33B: Notes

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Contents

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Format

Generally I'll spend the first part (at most 1/2) of class on exposition, recapping the material from class. The rest of class will be spent on working examples similar to those on your homework.

1 Introduction

Differential equations are just equations involving the derivatives of functions. There are partial differential equations - differential equations involving partial derivatives of functions, and ordinary differential equations - differential equations which just involve ordinary derivatives.

We can also classify differential equations by the order of the derivatives they involve:

Definition 1 (Order)**.** *The* order *of a differential equation is just the order of the highest order derivative that it involves.*

Differential equations don't specify just one solution, but rather a family of solutions. For instance consider $x' = 3$. This equation is solved by $x = 3t$ and $x = 3t + 5$. In general solutions are exactly those equations of the form $x = 3t + C$. We call this the *general solution.*

Definition 2 (General Solution)**.** *The* general solution *to an ordinary differential equation of order n is an equation containing n constants which describes all possible solutions to the equation.*

Often we're interested in one particular solution to a differential equation which satisfies constraints of the form $y^{(n)}(t) = y_0$. We call these constraints *initial conditions.* This sort of problem is called an *initial value problem.* It takes n conditions to fully specify a *particular solution* to a differential equation of order n.

Suppose in our example above that in addition to the differential equation $x' = 3$ we had specified that $x(5) = 4$, then we could find a particular solution by first finding the general solution $x = 3t + C$, and then plugging in the initial condition: $x(5) = 3 \cdot 5 + C = 4 \Rightarrow C = -1$. Our solution is then just $x(t) = 3t - 1$.

Examples

Give some examples of PDE / ODEs of different orders...

Differential equations come up all the time in practical applications. Here's a really simple example problem.

Example 1. *A ball is dropped from a height of 100m. It's position as a function of time satisfies the (second order linear) differential equation*

$$\frac{d^2x}{dt^2} = -10m/s^2$$

When does it hit the ground?

We should be able to solve this without any new techniques. Just integrate.

$$\frac{d^2x}{dt^2} = -10m/s^2 \Rightarrow \frac{dx}{dt} = -10t + C \Rightarrow x(t) = -5t^2 + Ct + D$$

Of course, this doesn't really specify the location of our ball. We need to use our initial conditions. Since we dropped ball, initial acceleration is $x'(0) = 0$, also we are given $x(0) = 100$. Substituting these into general solution we have:

$$x'(0) = -10t + C = C = 0$$

and

$$x(0) = -5t^2 + Ct + D = D = 100.$$

Thus, solve

$$0 = x(t) = -5t^2 + 100 \Rightarrow t = \pm\sqrt{20}.$$

Clearly our solution must be positive, so the answer is $2\sqrt{5}$ seconds.

Of course this is a sort of silly example since we can solve it by integration. We'll solve harder problems later using more sophisticated techniques.

2 Chapter 2

2.1 Section 1

If we have a purported solution $y(t)$ to some differential equation, we can check that it's actually valid by differentiating and checking to see that the derivatives satisfy whatever equation they are supposed to satisfy.

Definition 3 (Interval of Existence)**.** *The* interval of existence *for an initial value problem is the largest interval on which a solution exists and satisfies the differential equation.*

For instance, the interval of existence for $x' = 3; x(0) = 0$ is $(-\infty, \infty)$ since the solution $x = 3t$ exists, and satisfies the given differential equation for all time.

Definition 4 (Normal form)**.** *An order n differential equation is said to be in* normal form *if it is of the form $y^{(n)} = f(y^{(n-1)}, y^{(n-2)}, \ldots, y', y, t)$.*

Given a first order ODE in normal form $y' = f(y,t)$ we can draw a vector field describing solutions by drawing small lines of slope $f(y,t)$ at some collection of points (y,t).

Examples

Check

Example 2 (Exercise 2.1.3)**.** *Check that $y' = -ty$ is solved by $y = Ce^{-(1/2)t^2}$*

Just differentiate and compare:

$$y' = -tCe^{-(1/2)t^2} = -ty$$

so it is a solution to the differential equation.

Interval of Existence

Example 3 (Exercise 2.1.13)**.** *Find the interval of existence for the differential equation $y' = \frac{2}{3}t - \frac{5}{3t^2}$ satisfying initial condition $y(1) = 2$*

We can solve by integration:

$$\frac{dy}{dt} = \frac{2}{3}t - \frac{5}{3t^2}$$
$$dy = \left(\frac{2}{3}t - \frac{5}{3t^2}\right) dt$$
$$y = \frac{1}{3}t^2 + \frac{5}{3t} + C$$

This is the general solution. If we further demand $y(1) = 2$

$$y(1) = \frac{1}{3} + \frac{5}{3} + C = 2 \Rightarrow C = 0$$

Therefore

$$y(t) = \frac{1}{3}t^2 + \frac{5}{3t}$$

is the particular solution we're after.

Now that we have the solution it's easy to determine interval of existence. There's an asymptote at $t = 0$, so that is the lower end of the interval. The function is continuous as $t \to \infty$, so there is no upper limit. Interval of existence is therefore $(0, \infty)$.

Normal Form

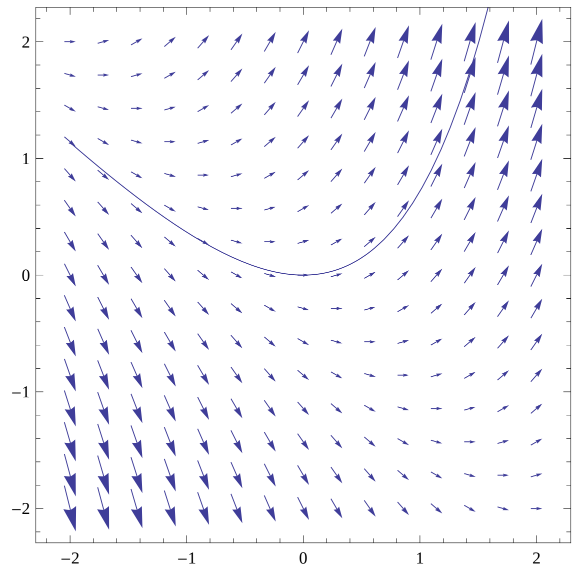
Example 4 (Exercise 2.1.1)**.** *Put $\phi(t, y, y') = t^2y' + (1+t)y = 0$ in normal form.*

Just solve for y':

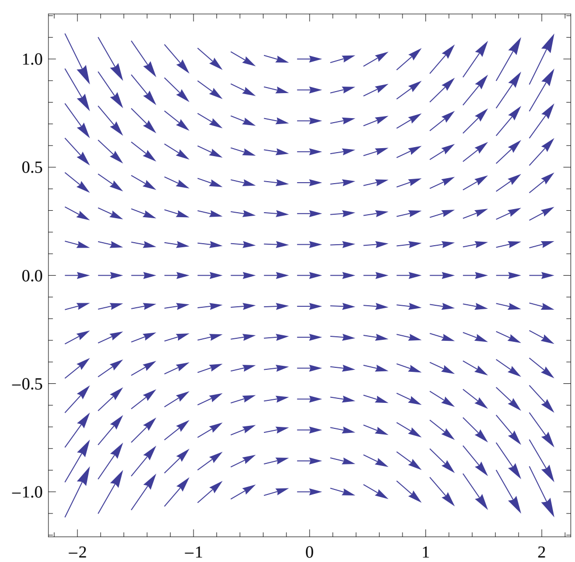
$$y' = -\frac{y(1+t)}{t^2}$$

Vector Field

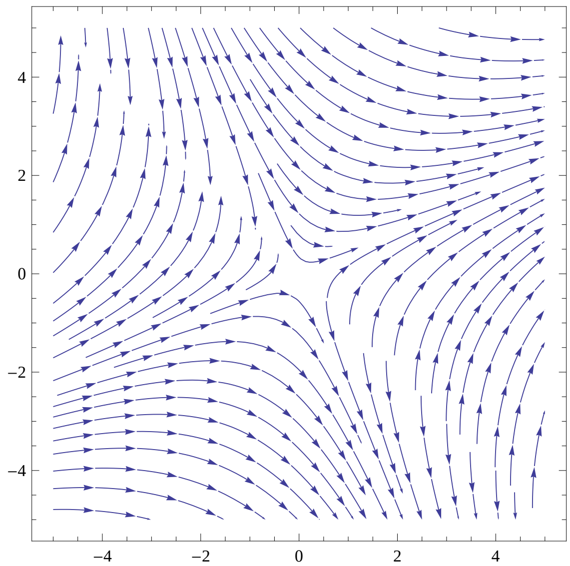
Example 5 (Exercise 2.1.17)**.** *Draw vector field for* $y' = y + t$*.*



Example 6 (Exercise 2.1.19)**.** *Draw vector field for* $y' = t \tan y/2$*.*



Example 7 (Exercise 2.1.19)**.** *Draw vector field for* $y' = \frac{x-y}{x+y}$*.*



2.2 Section 2 - Separation of Variables

Examples

Example 8 (2.2.5)**.** *Find the general solution for* $y' = y(x+1)$*.*

$$y = Ce^{1/2x^2+x}$$

Example 9 (2.2.13)**.** *Find the exact solution to the IVP. Indicate interval of existence* $\frac{dy}{dx} = y/x$*;* $y(1) = -2$*.*

$y = -2x$, $x \in (0, \infty)$ since DE undefined at $x = 0$.

Example 10 (2.2.15)**.** *Find the exact solution to the IVP. Indicate interval of existence* $\frac{dy}{dx} = \frac{\sin x}{y}$*;* $y(\pi/2) = 1$*.*

$$\begin{aligned} ydy &= \sin x dx \\ 1/2y^2 &= -\cos x + C \\ y^2 &= -2\cos x + C \\ y &= \pm\sqrt{-2\cos x + C} \end{aligned}$$

initial condition means that

$$y(\pi/2) = 1 = \sqrt{-2\cos \pi/2 + C}$$

so $C = 1$ and the solution is

$$y = \sqrt{1 - 2\cos x}.$$

The interval of existence is

$$2\cos x < 1 \Leftrightarrow \pi/3 < x < 5\pi/3.$$

There is no equality because original equation is undefined.

2.4 Section 4 - Linear Equations

Definition 5 (Linear Homogeneous)**.** *A* linear homogeneous *differential equation is one of the form*

$$x'(t) = a(t)x(t)$$

for some function $a(t)$.

We can solve these by separation of variables:

$$\begin{aligned} \frac{1}{x}dx &= a(t)dt \\ \ln x &= \int a(t)dt + C \\ x &= Ce^{\int a(t)dt} \end{aligned}$$

Definition 6 (Linear Inhomogeneous)**.** *A* linear inhomogeneous *differential equation is one of the form*

$$x'(t) = f(t)x(t) + g(t)$$

for some functions $f(t), g(t)$.

We have two main techniques for solving these: Integrating Factor, and Variation of Parameter.

Recipe 1. *Integrating Factor:*

1. *Write $x' - ax = f$.*
2. *Multiply by $u(t) = e^{-\int a(t)dt}$ to get $(ux)' = uf$.*
3. *Integrate to get $u(t)x(t) = \int u(t)f(t)dt + C$.*
4. *Solve for x.*

Recipe 2. *Variation of Parameter:*

1. *Put $y' = ay + f$, and solve associated homogeneous equation $y'_{hom} = ay_{hom}$.*
2. *Substitute guess $y = v(t)y_{hom}$ into original equation and solve for v.*
3. *Write general solution $y = v(t)y_{hom}$.*

Examples

Variation of Parameter

Example 11. *Solve $y' - 2y = t^2e^{2t}$ by variation of paramenter.*

First solve the associated homogeneous: $y'_{hom} = 2y_{hom}$.

$$\begin{aligned} \frac{1}{y}dy &= 2dt \\ lny &= 2t + C \\ y_{hom} &= Ce^{2t} \end{aligned}$$

Now we 'guess' $y = v(t)y_{hom}$. Since $y' = v'y_{hom} + vy'_{hom} = v'e^{2t} + 2ve^{2t}$ we have:

$$v'e^{2t} + 2ve^{2t} - 2(ve^{2t}) = t^2e^{2t} \Rightarrow v' = t^2 \Rightarrow v = \frac{1}{3}t^3 + C$$

Therefore, $y = e^{2t}\left(\frac{1}{3}t^3 + C\right)$.

Example 12. *Solve $y' + y/t = 3\cos(2t)$ by variation of parameter.*

First solve $y_{hom} + y_{hom}/t = 0$

$$\int \frac{1}{y} dy = -\int \frac{1}{t} dt$$
$$y_{hom} = C/t$$

So put $y = v(t)/t$. Compute $y' = v'y + vy' = v'/t - v/t^2$. Substituting back in to original equation gives us:

$$v'/t - v/t^2 + v/t^2 = 3\cos(2t) \Rightarrow v' = 3t\cos(2t)$$

We can compute $\int t\cos(2t)$ by parts. Taking $u = tdv = \cos 2t, du = 1, v = 1/2\sin 2t$ gives us

$$\int t\cos 2t = \frac{1}{2} t\sin 2t - \frac{1}{2}\int \sin 2t = \frac{t}{2}\sin 2t + \frac{1}{4}\cos(2t).$$

Thus, $v = \frac{3t}{2}\sin 2t + \frac{3}{4}\cos(2t) + C$ and the answer is

$$y = \frac{3}{2}\sin 2t + \frac{3}{4t}\cos(2t) + C/t$$

Integrating Factor

Example 13. *Solve $y' - 2y = t^2e^{2t}$ by integrating factor.*

Since this is already in the 'right' form, we can immediately compute the integrating factor

$$u = e^{-\int 2dt} = e^{-2t}.$$

Therefore,

$$ux = \int uf + C = \int t^2e^{2t}e^{-2t} + c = \frac{1}{3}t^3 + C \Rightarrow x = e^{2t}\left(\frac{1}{3}t^3 + C\right)$$

Example 14. *Solve $y' + y/t = 3\cos(2t)$ by integrating factor.*

Compute the integrating factor $a = e^{\int 1/t\ dt} = t$ and then write

$$ux = \int uf + C$$
$$tx = \int 3t\cos(2t) + C$$

integrating by parts as in example 12 gives us

$$x = \frac{3}{2}\sin 2t + \frac{3}{4t}\cos(2t) + \frac{C}{t}$$

2.5 Section 5 - Mixing Problems

Mixing problems are a class of examples of differential equations involving mixing liquids. We assume 'perfect mixing.' The most important equation is

$$\frac{dx}{dt} = \text{Rate In} - \text{Rate Out} \tag{1}$$

It's often helpful to look at the units and use 'dimensional analysis' as sanity check.

Examples

Example 15. *A 50-gallon tank initially contains 20 gallons of pure water. Salt water solution with concentration of 1/2 lb/gal is added at a rate of 4 gal/min. A drain allows salt water to leave at 2 gal/min. How much salt is in the tank when it fills?*

Use equation 1 (note: units should be lb/gal):

$$\frac{dx}{dt} = \text{Rate In} - \text{Rate Out}$$

First Rate in:

$$RI = \frac{1}{2}\frac{lb}{gal} \cdot 4\frac{gal}{min} = 2\frac{lb}{min}$$

For rate out we have:

$$RO = 2\frac{gal}{min} \cdot \frac{x(t)}{v(t)}\frac{lb}{gal}$$

where $v(t) = 20 + 2t$ gives the volume of water in the tank as a function of time.

All together we have

$$\frac{dx}{dt} = 2 - \frac{x}{10+t} \Leftrightarrow x' + \frac{1}{10+t}x = 2$$

We can solve by integrating factor:

$$u = e^{\int \frac{1}{10+t}dt} = 10 + t$$

$$(10+t)x = 2\int (10+t)dt + C \Rightarrow x = \frac{t^2 + 20t + C}{t+10}$$

Using initial value $x(0) = 0$ (since we started with pure water) we get $C = 0$, and

$$x = \frac{t(t+20)}{t+10}.$$

To find when the tank fills solve $50 = v(t) = 20 + 2t \Rightarrow t = 15$. Substituting gives us $x(15) = 21$.

2.6 Section 6 - Exact Differential Equations

Consider a differential equation

$$Pdx + Qdy = 0$$

Definition 7 (Exact)**.** *We call such a differential equation* exact *if*

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}.$$

Given an exact differential equation there exists some $\Psi(x, y)$ with

$$\frac{\partial \Psi}{\partial y} = Q, \frac{\partial \Psi}{\partial x} = P.$$

Solutions to the ODE are then given implicity by $\Psi = c$.

Sometimes we're given an eqution which is not exact, but which can be transformed into an exact equation by multiplying by an integrating factor. There are a lot of poitential choices of integrating factor, and in general finding an integrating factor is as hard as solving the original equation. However, we have formulas for finding integrating factors which depend on only one variable.

Recipe 3 (Integrating factor a funciton of x)**.** *Compute*

$$h = \frac{1}{Q}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right).$$

If h is a function of x alone then $\mu = e^{\int h dx}$ is an integrating factor.

Recipe 4 (Integrating factor a funciton of y)**.** *Compute*

$$g = \frac{1}{P}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right).$$

If g is a function of x alone then $\mu = e^{-\int g dy}$ is an integrating factor.

ADD HOMOGENEOUS HERE!

Examples

Example 16. *Is the equation*

$$(y\cos x + 2xe^y) + (\sin x + x^2e^y - 1)y' = 0$$

exact? If so, solve.

First we verify

$$\frac{\partial P}{\partial y} = \cos x + 2xe^y = \frac{\partial Q}{\partial x},$$

so the equation is exact.

Now, we need to find a Ψ with

$$\frac{\partial \Psi}{\partial y} = Q, \quad \frac{\partial \Psi}{\partial x} = P.$$

Thus integrate

$$\begin{aligned}\int \partial\Psi &= \int Q\partial y \\ \Psi &= \int \sin x + x^2e^y - 1\partial y \\ &= y\sin x + x^2e^y - y + h(x)\end{aligned}$$

for some unknown function h of x. Of course, we can find x by using the other equality:

$$\frac{\partial \Psi}{\partial x} = y\cos x + 2xe^y + h'(x) = y\cos x + 2xe^y = P.$$

So, $h'(x) = 0$ and $h(x) = C$ is a constant. Our solutions are therefore given implicity by

$$y\sin x + x^2e^y - y = C.$$

Example 17. *Solve the exact equation* $(2x+3) + (2y-2)y' = 0$.

$x^2 + 3x + y^2 - 2y = c$

Example 18. *Solve the exact equation* $(3x^2 - 2xy + 2)dx + (6y^2 - x^2 + 3)dy = 0$.

$x^3 - x^2y + 2x + 2y^3 + 3y = C$

Example 19. *Is the following equation exact?* $(x\log y + xy)dx + (y\log x + xy)dy = 0$.

no

Example 20. *Find b such that the following equation is exact, then solve.* $(xy^2 + bx^2y)dx + (x+y)x^2dy = 0$

$b = 3$ and $x^2y^2 + 2x^3y = C$

Integrating Factor

Example 21. *Show that the following equation is not exact then solve using the integrating factor* $\mu = 1/xy^3$

$$x^2y^3 + x(1 + y^2)y' = 0$$

$x^2 + 2\log|y| - y^{-2} = C$ and $y = 0$

Example 22. *Find an integrating factor, and solve.*

$$(3x^2y + 2xy + y^3)dx + (x^2 + y^2)dy = 0$$

$\mu = e^{3x}$ solution:$C = e^{3x}(3x^2y + y^3)$

Example 23. *Find an integrating factor, and solve.*

$$y' = e^{2x} + y - 1$$

$\mu = e^{-x}$ and $y = Ce^x + 1 + e^{2x}$

Example 24. *Solve using the integrating factor* $\mu = e^{2x}/y$

$$ydx + (2xy - e^{-2y})dy = 0$$

$y = 0$ and $xe^{2y} - \log y = C$

Homogeneous

ADD HOMOGENEOUS EXAMPLES

2.7 Section 7 - Existence and Uniqueness of Solutions

Theorem 1 (Existence)**.** *Consider the initial value problem*

$$x' = f(t, x) \quad f(t_0) = x_0.$$

If f *is defined and continuous near* (t_0, x_0) *then a solution to the IVP exists, near* t_0*.*

Theorem 2 (Uniqueness)**.** *Consider the initial value problem*

$$x' = f(t, x) \quad f(t_0) = x_0.$$

If f *and* $\frac{\partial f}{\partial x}$ *are defined and continuous near* (t_0, x_0) *then a solution to the IVP exists and is unique, near* t_0*.*

Examples

Example 25. *Show that the following both solve the IVP* $x' = x^{1/3}; x(0) = 0$

$$x(t) = 0 \quad x(t) = \begin{cases} (\frac{2}{3}t)^{3/2} & t > 0 \\ 0 & t \leq 0 \end{cases}$$

Why isn't the Uniqueness Theorem contradicted?

First, compute (by chain rule) $x' = (2t/3)^{1/2} = x^{1/3}$ and $x' = 0 = x^{1/3}$. Next, note that $\frac{\partial f}{\partial x} = 1/x^{2/3}$ is not continuous at $x = 0$, so the Uniqueness Theorem doesn't apply.

Example 26. *Find the general solution to the following differential equation. Is there a solution with* $y(0) = -3$*? If not, why isn't the existence theorem contradicted?*

$$ty' - y = t^2 \cos t$$

The general solution is (Integrating Factor)

$$y = t\sin t + Ct$$

Example 27. *Show that $y(t) = 0$ and $y(t) = t^3$ both solve $y' = 3y^{2/3}$ with $y(0) = 0$. Why doesn't this violate the uniqueness theorem?*

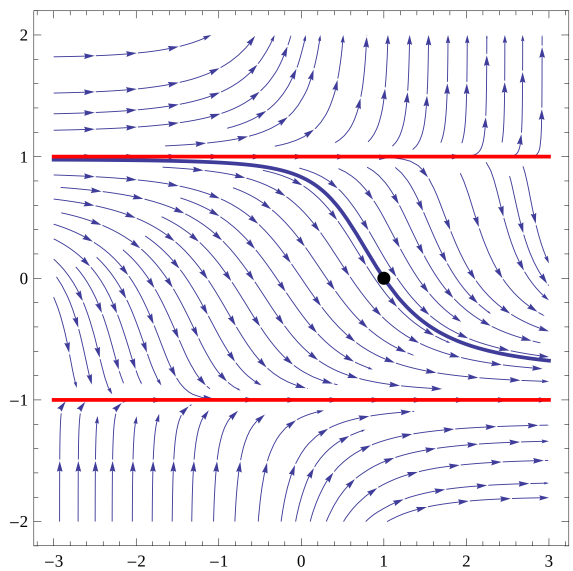
'Bounding Solutions'

Example 28 (Exercise 2.7.29)**.** *Let y be a solution to $y' = (y^2-1)e^{ty}$ with $y(1) = 0$. Prove that $-1 < y < 1$ for all t where y is defined.*

First we're going to establish two facts:

1. $y(t) = 1$ and $y(t) = -1$ are solutions to the differential equation
2. The uniqueness theorem applies along both of these solutions.

Since the uniqueness theorem says that 'solutions can't cross' and our solution starts between 1 and -1 this is enough to show that our solution is always between 1 and -1. You can think of these lines (shown in red below) as 'fences' bounding our solution.



So now we just verify

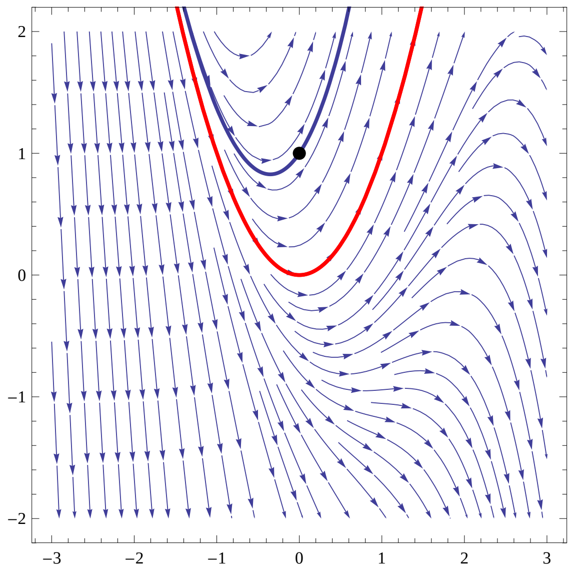
1. If $y = 1$ we have $y' = 0$, so $0 = y' = (y^2-1)e^{ty} = 0$. Similarly, if $y = -1$ we have $y' = 0$, so $0 = y' = (y^2-1)e^{ty} = 0$.
2. $f = (y^2-1)e^{ty}$ and $\frac{\partial f}{\partial y} = e^{ty}\left(2y + t\left(-1+y^2\right)\right)$ are both continuous when $y = 1$ or $y = -1$ (in fact, they're continuous everywhere). Therefore, the uniqueness theorem applies.

Example 29 (Exercise 2.7.31)**.** *Let x be a solution to $x' = x - t^2 + 2t$ with $x(0) = 1$. Prove that $x > t^2$ for all t where x is defined.*

As in the previous problem we're going to establish two facts:

1. $x(t) = t^2$ is a solution to the differential equation
2. The uniqueness theorem applies along this solution.

Again remember that the uniqueness theorem says that 'solutions can't cross.' Since our solution starts above $x(t) = t^2$ this is enough to show that our solution is always above it.



So now we just verify

1. If $x(t) = t^2$ we have $x'(t) = 2t$, so $x'(t) = 2t = t^2 - t^2 + 2t = x - t^2 + 2t$, and t^2 satisifies our differential equation.

2. $f = x - t^2 + 2t$ and $\frac{\partial f}{\partial x} = 1$ are both continuous when $x = t^2$ or (in fact, they're continuous everywhere). Therefore, the uniqueness theorem applies.

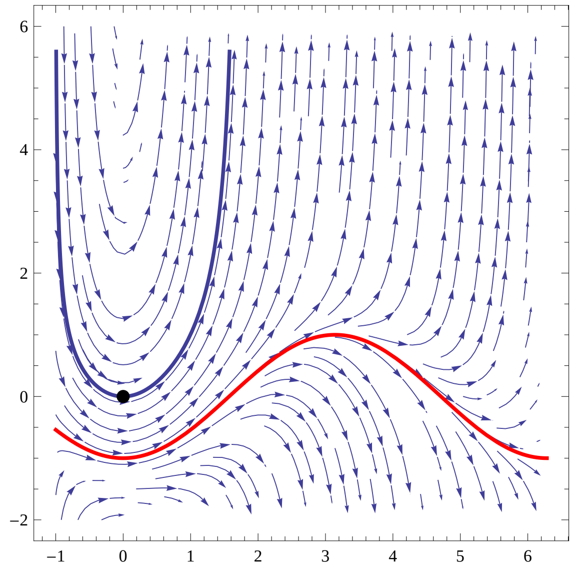
Example 30. *Consider the initial value problem*

$$y' = \sqrt{y^2 + 1}\ln(t+1)(y + \cos t) + \sin t, \quad y(0) = 0$$

1. *Do the theorems about the existence and uniqueness of solutions apply to this initial value problem? Explain.*

2. *Suppose $y(t)$ is a solution of the above initial value problem. Is it possible that $y(\pi) = 0$? Why or why not?*
   (Hint: Note that $y' - \sin t = (y + \cos t)'$.)

1. Since the function is smooth (for $t > -1$), both uniqueness and existence theorems apply.
2. Using the hint, note that $y = -\cos t$ is a solution. Thus, by uniqueness theorem, it bounds y below, and we can't have $y(\pi) = 0$.



2.8 Section 7 - Autonomous Equations

Definition 8 (Autonomous)**.** *A differential equation of the form*

$$x' = f(x)$$

is called autonomous*.*

Definition 9 (Equilibrium)**.** *A point x_0 with $f(x_0)$ is called an* equilibrium point*. The solution $f(t) = x_0$ is called an* equilibrium solution*.*

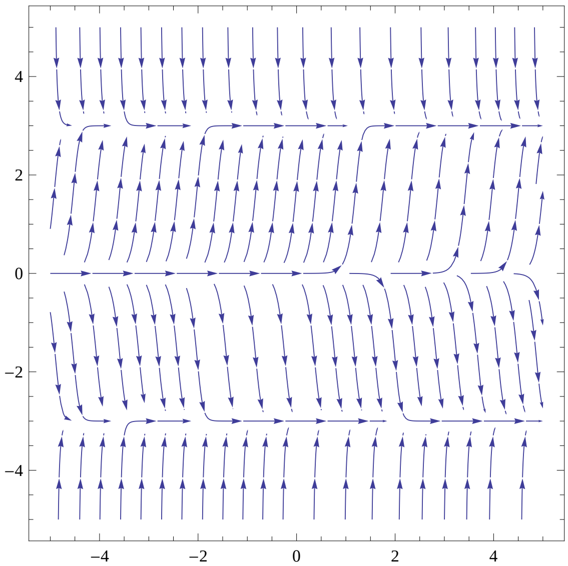
Definition 10 (Stable)**.** *An equilibrium point is called* asymptotically stable *if solutions starting nearby stay near as $t \to infty$. More formally, if for every $\epsilon > 0$ ther eexists $\delta > 0$ such that solutions starting within δ of x_0 remain within ϵ for all t.*

Definition 11 (Unstable)**.** *An equilibrium point is called* asymptotically unstable *if it is not stable.*

Definition 12 (Phase Line)**.** *A* phase line *is just a plot of f as a function of x.*

Examples

Example 31 (Exercise 2.9.19)**.** *Plot the vector field, and phase line for $y' = 9y - y^3$. Identify the equilibrium points and classify them:*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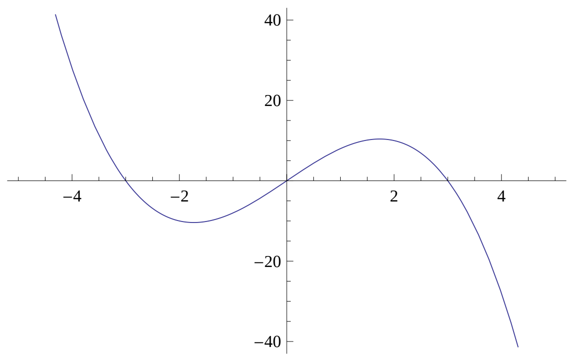
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