Math 33b, Winter 2013, Tonći Antunović - Homework 3 solutions

From the textbook solve the problems:
Section 2.7: 2, 4, 6, 8, 10, 18 part(ii), 20 part(ii), 28, 30, 32.

And also the problems below:

Problem 1. Consider the initial value problem

$$x' = 2x - \tan t, \quad x(0) = 1.$$

Without solving the equation, explain why this problem has a unique solution and determine it's interval of existence.

Solution: The function on the right hand side is defined and continuous on the rectangle $(-\pi/2, \pi/2) \times \mathbb{R}$ which contains $(0, 0)$ and thus the initial value problem has a solution. As this is a linear equation the interval of existence contains the interval $(-\pi/2, \pi/2)$ (remark on page 79). But this is the whole existence interval since the equation is not defined on a larger interval. Moreover, since the partial derivative of $2x - \tan t$ with respect to x (equal to 2) is defined and continuous everywhere, the solution is also unique.

Problem 2. Consider the initial value problem

$$x' = e^t x^2 - 2x, \quad x(0) = 1/2.$$

Explain why this problem has a unique solution x and show that it satisfies $0 < x(t) < e^{-t}$.

Solution: The right hand side is continuous and the partial derivative with respect to x is also continuous. Therefore, any two solutions must agree on their intervals of existence. To show the inequalities observe that $x(0) = 1/2$ lies between 0 and e^0, so the inequality is true for $t = 0$. If one of the inequalities would fail for some t, say $x(t) \geq e^t$ then solution $x(t)$ and e^t would be equal at some point which can't happen by Theorem 7.16.

Problem 3. In the setting of questions 17-20 in the textbook solve the problem with

$$E(t) = \begin{cases} t, & 0 < t < 2, \\ -t, & t \geq 2. \end{cases}$$

Solution: Since the initial condition is given at $t = 0$ we first solve

$$q' + q = t, \qquad q(0) = 0.$$

The integrating factor is e^t and the solution is obtained as

$$e^t q = \int te^t \, dt = e^t(t-1) + C \quad \Rightarrow \quad q = t - 1 + Ce^{-t}.$$

By the initial condition $q(0) = 0$ we have $-1 + C = 0$ and $C = 1$. Therefore, we have

$$q(t) = t - 1 + e^{-t},$$

for $0 \leq t < 2$. For $t \geq 2$ we need to solve $q' + q = -t$ and as the initial condition we take the value of q at the border of two phases that is $q(2) = 2 - 1 + e^{-2} = 1 + e^{-2}$. The equation has the same integrating factor and to solve it

$$e^t q = \int -te^t \, dt = e^t(1-t) + C \quad \Rightarrow \quad q = 1 - t + Ce^{-t}.$$

The condition $q(2) = 1 + e^{-2}$ gives $1 + e^{-2} = -1 + Ce^{-2}$ and $C = 1 + 2e^2$, which gives

$$q = 1 - t + e^{-t} + 2e^{2-t}.$$

Therefore, the solution is

$$q = \begin{cases} t - 1 + e^{-t}, & 0 < t < 2, \\ 1 - t + e^{-t} + 2e^{2-t}, & t \geq 2. \end{cases}$$

Problem 4. Consider the initial value problem

$$x' = 2x + f(t), \qquad x(0) = 1,$$

where

$$f(t) = \begin{cases} 0, & t \leq 1, \\ ae^{2t}, & t > 1, \end{cases}$$

where a is a real parameter. Does there exist a non-zero value of the parameter a such that the solution to this equation is differentiable at $t = 1$?

Solution: First we solve $x' = 2x + 0$ which gives $x = Ce^{2t}$ and by the initial condition $x(0) = 1$ we have $x = e^{2t}$. Then we solve

$$x' = 2x + ae^{2t},$$

and as the initial condition take the value of x at the border of two phases $x(1) = e^2$. The integrating factor for the equation is e^{-2t} and we get

$$e^{-2t}x = \int a\ dt = at + C, \quad \Rightarrow \quad x = (at + C)e^{2t}.$$

The constant C comes from the initial condition $x(1) = e^2$

$$(a + C)e^2 = e^2 \quad \Rightarrow \quad C = 1 - a,$$

so

$$x = (at + 1 - a)e^{2t}.$$

Therefore, the solution is

$$x = \begin{cases} e^{2t}, & t \leq 1, \\ (at + 1 - a)e^{2t}, & t > 1. \end{cases}$$

For the function to be differentiable at 1 we need the derivatives of e^{2t} and of $(at+1-a)e^{2t}$ to agree at $t = 1$. That is we need $2e^{2t}$ and $(2at+2-a)e^{2t}$ to agree at $t = 1$ which means $2e^2 = (a + 2)e^2$, which is possible, only for $a = 0$. So there are no non-zero values of the parameter a with this property.