HW5 - 1

Homework 5 - Worked-out solutions. May 2013

Problem 1

(i) $y'' - 3y' + 2y = 14\sin 2x - 18\cos 2x$. (1)

The G.S takes the form $y(x) = y_h + y_p$.

where y_h is a soln to the homogeneous eqn:

$$y'' - 3y' + 2y = 0. \quad (2)$$

and y_p is a particular soln to (1).

Find y_h

To determine y_h, we solve (2) using the method of the characteristic eqn:

$$y'' - 3y' + 2y = 0.$$

char. eqn. $m^2 - 3m + 2 = 0$
$(m-1)(m-2) = 0$

The roots are $m_1 = 1$, $m_2 = 2$ $\Rightarrow$ the homog. soln is:

$$\boxed{y_h = c_1 e^{x} + c_2 e^{2x}} \quad (3)$$

Find y_p

A particular soln should solve the nonhomogeneous eqn. Since the RHS takes a 'simple' form, we can use the method of undetermined coefficients.

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In this case, $f(x) = 14\sin 2x - 18\cos 2x$.

Whenever $f(x)$ is a sine or cosine or sine + cosine, we try:

$$y_p = A\sin 2x + B\cos 2x. \quad (4)$$

Note: since our guess for y_p doesn't clash with any terms in the homogeneous solution, then should work.

A & B are the undetermined coefficient which we need to determine.

Find undetermined coefficients.

y_p satisfies: $y_p'' - 3y_p' + 2y_p = 14\sin 2x - 18\cos 2x$ (5).

Diff. (4):

$y_p' = 2A\cos 2x - 2B\sin 2x$ & $y_p'' = -4A\sin 2x - 4B\cos 2x$.

Sub. y_p, y_p', y_p'' in (5)

$$(-4A\sin 2x - 4B\cos 2x) - 3(2A\cos 2x - 2B\sin 2x) + 2(A\sin 2x + B\cos 2x) = 14\sin 2x - 18\cos 2x.$$

Collect like terms together:

$$\sin 2x(-4A + 6B + 2A) + \cos 2x(-4B - 6A + 2B) = 14\sin 2x - 18\cos 2x. \quad (6)$$

From (6), we obtain 2 eqns to solve for A & B:

$-2A + 6B = 14$. (by comparing coefs of $\sin 2x$)

& $-2B - 6A = -18$ (by comparing coefs of $\cos 2x$)

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$$\begin{array}{r} \cancel{6}B - 2A = 14 \\ + \; -\cancel{6}B - 18A = -54 \\ \hline -20A = 40 \end{array}$$ (multiplied orig. eqn by 3 to eliminate B).

$\Rightarrow A = 2$.

$\therefore$ $6B = 14 + 2A = 18 \Rightarrow B = 3$.

In ④, $y_p = 2\sin 2x + 3\cos 2x$.

The G.S. to ① is $\boxed{y(x) = c_1 e^x + c_2 e^{2x} + 2\sin 2x + 3\cos 2x.}$

(ii) $y'' - 2y' + y = 6e^x$. ①

Again, we're looking for a soln in the form: $y(x) = y_h + y_p$.

Find y_h

y_h solves: $y'' - 2y' + y = 0 \longrightarrow m^2 - 2m + 1 = 0$.

$(m-1)^2 = 0$

and hence, $y_1 = e^x$

The only root is $m = 1$. For equal + repeated root, the G.S. to the homogeneous eqn is:

$$y_h = c_1 \underbrace{\boxed{e^x}}_{=y_1} + c_2 \underbrace{\boxed{x e^x}}_{=y_2}. \quad ②$$

multiply y_1 by x.

Find y_p.

Our guess for an exponential $f(x)$, would normally be $y_p = Ae^x$. Since e^x already appears in the homog. soln, we multiply our original guess by the independent variable (here, this is x) so the new guess is $y_p = Axe^x$. Since Axe^x also appears in y_h, we proceed by multiplying y_p by x again.

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Therefore, the appropriate form of the particular soln we are looking for, is:

$$y_p = Ax^2e^x \quad (3).$$

and A is the undetermined coefficient.

y_p should solve (1): $\quad y_p'' - 2y_p' + y_p =$.

Diff y_p (ie eq. (3)):

$y_p' = A(x^2e^x + 2xe^x)$

$y_p'' = A(x^2e^x + 2xe^x + 2xe^x + 2e^x)$

Sub. y_p, y_p', y_p'' in (1)

$$A\left[x^2e^x + 4xe^x + 2e^x - 2(x^2e^x + 2xe^x) + x^2e^x\right] = 6e^x.$$

Collect like-terms together:

$$e^x A\left[x^2(1-2+1) + x(4-4) + 2\right] = 6e^x.$$

$$\Rightarrow \quad 2A = 6 \Rightarrow A = 3.$$

In (3), $\quad y_p = 3x^2e^x$.

The G.S. to (1) is $\boxed{y(x) = c_1e^x + c_2xe^x + 3x^2e^x.}$

Problem 2.

$$y'' + k^2y = \sin bx. \quad (1).$$

HWS-3

The G.S to ① is $y(x) = y_h + y_p$.

The homog. soln is a soln to: $y'' + k^2 y = 0$.

Char. eqn: $m^2 + k^2 = 0 \Rightarrow m = \pm ki$

for complex roots the 2 fundamental soln we're looking for are $y_1 = \cos kx$ and $y_2 = \sin kx$

$\Rightarrow y_h = c_1 \cos kx + c_2 \sin kx$. ②.

We now seek a particular soln that solves ①.

(i) We use the method of undetermined coefficient again since $f(x) = \sin bx$ and it's one of the 'simple' forms we discussed. which work with this method.

Our guess therefore is $y_p = A\sin bx + B\cos bx$. ③.

We note that since $b \neq k$, then y_p does not clash with y_h; $\sin bx$ & $\sin kx$ are linearly independent if $b \neq k$.

Diff. ③ twice: $y_p' = Ab\cos bx - Bb\sin bx$

$y_p'' = -Ab^2 \sin bx - Bb^2 \cos bx$.

Sub. y_p, y_p'' in ①

$(-Ab^2 \sin bx - Bb^2 \cos bx) + k^2(A\sin bx + B\cos bx) = \sin bx$.

$\sin bx(-Ab^2 + Ak^2) + \cos bx(-Bb^2 + k^2 B) = \sin bx$

The 2 indep. eqns that help us solve for A & B are: $A(k^2 - b^2) = 1$
$B(k^2 - b^2) = 0$

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$\Rightarrow A = \dfrac{1}{k^2-b^2}$
Also, since $b \neq k$, $k^2-b^2 \neq 0 \Rightarrow B=0$

$\Rightarrow y_p = \dfrac{1}{k^2-b^2} \sin bx.$

The G.S. to ① is $\boxed{y(x) = C_1 \cos kx + C_2 \sin kx + \dfrac{1}{k^2-b^2} \sin bx.}$

(ii) Now, $b=k$. So our guess (Eq. ③) becomes:

$$y_p = A \sin kx + B \cos kx \quad ④$$

Since ④ clashes with the homog. soln. [i.e. ④ & ② are essentially the same function, they only differ through the constants], we need to multiply our guess by x:

$$y_p = x(A \sin kx + B \cos kx). \quad ⑤$$

Diff y_p twice to obtain y_p'': $\quad y_p' = x(Ak\cos kx - Bk\sin kx) + (A\sin kx + B\cos kx)$

$$y_p'' = x(-Ak^2\sin kx - Bk^2\cos kx) + (Ak\cos kx - Bk\sin kx) + (Ak\cos kx - Bk\sin kx)$$

Sub y_p, y_p'' in ① with $b=k$.

$$x(-Ak^2\sin kx - Bk^2\cos kx) + 2Ak\cos kx - 2Bk\sin kx + k^2x(A\sin kx + B\cos kx) = \sin kx$$

$$x\sin kx\cancelto{0}{(-Ak^2+Ak^2)} + x\cos kx\cancelto{0}{(-Bk^2+Bk^2)} + 2Ak\cos kx - 2Bk\sin kx = \sin kx$$

Indep. eqn for A & B: $2Ak\cos kx = 0 \quad \Rightarrow A = 0$

$-2Bk\sin kx = \sin kx \quad \Rightarrow B = \dfrac{1}{-2k}$

In ⑤, $y_p = \dfrac{-x}{2k} \cdot \cos kx.$

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The G.S. to ① when $b=k$ is:

$$\boxed{y(x) = C_1 \cos kx + C_2 \sin kx - \frac{x}{2k}\cos kx.}$$

Problem 3

$$x^2 y'' + xy' + \left(x^2 - \tfrac{1}{4}\right)y = 0.$$

In standard form, $\quad y'' + \frac{1}{x}y' + \left(\frac{x^2 - 1/4}{x^2}\right)y = 0. \quad$ ①.

(i) to show that $y_1 = x^{-1/2}\sin x$ is a soln to ①, we need to show that y_1 satisfies:

$$y_1'' + \frac{1}{x}y_1' + \left(1 - \frac{1}{4x^2}\right)y_1 = 0. \quad ②$$

Diff. y_1: $\quad y_1' = x^{-1/2}.\cos x + \sin x . \left(\frac{-1}{2}x^{-3/2}\right).$

$$y_1'' = -x^{-1/2}.\sin x + \cos x\left(\frac{-1}{2}x^{-3/2}\right) + \sin x\left(\frac{3}{4}x^{-5/2}\right) - \frac{1}{2}x^{-3/2}.\cos x$$

In ②, $\left[-\frac{\sin x}{x^{1/2}} - \frac{\cos x}{x^{3/2}} + \frac{3}{4}\frac{\sin x}{x^{5/2}}\right] + \frac{1}{x}\left(\frac{\cos x}{x^{1/2}} - \frac{1}{2}\frac{\sin x}{x^{3/2}}\right) + \left(1 - \frac{1}{4x^2}\right)\frac{\sin x}{x^{1/2}}$

$$= \sin x\left(\cancel{\frac{-1}{x^{1/2}}} + \cancel{\frac{3}{4x^{5/2}}} - \cancel{\frac{1}{2x^{5/2}}} + \cancel{\frac{1}{x^{1/2}}} - \cancel{\frac{1}{4x^{5/2}}}\right) + \cos x\left(\cancel{\frac{-1}{x^{3/2}}} + \cancel{\frac{1}{x^{3/2}}}\right)$$

$$= 0$$

$\Rightarrow$ $y_1 = x^{-1/2}\sin x$ is a soln to ①.

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(ii) The second soln is: $y_2 = v(x)y_1$ (3) where $v(x)$ is unknown.

To find $v(x)$ we can either use the formula derived in class [see Lecture 15 and Review 10]. or we can determine $v(x)$ through solving:

$$y_2'' + \frac{1}{x}y_2' + \left(1 - \frac{1}{4x^2}\right)y_2 = 0. \quad (4).$$

Let's do it from first principles first.

Since $y_2 = V(x)y_1(x)$ is made up of 3 fcts of x, let's just leave y_2 in its more arbitrary form, (i.e. $y_2 = v(x)y_1$)

Diff (3): $y_2' = v \cdot y_1' + y_1 v'$

$$y_2'' = vy_1'' + 2y_1'v' + y_1 v''.$$

Sub. y_2, y_2', y_2'' in (4)

$$(vy_1'' + 2y_1'v' + y_1v'') + \frac{1}{x}(vy_1' + y_1v') + \left(1 - \frac{1}{4x^2}\right)(v \cdot y_1) = 0.$$

Factorize v, v', v''

$$v\underbrace{\left[y_1'' + \frac{1}{x}y_1' + \left(1 - \frac{1}{4x^2}\right)y_1\right]}_{=0 \text{ since } y_1 \text{ solves } (1)} + v'\left(2y_1' + \frac{1}{x}y_1\right) + y_1v'' = 0.$$

We are left with: $v'\left(2y_1' + \frac{1}{x}y_1\right) = -y_1v''$

Rearranging: $$\frac{v''}{v'} = \frac{-2y_1' - \frac{1}{x}y_1}{y_1}$$

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$$\frac{v''}{v'} = -\frac{2y_1'}{y_1} - \frac{1}{x}.$$

Integrating both sides wrt x:

$\ln v' = -2\ln y_1 - \ln x.$

$$v' = \frac{1}{xy_1^2}.$$

Integ. once more:

$$v = \int \frac{1}{xy_1^2}\,dx. \quad (5)$$

Now, we can sub. $y_1 = x^{-1/2}.\sin x$ in (5):

$$V(x) = \int \frac{1}{x} \cdot \frac{1}{(x^{-1/2}\sin x)^2}\,dx.$$

$$= \int \frac{1}{\sin^2 x}\,dx.$$

To integrate (6), divide numerator & denominator by $\cos^2 x$:

$$V(x) = \int \frac{1/\cos^2 x}{\sin^2 x / \cos^2 x}\,dx = \int \frac{\sec^2 x}{\tan^2 x}\,dx.$$

Now use u-substitution: let $u = \tan x$

$$\frac{du}{dx} = \sec^2 x.$$

$$\therefore I = \int \frac{\sec^2 x}{\tan^2 x}\,dx = \int \frac{\cancel{\sec^2 x}}{u^2} \cdot \frac{du}{\cancel{\sec^2 x}} = \int \frac{1}{u^2}\,du = -u^{-1}$$

$$I = -\frac{1}{u} = -\frac{1}{\tan x}.$$

$\Rightarrow v(x) = -\cot x.$

Since $y_2 = v(x)y_1(x)$ then $y_2 = -\cot x \cdot x^{-1/2} \cdot \sin x$

$$\boxed{y_2 = -x^{-1/2}\cos x}$$

Alternatively,

the formula we derived for $v(x)$ is:

$$v(x) = \int \frac{1}{y_1^2} \cdot e^{-\int p(x)dx} dx.$$

$$\therefore e^{-\int p(x)dx} = e^{-\int \frac{1}{x}dx} = \frac{1}{x}$$

(note $p(x)$ corresponds to the coef. of y' when the ODE is in SF).

$$\Rightarrow v(x) = \int \frac{1}{(x^{-1/2}\sin x)^2} \cdot \frac{1}{x} dx$$

$$= \int \frac{1}{\sin^2 x} dx = -\cot x$$

$$\Rightarrow y_2 = -\cot x \cdot x^{-1/2} \sin x$$

$$\Rightarrow \boxed{y_2 = -x^{-1/2}\cos x}.$$

Problem 4

$t^2y'' - 2y = 3t^2 - 1, \quad t > 0.$

This is a nonconstant coef. ODE so we'll use the method of variations of parameters.

HW5-11

The $\mathcal{D}\varepsilon$ in SF is: $y'' - \frac{2}{t^2} y = 3 - \frac{1}{t^2}$ (1).

The homog. soln is: $y_h(t) = c_1 t^2 + c_2 t^{-1}$ (2) (note: the fundamental set y_1, y_2 are given in the problem statements).

To find the desired particular soln, replace c_1 with $v_1(t)$ and c_2 with $v_2(t)$ in (2):

$$y_p = v_1(t) t^2 + v_2(t) t^{-1}. \quad (3)$$

Where $v_1(t)$ & $v_2(t)$ are unknown fcn of t.

A particular soln to the nonhomog. problem should satisfy:

$$y_p'' - \frac{2}{t^2} y_p = 3 - \frac{1}{t^2} \quad (4).$$

Now, diff. (3) wrt t to obtain y_p''

$$y_p' = v_1 \cdot 2t + \underline{t^2 v_1'} + v_2 \cdot \frac{-1}{t^2} + \underline{t^{-1} v_2'}.$$

Now we claim the following relation is true: $t^2 v_1' + t^{-1} v_2' = 0$ (5)

(5) reduces the form for y_p' to: $y_p' = 2t v_1 - \frac{1}{t^2} v_2$. (6)

and from (6) $y_p'' = 2t v_1' + 2v_1 - \frac{1}{t^2} v_2' + \frac{2 v_2}{t^3}$ (7)

Sub. (3) & (7) in (4)

$$\left(2t v_1' + 2v_1 - \frac{1}{t^2} v_2' + \frac{2v_2}{t^3}\right) - \frac{2}{t^2}\left(v_1 t^2 + v_2 t^{-1}\right) = 3 - \frac{1}{t^2}$$

HW5-12

Factorize v_1, v_2:

$$v_1\left(\cancel{2-\frac{2t^2}{t^2}}\right) + v_2\left(\cancel{\frac{2}{t^3}-\frac{2t^{-1}}{t^2}}\right) + 2tv_1' - \frac{1}{t^2}v_2' = 3 - \frac{1}{t^2}$$

$$2tv_1' - \frac{1}{t^2}v_2' = 3 - \frac{1}{t^2}. \quad (8)$$

Eq. (8) gives an eqn for v_1', v_2'. Since we have two unknowns, we need another independent eqn for v_1' & v_2'.

Eq. (5) was something we set ourselves and we demanded that whatever v_1, v_2 are, they should satisfy that relation.

$\therefore$ 2 independent eqns: $2t^3v_1' - v_2' = 3t^2 - 1$. (9) (from (8))

& $t^3v_1' + v_2' = 0$ (10) (from (5))

Eqs. (9) & (10) help us solve for v_1', v_2':

from (10) $v_2' = -t^3v_1'$ (11)

in (9), $2t^3v_1' + t^3v_1' = 3t^2 - 1$

$$v_1' = \frac{3t^2-1}{3t^3} = \frac{1}{t} - \frac{1}{3t^3}.$$

$$\therefore v_1 = \int \frac{1}{t} - \frac{1}{3t^3}\,dt$$

$$\boxed{v_1 = \ln t - \frac{1}{3}\left(\frac{-1}{2}t^{-2}\right)} \quad (12)$$

(no need for a constant of integration since we're looking for a particular solution)

HW5-19

Using (12) in (11):

$$v_2' = -t^3\left(\frac{1}{t} - \frac{1}{3t^3}\right)$$

$$v_2 = -\int t^2 - \frac{1}{3}\, dt$$

$$\boxed{v_2 = -\frac{t^3}{3} + \frac{1}{3}t}$$

In (3), $y_p = v_1 t^2 + v_2 t^{-1}$

$$\therefore y_p = \left(\ln t + \frac{1}{6t^2}\right)t^2 + \left(\frac{t - t^3}{3}\right)\cdot\frac{1}{t}$$

$$= t^2 \ln t + \frac{1}{6} + \frac{1}{3} - \frac{t^2}{3}$$

$$y_p = t^2 \ln t + \frac{1}{2} - \frac{t^2}{3}$$

The G.S. is: $y(x) = c_1 t^2 + c_2 t^{-1} + t^2 \ln t + \frac{1}{2} - \frac{t^2}{3}$

or $\boxed{y(x) = \overline{c_1}\, t^2 + c_2 t^{-1} + t^2 \ln t + \frac{1}{2}}$

where $\overline{c_1} = \left(c_1 - \frac{1}{3}\right)$

Problem 5

$ty'' - (1+t)y' + y = 0$ (1) $\quad y_1 = 1+t;\ y_2 = e^t.$

HW5-14

In S.F. ① is: $y'' - \left(\frac{1+t}{t}\right)y' + \frac{1}{t}y = 0$. ②

y_1, y_2 must satisfy ② in order to be soln to the differential equation.

$y_1 = 1+t \Rightarrow y_1' = 1 \Rightarrow y_1'' = 0.$

In ②: $0 - \left(\frac{1}{t}+1\right) \cdot + \frac{1}{t}(1+t)$

$= \frac{1}{t}(-1+1) - 1 + \frac{t}{t}$

$= 0$ //

$y_2 = e^t \Rightarrow y_2' = e^t \Rightarrow y_2'' = e^t$

In ② $e^t - \left(\frac{1}{t}+1\right)e^t + \frac{1}{t}e^t$

$= e^t - e^t + \frac{1}{t}(e^t - e^t)$

$= 0$ //.

The G.S to ② is $\boxed{y_h = c_1(1+t) + c_2 e^t}$

To find a particular soln to the nonhomog. eqn, use:

$$y_p = v_1(t)(1+t) + v_2(t)e^t. \quad ③$$

Determine $v_1(t)$ & $v_2(t)$ by solving:

$$y_p'' - \left(\frac{1}{t}+1\right)y_p' + \frac{1}{t}y_p = t\,e^{2t} \quad ④$$

HW5-15

Diff ③ once:

$y_p' = v_1 + (1+t)v_1' + v_2 e^t + v_2' e^t$ ⑤

Indep. eqn: $(1+t)v_1' + e^t v_2' = 0$ (set the sum of the terms with derivatives of v_1', v_2 to 0). ⑥

⑥ reduces ⑤ to: $y_p' = v_1 + v_2 e^t$ ⑦

$\Rightarrow y_p'' = v_1' + v_2 e^t + v_2' e^t$ ⑧.

Sub. y_p, y_p', y_p'' in ④

$$\left(v_1' + v_2 e^t + v_2' e^t\right) - \left(\frac{1}{t}+1\right)\left(v_1 + v_2 e^t\right) + \frac{1}{t}\left[v_1(1+t) + v_2 e^t\right] = t\,e^{2t}.$$

Factorize v_1, v_2

$$v_1\left[\cancel{\frac{-1}{t} - 1 + \frac{1}{t}(1+t)}\right] + v_2\left[\cancel{e^t - \frac{e^t}{t} - e^t + \frac{e^t}{t}}\right] + v_1' + v_2' e^t = t\,e^{2t}$$

$$\Rightarrow v_1' + v_2' e^t = t\,e^{2t} \quad ⑨$$

The 2 independent equations for v_1' & v_2' are given by ⑥ & ⑨.

from ⑥, $v_2' = \frac{-(1+t)v_1'}{e^t}$

In ⑨, $v_1' - \frac{v_1'(1+t)e^t}{e^t} = t\,e^{2t}$

$= t\,e^{2t}$

$$v_1' = \frac{-t e^{2t}}{t} = -e^{2t} \Rightarrow \boxed{v_1 = -\frac{1}{2}e^{2t}}$$

HW5-16

Back in v_2' expression:

$$v_2' = -\frac{(1+t)\cdot -e^{2t}}{e^t} = (1+t)e^t \Rightarrow v_2 = \int (1+t)e^t\,dt$$

$$= \int e^t dt + \int e^t t\,dt$$

$$\boxed{v_2 = \cancel{e^t} + te^t - \cancel{e^t}}$$

Integ. by parts

The desired particular soln is:

$$y_p = -\frac{1}{2}e^{2t}(1+t) + te^t \cdot e^t$$

$$= -\frac{1}{2}e^{2t} - \frac{1}{2}te^{2t} + te^{2t}$$

$$y_p = \frac{1}{2}e^{2t}(t-1)$$

The G.S. to the nonhomogeneous eqn is:

$$\boxed{y(t) = c_1(1+t) + c_2 e^t + \frac{1}{2}e^{2t}(t-1)}$$