Math 33B: Differential Equations

Review 3: First order linear ODEs

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1 Standard form

The most general first order linear ODE is expressed in standard form as follows:

$$\frac{dy}{dx} + p(x)y = f(x), \tag{1}$$

where y is the dependent variable and x is the independent variable.

A first order ODE is *linear* if:

- the function $y(x)$ and its derivative, $y'(x)$, are **not** present at any power higher than 1;
- there are **no** products of the function $y(x)$ and its derivative, $y'(x)$.

We note that the x-dependent functions, $p(x)$ and $f(x)$ should be continuous but they may be represented by a constant, nonconstant, a linear or nonlinear function.

In situations where the function on the RHS of Eq. (1), $f(x)$, is zero then, Eq. (1) is *separable*.

If $f(x) = 0$, Eq. (1) reduces to

$$\frac{dy}{dx} + p(x)y = 0. \tag{2}$$

This is now a separable ODE, which we can easily solve using the technique discussed in **Review 2**.

2 General solution formula

For first order linear ODEs the solution to the differential equation may be expressed by the following general formula,

$$y(x) = \frac{\int \mu(x) f(x)\, dx}{\mu(x)} \tag{3}$$

where $\mu(x)$ is the *integrating factor* (I.F.). The I.F. is a function of x which we introduce in the ODE we are trying to solve to help us follow a *solution technique* which is guaranteed to work for first order linear ODEs. This technique is outlined in Section 3.

2.1 Derivation

In this section, we derive the general solution formula given by (3) [note: the derivation of the formula is ***not*** examinable].

1. Assume there exists an I.F. given by $\mu(x)$ where $\mu(x) \neq 0$. Starting from the ODE given in standard form [Eq. (1)], multiply the ODE by $\mu(x)$ such that Eq. (1) becomes,
$$\mu(x)\frac{dy}{dx} + \mu(x)p(x)y = \mu(x)f(x). \tag{4}$$
2. Our objective is to find an I.F. that allows us to express the LHS of Eq. (4) as the derivative of *something*. We observe that we can reduce the LHS of (4) as follows
$$\text{LHS of (4)} = \frac{d}{dx}\Big[\mu(x)y(x)\Big], \tag{5}$$
as long as the following ODE is satisfied,
$$\frac{d\mu}{dx} = \mu(x)p(x). \tag{6}$$
Note that Eq. (6) is an ODE for $\mu(x)$ from which we can solve for the I.F. - see Subsection 2.2.
3. Reducing the LHS of (4) to the form given by (5), we can express Eq. (4) as
$$\frac{d}{dx}\Big[\mu(x)y(x)\Big] = \mu(x)f(x); \tag{7}$$
where $y(x)$ is what we want to solve for.
4. We now integrate both sides of Eq. (7) wrt x. Note that, as the LHS of (7) is the x-derivative of a product, integrating wrt x simply gives the product itself:
$$\mu(x)y(x) = \int \mu(x)f(x)\, dx. \tag{8}$$
5. Solving for $y(x)$ gives,
$$y(x) = \frac{\int \mu(x)f(x)\, dx}{\mu(x)}. \tag{9}$$

2.2 Integrating factor (I.F.)

The derivation of the solution given by (9) is subject to the condition given by Eq. (6). Equation (6) is a first order ODE for $\mu(x)$ (x is the independent variable and μ is the dependent variable) and it is *separable*. Therefore, by separating the variables in (6),

$$\frac{1}{\mu}\, d\mu = p(x)\, dx. \tag{10}$$

Integrating both sides of (10),

$$\ln \mu = \int p(x)\, dx. \tag{11}$$

Exponentiating both sides of (11),

$$\mu(x) = e^{\int p(x)\, dx}. \tag{12}$$

Note: for a linear, first order ODE expressed in standard form, you should know that the I.F. is defined by (12). You can then use (12) to find the I.F. without proof.

3 General solution method

We will now outline a solution technique to solve ODEs of the form given by (1) (or any ODE that can be put in that form).

Solution process

1. **Step 1:** Check if the ODE is in standard form; if not, put it in that form.
2. **Step 2:** Find the integrating factor using $\mu(x) = e^{\int p(x)\, dx}$. First, integrate $p(x)$ wrt x (or, more accurately, find *any* function whose derivative is $p(x)$: in other words, you don't have to bother about constants of integration), then exponentiate the integration result to obtain $\mu(x)$.
3. **Step 3:** Multiply the ODE in standard form throughout by the I.F. [like in Eq. (4)].
4. **Step 4:** The LHS of the ODE obtained in **Step 3** is *always* equivalent to $\dfrac{d}{dx}\Big[\mu(x)y(x)\Big]$ (make sure you always check that this is true; this will verify that you have found the I.F. correctly).

   Rewrite the ODE so that the LHS is written in the reduced form [like in Eq. (7)].

5. **Step 5:** Integrate both sides wrt x to obtain the solution [like Eqs. (8) and (9)].

The idea behind the solution technique outlined in this section is that, given the appropriate integrating factor, it possible to express the LHS of Eq. (1) as the x-derivative of a product of functions; this technique makes it exceptionally easy to solve linear ODEs which at first sight may look tedious and complicated.

Now let us look at a specific example to demonstrate the ideas communicated in the solution process in the previous section.

Example

Find the general solution to the following ODE,

$$\frac{dy}{dx} - \frac{y}{x} = x^2. \tag{13}$$

Solution process

1. **Step 1:** The ODE is already given in standard form with $p(x) = -1/x$ and $f(x) = x^2$.
2. **Step 2:** Find the I.F. using $\mu(x) = e^{\int p(x)\,dx}$

   First find the integral of $p(x)$:

$$\begin{aligned} \int p(x)\,dx &= -\int \frac{1}{x}\,dx \\ &= -\ln x \\ &= \ln x^{-1} \end{aligned}$$

   Then, exponentiate the result to find $\mu(x)$:

$$\begin{aligned} \mu(x) &= e^{\ln x^{-1}} \\ &= \frac{1}{x} \end{aligned}$$

3. **Step 3:** Multiply (13) by $\mu(x) = \dfrac{1}{x}$:

$$\frac{1}{x}\frac{dy}{dx} - \frac{1}{x}\frac{y}{x} = \frac{1}{x}x^2.$$

Simplifying,

$$\frac{1}{x}\frac{dy}{dx} - \frac{y}{x^2} = x. \tag{14}$$

4. **Step 4:** The LHS of (14) is equivalent to $\dfrac{d}{dx}\left[\dfrac{1}{x}y(x)\right]$.

   Check:

$$\frac{d}{dx}\left[\frac{1}{x}y(x)\right] = \frac{1}{x}\frac{dy}{dx} + y\left(\frac{-1}{x^2}\right) \qquad \checkmark$$

   Rewrite (14) using the reduced form of the LHS:

$$\frac{d}{dx}\left[\frac{1}{x}y(x)\right] = x. \tag{15}$$

5. **Step 5:** Integrate both sides of (15) wrt x:

$$\begin{aligned} \frac{1}{x}y(x) &= \int x\,dx \\ &= \frac{x^2}{2} + k \end{aligned}$$

   The general solution is given by,

$$y(x) = \frac{x^3}{2} + kx. \tag{16}$$

4 Physical applications

In this course we will deal with so-called *mixing problems* (see Lectures 4 and 5) which typically involve a 'tank' into which a certain substance is added at a certain input rate $\left(\text{measured in units of } \dfrac{[\text{mass}]}{[\text{time}]}\right)$ and the substance leaves the system at a certain output rate $\left(\text{also measured in units of } \dfrac{[\text{mass}]}{[\text{time}]}\right)$.

The ODE describing this kind of process is given by a *material balance* as follows:

$$\frac{dy}{dt} = \text{Rate in} - \text{Rate out}, \tag{17}$$

where $y(t)$ denotes the amount of substance in the tank at any time t.

These types of problems are often described by first order, linear ODEs which may be solved by the technique outlined in these notes.