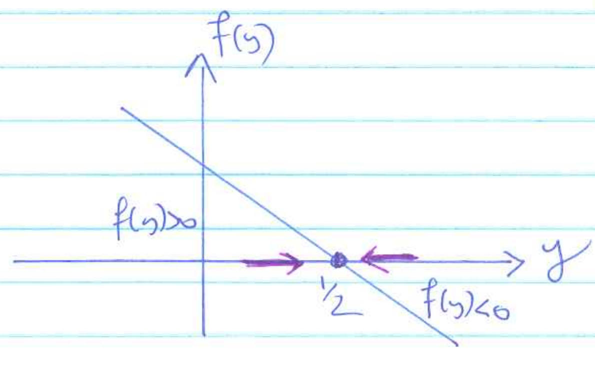
$$\frac{dy}{dx} + 2y = 1$$

(1) Critical pts at $f(y) = 1 - 2y = 0$

$\Rightarrow$ $y = \frac{1}{2}$ is a critical pt.

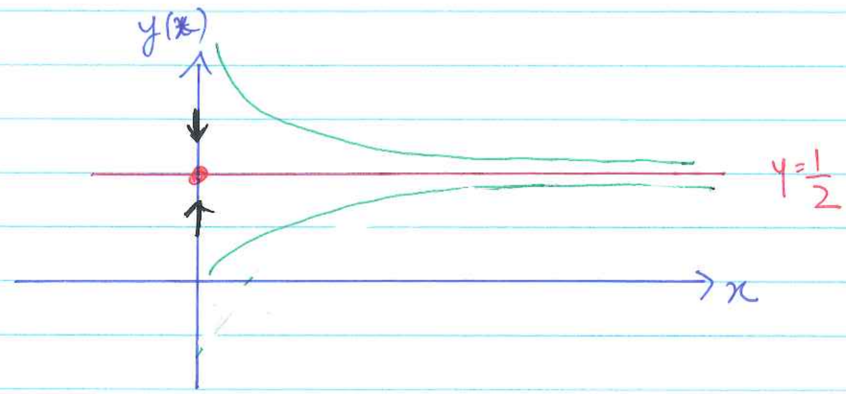
$\frac{dy}{dx} = f(y)$ >0 incr. solns; <0 decr. solns

Plot $f(y)$ vs y



The c.p. $\underset{y=\frac{1}{2}}{\rightarrow \bullet \leftarrow}$ is asymptotically stable

(ii) Sketch of eqm & non-eqm solns



(iii) $\dfrac{dy}{dx} = 1 - 2y$ *

By sep. variables,

$$\int \frac{1}{1-2y}\,dy = \int dx .$$

$$-\frac{1}{2}\ln(1-2y) = x + \ln k .$$

$$\ln\left(\frac{(1-2y)^{-1/2}}{k}\right) = x .$$

$$(1-2y)^{-1/2} = k e^{x} \Rightarrow \boxed{y(x) = \frac{1+Ce^{-2x}}{2}}$$

where $C = -k^{-2}$

The G.S. verifies that as $x \to \infty$, $Ce^{-2x} \to 0$ (for any C), $y \to \frac{1}{2}$ which means that all solns approach $y = \frac{1}{2}$.

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PROBLEM 6

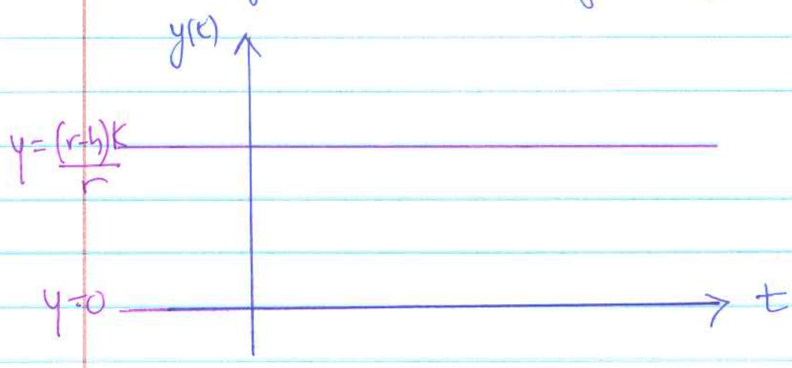
$$\frac{dy}{dt} = r\left(1 - \frac{y}{K}\right)y - hy.$$

(i) $f(y) = r\left(1 - \frac{y}{K}\right)y - hy.$

c.p.s occur at $f(y) = 0 \Rightarrow y\left[r - \frac{r}{K}y - h\right] = 0$

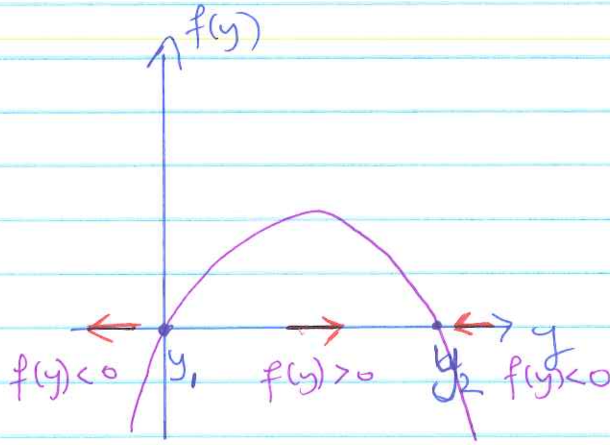
$\therefore\ y_1 = 0$ and $y_2 = \frac{(r-h)K}{r}.$

The eqm soln y_2 has physical significance if $h < r$. (we need y to be nonnegative)



(ii) Stability:

Plot f(y) vs y



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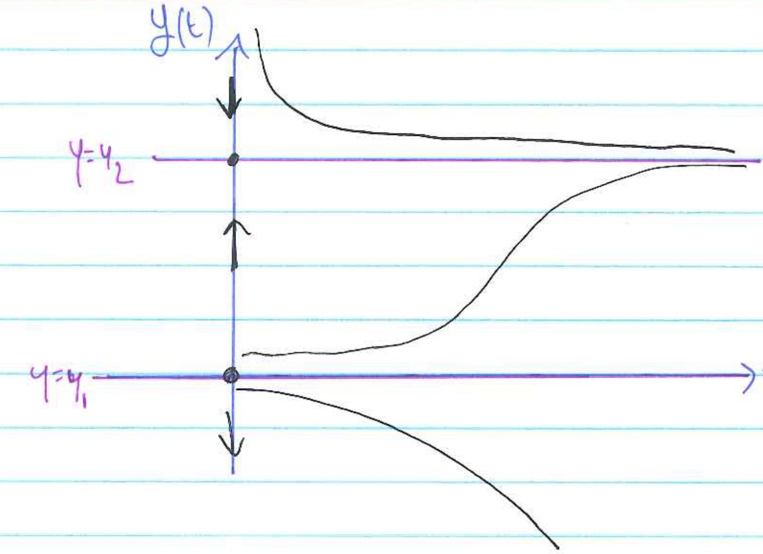
$y_1 = 0$ is $\leftarrow \bullet \rightarrow$ is asymptotically unstable

$y_2 = \frac{(r-h)K}{r}$ is $\rightarrow \bullet \leftarrow$ is asymptotically stable.



STABLE:
Solns above + below $y=y_2$ approach y_2

UNSTABLE:
Solns above + below $y=y_1$ move away from y_1.

The sustainable yield rate, Y is given by:

$$Y = h\,y_2 \quad \text{(stable soln)}$$

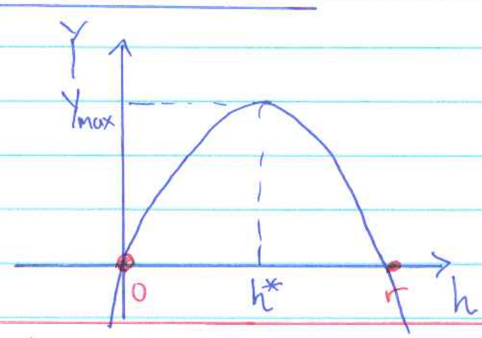
$$Y = \frac{h(r-h)K}{r} \qquad (1)$$

Equation (1) gives a relationship of Y as a fct of h, r, K. To determine how Y changes with h, we assume r & K are fixed and (1) is therefore

$$\boxed{Y = Y(h)}$$

Plot of Y vs h

$Y=0$ at $h=0$, $h=r$



To find h^* (where $Y = Y_{max}$), we look for stationary pts of (1):

$$\frac{dY}{dh} = \left(1 - \frac{2h}{r}\right)K = 0$$

$$\therefore\ h = \frac{r}{2} = h^*$$

$\Rightarrow$ $\boxed{\text{At } h = r/2 \text{, the sustainable yield rate is maximized.}}$