Midterm 2 practice, Math 33b, Winter 2013
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Name and student ID: ______________________________

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
Total:	50	

1. (a) (2 points) Specify all values x_0 for which the initial value problem

$$y' = \frac{y^2 - xy}{x^2 - x}, \quad y(x_0) = 3,$$

may fail to have any solutions.

Solution: The only values of x where the right hand side is not defined or continuous are 0 and 1 so these are only values of x_0 where the problem may have no solutions.

(b) (2 points) Write down an autonomous equation $x' = f(x)$ for which $x = 1$ is an unstable equilibrium solution and such that for any $x_0 < 1$ the solution of the initial value problem

$$x' = f(x), \quad x(0) = x_0,$$

satisfies $\lim_{t\to\infty} x(t) = 1$.

Solution: You want f to be positive on $(-\infty, 1)$, have zero value at $x = 1$ and then be positive on $(1, \infty)$. So $f(x) = (x - 1)^2$ will work.

(c) (2 points) Write down the solution of any second homogeneous order linear differential equation with constant coefficients which satisfies $y(0) = 0$ and $y'(0) = 0$.

Solution: An obvious solution is $y(t) = 0$ and since there is only one solution this is it.

(d) (2 points) Check that $y = 1/t$ and $y = t^3$ are solutions of

$$t^2y'' - ty' - 3y = 0,$$

and write down the general solution of this equation.

Solution: Plug this functions in the equation and see that it is satisfied. Since these functions are independent on $(0, \infty)$ and $(-\infty, 0)$ they form a fundamental set and the solution is given by $y = C_1/t + C_2t^3$.

(e) (2 points) The solution to the initial value problem

$$y'' - qy = 0, \quad y(0) = 1, \quad y'(0) = y_1,$$

satisfies $\lim_{t\to\infty} y(t) = 0$. If $q > 0$ is a positive constant find y_1.

Solution: Since the general solution is $y = C_1e^{-\sqrt{q}t} + C_2e^{\sqrt{q}t}$ and since $\lim_{t\to\infty} e^{\sqrt{q}t} = \infty$ and $\lim_{t\to\infty} e^{-\sqrt{q}t} = 0$ we need to have $C_2 = 0$. Then $y(0) = C_1 = 1$, so $y'(0) = -C_1\sqrt{q}e^{-\sqrt{q}\cdot 0} = -C_1\sqrt{q} = -\sqrt{q}$.

2. (10 points) Find all equilibrium solutions of the autonomous equation $y' = (e^y - 1)(y-1)^2(y-2)$ and for each of them determine whether it's stable or unstable. Also determine all the possible values of the limit $\lim_{t\to\infty} y(t)$.

   Solution: The equilibrium solutions $y = y_0$ happen when y_0 is a zero of the function $y \mapsto (e^y - 1)(y-1)^2(y-2)$ and are three zeros $0, 1, 2$. So the equilibrium solutions are $y = 0$, $y = 1$ and $y = 2$. At $y = 0$ the function $f(y) = (e^y - 1)(y-1)^2(y-2)$ changes the value from positive to negative, at $y = 1$ it stays negative and at $y = 2$ changes the sign from negative to positive. Therefore $y = 0$ is stable and other two are unstable. A brief look at the direction field reveals, when $y(0)$ is negative the limit $\lim_{t\to\infty} y(t)$ is one, as well as when $y(0)$ is between 0 and 1, when $y(0)$ is between 1 and 2 the limit is 1 and when $y(0)$ is larger than 1 the limit is $+\infty$ (not that when $y(0) = 2$ the limit is trivially equal to 2). Therefore, possible limits are 0, 1, 2 (and $+\infty$).

3. (10 points) Find the solution of the initial value problem

$$y'' - 2y' + 5y = 0, \quad y(0) = 1, \quad y'(0) = 2.$$

Solution: The characteristic equation is $\lambda^2 - 2\lambda + 5 = 0$ and roots are $1 + 2i$ and $1 - 2i$ so the general solution is $y = e^t(C_1 \sin(2t) + C_2 \cos(2t))$. Since $y'(t) = e^t((C_1 - 2C_2)\sin(2t) + (2C_1 + C_2)\cos(2t))$ we get $C_2 = 1$ and $2C_1 + C_2 = 2$ so $C_1 = 1/2$ and $C_2 = 1$ and the solution is $y = e^t(\frac{1}{2}\sin(2t) + \cos(2t))$

4. (10 points) Find the solution of the initial value problem

$$y'' + 4y' + 4y = 2e^{-2t} + t, \quad y(0) = 0, \quad y'(0) = 1.$$

Solution: The characteristic equation is $\lambda^2 + 4\lambda + 4 = 0$ and the only root is -2 so the general solution to the homogeneous equation $y'' + 4y' + 4y = 0$ is $y = C_1 e^{-2t} + C_2 te^{-2t}$. A particular solution $y_{p,1}$ to $y'' + 4y' + 4y = 2e^{-2t}$ can't be in the form ae^{-2t} or ate^{-2t} so set $y_{p,1} = at^2e^{-2t}$ which gives $y'_{p,1} = ae^{-2t}(-2t^2 + 2t)$ and $y''_{p,1} = ae^{-2t}(4t^2 - 8t + 2)$ so

$$ae^{2t}(4t^2 - 8t + 2 - 8t^2 + 8t + 4t^2) = 2e^{-2t}, \quad \Rightarrow \quad a = 1,$$

so $y_{p,1} = t^2e^{-2t}$. Look for a particular solution of $y'' + 4y' + 4y = t$ in the form $y = bt + c$ so

$$4b + 4bt + 4c = t, \quad \Rightarrow \quad b = 1/4,\ c = -1/4,$$

and $y_{p,2} = \frac{1}{4}t - \frac{1}{4}$. The general solution to $y'' + 4y' + 4y = 2e^{-2t} + t$ is then

$$y = C_1e^{-2t} + C_2te^{-2t} + t^2e^{-2t} + \frac{1}{4}t - \frac{1}{4}.$$

Since $y'(t) = (-2C_1 + C_2 - 2C_2t + 2t - 2t^2)e^{-2t} + \frac{1}{4}$ the initial conditions give

$$C_1 - 1/4 = 0, \quad -2C_1 + C_2 + 1/4 = 1, \quad \Rightarrow \quad C_1 = 1/4, \quad C_2 = 5/4$$

which gives

$$y = \frac{1}{4}e^{-2t} + \frac{5}{4}te^{-2t} + t^2e^{-2t} + \frac{1}{4}t - \frac{1}{4}.$$

5. (10 points) An object of mass m is attached to a spring of constant k. It is determined that to make the system critically damped one would need to set the damping constant to $\mu = 4$. When there is no damping the system oscillates with period 4π. Find the values of m and k. In the case when there is no damping the object is removed from the equilibrium is to $y(0) = -\sqrt{2}$ and is given an initial velocity $y'(0) = -1/\sqrt{2}$. Find the amplitude and the phase of oscillations.

   Solution: The equation with damping μ is

$$mx'' + \mu y' + ky = 0,$$

   and the characteristic equation $m\lambda^2+\mu\lambda+k = 0$ shows that the critical damping (only one root) happens $\mu^2-4mk = 0$, so $2\sqrt{mk} = 4$ and $mk = 4$. When there is no damping the solution is $C_1 \sin(\omega t)+C_2 \cos(\omega t)$ where $\omega = \sqrt{k/m}$. The period is $T = 2\pi/\omega = 4\pi$ so $\omega = 1/2$ and $\sqrt{k/m} = 2$ and $k/m = 4$. From the equation $mk = 4$ and $k/m = 4$ we get $k = 4$, $m = 1$. Since $\omega = 1/2$ the solution can be written as

$$y(t) = C_1 \sin(t/2) + C_2 \cos(t/2).$$

   Since $y'(t) = \frac{C_1}{2} \cos(t/2) - \frac{C_2}{2} \sin(t/2)$ the initial conditions give $C_1 = C_2 = -\sqrt{2}$. The amplitude is

$$A = \sqrt{C_1^2 + C_2^2} = 2,$$

   and the phase satisfies

$$\sin(\phi) = \cos(\phi) = -1/\sqrt{2},$$

   so $\phi = -3\pi/4$.