33B: Notes

Frederick Robinson

January 22, 2014

Contents

Frederick Robinson
frobinson@math.ucla.edu
math.ucla.edu/∼frobinson
Office Hours: —– (or by appointment) MS 3969

Format

Generally I'll spend the first part (at most 1/2) of class on exposition, recapping the material from class. The rest of class will be spent on working examples similar to those on your homework.

1 Introduction

Differential equations are just equations involving the derivatives of functions. There are partial differential equations - differential equations involving partial derivatives of functions, and ordinary differential equations - differential equations which just involve ordinary derivatives.

We can also classify differential equations by the order of the derivatives they involve:

Definition 1 (Order)**.** *The* order *of a differential equation is just the order of the highest order derivative that it involves.*

Differential equations don't specify just one solution, but rather a family of solutions. For instance consider $x' = 3$. This equation is solved by $x = 3t$ and $x = 3t + 5$. In general solutions are exactly those equations of the form $x = 3t + C$. We call this the *general solution.*

Definition 2 (General Solution)**.** *The* general solution *to an ordinary differential equation of order n is an equation containing n constants which describes all possible solutions to the equation.*

Often we're interested in one particular solution to a differential equation which satisfies constraints of the form $y^{(n)}(t) = y_0$. We call these constraints *initial conditions.* This sort of problem is called an *initial value problem.* It takes n conditions to fully specify a *particular solution* to a differential equation of order n.

Suppose in our example above that in addition to the differential equation $x' = 3$ we had specified that $x(5) = 4$, then we could find a particular solution by first finding the general solution $x = 3t + C$, and then plugging in the initial condition: $x(5) = 3 \cdot 5 + C = 4 \Rightarrow C = -1$. Our solution is then just $x(t) = 3t - 1$.

Examples

Give some examples of PDE / ODEs of different orders...

Differential equations come up all the time in practical applications. Here's a really simple example problem.

Example 1. *A ball is dropped from a height of 100m. It's position as a function of time satisfies the (second order linear) differential equation*

$$\frac{d^2x}{dt^2} = -10m/s^2$$

When does it hit the ground?

We should be able to solve this without any new techniques. Just integrate.

$$\frac{d^2x}{dt^2} = -10m/s^2 \Rightarrow \frac{dx}{dt} = -10t + C \Rightarrow x(t) = -5t^2 + Ct + D$$

Of course, this doesn't really specify the location of our ball. We need to use our initial conditions. Since we dropped ball, initial acceleration is $x'(0) = 0$, also we are given $x(0) = 100$. Substituting these into general solution we have:

$$x'(0) = -10t + C = C = 0$$

and

$$x(0) = -5t^2 + Ct + D = D = 100.$$

Thus, solve

$$0 = x(t) = -5t^2 + 100 \Rightarrow t = \pm\sqrt{20}.$$

Clearly our solution must be positive, so the answer is $2\sqrt{5}$ seconds.

Of course this is a sort of silly example since we can solve it by integration. We'll solve harder problems later using more sophisticated techniques.

2 Chapter 2

2.1 Section 1

If we have a purported solution $y(t)$ to some differential equation, we can check that it's actually valid by differentiating and checking to see that the derivatives satisfy whatever equation they are supposed to satisfy.

Definition 3 (Interval of Existence). *The* interval of existence *for an initial value problem is the largest interval on which a solution exists and satisfies the differential equation.*

For instance, the interval of existence for $x' = 3; x(0) = 0$ is $(-\infty, \infty)$ since the solution $x = 3t$ exists, and satisfies the given differential equation for all time.

Definition 4 (Normal form). *An order n differential equation is said to be in* normal form *if it is of the form $y^{(n)} = f(y^{(n-1)}, y^{(n-2)}, \ldots, y', y, t)$.*

Given a first order ODE in normal form $y' = f(y, t)$ we can draw a vector field describing solutions by drawing small lines of slope $f(y, t)$ at some collection of points (y, t).

Examples

Check

Example 2 (Exercise 2.1.3). *Check that $y' = -ty$ is solved by $y = Ce^{-(1/2)t^2}$*

Just differentiate and compare:

$$y' = -tCe^{-(1/2)t^2} = -ty$$

so it is a solution to the differential equation.

Interval of Existence

Example 3 (Exercise 2.1.13)**.** *Find the interval of existence for the differential equation* $y' = \frac{2}{3}t - \frac{5}{3t^2}$ *satisfying initial condition* $y(1) = 2$

We can solve by integration:

$$\begin{aligned}\frac{dy}{dt} &= \frac{2}{3}t - \frac{5}{3t^2} \\ dy &= \left(\frac{2}{3}t - \frac{5}{3t^2}\right) dt \\ y &= \frac{1}{3}t^2 + \frac{5}{3t} + C\end{aligned}$$

This is the general solution. If we further demand $y(1) = 2$

$$y(1) = \frac{1}{3} + \frac{5}{3} + C = 2 \Rightarrow C = 0$$

Therefore

$$y(t) = \frac{1}{3}t^2 + \frac{5}{3t}$$

is the particular solution we're after.

Now that we have the solution it's easy to determine interval of existence. There's an asymptote at $t = 0$, so that is the lower end of the interval. The function is continuous as $t \to \infty$, so there is no upper limit. Interval of existence is therefore $(0, \infty)$.

Normal Form

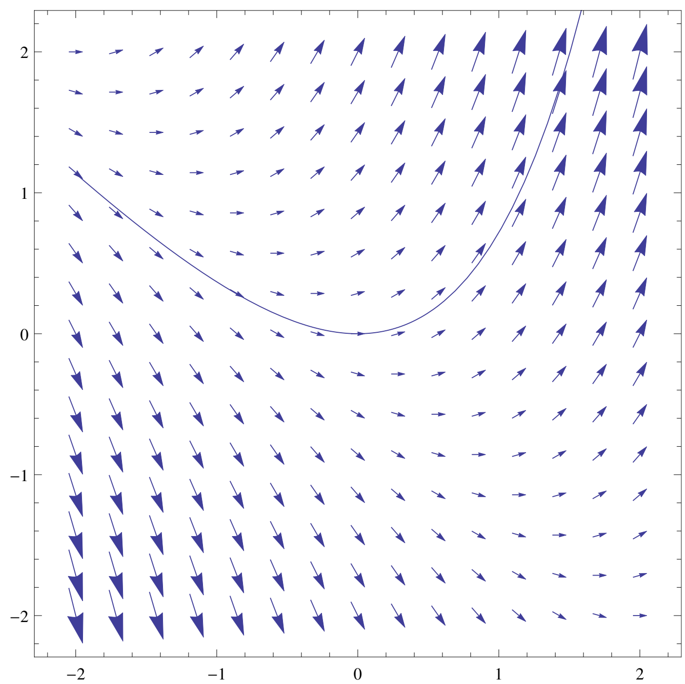
Example 4 (Exercise 2.1.1)**.** *Put* $\phi(t, y, y') = t^2y' + (1 + t)y = 0$ *in normal form.*

Just solve for y':

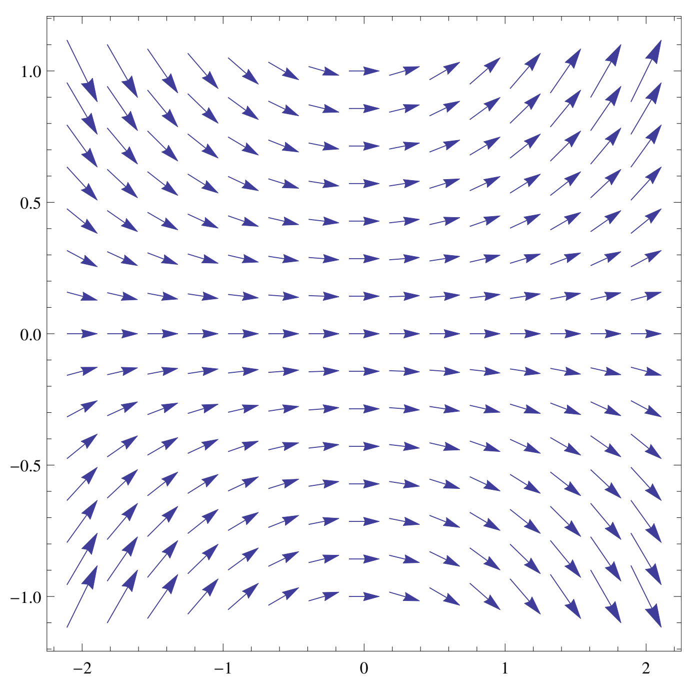
$$y' = -\frac{y(1 + t)}{t^2}$$

Vector Field

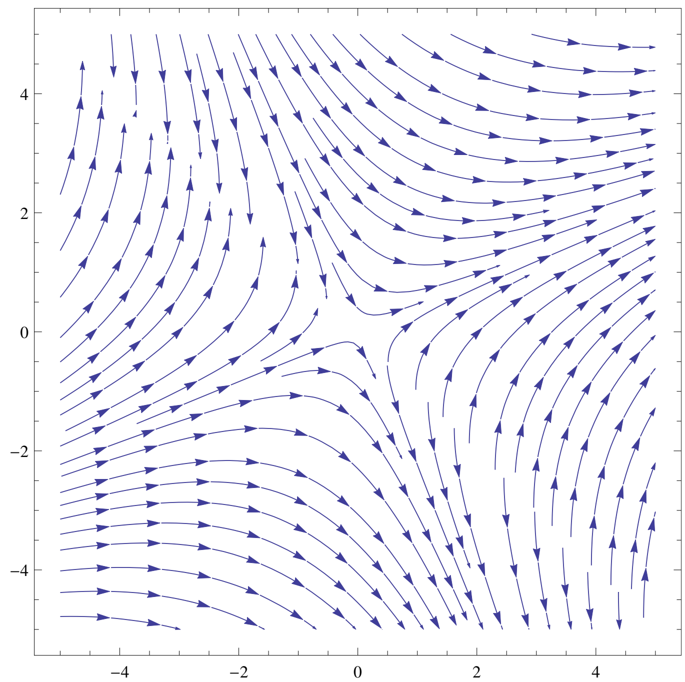
Example 5 (Exercise 2.1.17). *Draw vector field for $y' = y + t$.*



Example 6 (Exercise 2.1.19). *Draw vector field for $y' = t \tan y/2$.*



Example 7 (Exercise 2.1.19). *Draw vector field for* $y' = \frac{x-y}{x+y}$.



2.2 Section 2 - Separation of Variables

Examples

Example 8 (2.2.5). *Find the general solution for* $y' = y(x+1)$.

$$y = Ce^{1/2x^2+x}$$

Example 9 (2.2.13). *Find the exact solution to the IVP. Indicate interval of existence* $\frac{dy}{dx} = y/x$; $y(1) = -2$.

$y = -2x$, $x \in (0, \infty)$ since DE undefined at $x = 0$.

Example 10 (2.2.15). *Find the exact solution to the IVP. Indicate interval of existence* $\frac{dy}{dx} = \frac{\sin x}{y}$; $y(\pi/2) = 1$.

$$
\begin{aligned}
ydy &= \sin x dx \\
1/2y^2 &= -\cos x + C \\
y^2 &= -2\cos x + C \\
y &= \pm\sqrt{-2\cos x + C}
\end{aligned}
$$

initial condition means that

$$y(\pi/2) = 1 = \sqrt{-2\cos \pi/2 + C}$$

so $C = 1$ and the solution is

$$y = \sqrt{1 - 2\cos x}.$$

The interval of existence is

$$2\cos x < 1 \Leftrightarrow \pi/3 < x < 5\pi/3.$$

There is no equality because original equation is undefined.

2.3 Section 4 - Linear Equations

Definition 5 (Linear Homogeneous)**.** *A* linear homogeneous *differential equation is one of the form*

$$x'(t) = a(t)x(t)$$

for some function $a(t)$.

We can solve these by separation of variables:

$$
\begin{aligned}
\frac{1}{x}dx &= a(t)dt \\
\ln x &= \int a(t)dt + C \\
x &= Ce^{\int a(t)dt}
\end{aligned}
$$

Definition 6 (Linear Inhomogeneous)**.** *A* linear inhomogeneous *differential equation is one of the form*

$$x'(t) = f(t)x(t) + g(t)$$

for some functions $f(t), g(t)$.

We have two main techniques for solving these: Integrating Factor, and Variation of Parameter.

Recipe 1. *Integrating Factor:*

1. *Write* $x' - ax = f$.
2. *Multiply by* $u(t) = e^{-\int a(t)dt}$ *to get* $(ux)' = uf$.
3. *Integrate to get* $u(t)x(t) = \int u(t)f(t)dt + C$.
4. *Solve for* x.

Recipe 2. *Variation of Parameter:*

1. *Put* $y' = ay+f$, *and solve associated homogeneous equation* $y'_{hom} = ay_{hom}$.
2. *Substitute guess* $y = v(t)y_{hom}$ *into original equation and solve for* v.
3. *Write general solution* $y = v(t)y_{hom}$.

Examples

Variation of Parameter

Example 11. *Solve* $y' - 2y = t^2e^{2t}$ *by variation of paramenter.*

First solve the associated homogeneous: $y'_{hom} = 2y_{hom}$.

$$\begin{aligned}\frac{1}{y}dy &= 2dt\\ lny &= 2t + C\\ y_{hom} &= Ce^{2t}\end{aligned}$$

Now we 'guess' $y = v(t)y_{hom}$. Since $y' = v'y_{hom} + vy'_{hom} = v'e^{2t} + 2ve^{2t}$ we have:

$$v'e^{2t} + 2ve^{2t} - 2(ve^{2t}) = t^2e^{2t} \Rightarrow v' = t^2 \Rightarrow v = \frac{1}{3}t^3 + C$$

Therefore, $y = e^{2t}\left(\frac{1}{3}t^3 + C\right)$.

Example 12. *Solve* $y' + y/t = 3\cos(2t)$ *by variation of parameter.*

First solve $y_{hom} + y_{hom}/t = 0$

$$\begin{aligned}\int \frac{1}{y}dy &= -\int \frac{1}{t}dt\\ y_{hom} &= C/t\end{aligned}$$

So put $y = v(t)/t$. Compute $y' = v'y + vy' = v'/t - v/t^2$. Substituting back in to original equation gives us:

$$v'/t - v/t^2 + v/t^2 = 3\cos(2t) \Rightarrow v' = 3t\cos(2t)$$

We can compute $\int t\cos(2t)$ by parts. Taking $u = tdv = \cos 2t, du = 1, v = 1/2\sin 2t$ gives us

$$\int t\cos 2t = \frac{1}{2}t\sin 2t - \frac{1}{2}\int \sin 2t = \frac{t}{2}\sin 2t + \frac{1}{4}\cos(2t).$$

Thus, $v = \frac{3t}{2}\sin 2t + \frac{3}{4}\cos(2t) + C$ and the answer is

$$y = \frac{3}{2}\sin 2t + \frac{3}{4t}\cos(2t) + C/t$$

Integrating Factor

Example 13. *Solve $y' - 2y = t^2e^{2t}$ by integrating factor.*

Since this is already in the 'right' form, we can immediately compute the integrating factor

$$u = e^{-\int 2dt} = e^{-2t}.$$

Therefore,

$$ux = \int uf + C = \int t^2e^{2t}e^{-2t} + c = \frac{1}{3}t^3 + C \Rightarrow x = e^{2t}\left(\frac{1}{3}t^3 + C\right)$$

Example 14. *Solve $y' + y/t = 3\cos(2t)$ by integrating factor.*

Compute the integrating factor $a = e^{\int 1/t\ dt} = t$ and then write

$$\begin{aligned} ux &= \int uf + C \\ tx &= \int 3t\cos(2t) + C \end{aligned}$$

integrating by parts as in example 12 gives us

$$x = \frac{3}{2}\sin 2t + \frac{3}{4t}\cos(2t) + \frac{C}{t}$$

2.4 Section 5 - Mixing Problems

Mixing problems are a class of examples of differential equations involving mixing liquids. We assume 'perfect mixing.' The most important equation is

$$\frac{dx}{dt} = \text{Rate In} - \text{Rate Out} \tag{1}$$

It's often helpful to look at the units and use 'dimensional analysis' as sanity check.

Examples

Example 15. *A 50-gallon tank initially contains 20 gallons of pure water. Salt water solution with concentration of 1/2 lb/gal is added at a rate of 4 gal/min. A drain allows salt water to leave at 2 gal/min. How much salt is in the tank when it fills?*

Use equation 1 (note: units should be lb/gal):

$$\frac{dx}{dt} = \text{Rate In} - \text{Rate Out}$$

First Rate in:

$$RI = \frac{1}{2}\frac{lb}{gal} \cdot 4\frac{gal}{min} = 2\frac{lb}{min}$$

For rate out we have:

$$RO = 2\frac{gal}{min} \cdot \frac{x(t)}{v(t)}\frac{lb}{gal}$$

where $v(t) = 20 + 2t$ gives the volume of water in the tank as a function of time.

All together we have

$$\frac{dx}{dt} = 2 - \frac{x}{10+t} \Leftrightarrow x' + \frac{1}{10+t}x = 2$$

We can solve by integrating factor:

$$u = e^{\int \frac{1}{10+t}dt} = 10 + t$$

$$(10+t)x = 2\int (10+t)dt + C \Rightarrow x = \frac{t^2 + 20t + C}{t+10}$$

Using initial value $x(0) = 0$ (since we started with pure water) we get $C = 0$, and

$$x = \frac{t(t+20)}{t+10}.$$

To find when the tank fills solve $50 = v(t) = 20 + 2t \Rightarrow t = 15$. Substituting gives us $x(15) = 21$.