We solve

$$\vec{y}' = A\vec{y} \quad \text{for} \quad A = \begin{pmatrix} 0 & 4 \\ -2 & -4 \end{pmatrix} \quad \text{satisfying} \quad y(0) = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

First, compute the characteristic polynomial

$$|A - \lambda I| = \lambda^2 + 4\lambda + 8$$

and use the quadratic equation to find the roots $\lambda = -2 + 2i$ and $\bar{\lambda} = -2 - 2i$. Now we must find an eigenvector corresponding to each eigenvalue. If v is an eigenvector for λ, then $\bar{v}$ is an eigenvector for $\bar{\lambda}$, so we can just compute

$$A\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \lambda \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \iff \begin{matrix} 4v_2 = (-2+2i)v_1 \\ -2v_1 - 4v_2 = (-2+2i)v_2 \end{matrix} \iff \begin{matrix} (2-2i)v_1 + 4v_2 = 0 \\ -2v_1 + (-2-2i)v_2 = 0 \end{matrix}$$

but since $\lambda = -2 + i$ is an eigenvalue, these two equations are equivalent. This may be a bit hard to see because of the complex numbers, so let's check:

$$\left(-\frac{1}{2} - \frac{i}{2}\right)((2-2i)v_1 + 4v_2) = \left(-\frac{1}{2} - \frac{i}{2}\right)(2-2i)v_1 + 4\left(-\frac{1}{2} - \frac{i}{2}\right)v_2 = -2v_1 + (-2-2i)v_2 = 0$$

Now all we have to do is come up with some eigenvector which satisfies one (both) of these equations. Let's choose

$$v = \begin{pmatrix} -1-i \\ 1 \end{pmatrix}$$

Recall that the general solution to this sort of equation is given by

$$y = C_1 e^{\lambda t} v + C_2 e^{\bar{\lambda} t}\bar{v} \rightsquigarrow C_1 e^{(-2+2i)t}\begin{pmatrix} -1-i \\ 1 \end{pmatrix} + C_2 e^{(-2-2i)t}\begin{pmatrix} -1+i \\ 1 \end{pmatrix}$$

So, in order to satisfy our initial conditions, we must have

$$y(0) = C_1 \begin{pmatrix} -1-i \\ 1 \end{pmatrix} + C_2 \begin{pmatrix} -1+i \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

so, $C_1 = -\frac{1}{2} - \frac{i}{2}, C_2 = -\frac{1}{2} + \frac{i}{2}$. Therefore, the solution is just

$$\left(-\frac{1}{2} - \frac{i}{2}\right) e^{(-2+2i)t}\begin{pmatrix} -1-i \\ 1 \end{pmatrix} + \left(-\frac{1}{2} + \frac{i}{2}\right) e^{(-2-2i)t}\begin{pmatrix} -1+i \\ 1 \end{pmatrix}$$

Let's find the real part

$$\begin{aligned}
y &= \Re\left[\left(-\frac{1}{2} - \frac{i}{2}\right) e^{(-2+2i)t}\begin{pmatrix} -1-i \\ 1 \end{pmatrix} + \left(-\frac{1}{2} + \frac{i}{2}\right) e^{(-2-2i)t}\begin{pmatrix} -1+i \\ 1 \end{pmatrix}\right] \\
&= e^{-2t}\Re\left[\left(-\frac{1}{2} - \frac{i}{2}\right) e^{2it}\begin{pmatrix} -1-i \\ 1 \end{pmatrix} + \left(-\frac{1}{2} + \frac{i}{2}\right) e^{-2it}\begin{pmatrix} -1+i \\ 1 \end{pmatrix}\right] \\
&= e^{-2t}\Re\left[e^{2it}\begin{pmatrix} \left(-\frac{1}{2} - \frac{i}{2}\right)(-1-i) \\ -\frac{1}{2} - \frac{i}{2} \end{pmatrix} + e^{-2it}\begin{pmatrix} \left(-\frac{1}{2} + \frac{i}{2}\right)(-1+i) \\ -\frac{1}{2} + \frac{i}{2} \end{pmatrix}\right] \\
&= e^{-2t}\Re\left[(\cos 2t + i\sin 2t)\begin{pmatrix} i \\ -\frac{1}{2} - \frac{i}{2} \end{pmatrix} + (\cos(-2t) + i\sin(-2t))\begin{pmatrix} -i \\ -\frac{1}{2} + \frac{i}{2} \end{pmatrix}\right] \\
&= e^{-2t}\left(\begin{pmatrix} -\sin 2t \\ -\frac{1}{2}\cos 2t + \frac{1}{2}\sin 2t \end{pmatrix} + \begin{pmatrix} \sin(-2t) \\ -\frac{1}{2}\cos(-2t) - \frac{1}{2}\sin(-2t) \end{pmatrix}\right) \\
&= e^{-2t}\begin{pmatrix} -2\sin 2t \\ -\cos 2t + \sin 2t \end{pmatrix}
\end{aligned}$$