Math 33b, Winter 2013, Tonći Antunović - Homework 1

From the textbook solve the problems:
Section 2.1: 10
Section 2.2: 10, 14, 16, 18, 36, 40
Section 2.4: 8, 16, 20, 24, 30, 34.

And also the problems below:

Problem 1. Find the general solution of the equation

$$y' - y^2 = (yt)^2.$$

Solution: Since we can write $y' = y^2(t^2+1)$ this is a separable equation and so

$$\int \frac{dy}{y^2} = \int (t^2+1)\ dt$$

so

$$-\frac{1}{y} = t^3/3 + t + C \quad \Rightarrow \quad y = -\frac{1}{t^3/3+t+C}.$$

Since we divided by y we also check $y = 0$ is indeed another solution to the equation.

Problem 2. Solve the initial value problem

$$y' = e^y t, \quad y(0) = 0.$$

Solution: This is separable and

$$\int e^{-y}\ dy = \int t\ dt,$$

which gives

$$-e^{-y} = \frac{t^2}{2} + C \quad \Rightarrow \quad y = -\ln\Big(-\big(\frac{t^2}{2}+C\big)\Big).$$

The initial condition $y(0) = -\ln(-C) = 0$ gives $C = -1$ and so the solution is $y = -\ln\left(1 - \frac{t^2}{2}\right)$.

Problem 3. Find the general solution of the equation

$$y' = y + e^{2t}.$$

Solution: This is a linear equation $y' - y = e^{2t}$ and the integrating factor is $u(t) = e^{\int(-1)\ dt} = e^{-t}$. Multiplying the equation with $u(t)$ we get

$$e^{-t}y' - e^{-t}y = e^{t}.$$

The left hand side is the derivative of $e^{-t}y$ so we get

$$e^{-t}y = \int e^{t}\ dt = e^{t} + C \quad \Rightarrow \quad y = e^{2t} + Ce^{t}.$$

Problem 4. Using the substitution $z = y^2$ solve the initial value problem

$$yy' + y^2 = t, \quad y(0) = 1.$$

Solution: Since $z' = 2yy'$ the equation becomes

$$\frac{1}{2}z' + z = t, \quad z(0) = 1^2 = 1.$$

This is a linear equation with the integrating factor $u(t) = e^{2t}$ and the equation becomes

$$e^{2t}z' + 2e^{2t}z = 2te^{2t}.$$

The left hand side is the derivative of $e^{2t}z$ and so

$$e^{2t}z = \int 2te^{2t}\ dt = te^{2t} - \frac{1}{2}e^{2t} + C \quad \Rightarrow \quad z = Ce^{-2t} + t - \frac{1}{2}.$$

The initial condition gives

$$z(0) = C - \frac{1}{2} = 1 \quad \Rightarrow \quad C = \frac{3}{2}.$$

Therefore, $z = 3e^{-2t}/2 + t - 1/2$ which gives (since $y(0) = 1 > 0$)

$$y = \sqrt{3e^{-2t}/2 + t - 1/2}.$$