1 Problem/Solution

$$y' = Ay \qquad A = \begin{pmatrix} 2 & -4 \\ 8 & -6 \end{pmatrix}$$

First compute

$$\text{Trace}(A) = 2 - 6 = -4 \quad det(A) = (2 \cdot -6) - (-4 \cdot 8) = -12 + 32 = 20.$$

Thus, the characteristic polynomial is given by

$$\lambda^2 + 4\lambda + 20,$$

and the eigenvalues are

$$\lambda = \frac{-4 \pm \sqrt{16 - 4 \cdot 20}}{2} = -2 \pm 4i.$$

Let's choose to work with $\lambda = -2 + 4i$. We know that the solution is $y(t) = e^{\lambda t}v$ for v the associated eigenvector, so we need to find v. Let $v = (a\ b)^T$.

$$Av = \lambda v \Rightarrow 2a - 4b = (-2 + 4i)a \Rightarrow (4 - 4i)a = 4b \Rightarrow (1 - i)a = b.$$

So, we conclude that $v = (1\ 1 - i)^T$ is an eigenvector corresponding to v.

Solutions are therefore (plugging in to above)

$$y(t) = e^{(-2+4i)t} \begin{pmatrix} 1 \\ 1 - i \end{pmatrix}.$$

But we don't like complex numbers, so let's separate this into real/complex parts.

$$y(t) = e^{(-2+4i)t} \begin{pmatrix} 1 \\ 1 - i \end{pmatrix} = e^{-2t}(\cos(4t) + i\sin(4t)) \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix} + i \begin{pmatrix} 0 \\ -1 \end{pmatrix} \right],$$

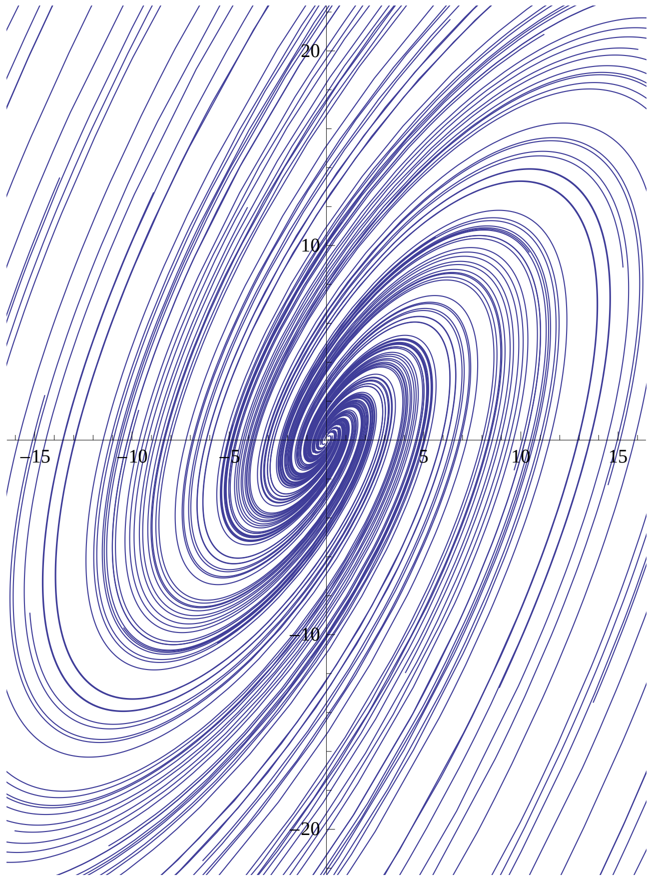
so

$$y_1(t) = \Re(y(t)) = e^{-2t} \left(\cos(4t) \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \sin(4t) \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right),$$

$$y_2(t) = \Im(y(t)) = e^{-2t} \left(\sin(4t) \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \cos(4t) \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right),$$

2 Analysis

Here's a plot of some solutions:



Observe that

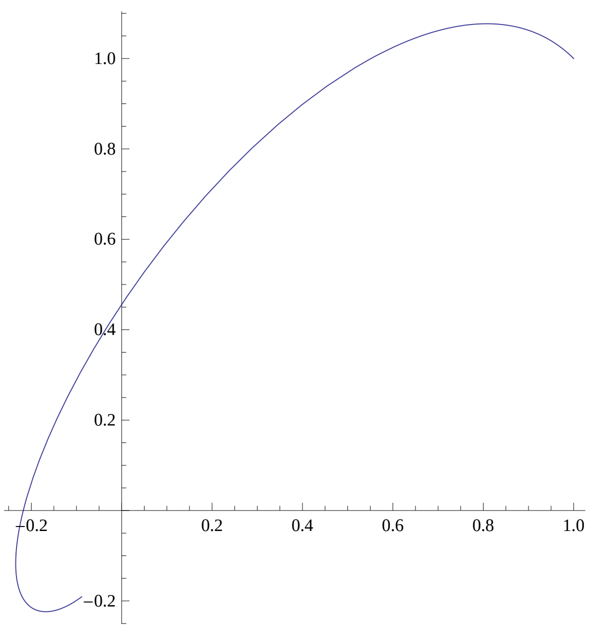
$$y_1(t - \pi/8) = \begin{pmatrix} e^{\frac{\pi}{4}-2t}\sin(4t) \\ e^{\frac{\pi}{4}-2t}(-\cos(4t)+\sin(4t)) \end{pmatrix} = e^{\pi/4}e^{-2t}\begin{pmatrix} \sin(4t) \\ -\cos(4t)+\sin(4t) \end{pmatrix} = e^{\pi/4}\cdot y_2(t)$$

so, in fact, any curve that we can write as a multiple of y_1 can also be written as a multiple of y_2, albeit for a different range of t.

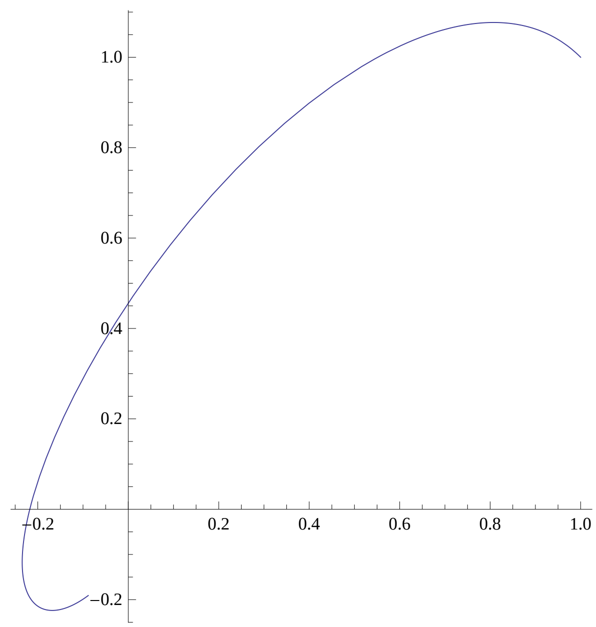
This is why if we just want to know what the plane looks like, we can work with just one of y_1, y_2. However, if we want to solve some IVP, say $y(0) = (a\ b)^T$, then we need both.

$$\begin{aligned} ay_1(0) + by_2(0) &= ae^{-2t}\left(\cos(4t)\begin{pmatrix}1\\1\end{pmatrix} + \sin(4t)\begin{pmatrix}0\\1\end{pmatrix}\right) + be^{-2t}\left(\sin(4t)\begin{pmatrix}1\\1\end{pmatrix} - \cos(4t)\begin{pmatrix}0\\1\end{pmatrix}\right) \\ &= a\begin{pmatrix}1\\1\end{pmatrix} - b\begin{pmatrix}0\\1\end{pmatrix} \end{aligned}$$

So, we can only handle initial conditions of the form $y(0) = (a\ a)^T$ with y_1, and $y(0) = (0\ b)$ with y_2.



(a) $y_1(t), 0 \leq t \leq 1$



(b) $e^{\pi/4} \cdot y_2, \frac{\pi}{8} \leq t \leq 1 + \frac{\pi}{8}$