Math 33b, Winter 2013, Tonći Antunović - Homework 2

From the textbook solve the problems:

Section 2.5: 2, 4, 8, 12

Section 2.6: 2, 6, 10, 14, 18, 22, 24, 26, 28

And also the problems below:

Problem 1. A 10 gallon tank contains salt-water of concentration 1 gram per gallon. Water is pumped out of the tank at the rate of 0.1 gallon per second and sent through a filter and then back to the tank. The filter reduces the concentration in the water that goes through it: the salt concentration in the water coming out of the filter is half of the salt concentration of the water coming into the filter. Find the concentration of the salt in the tank after t seconds.

Solution: If $x(t)$ is the mass of thesalt in the tank after time t the concentration is $x(t)/10$. The rate out is $0.1x(t)$ and the rate in is $0.1x(t)/2$ so we have

$$x'(t) = -0.1x(t) + 0.05x(t) = -0.05x(t).$$

Solutions are

$$\frac{dx}{x} = -0.05\ dt \quad \Rightarrow \quad x = Ce^{-0.05t}.$$

Initial condition $x(0) = 10$ gives $C = 10$ so $x = 10e^{-0.05t}$

Problem 2. Show that the differential equation

$$4x^3y^3\ dx + 3x^4y^2\ dy = 0$$

is exact. Find the general solution.

Solution: Partial derivative of $4x^3y^3$ with respect to y and of $3x^4y^2$ with respect to x are both equal to $12x^3y^2$ and since they are defined for all x and y the differential equation is exact. To solve it integrate first $4x^3y^3$ with respect to x

$$F(x, y) = x^4y^3 + \phi(y).$$

Since $\frac{\partial F}{\partial y} = 3x^4y^2 + \phi'(y)$ has to be equal to $3x^4y^2$ we can take $\phi(y) = 0$ and then $F(x, y) = x^4y^3$. The general solution is

$$x^4y^3 = C \quad \Rightarrow \quad y = C^{4/3}x^{-4/3} = Dx^{-4/3}.$$

Problem 3. Solve the initial problem

$$x^3\ dx + y\ dy = 0, \quad y(0) = 1.$$

Solution: The differential equation is exact and the solution is obtained by integration

$$\int x^3\ dx + y\ dy = \frac{x^4}{4} + \frac{y^2}{2} = C$$

Plugging in $y(0) = 1$ gives $C = \frac{0^4}{4} + \frac{1^2}{2} = \frac{1}{2}$, and since $y(0) > 0$ we have

$$y = \sqrt{1 - x^4/2}.$$

Problem 4. Explain how to find an integrating factor for the equation

$$(y - f(x))\ dx + g(x)\ dy = 0,$$

where f and g are differentiable functions.

Solution: This is a differential equation of the form $P\ dx + Q\ dy = 0$ where

$$\frac{1}{Q}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = \frac{1}{g(x)}\Big(1 - g'(x)\Big),$$

which depends on x only. Therefore, we will search for the integrating factor which is a function of x.