Solutions to Hour Exam I

The problems on this exam came in multiple versions and were scrambled, so I have just labelled them A, B, C,... here. The problems from your exam are here somewhere.

A. Consider the differential equation $y' = f(t,y)$, where $f(t,y) = f(t+2\pi, y)$ for all t and y. This says f is "2π-periodic in t". One of our 'population with harvesting' examples had this property. Assume that f and $\partial f/\partial y$ are continuous everywhere.

a) (5 pts.) Show that if $y_1(t)$ is a solution to $y' = f(t,y)$, then $y_2(t) = y_1(t+2\pi)$ is also a solution to $y' = f(t,y)$.

Here you just have to check that $y_2(t)$ satisfies the equation:

$$y_2'(t) = y_1'(t+2\pi) = f(t+2\pi, y_1(t+2\pi)) = f(t, y_1(t+2\pi)) = f(t, y_2(t)).$$

So, yes, it does satisfy the equation.

b) (5 pts.) Suppose that $y_1(t)$ from part a) happens to satisfy $y_1(0) = y_1(2\pi)$. Explain why $y_1(t)$ must equal $y_2(t)$ for all t in this case. This means that $y_1(t)$ is a "periodic solution" to the differential equation.

Note that the assumption that f and $\partial f/\partial y$ are continuous everywhere means that we can use the uniqueness theorem here. We have $y_1(0) = y_1(2\pi) = y_2(0)$, and by part a) $y_1(t)$ and $y_2(t)$ are both solutions to $y' = f(t,y)$. Since they take the same value at $t = 0$, they agree everywhere: $y_1(t) = y_2(t) = y_1(t+2\pi)$ for all t.

This problem mystified most of you. Somehow almost everyone thought that part a) had something to do with the existence and uniqueness theorem. As you can see, it does not. Many people claimed that $y_2(t) = y_1(t)$. This is not necessarily so: a simple example is $f(t,y) = 1 + \cos t$. That certainly satisfies $f(t,y) = f(t+2\pi, y)$, and $y' = f(t,y)$ has the solution $y_1(t) = t + \sin t$. Note that $y_2(t) = y_1(t+2\pi) = t + 2\pi + \sin t$ is also a solution, but $y_2(t) \neq y_1(t)$. Perhaps the difficulty was that people forgot that $y(t)$ is a solution of $y' = f(t,y)$ *means* $y'(t) = f(t, y(t))$ for all t in the interval of existence.

B. Explain why the equation

$$e^{A(x)}\frac{dy}{dx} + a(x)e^{A(x)}y = f(x)e^{A(x)},$$

where $A(x) = \int a(x)dx$, implies that

$$y(x) = e^{-A(x)}\int f(x)e^{A(x)}dx + Ce^{-A(x)} \text{ for some constant } C.$$

Well, $A(x) = \int a(x)dx$ means $A'(x) = a(x)$, and so by the chain rule $(e^{A(x)})' = a(x)e^{A(x)}$. Then by the product rule

$$e^{A(x)}\frac{dy}{dx} + a(x)e^{A(x)}y = \frac{d}{dx}(e^{A(x)}y).$$

So we can rewrite the equation in the statement of the problem as

$$\frac{d}{dx}(e^{A(x)}y) = f(x)e^{A(x)}.$$

That just says that $e^{A(x)}y$ is an anti-derivative of $f(x)e^{A(x)}$. Since any two anti-derivatives of a function differ by a constant, we have

$$e^{A(x)}y = \int f(x)e^{A(x)}dx + C.$$

Now, dividing both sides of the equation by $e^{A(x)}$ and using $1/e^{A(x)} = e^{-A(x)}$, gives the formula you are asked to explain.

There was a lot of confusion about what one was supposed to do on this problem. Many people said "it works because $e^{A(x)}$ is an integrating factor." That is true, but as you can see above what I was asking for was the explanation of why multiplying by $e^{A(x)}$ leads to the formula for the solution. I intended "Explain" to mean "take me through the steps in the derivation."

C. A baked potato is removed from a 470^o oven, and 5 minutes later its temperature is 300^o. Assuming Newton's law of cooling, if room temperature is 70^o, how long will it take for the potato to be cool enough to eat? Interpret "cool enough to eat" as 100^o. Since calculators are not allowed, leave your answer in logarithms. Remember that Newton's law of cooling says $dT/dt = -k(T - A)$, where T is temperature, t is time, A is surrounding temperature and k is a constant.

Here you are given $T(0) = 470$, $T(5) = 300$ and $A = 70$. So solving $dT/dt = -k(T - A)$ will get you

$$T(t) = A + (T(0) - A)e^{-kt} = 70 + 400e^{-kt},$$

and $T(5) = 300$ gives

$$300 = 70 + 400e^{-5k} \text{ which you should solve to get } k = \frac{1}{5}\ln(40/23).$$

Then you have to solve $T(t) = 100$ for t, i.e. solve $100 = 70 + 400e^{-kt}$. That gives

$$t = \frac{1}{k}\ln(40/3) = 5\frac{\ln(40/3)}{\ln(40/23)} = 23.4\,\text{min}\,.$$

That was a very hot potato. Note that 23.4 minutes is the total time it has been out of the oven.

D. Suppose that you know $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$. Explain why this fact is enough to make $F(x, y)$ of the form

$$F(x, y) = \int P(x, y)dx + g(y), \text{ have } \frac{\partial F}{\partial y} = Q(x, y)$$

for the right choice of $g(y)$.

Differentiating the given $F(x,y)$ with respect to y,

$$\frac{\partial F}{\partial y}(x,y) = \int \frac{\partial P}{\partial y}(x,y)dx + g'(y),$$

and, substituting $\partial Q/\partial x$ for $\partial P/\partial y$,

$$\frac{\partial F}{\partial y}(x,y) = \int \frac{\partial Q}{\partial x}(x,y)dx + g'(y).$$

Now $Q(x,y)$ is certainly an anti-derivative of $\partial Q/\partial x$ with respect to x (since $G(x,y)$ is an "anti-derivative with respect to x of $\partial Q/\partial x$" just means $(\partial/\partial x)G = \partial Q/\partial x$, and any other anti-derivative can only differ from it by a function of y alone. So we have

$$\frac{\partial F}{\partial y}(x,y) = Q(x,y) + h(y) + g'(y),$$

and, choosing $g(y)$ so that $g'(y) = -h(y)$, we get $\partial F/\partial y(x,y) = Q(x,y)$.

The comment following problem B applies here, too. The answer "this works because the form is exact" is true, but the derivation of why it works was what I intended to ask for.

E. A roast is removed from the oven when its temperature is 170^o, and 5 minutes later its temperature is 150^o. Assume Newton's law of cooling, and a room temperature is 70^o. How much longer should you allow the roast to cool, if you want to serve it when its temperature is 110^o? Since calculators are not allowed, leave your answer in logarithms. Remember that Newton's law of cooling says $dT/dt = -k(T-A)$, where T is temperature, t is time, A is surrounding temperature and k is a constant.

Here you are given $T(0) = 170$, $T(5) = 150$ and $A = 70$. So solving $dT/dt = -k(T-A)$ will get you

$$T(t) = A + (T(0) - A)e^{-kt} = 70 + 100e^{-kt},$$

and $T(5) = 150$ gives

$$150 = 70 + 100e^{-5k} \text{ which you should solve to get } k = \frac{1}{5}\ln(5/4).$$

Then you have to solve $T(t) = 110$ for t, i.e. solve $110 = 70 + 100e^{-kt}$. That gives

$$t = \frac{1}{k}\ln(5/2) = 5\frac{\ln(5/2)}{\ln(5/4)} = 20.5\,\text{min}\,.$$

Note that 20.5 minutes is the total time it has been out of the oven.

F. Consider the equation $y' = y(y-1)^2(y-2)^2$. This has no stable equilibrium, but the phase line will still allow you to make a sketch of what the solutions have to look like. Draw the phase line, and make a sketch.

Equilibria are $y(t) \equiv 0$, $y(t) \equiv 1$ and $y(t) \equiv 2$. Solutions are decreasing when $y(t) < 0$, increasing when $0 < y(t) < 1$, increasing when $1 < y(t) < 2$ and increasing when $2 < y(t)$. [Do not know how to draw phase line or sketch in TeX.]

G. Consider the equation $y' = -2y(y+1)^2(y-2)^2$. This has just one stable equilibrium, but the phase line will still allow you to make a sketch of what the solutions have to look like. Draw the phase line, and make a sketch.

Equilibria are $y(t) \equiv -1$, $y(t) \equiv 0$ and $y(t) \equiv 2$. Solutions are increasing when $y(t) < -1$, increasing when $-1 < y(t) < 0$, decreasing when $0 < y(t) < 2$ and decreasing when $2 < y(t)$. $y(t) \equiv 0$ is the stable equilibrium. [Do not know how to draw phase line or sketch in TeX.]

H. a) (7 pts.)The equation
$$\frac{dy}{dx} = \frac{x^2 - xy + y^2}{x^2}$$
is homogeneous (of degree 0). Find the general solution to this equation.

Using the substitution $y(x) = xv(x)$, this equation becomes
$$v + x\frac{dv}{dx} = 1 - v + v^2 \text{ or } x\frac{dv}{dx} = (v-1)^2$$
Note that $v(x) \equiv 1$ is a solution to that. To get the rest of the solutions of this separable equation
$$\int \frac{dv}{(v-1)^2} = \int \frac{dx}{x} + C.$$
So
$$\frac{-1}{v-1} = \ln|x| + C \text{ and } v = 1 - \frac{1}{\ln|x| + C}.$$
is the general solution. That makes the general solution
$$y = x - \frac{x}{\ln|x| + C} \text{ and } y = x.$$

Note that $y = x$ corresponds to $C = \infty$, as often happens for separable equations.

b) (3 pts.)Find the solution to the equation in part a) satisfying $y(2) = 2$. Be careful this is a solution that you might have missed in part a).

The answer is $y(x) = x$.

I. a) (7 pts.)The equation
$$\frac{dy}{dx} = \frac{x^2 + 3xy + y^2}{}$$

is homogeneous (of degree 0). Find the general solution to this equation.

Using the substitution $y(x) = xv(x)$, this equation becomes

$$v + x\frac{dv}{dx} = 1 + 3v + v^2 \text{ or } x\frac{dv}{dx} = (v+1)^2$$

Note that $v(x) \equiv -1$ is a solution to that. To get the rest of the solutions of this separable equation

$$\int \frac{dv}{(v+1)^2} = \int \frac{dx}{x} + C.$$

So

$$\frac{-1}{v+1} = \ln|x| + C \text{ and } v = -1 - \frac{1}{\ln|x| + C}.$$

is the general solution. That makes the general solution

$$y = -x - \frac{x}{\ln|x| + C} \text{ and } y = -x.$$

Note that $y = x$ corresponds to $C = \infty$, as often happens for separable equations.

b) (3 pts.) Find the solution to the equation in part a) satisfying $y(2) = -2$. Be careful this is a solution that you might have missed in part a).

The solution is $y(x) = -x$.