Midterm 1 practice, Math 33b, Winter 2013
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Name and student ID: ______________________________

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
Total:	50	

1. (a) (2 points) Verify that $x = \frac{1}{t-1}$ is a particular solution of the equation

$$tx' + x^2 + x = 0.$$

Solution: Since $x' = -\frac{1}{(t-1)^2}$ we have

$$tx' + x^2 + x = -\frac{t}{(t-1)^2} + \frac{1}{(t-1)^2} + \frac{t-1}{(t-1)^2} = 0.$$

(b) (2 points) General solution of the equation $y' = y\cos t$ is given by

$$y = Ce^{\sin t}.$$

Find the solution of the initial value problem $y' = y\cos t$, $y(\pi/2) = 2$.

Solution: Since $y(\pi/2) = Ce^{\sin(\pi/2)} = Ce^1 = Ce$ we have to choose $C = 2/e$ which gives $y = 2e^{\sin t - 1}$.

(c) (2 points) The graph of the function $y(x)$, $x \geq 0$ passes through the origin $(0,0)$. The slope of the tangent line at the point $(x, y(x))$ is equal to the area of the rectangle whose opposite vertices are $(0,0)$ and $(x, y(x))$. Write the initial value problem that $y(x)$ satisfies (you don't have to solve it).

Solution: The differential equation is $y' = xy$ and the initial condition $y(0) = 0$.

(d) (2 points) Does there exist a solution of the equation $y' = e^x y^2 + e^{y^5}$ such that $y'(0) = -1$? No need to solve to equation, but explain your answer.

Solution: Since $e^x y^2 + e^{y^5}$ is always positive, y' always has to be positive for any solution y and so we can't have $y'(0) = -1$.

(e) (2 points) Is the following differential equation exact

$$(x + x\sin y)\ dx + \cos y\ dy = 0.$$

Solution: The derivative $x + x\sin y$ with respect to y is $x\cos y$ and the derivative os $\cos y$ with respect to x is zero. They are different so the equation is not exact.

2. (10 points) Find the solution of the initial value problem

$$y^2y' - e^{y^3} = te^{y^3}, \quad y(0) = 0.$$

Solution: This equation is actually separable since we can write it as

$$y^2 e^{-y^3} y' = t + 1.$$

Thus

$$\int y^2 e^{-y^3}\ dy = \int (t+1)\ dt$$

which gives

$$-\frac{1}{3}e^{-y^3} = t^2/2 + t + C$$

and

$$y = -\Big(\ln\Big(-3t^2/2 - 3t - 3C\Big)\Big)^{1/3}.$$

Since $y(0) = 0$ we need $C = -1/3$ so that

$$y = -\Big(\ln\Big(1 - 3t^2/2 - 3t\Big)\Big)^{1/3}.$$

3. (10 points) Find the general solution of the equation

$$y' + y\sin t = e^{\cos t}\sin t.$$

Solution: The muliplicative factor is

$$u = e^{\int \sin t \ dt} = e^{-\cos t},$$

which gives

$$e^{-\cos t}y' + ye^{-\cos t}\sin t = \sin t.$$

The left hand side is the derivative of $e^{-\cos t}y$ and so

$$e^{-\cos t}y = \int \sin t \ dt = -\cos t + C,$$

which gives

$$y = Ce^{\cos t} - e^{\cos t}\cos t.$$

4. (10 points) A 10 gallon tank contains a mixture of water and a pound of salt. A pure water is entering the tank at the rate of 1 gallon per second and the mixture is leaving the tank at the same rate. Find the amount of salt in the tank after time t.

   Solution: If $x(t)$ denotes the mass of salt in the tank after time t then the salt rate in is zero and the rate out is $x/10$. Therefore,

   $$x' = -x/10 \quad \Rightarrow \quad x = Ce^{-t/10}.$$

   Since $x(0) = 1$ (in pounds) so $C = 1$ and $x = e^{-t/10}$.

5. (10 points) Show that the following differential equation is exact and find the general solution.

$$(x + y\sin x)\ dx - \cos x\ dy = 0.$$

Solution: The partial derivative of $x + y\sin x$ with respect to y is $\sin x$ just like the partial derivative $-\cos x$ with respect to x. Since both functions are defined on the whole plane $\mathbb{R}^2$ the equation is exact. Now integrate

$$F(x, y) = \int -\cos x\ dy = -y\cos x + \phi(x),$$

which gives

$$\frac{\partial F}{\partial x} = y\sin x + \phi'(x) = x + y\sin x,$$

which gives $\phi'(x) = x$ and so $\phi(x) = x^2/2$. This gives $F(x, y) = -y\cos x + x^2/2$, and the general solution is given in the implicit form as

$$-y\cos x + x^2/2 = C,$$

or explicitely

$$y = \frac{x^2/2 - C}{\cos x}.$$