Q1 (25 pts).
(a). Solve the initial value problem

$$\frac{dy}{dx} = \frac{e^{-y}\cos^2 x}{1+y} \qquad \text{with} \quad y(0) = 1.$$

$$(1+y)e^{y}\,dy = \cos^2 x\,dx$$

$$\int (1+y)e^{y}\,dy = \int \cos^2 x\,dx$$

$$(1+y)e^{y} - e^{y} = \frac{1}{2}\int(1+\cos 2x)\,dx$$

$$ye^{y} = \frac{1}{2}x + \frac{1}{4}\sin 2x + C$$

$x=0,\ y=1$

$$\Rightarrow \quad e = 0+0+C \quad \Rightarrow \quad C = e$$

$$ye^{y} = \frac{1}{2}x + \frac{1}{4}\sin 2x + e$$

(b). Suppose that a radioactive substance decays according to the model $\frac{dN}{dt} = -\lambda N$. Show that the half-life of this radioactive substance is given by

$$T_{1/2} = \frac{\ln 2}{\lambda}.$$

$$\frac{dN}{dt} = -\lambda N$$

$$\frac{1}{N}\,dN = -\lambda\,dt$$

$$\ln N = -\lambda t + C$$

$$N = N_0 e^{-\lambda t}$$

at $t = T_{1/2}$,

$N = \frac{N_0}{2}$

$$\Rightarrow \quad \frac{N_0}{2} = N_0 e^{-\lambda T_{\frac{1}{2}}}$$

$$T_{\frac{1}{2}} = \frac{\ln 2}{\lambda}$$

Q2 (20 pts). Find the general solution of the homogeneous equation

$$\frac{dy}{dx} = \frac{y + xe^{-y/x}}{x}.$$

Let $y = xv$, then $dy = x\,dv + v\,dx$

$$dy = \frac{y + xe^{-y/x}}{x}\,dx$$

$$x\,dv + v\,dx = (v + e^{-v})\,dx$$

$$x\,dv = e^{-v}\,dx$$

$$e^{v}\,dv = \frac{1}{x}\,dx$$

$$e^{v} = \ln|x| + c$$

$$v = \ln(\ln|x| + c)$$

$$y = xv = x\ln(\ln|x| + c)$$

Q3 (25 pts). Consider the differential equation

$$2\cos 2x\, \mathrm{d}x + \left(\frac{e^y}{1+y^2} - \sin 2x\right) \mathrm{d}y = 0.$$

(a). This differential equation is not exact. Find an integration factor μ which depends on only one variable.

$$P = 2\cos 2x$$

$$Q = \frac{e^y}{1+y^2} - \sin 2x$$

$$\frac{\partial P}{\partial y} = 0 \qquad \frac{\partial Q}{\partial x} = 2\cos 2x$$

$$h = \frac{1}{P}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = 1 \quad \text{is a function of } y \text{ only}$$

there $$\mu = e^{-\int h(y)\,dy} = e^{-y}$$

is an integration factor.

(b). Solve this equation.

$$e^{-y}\left(2\cos 2x\,dx + \left(\frac{e^{y}}{1+y^2} - \sin 2x\right)dy\right) = 0$$

$$2\cos 2x \cdot e^{-y}\,dx + \left(\frac{1}{1+y^2} - \sin 2x \cdot e^{-y}\right)dy = 0$$

let $\frac{\partial F}{\partial x} = 2\cos 2x \cdot e^{-y}$

then $F = \sin 2x \cdot e^{-y} + g(y)$

$$\frac{\partial F}{\partial y} = \frac{1}{1+y^2} - \sin 2x \cdot e^{-y}$$

$$-\sin 2x \cdot e^{-y} + g'(y) = \frac{1}{1+y^2} - \sin 2x \cdot e^{-y}$$

$$g'(y) = \frac{1}{1+y^2}$$

$$g(y) = \tan^{-1} y$$

The solution is $F(x, y) = C$

i.e.,

$$e^{-y}\sin 2x + \tan^{-1} y = C.$$

Q4 (30 pts). A 40-gal tank initially contains 20 gal of pure water. Sugar water solution containing 2 lb of sugar for each gal of water begins entering the tank at a rate of 4 gal/min. Simultaneously, a drain is opened at the bottom of the tank, allowing the sugar water solution to leave the tank at a rate of 2 gal/min.

(a). Write down the function of the volume $V(t)$ of sugar water solution in the tank, up to the time when the tank is full.

$$V(t) = 20 + 2t$$

$$0 \le t \le 10.$$

(b). Write down the initial value problem for the weight $x(t)$ of the sugar content in the tank.

$$\frac{dx}{dt} = 4 \cdot 2 - 2 \cdot \frac{x}{V(t)}$$

$$\begin{cases} \dfrac{dx}{dt} = 8 - \dfrac{2x}{20+2t} \\ x(0) = 0 \end{cases}$$

(c). Solve the initial value problem in (b).

$$\frac{dx}{dt} + \frac{1}{10+t} x = 8$$

let $v(t) = e^{\int \frac{1}{10+t} dt} = e^{\ln(10+t)} = (10+t)$

Multiply $v(t)$ on both sides

$$\cancel{(10+t)\frac{dx}{dt}}$$

$$(10+t)\frac{dx}{dt} + x = 8 \cdot (10+t)$$

$$\frac{d}{dt}\left((10+t)x\right) = 8(10+t)$$

$$(10+t)\,x = 80t + 4t^2 + C$$

$t=0,\ x=0 \Rightarrow C=0$

$$x = \frac{80t+4t^2}{10+t}$$

(d). How much sugar is in the tank at the precise moment that the tank is full?

$V(t) = 40 \Rightarrow t = 10$

$x = 60$ lb.