Math 33B: Differential Equations

Practice problems I: 1^{st} order ODEs

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Review list

- Midterm 1 will focus on solving first order ODEs using the techniques covered in Lectures 1-9:
  - separating variables [Lectures 1 & 2]
  - integrating factors for linear ODEs [Lectures 3 & 4]
  - exact ODEs [Lectures 5 - 7]
- You should be able to identify which technique to use depending on the ODE. If no solution method is proposed in the question, then you may use any technique that applies.
- Know how to identify an interval of existence within which the solutions to the ODEs are valid [Lecture 2].
- You should be able to work your way through all the problems covered in the homework assignments # 1 and # 2 (and parts of # 3 - note: the existence and uniqueness theorems will *not* be examined in Midterm 1).
- Know how to qualitatively describe the behavior of differential equations using the direction field and how integral curves may be used to determine the late-time behavior of the solutions (e.g. as the solutions approach a limiting value as $t \to \infty$) [see Lectures 1 & 2]

- Understand how first order IVPs are formulated for physical applications (e.g. mixing problems) [Lectures 4 & 5].
- Know how to obtain qualitative information from first order autonomous differential equations. Identify critical points, equilibrium and non-equilibrium solutions [Lectures 8 & 9].
- Textbook *reference*: Chapter 2 - sections 2.1, 2.2, 2.4-2.6 and 2.9.
- Note that the logistic equation we looked at in lectures 8 & 9 is actually covered in section 3.1 in the textbook.
- Everything we covered up to and including **Lecture 9** [excluding existence and uniqueness in Lecture 7] is included in the first midterm exam.
- There are 10 problems in this handout covering the material of Lectures 1 - 9.
- The midterm exam will consist of 5 problems each worth 10 points for a maximum of 50.

Problem 1

Solve the following initial value problem,

$$\frac{dy}{dx} = \frac{3x^2}{3y^2 - 4}, \qquad y(1) = 0.$$

Determine the interval in which the solution is defined.

Hint: To find the interval in which the solution is defined, look for points where the integral curve, represented by the solution to the IVP, has a vertical tangent.

Problem 2

Consider the following differential equation

$$\frac{dy}{dx} = \frac{x(x^2+1)}{4y^3},$$

with $y(0) = -1/\sqrt{2}$ as the initial condition.

(i) Find the particular solution of the given IVP in explicit form.

(ii) Determine the interval in which the solution is defined.

Problem 3

Find the critical points of the following ODE:

$$\frac{dy}{dx} = 2y(1-y).$$

Plot the equilibrium solutions and draw a direction field for the ODE.

Problem 4

By substituting $y(x) = \dfrac{1}{u(x)}$, show that the nonlinear differential equation,

$$\frac{dy}{dx} + 2xy = xy^2$$

reduces to the linear equation,

$$\frac{du}{dx} - 2xu = -x.$$

Hence find the particular solution of the *nonlinear* equation if $y(0) = 1$.

Problem 5

A tank contains 50 gallons of brine in which 75 lbs of salt are dissolved initially. A mixture of brine containing 3 lbs of salt per gallon flows in the tank at a rate of 2 gal/min. Brine flows out of the tank at the same rate.

Let $y(t)$ denote the amount of salt in the tank at any time t.

(i) Formulate the IVP that describes the above mixing process.

(ii) When will there be 125 lbs of salt dissolved in the tank?

(iii) How much dissolved salt is in the tank after a long time (i.e. as $t \to \infty$)?

Problem 6

A party is held in a room that contains 1800 cubic feet of air which is originally free of carbon monoxide (CO). Several people start smoking cigarettes resulting into smoke containing 6% CO entering the room at a rate of 0.15 cubic feet per minute. The well-circulated mixture leaves the room at the same rate through a small open window.

(i) Formulate and solve the IVP that describes the above process using $y(t)$ to denote the amount of smoke in the room (in units of cubic feet of smoke).

(ii) Extended exposure to a CO concentration as low as 0.00018 can be dangerous. When should a prudent person leave the party?

Problem 7

Consider the equation,

$$a\frac{dy}{dx} + by = ke^{-\lambda x}$$

where a, b and k are positive constant and λ is a nonnegative[1] constant.

(i) Find the general solution for $y(x)$.

(ii) Show that if $\lambda = 0$, every solution approaches k/b as $x \to \infty$.

(iii) What is the limiting value of the solution as $x \to \infty$ if $\lambda > 0$?

[1] note: the term *nonnegative* includes zero while the term *positive* does not.

Problem 8

Find the value of a for which the following differential equation is exact,

$$ye^{2xy} + x + axe^{2xy}\frac{dy}{dx} = 0.$$

Solve the ODE using that value of a.

Problem 9

Consider the following autonomous first order ODE:

$$\frac{dy}{dt} = ay - y^3.$$

(i) Find the critical points and classify each one as asymptotically stable, unstable or semistable for

- $a > 0$;
- $a < 0$;
- $a = 0$.

(ii) In each case, sketch the equilibrium and several non-equilibrium solutions on the $t - y$ plane.

Problem 10

The following initial value problem describes the spread of a contagious disease

$$\frac{dy}{dt} = ay(1-y), \quad y(0) = y_0, \tag{1}$$

where a is a positive constant and y_0 is the initial proportion of infectious individuals. In (1) y denotes the proportion of infectious individuals while $1 - y$ is the proportion of susceptible individuals.

(i) Find the critical points and determine whether each is asymptotically stable, unstable or semistable.

(ii) Sketch the equilibrium and several non-equilibrium solutions on the $t - y$ plane.

(iii) What does the late-time behavior of the solutions represent physically?