MATH 33B: DIFFERENTIAL EQUATIONS APRIL 2013

Example: autonomous equations & Stability.

Consider the modified logistic growth equation,

$$\frac{dy}{dt} = (a-by)y - h \qquad (1).$$

Which describes the rate of change of a population species (say, tuna) by taking into account harvesting effects.

Note: this is the example we looked at in class in Lecture 9.

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firstly, $a-by$ describes a logistic growth rate which takes into account that, as the population, $y(t)$ increases, the growth rate declines
(a, b are +ve constants).

then, $(a-by)y$ describes the logistic growth of the population.

h is a constant harvesting rate ($h>0$).

We are not interested in solving (1); we simply want to obtain some qualitative information on the solutions.

Step 1: Find critical pts

These occur at $f(y)=0$. From (1), $f(y)$ is the RHS of the ODE. Therefore:

$$f(y) = (a-by)y - h = 0$$

$$f(y) = ay - by^2 - h = 0 \qquad \text{(use quad. formula)}.$$

Using the quadratic formula,

$$y = \frac{-a \pm \sqrt{a^2 - 4bh}}{-2b}$$

which is,

$$y = \frac{a}{2b} \mp \frac{1}{2b}\sqrt{a^2 - 4bh} \qquad ②$$

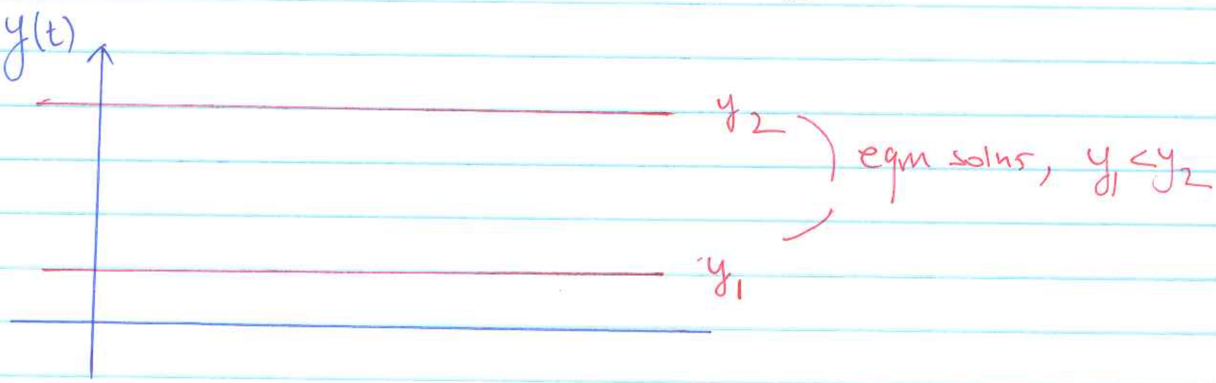
Real values to ② exist for $h \leq \frac{a^2}{4b}$.

Let us consider $h < a^2/4b$ first

In ② the 2 cps are $y_1 = \frac{a}{2b} - \frac{1}{2b}\sqrt{a^2-4bh}$

and $y_2 = \frac{a}{2b} + \frac{1}{2b}\sqrt{a^2-4bh}$

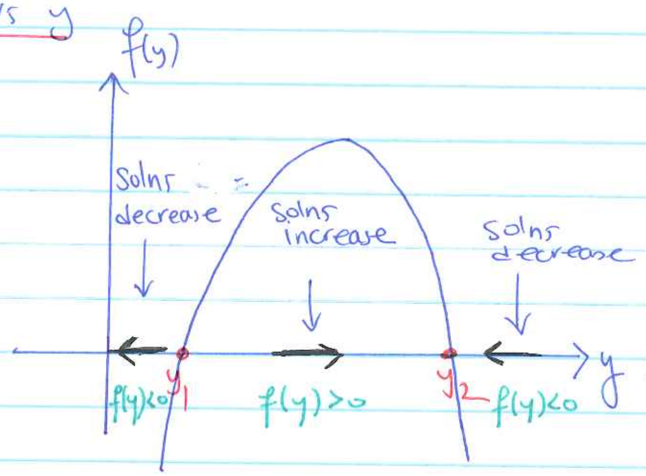
The corresponding eqm solns are $y_{1,2}(t) = \frac{a}{2b} \mp \frac{1}{2b}\sqrt{a^2-4bh}$ for all t.



Step 2: Find where $f(y)<0$, $f(y)>0$

This information will tell us where solns are decreasing and where they're increasing.

Plot $f(y)$ vs y



$\frac{dy}{dt} = f(y) < 0 \Rightarrow$ solns are decreasing

$\frac{dy}{dt} = f(y) > 0 \Rightarrow$ solns are increasing.

Now, $\leftarrow \underset{y_1}{\circ} \rightarrow$: the direction of the solns shows that all solns that start at $y=y_1^-$ and $y=y_1^+$ want to move away from $y=y_1$

Also, $\rightarrow \underset{y_2}{\circ} \leftarrow$: the direction of the solns shows that all solns that start at $y=y_2^-$ and $y=y_2^+$ want to move toward $y=y_2$

$\Rightarrow$ y_1 is asymptotically unstable while y_2 is asymptotically stable.

Show behavior of non-eqm solns

Physical interpretation

- If the initial population is $y < y_1$, then the plot on the bottom of page 3 implies that the species will become extinct no matter what.

- If the initial population is within $y_1 < y < y_2$, then the species population will increase and asymptotically approach the eqm soln, y_2.

- If the initial population is $y > y_2$, then the fish will begin to die (possibly due to lack of food in the ocean) until the population decreases and approaches $y = y_2$.

Note that results for $y<0$ have no physical significance but the solns exist mathematically.

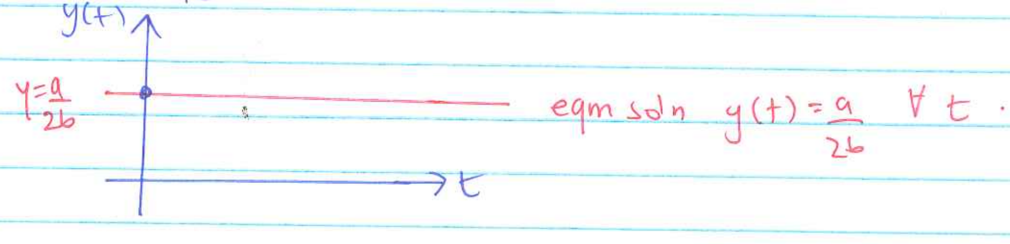
What happens as h increases?

As $h \to \frac{a^2}{4b}$, from ②, we have that $y \to \frac{a}{2b}$.

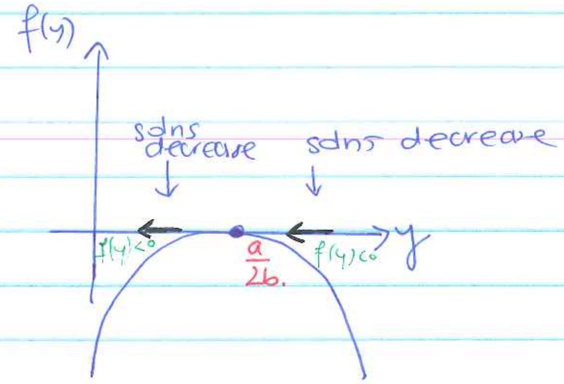
This means that as h increases, the 2 eqm solutions y_1 and y_2 get closer and closer to each other (y_1 increases while y_2 decreases) until they become the same eqm solution,

$$y_1 \to y_2 \to \frac{a}{2b}.$$

Hence, if $h = \frac{a^2}{4b}$, then we only have one c.p.



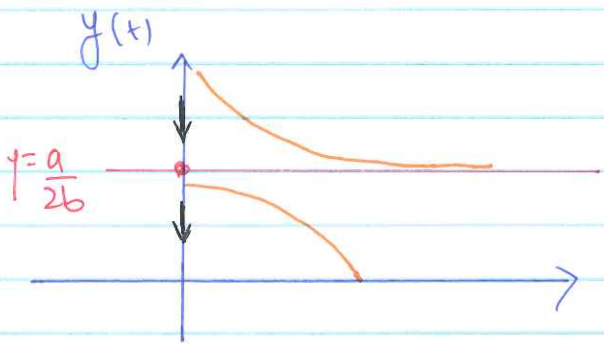
We then plot $\underline{f(y) \text{ vs } y}$ again to find where $f(y)<0$, $f(y)>0$.



The c.p $y=\frac{a}{2b}$ is $\longleftarrow \bullet \longleftarrow$ semi-stable

Sdlns that start at y^- move away from $y=\frac{a}{2b}$ but solns that start at y^+, move toward it.

Behavior of non-eqm solns



Physical interpretation

- If the initial population is below $y=\frac{a}{2b}$, the species will become extinct no matter what.

- if the population is above $y=\frac{a}{2b}$, the population will decline until it reaches the eqm sdn.

General notes

As h increased and approached the value of $\frac{a^2}{4b}$, the stable pt, y_2 went from being asymptotically stable to semi-stable. This phenomenon where stability is "lost" is called bifurcation.