Math 33B: Differential Equations

Homework 6: Second orde ODEs & Applications

***Due on*: Fri., May 17, 2013 - 9:00 AM** *Instructor: Aliki M.*

Please include your *name*, *UID* and *discussion section* on the submitted homework.

Problem 1

Each of the following expressions represents the solution of the displacement of a mass attached on a spring which executes simple harmonic motion. Determine the frequency of motion, ω_0, the amplitude A and the phase angle ϕ by putting the given expressions in the form $y(t) = A\cos(\omega_0 t - \phi)$.

(i) $y(t) = 3\cos 2t + 4\sin 2t$

(ii) $y(t) = 4\cos 3t - 2\sin 3t$

(iii) $y(t) = -2\cos \pi t - 3\sin \pi t$

[Answers: (i) $\omega_0 = 2,\ A = 5,\ \phi \approx 0.93$
(ii) $\omega_0 = 3,\ A = 2\sqrt{5},\ \phi \approx -0.46$
(iii) $\omega_0 = \pi,,\ A = \sqrt{13},\ \phi \approx -2.16$[1]]

Problem 2

A mass of 100 g stretches a spring 5 cm. Suppose that the mass is set in motion from its equilibrium position with a velocity of 10 cm/s downwards. If there is no damping, find the position y of the mass at any time t.

When does the mass *first* return to its equilibrium position?

[Answers: $y(t) = \dfrac{5}{7}\sin 14t$; $t = \pi/14$ seconds.]

[1] I had the answer to this previously as $\phi \approx 4.12$; note that both $\phi \approx 4.12$ and $\phi \approx -2.16$ satisfy the values for c_1 and c_2 but the value we choose depends on how ϕ is defined. In the textbook, ϕ is defined within $-\pi < \phi < \pi$ which means that $\phi \approx -2.16$ is the appropriate answer.

Problem 3

Show that the period of motion of an undamped vibration of a mass hanging from a vertical spring is given by,

$$T = 2\pi\sqrt{\frac{L}{g}},$$

where L is the elongation of the spring due to the mass and g is the acceleration due to gravity.

Problem 4

The motion of a spring-mass system is described by,

$$y'' + 0.125y' + y = 0,$$

and it satisfies $y(0) = 2$ and $y'(0) = 0$. The term $0.125y'$ on the LHS is due to damping.

(i) Show that the position of the mass, y at any time t is given by:

$$y(t) = e^{-t/16}\Big(c_1 \cos \omega t + c_2 \sin \omega t\Big),$$

and find c_1 and c_2 and ω.

(ii) Put the solution in the form $y(t) = Ae^{-t/16} \cos\Big(\omega t - \phi\Big)$ and find A and ϕ.

(iii) On the same graph, plot the solution $y(t) = Ae^{-t/16} \cos\Big(\omega t - \phi\Big)$, $y(t) = Ae^{-t/16}$ and $y(t) = -Ae^{-t/16}$.

[Answers: (i) $c_1 = 2$, $c_2 = \dfrac{2}{\sqrt{255}}$, $\omega = \dfrac{\sqrt{255}}{16}$
(ii) $A = \dfrac{32}{\sqrt{255}}$ and $\phi \approx 0.06$.]

Problem 5

A 10-kg mass stretches a spring 1 m. The system is placed in a viscous medium that provides a damping constant $\mu = 20$ kg/s. The system is allowed to attain equilibrium. Then, a sharp tap to the mass imparts an instantaneous downwards velocity of 1.2 m/s. Find the amplitude, frequency and phase of the resulting motion. Plot the solution.

[Answers: $A = 0.404$, $\omega_0 = 2.97$ and $\phi = \pi/2$]

Problem 6

An object of mass 30 kg stretches a spring by 125 cm when attached to it. The mass has a damper that will exert critical damping. The mass is initially displaced 25 cm upwards from its equilibrium position with an initial velocity of 40 cm/s upwards. Determine the displacement at any time t.

[Answer: $y(t) = -0.25\, e^{-2.8t} - 1.1\, t\, e^{-2.8t}$]