1 Problem

A 5 kg object is suspended in a jar by a spring which exerts 5 N force when extende 2.5 m past its equilibrium. The jar is filled with a liquid chosen so as to critically damp the system. The object is displaced 1m from equilibrium, and released. Find t_0 so that $|x(t)| < .01$ for all $t > t_0$.

2 Solution

Recall the relevant differential equation:

$$mx'' + \mu x' + kx = f(t)$$

There is no forcing term. We are given $m = 5$. Compute k with Hooke's Law

$$F = -kx \Rightarrow 5 = k \cdot 2.5 \Rightarrow k = 2.$$

Since the system is critically damped we have

$$0 = \Delta = \mu^2 - 4mk \Rightarrow \mu = \sqrt{4 \cdot 5 \cdot 2} = 2\sqrt{10}$$

So our differential equation is just

$$5x'' + 2\sqrt{10}x' + 2x = 0$$

Solve the corresponding characteristic equation with the quadratic formula to get

$$\lambda = \frac{-2\sqrt{10}}{2 \cdot 5} = -\frac{\sqrt{10}}{5}$$

with multiplicity two. The solution is therefore

$$x(t) = Ae^{-\sqrt{10}t/5} + Bte^{-\sqrt{10}t/5}$$

with derivative

$$\begin{aligned} x'(t) &= -A\frac{\sqrt{10}}{5}e^{-\sqrt{10}t/5} + B\left(e^{-\sqrt{10}t/5} - t\frac{\sqrt{10}}{5}e^{-\sqrt{10}t/5}\right) \\ &= -A\frac{\sqrt{10}}{5}e^{-\sqrt{10}t/5} + Be^{-\sqrt{10}t/5}\left(1 - t\frac{\sqrt{10}}{5}\right) \end{aligned}$$

We have initial conditions $x(0) = 1/2, x'(0) = 0$, so

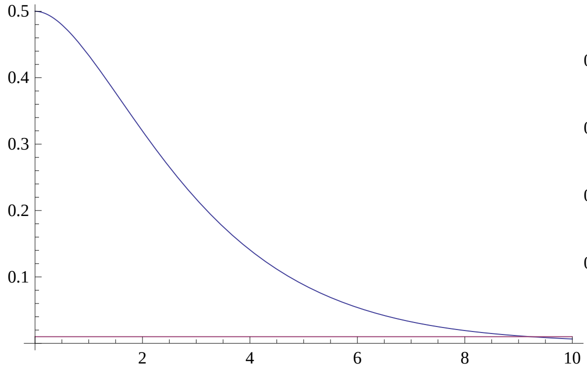
$$x(0) = 1/2 = A$$

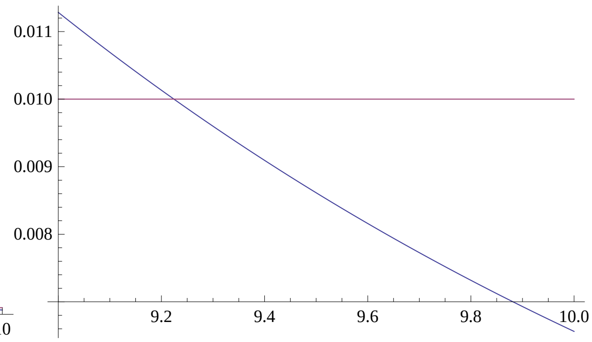
$$x'(0) = 0 = -A\frac{\sqrt{10}}{5} + B = -\frac{1}{2}\frac{\sqrt{10}}{5} + B \Rightarrow B = \frac{\sqrt{10}}{10}$$

and the final solution is

$$x(t) = \frac{1}{2}e^{-\sqrt{10}t/5} + \frac{\sqrt{10}}{10}te^{-\sqrt{10}t/5} = \frac{1}{10}e^{-\sqrt{10}t/5}\left(5 + \sqrt{10}t\right)$$

3 Analysis





Any candidate for t_0 must have $|x(t)| = .01$. So, since $x(t) > 0$ for $t > 0$ we solve

$$\frac{1}{10}e^{-\sqrt{10}t/5}\left(5+\sqrt{10}t\right) = .01$$

finding $t = -1.56942$, or $t = 9.22424$. We can discard the negative solution, so the only candidate is

$$t_0 = 9.22424$$

To prove that we have $|x(t)| < .01$ for $t > t_0$ it suffices to note that $x(t) > 0$ for $t > 0$, and

$$x'(t) = -\frac{1}{2}\frac{\sqrt{10}}{5}e^{-\sqrt{10}t/5} + \frac{\sqrt{10}}{10}e^{-\sqrt{10}t/5}\left(1 - t\frac{\sqrt{10}}{5}\right) < 0 \text{ for } t > 9.22424$$