Math 33B: Differential Equations

Review 8: Homogeneous 2$^{\text{nd}}$ order constant coefficient ODEs

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1 Homogeneous linear equations with constant coefficients

As seen in **Review 7**, the second order, linear differential equation is given in standard form as,

$$y'' + p(x)y' + q(x)y = f(x). \tag{1}$$

In this review, we are interested in finding solutions to the *homogeneous* version of Eq. (1) i.e. when $f(x) = 0$ and we'll also be looking at the case where the coefficients are constants (i.e. when $p(x)$ and $q(x)$ are constants):

$$y'' + py' + qy = 0. \tag{2}$$

Equation (2) will be our main concern here. We recall that Eq. (2) is a *linear* second order ODE and by the principle of superposition, the general solution is made up of a *linear combination* of two linearly independent solutions, $y_1(x)$ and $y_2(x)$:

$$y(x) = c_1 y_1(x) + c_2 y_2(x). \tag{3}$$

Our objective here is to find the pair of fundamental solutions [$y_1(x)$ and $y_2(x)$] and use them in Eq. (3) to construct the general solution to Eq. (2).

2 Characteristic equation

For linear ODEs with constant coefficients [i.e. like Eq. (2)], it is easy to see that the equation's solution is a function which, when differentiated, gives us constant multiples of the original function.

The statement above motivates us to 'guess' a solution to the ODE that takes the form $y = e^{mx}$ where m is unknown. Our objective is to find m.

Differentiating $y = e^{mx}$ wrt x to obtain y' and y'', we get $y' = me^{mx}$ and $y'' = m^2 e^{mx}$.

Substituting the expressions for y, y' and y'' in Eq. (2) and simplifying, we obtain:

$$e^{mx}(m^2 + pm + q) = 0. \tag{4}$$

Equation (4) is satisfied if $e^{mx} = 0$ or if $m^2 + pm + q = 0$. Since $e^{mx} \neq 0$, we use the following equation to determine m:

$$m^2 + pm + q = 0. \tag{5}$$

Equation (5) is known as the characteristic equation of the ODE (2) which needs to be satisfied if $y = e^{mx}$ is to be a solution to ODE (2). The solution to the quadratic equation gives us the roots, m as follows:

$$m = \frac{-p \pm \sqrt{[p^2 - 4q]}}{2}. \tag{6}$$

Assuming that p and q are real numbers, we consider **three** cases:

1. Case 1: The characteristic equation has **two** real and distinct roots.
2. Case 2: The characteristic equation has **two** complex and distinct roots.
3. Case 3: The characteristic equation has **one** repeated root.

We will look at each case separately.

Before proceeding we note the following:

- When looking for the solution to a linear ODE with constant coefficients, the problem is reduced from *solving an ODE* to *solving an algebraic equation* [this is the characteristic equation given by (5)].
- For second order ODEs we are looking for two fundamental solutions, y_1 and y_2.
- The solutions take the form $y_1 = e^{m_1 x}$ and $y_2 = e^{m_2 x}$ where m_1 and m_2 denote the roots to the characteristic equation.
- If there exist two roots to the characteristic equation, the general solution is,

$$y(x) = c_1 e^{m_1 x} + c_2 e^{m_2 x}. \tag{7}$$

2.1 Case 1: Roots are real and distinct

If $p^2 - 4q > 0$, then from Eq. (6), we obtain two real roots, let's call them m_1 and m_2.

Fundamental solutions

The next step is to form the fundamental solutions. Recall that the characteristic equation is a result of *guessing* that the solution to the ODE is $y = e^{mx}$ where m is what we were looking for (and found).

The first solution is $y_1 = e^{m_1 x}$ and the second is $y_2 = e^{m_2 x}$; of course any constant multiple of y_1 (say, $c_1 y_1$) and any constant multiple of y_2 (say, $c_2 y_2$) are also solutions to the ODE.

General solution

By taking the pair of fundamental solutions, we can construct the general solution, $y(x) = c_1 e^{m_1 x} + c_2 e^{m_2 x}$.

If the roots of the characteristic equation (5), m_1 and m_2 are **real and distinct**, then the general solution to (2), is given by

$$y(x) = c_1 e^{m_1 x} + c_2 e^{m_2 x}. \tag{8}$$

2.2 Case 2: Roots are complex and distinct

If $p^2 - 4q < 0$, then from Eq. (6), we obtain two complex roots, *again*, let's call them m_1 and m_2. These two roots are conjugates:

$$m_1 = a + bi \quad \text{and} \quad m_2 = a - bi, \tag{9}$$

where a and b are real numbers.

As in **Case 1**, this gives two linearly independent solutions: $y_1 = e^{m_1 x}$ and $y_2 = e^{m_2 x}$. Consequently, the linear combination of y_1 and y_2 make up the general solution $y(x) = c_1 e^{m_1 x} + c_2 e^{m_2 x}$.

Of course a natural question that arises is what becomes of the exponential function if the exponents are complex numbers? The answer to that question is given by **Euler's formula** defined as follows:

Euler's formula: for any real number x,

$$e^{ix} = \cos x + i \sin x. \tag{10}$$

Equation (9) gives the form of the complex roots, m_1 and m_2. So, for $m_1 = a + ib$, the exponential function $y_1 = e^{m_1 x}$ becomes:

$$e^{m_1 x} = e^{(a+ib)x} = e^{ax} e^{ibx} = e^{ax}[\cos(bx) + i \sin(bx)]. \tag{11}$$

Similarly, for $m_2 = a - ib$, we have:

$$e^{m_2 x} = e^{(a-ib)x} = e^{ax} e^{-ibx} = e^{ax}[\cos(bx) - i \sin(bx)]. \tag{12}$$

From Eqs. (11) and (12), we can construct the general solution:

$$y(x) = k_1 e^{ax}[\cos(bx) + i \sin(bx)] + k_2 e^{ax}[\cos(bx) - i \sin(bx)]. \tag{13}$$

However, the general solution given by (13) is a complex-valued function; this means that, given a real number x, the value of the function $y(x)$ could be complex. Equation (13) represents the *most* general form: it represents all solutions with either real or complex number coefficients.

As usual, for the purposes of this differential equations course, what we look for, are **real-valued** functions. This means that we want to 'filter out' all functions containing coefficients with an imaginary part, keeping only those whose coefficients are real numbers.

We choose to write down the general solution in the following form:

$$y(x) = e^{ax}[c_1 \cos(bx) + c_2 \sin(bx)], \tag{14}$$

The two real-valued solutions that we seek therefore are given by

$$\text{(i)} \;\; e^{ax}\cos(bx) \quad \text{and} \quad \text{(ii)} \;\; e^{ax}\sin(bx).$$

Their Wronskian determinant is $W(x) = be^{2ax}$ which is never zero; therefore, the two functions are linearly independent and they constitute a fundamental set of solutions which leads to the general solution given by Eq. (14).

If the roots of the characteristic equation (5), are **complex and distinct** i.e. $m_1 = a+ib$ and $m_2 = a - ib$, then the general solution to (2), is given by

$$y(x) = e^{ax}[c_1 \cos(bx) + c_2 \sin(bx)] \tag{15}$$

2.3 Case 3: Repeated root

If $p^2 - 4q = 0$, then from Eq. (6), we obtain one (real) root, m which is equal to $-p/2$. Recall our 'guess' of a solution to (2) was $y = e^{mx}$ and with $m = -p/2$, we have one solution given as,

$$y_1(x) = e^{-px/2}. \tag{16}$$

We now have a problem because unlike **Case 1** and **Case 2**, where the solution to the characteristic equation gave us two distinct solutions, now we only have one and this is not enough to construct the general solution. Therefore, in order to be able to write down the general solution for this case, we need to come up with a second solution (which is, of course, linearly independent to the first one) on our own.

Using a known solution to find another

Here, the known solution is y_1 given by Eq. (16). We are looking for y_2, a linearly independent function to y_1 which satisfies the ODE (2).

We assume that the second solution takes the following form,

$$y_2(x) = v(x)y_1(x), \tag{17}$$

where $v(x)$ is an unknown **nonconstant** function of x, and y_1 is the solution we already have (from the characteristic equation). The motivation behind this is that if two functions are linearly independent, then (as stated in **Review 7**), they should not be constant multiples of each other:

$$\frac{y_2(x)}{y_1(x)} = \frac{v(x)y_1(x)}{y_1(x)} = v(x) = \text{nonconstant}. \tag{18}$$

The fact that y_1 and y_2 are not constant multiples of each other is an encouraging first step in determining the second fundamental solution.

The *objective* here is to determine $v(x)$. If we have $v(x)$ then, using Eq. (17), we have $y_2(x)$.

Determine $v(x)$

- If y_2 is a solution to the ODE (2) then obviously it must satisfy Eq. (2). Our next step therefore is to take Eq. (17) and differentiate wrt x (to obtain y_2' and y_2'') and substitute back in the original ODE:

$$\begin{aligned} y_2' &= y_1'v + v'y_1 \quad (19) \\ y_2'' &= y_1''v + 2v'y_1' + v''y_1 \quad (20) \end{aligned}$$

- Take Eqs. (17), (19), (20) and substitute in (2):

$$y_2'' + py_2' + qy_2 = 0$$

$$y_1''v + 2v'y_1' + v''y_1 + p\left(y_1'v + v'y_1\right) + q\,vy_1 = 0 \tag{21}$$

- Factorize v, v' and v'' in Eq. (21):

$$v(y_1'' + py_1' + qy_1) + v'(2y_1' + py_1) + v''y_1 = 0. \tag{22}$$

  Now, Eq. (22) is made up of three terms which (from left to right) are: the coefficients of v (blue terms), of v' (green terms) and of v''.

- Let us start with the coefficients of v:

  If y_1 satisfies the ODE (2), then:

$$y_1'' + py_1' + qy_1 = 0; \tag{23}$$

  which means that the coefficient of v is 0 and hence the first term on the LHS of (22) disappears:

$$\cancel{v(y_1'' + py_1' + qy_1)} + v'(2y_1' + py_1) + v''y_1 = 0.$$

- Moving on to the coefficient of v'. This is given by:

$$2y_1' + py_1 = 2\Big(-\frac{p}{2}e^{-px/2}\Big) + pe^{-px/2} = 0; \tag{24}$$

so, the coefficient of v' also disappears:

$$v'\cancel{(2y_1' + py_1)} + v''y_1 = 0.$$

- We are left with $v''y_1 = 0$ and, since $y_1 = e^{-px/2} \neq 0$ then, what we are really left with is $v'' = 0$. This is a second order ODE for $v(x)$ which can be easily solved by integrating twice wrt x:

$$v(x) = k_1 x + k_2, \tag{25}$$

for any *nonzero* k_1 and any k_2. Note that the constants k_1 and k_2 can be arbitrarily chosen to construct the general solution since we ultimately solve for the constants appearing in the solution through application of initial conditions. As long as we have $k_1 \neq 0$ then, $v(x)$ will be a nonconstant function of x which, in turn, guarantees that y_1 and y_2 are linearly independent.

- The simplest form $v(x)$ can take therefore is $v(x) = x$ where we chose $k_1 = 1$ and $k_2 = 0$. This implies that the second solution is given by,

$$y_2(x) = xy_1(x). \tag{26}$$

Note that for constant coefficient, homogeneous ODEs whose characteristic equation gives a single, repeated root the pair of fundamental solutions is *always* given by

$$y_1(x) = e^{-px/2} \quad \text{and} \quad y_2 = xe^{-px/2}, \tag{27}$$

where p is the coefficient of y' when the ODE is written in standard form.

If the characteristic equation (5) gives a repeated, real root i.e. $m = -p/2$, then the general solution to (2), is given by

$$y(x) = c_1 e^{-px/2} + c_2 x e^{-px/2}. \tag{28}$$

3 Summary

Given a second order linear, homogeneous equation with constant coefficients,

$$y'' + py' + qy = 0, \tag{29}$$

solve its characteristic equation,

$$m^2 + pm + q = 0. \tag{30}$$

The general solution to (29), depends on the type of roots obtained in (30).

1. When $p^2 - 4q > 0$, the roots m_1 and m_2 are **real + distinct** and the general solution is,
$$y(x) = c_1 e^{m_1 x} + c_2 e^{m_2 x}. \tag{31}$$

2. When $p^2 - 4q < 0$, the roots are **complex + distinct** and they take the form $m = a \pm ib$. The general solution is,
$$y(x) = e^{ax}[c_1 \cos(bx) + c_2 \sin(bx)]. \tag{32}$$

3. When $p^2 - 4q = 0$, there is **only one real** root given by $m = -p/2$. The general solution is,
$$y(x) = c_1 e^{-px/2} + c_2 x e^{-px/2}. \tag{33}$$

Notes

- You do *not* need to set the ODE in standard form when solving it using the characteristic equation method.

- That said, note that in Eq. (33), the root given as $m = -p/2$ is a result of having the ODE in standard form. Had we started off using the following ODE (not in standard form):
$$\alpha y'' + \beta y' + \gamma y = 0, \tag{34}$$
then, the characteristic equation solution would be given by,
$$m = \frac{-\beta \pm \sqrt{\beta^2 - 4\alpha\gamma}}{2\alpha}. \tag{35}$$
If $\beta^2 - 4\alpha\gamma = 0$ then the only root is $m = \dfrac{-\beta}{2\alpha}$.