Math 33B Sample Midterm Questions

These are questions from previous years' midterms. They are intended to give you an idea of the types of questions to expect on the exam. They should not be considered as a comprehensive study guide. Note that the exam is 50 minutes long, so an appropriate number of questions will be asked (not this many).

In addition, I might ask some conceptual questions like if any equation is an ODE or PDE, linear vs. nonlinear, etc.

1. Give general solutions to the following differential equations:

   (a)
   $$z' = \frac{x^2}{cosz}$$

   (b)
   $$tu' = e^u$$

2. (a) Solve the following initial value problem
   $$t^2u' = e^u, \quad u(1) = 0$$

   (b) What is the interval of existence of this solution?

3. Indiana Jones has fallen into a deep pit containing 100m^3 of pure water. Corrosive acid is pouring into the pit at a rate of $3\text{m}^3/\text{min}$, while $2\text{m}^3/\text{min}$ of liquid drains from the pit.

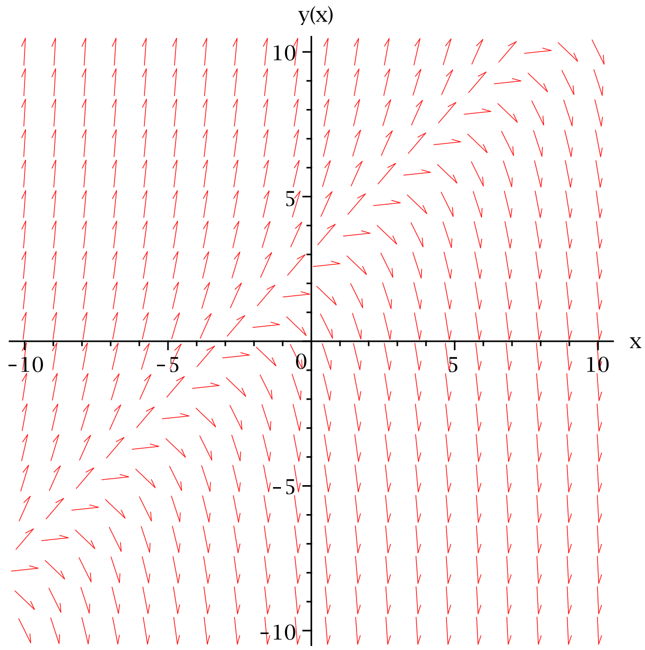
   (a) Let $A(t)$ be the amount of acid in the pit after t minutes. Write a differential equation expressing $A'(t)$ in terms of $A(t)$ and t (assuming instantaneous mixing).

   (b) Find a general solution of the equation in part (a).

   (c) Find the particular solution to the equation in part (a), taking into account information about the initial situation.

   (d) When the concentration of the acid (that is, $A(t)$ divided by $V(t)$) reaches $1/4$, Indiana Jones will die. How long does he have to escape? (That is, at what t does $A(t)/V(t) = 1/4$. You need not evaluate any numerical calculations; it suffices to solve the appropriate equation for t.)

4. Consider the following direction field of an unknown differential equation:



(a) Draw (on the direction field above) the solution through $(0, 0)$.

(b) Is this differential equation autonomous? How can you tell? (No credit without an explanation.)

5. Consider the differential equation $y' = f(t, y)$, where $f(t, y) = f(t + 2, y)$ for all t and y. This says f is "2π-periodic in t". One of our 'population with harvesting' examples had this property. Assume that f and $\partial f / \partial y$ are continuous everywhere.

(a) Show that if $y_1(t)$ is a solution to $y' = f(t, y)$, then $y_2(t) = y1(t + 2)$ is also a solution to $y' = f(t, y)$.

(b) Suppose that $y_1(t)$ from part (a) happens to satisfy $y_1(0) = y_1(2\pi)$. Explain why $y_1(t)$ must equal $y_2(t)$ for all t in this case. This means that $y_1(t)$ is a"periodic solution" to the differential equation.

6. Explain why the equation

$$e^{A(x)} \frac{dy}{dx} + a(x) e^{A(x)} y = f(x) e^{A(x)},$$

where $A(x) = \int a(x) dx$ implies that

$$y(x) = e^{-A(x)} \int f(x) e^{A(x)} dx + C e^{-A(x)} \text{ for some constant } C.$$

7. A baked potato is removed from a 470° oven, and 5 minutes later its temperature is 300°. Assuming Newtons law of cooling, if room temperature is 70°, how long will it take for the potato to be cool enough to eat? Interpret "cool enough to eat" as 100°. Since calculators are not allowed, leave your answer in logarithms. Remember that Newtons law of cooling says $dT/dt = -k(T - A)$, where T is temperature, t is time, A is surrounding temperature and k is a constant.

8. Suppose that you know $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$. Explain why this fact is enough to make $F(x, y)$ of the form
$$F(x, y) = \int P(x, y)dx + g(y),$$
have $\frac{\partial F}{\partial y} = Q(x, y)$ for the right choice of $g(y)$.

9. A roast is removed from the oven when its temperature is 170°, and 5 minutes later its temperature is 150°. Assume Newtons law of cooling, and a room temperature of 70°. How much longer should you allow the roast to cool if you want to serve it when its temperature is 110°? Since calculators are not allowed, leave your answer in logarithms. Remember that Newtons law of cooling says $dT/dt = -k(T - A)$, where T is temperature, t is time, A is surrounding temperature and k is a constant.

10. Consider the equation $y' = y(y - 1)^2(y - 2)^2$. This has no stable equilibrium, but the phase line will still allow you to make a sketch of what the solutions have to look like. Draw the phase line, and make a sketch.

11. (a) The equation
$$\frac{dy}{dx} = \frac{x^2 - xy + y^2}{x^2}$$
is homogeneous (of degree 0). Find the general solution to this equation.

(b) Find the solution to the equation in part (a) satisfying $y(2) = 2$. Be careful this is a solution that you might have missed in part (a).