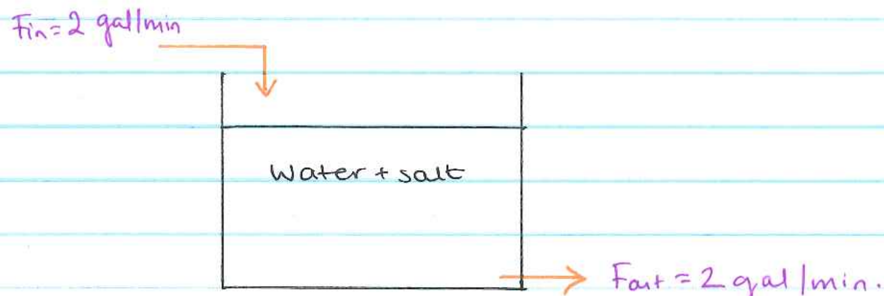
MATH 33B: DIFFERENTIAL EQUATIONS — APRIL 2013

Example: modeling with first order ODEs (mixing problems)



Additional information

- At $t=0$, the volume of liquid in tank is 100 gallons and the amount of salt is 50 oz.
- The concentration of salt at the inlet is given by $c_{in} = \frac{1}{4}\left(1 + \frac{1}{2}\sin t\right) \frac{oz}{gal.}$

Formulate the IVP and determine the amount of salt in the tank at any time, t.

A material balance around the tank gives:

$$\frac{dy}{dt} = F_{in} \times c_{in} - F_{out} \times c_{out} \qquad (1)$$

where c_{in} = salt concentration in inlet stream

c_{out} = salt concentration in outlet stream.

c_{out} is defined as: $c_{out} = \dfrac{\text{amount of salt in tank at any time } t}{\text{volume of liquid in tank at any time } t} = \dfrac{y(t)}{V(t)}$

$\therefore\ C_{out} = \dfrac{y(t)}{V(t)}$ (2)

Now, the volume of liquid in tank is:

$$V(t) = \underbrace{V_0}_{\text{initial volume}} + \underbrace{(F_{in} - F_{out})\cdot t}$$ (3)

↳ this dictates whether the volume will:
(i) increase: if $F_{in} > F_{out}$
(ii) decrease: if $F_{out} > F_{in}$
(iii) remain the same: if $F_{in} = F_{out}$.

$V_0 = 100$ gallons
$F_{in} = F_{out} = 2\ \dfrac{gal}{min}$
} stated in the problem description

$\Rightarrow$ (3) becomes $V(t) = 100 + (2-2)\cdot t = 100$ (volume is kept constant)

$\Rightarrow$ (2) is $C_{out} = \dfrac{y(t)}{100}$.

Back in (1), $\dfrac{dy}{dt} = 2\underbrace{\boxed{\left[\frac{1}{4}\left(1+\frac{1}{2}\sin t\right)\right]}}_{=C_{in}} - 2\underbrace{\boxed{\frac{y}{100}}}_{=C_{out}}$ (4)

The I.C is $y(0) = 50$ (5) (stated in problem descr.)

The IVP is given by Eq. (4) and Eq. (5)

- Solving the IVP

Put (4) in S.F. $\boxed{\dfrac{dy}{dt} + \dfrac{2}{100}\cdot y = \dfrac{1}{2}\left(1+\dfrac{1}{2}\sin t\right)}$ linear, 1st order ODE

(6)

Eq. 6 may be solved using integrating factors. (I.F)

In Eq. 6 $\quad p = \frac{2}{100} = \frac{1}{50}$ (note, here, the coefficient of y is a constant)

$\Rightarrow$ the I.F. is $\quad \mu(t) = e^{\int p\,dt} = e^{\int \frac{1}{50}\,dt} = e^{\frac{1}{50}t}$

$$\therefore \boxed{\mu(t) = e^{1/50\,t}}$$

Multiplying 6 by the I.F.

$$e^{1/50\,t} \cdot \frac{dy}{dt} + \frac{\cancel{2}\,1}{\cancel{100}\,50} e^{1/50\,t}\, y = \frac{1}{2}\left(1 + \frac{1}{2}\sin t\right) e^{1/50\,t}. \qquad (7)$$

Reducing the LHS of 7 gives,

$$\text{LHS of } (7) = \frac{d}{dt}\left[e^{1/50\,t} \cdot y\right]$$

Check! $\quad \frac{d}{dt}\left[e^{1/50\,t} \cdot y\right] = e^{1/50\,t} \cdot \frac{dy}{dt} + y\,e^{1/50\,t} \cdot \frac{1}{50}$ ✓

So, 7 becomes.

$$\frac{d}{dt}\left[e^{\frac{1}{50}t} \cdot y\right] = \frac{1}{2} e^{1/50\,t} + \frac{1}{4} \sin t\, e^{1/50\,t} \qquad (8)$$

Integrating both sides of 8 wrt t,

$$e^{\frac{1}{50}t} \cdot y = \frac{1}{2} \int e^{1/50\,t} + \underbrace{\frac{1}{2} \sin t\, e^{1/50\,t}}_{\text{integrate by parts!}}\, dt \qquad (9)$$

From 9 the soln, $y(t)$ is

$$\boxed{y(t) = \frac{1}{2e^{1/50\,t}} \int e^{1/50\,t}\,dt + \frac{1}{4e^{1/50\,t}} \int \sin t\, e^{1/50\,t}\,dt} \qquad (10)$$

Let's look at the integrals in (10) separately:

the first one is

$$I_1 = \frac{1}{2e^{1/50t}} \int e^{1/50t}\, dt$$

$$\boxed{I_1 = \frac{1}{2e^{1/50t}}\left\{\left[e^{1/50t} \cdot 50\right] + k_1\right\}.} \quad (11)$$

and the second one is:

$$I_2 = \frac{1}{4e^{1/50t}} \underbrace{\boxed{\int \sin t \cdot e^{1/50t}\, dt.}}_{=I_3} \quad (12)$$

the integrand in (12) is a product of functions of t so we need to use integration by parts (sadly) and we need to do this twice.

let $I_3 = \int \sin t\, e^{1/50t}\, dt$

then, choose $u = e^{1/50t}$ and $\frac{dv}{dt} = \sin t$ (for trig/exp functions, the choice for u & $\frac{dv}{dt}$ doesn't matter)

now $u = e^{1/50t}$ and $du = \frac{1}{50} e^{1/50t}\, dt$

$dv = \sin t\, dt$ and $v = -\cos t$

By integration by parts, $\boxed{I_3 = u \cdot v - \int v\, du + k}$ note: for definite integrals, the constant of integration, k will cancel out.

So, $I_3 = -e^{1/50t} \cos t - \int \frac{1}{50} e^{1/50t}(-\cos t)\, dt$.

$$I_3 = -e^{1/50t} \cos t + \frac{1}{50}\boxed{\int e^{1/50t} \cos t\, dt} + k_2 \quad (13)$$

integrate by parts again.

For the integrand in (13), we use $u = e^{1/50\,t}$ $\quad du = \frac{1}{50} e^{1/50\,t}\, dt$

$dv = \cos t\, dt \quad v = \sin t$

Then (13) becomes,

$$I_3 = -e^{-1/50\,t}\cos t + \frac{1}{50}\left\{ e^{1/50\,t}\cdot \sin t - \frac{1}{50}\boxed{\int e^{1/50\,t}\sin t\, dt}\right\} + k_2 + k_3 \quad (14)$$

this is what we started with and it is equal to I_3

(14) becomes:

$$I_3 = -e^{1/50\,t}\cos t + \frac{1}{50}e^{1/50\,t}\sin t - \frac{1}{50^2}\, I_3 + k_2 + k_3$$

Collecting the I_3 terms together,

$$I_3 + \frac{1}{50^2} I_3 = -e^{1/50\,t}\cos t + \frac{1}{50}e^{1/50\,t}\sin t + k_2 + k_3$$

$$I_3\left(1 + \frac{1}{50^2}\right) = e^{1/50\,t}\left(\frac{1}{50}\sin t - \cos t\right) + k_2 + k_3$$

$$\Rightarrow I_3 = \frac{50^2}{(1+50^2)}\left\{e^{1/50\,t}\left(\frac{1}{50}\sin t - \cos t\right) + k_2 + k_3\right\} \quad (15)$$

Sub (15) in (12)

$$I_2 = \frac{1}{4e^{1/50\,t}} I_3 = \frac{1}{4e^{1/50\,t}}\left(\frac{50^2}{1+50^2}\right)\left\{e^{1/50\,t}\left(\frac{1}{50}\sin t - \cos t\right)\right\} + \frac{K}{4e^{1/50\,t}} \quad (16)$$

Where $K = (k_2 + k_3)\dfrac{50^2}{(1+50^2)}$

∴ Now, back to (10) the soln is given by

$$y(t) = \boxed{\frac{1}{2e^{1/50 t}} \int e^{1/50 t}\, dt} + \boxed{\frac{1}{4e^{1/50 t}} \int \sin t\, e^{1/50 t}\, dt}$$

↑ given by Eq. (11) ↑ given by Eq. (16)

Sub. (11) & (16) in the above eqn gives:

$$y(t) = \frac{50}{2e^{1/50 t}} e^{1/50 t} + \frac{50^2 \cancel{e^{1/50 t}}}{4(1+50^2)\cancel{e^{1/50 t}}}\left(\frac{1}{50}\sin t - \cos t\right) + \frac{C}{2} e^{-1/50 t}$$

where $C = K_1 + \frac{K}{2}$ (absorb all constants of integration into one to keep things as simple as possible)

Simplifying,

$$y(t) = 25 + \frac{25}{5002}\sin t - \frac{625}{2501}\cos t + \frac{C}{2} e^{-1/50 t}$$ general soln

Apply I.C [∴ $y(0) = 50$]

$$\therefore y(0) = 25 + 0 - \frac{625}{2501} + \frac{C}{2} = 50$$

$$\Rightarrow C = \frac{126300}{2501}$$

The particular soln is $\boxed{y(t) = 25 + \frac{25}{5002}\sin t - \frac{625\cos t}{2501} + \frac{63150}{2501} e^{-1/50 t}}$

which gives the amount of salt in the tank at any time t.