Homework 1: Worked out solutions — April 2013

Problem 1

For the DE $\frac{dy}{dx} = 2x$, using the method of isoclines, we set the DE equal to some constant, C.

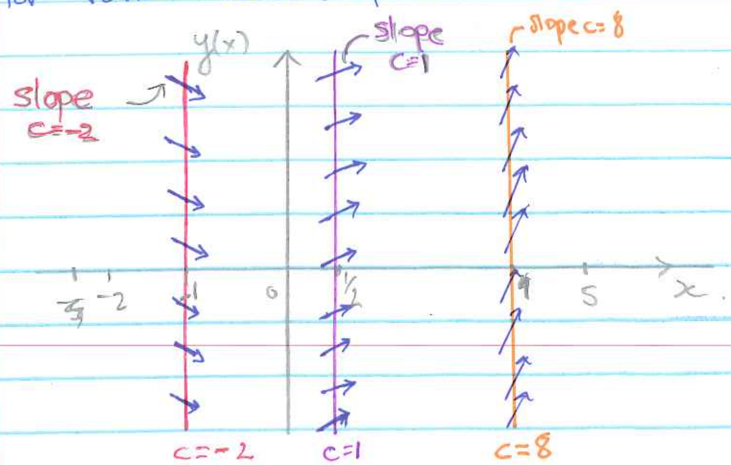
i.e. $\frac{dy}{dx} = 2x = C$.

The equation for the isoclines is given by:

$2x = C$. or $x = \frac{c}{2}$ ①.

The isoclines are all <u>vertical</u> and equal to $\frac{c}{2}$.

Choosing an interval of $-2 \leq x \leq 5$, we start plotting isoclines for various values of c:



For $c=-2$ the isocline is $x=-1$ (red line). The direction field arrows on this isocline should have a slope of -2 (= the value of c)

For $c=1$, the isocline is $x = \frac{1}{2}$ (purple line) with the arrows on

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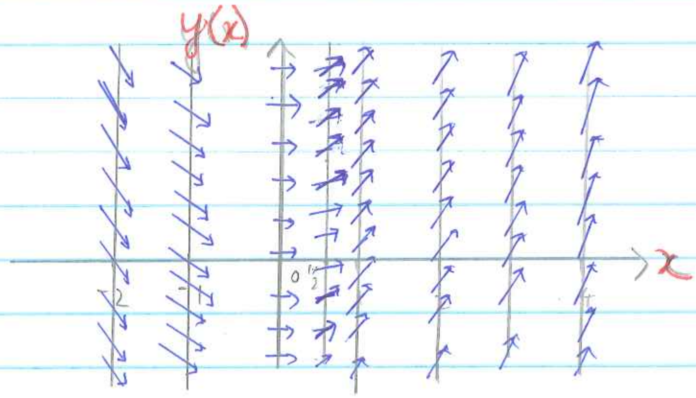
the isocline having a slope of c=1.

Similarly, for c=8, the isocline is x=4 (orange line) and the arrows have a slope of c=8 (so, pretty steep).

We continue to plot isoclines and arrows to get a more complete view of the direction field:

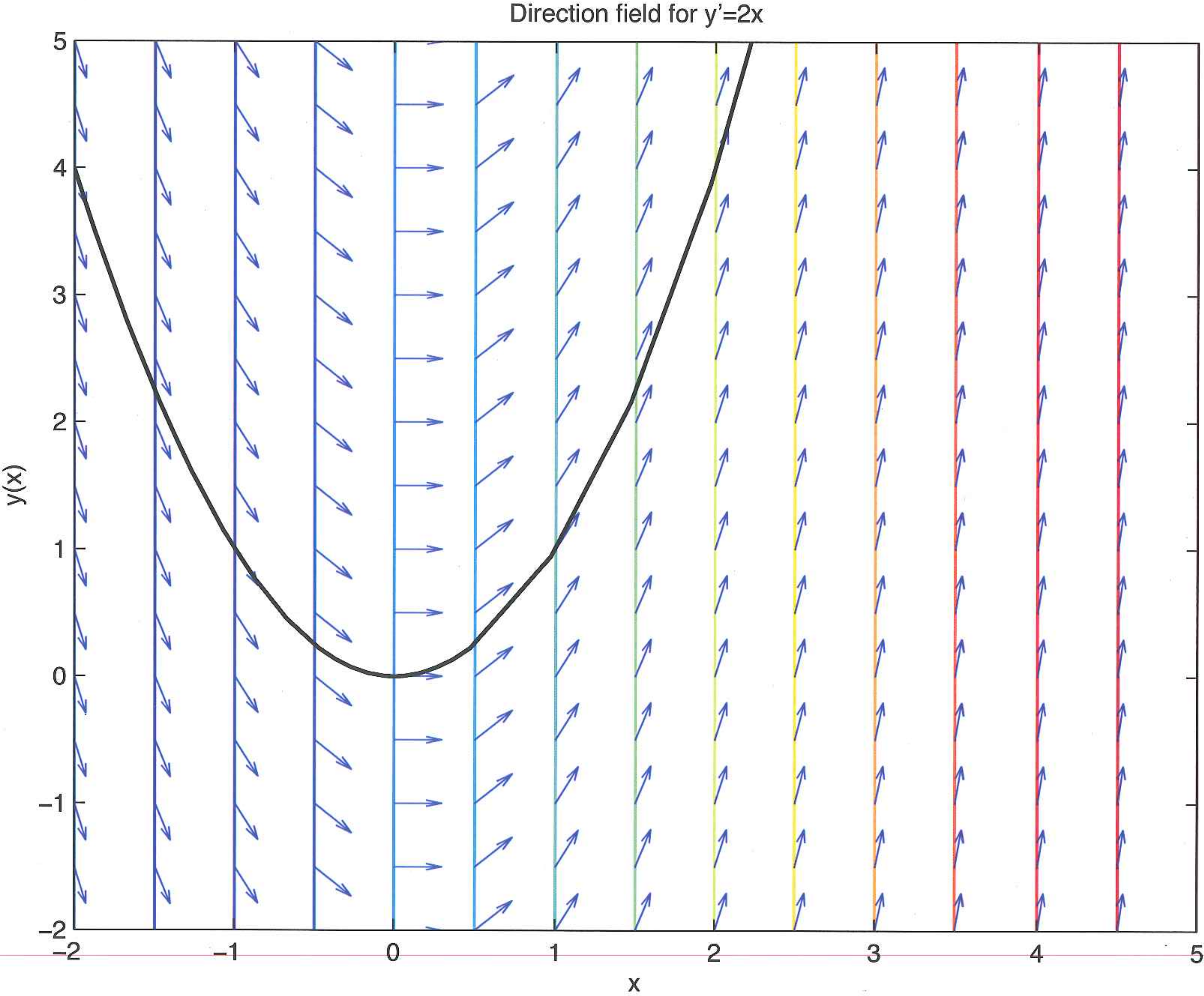


The direction field as generated in MATLAB is shown in the next page.

The isoclines are shown by colourful, vertical lines.

An integral curve passing through the origin (0,0) is shown by a black curve (note: there was no need to plot this for problem 1)

HW 1-3



HW1-4

Problem 2

$$\frac{dy}{dt} = y + 2 \quad (1).$$

Again, using the method of isoclines, we set the derivative, $\frac{dy}{dt}$, equal to some constant c.

$$\rightarrow \frac{dy}{dt} = y + 2 = c \quad (2)$$

And, from (2), the eqn for the isoclines is given by:

$$\boxed{y = c - 2} \quad (3)$$

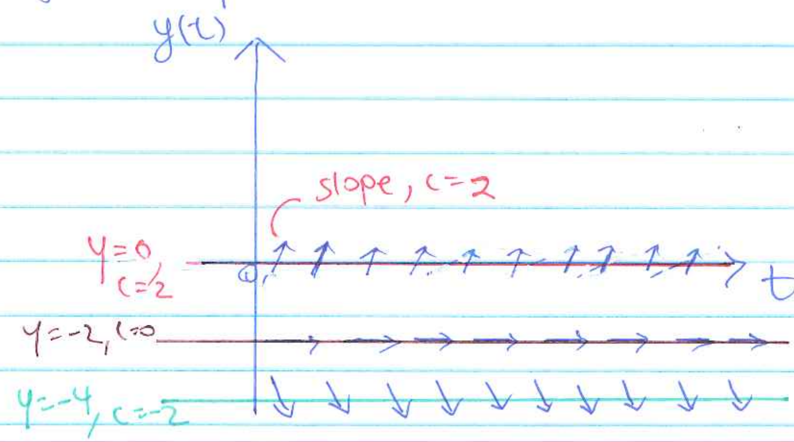
Here, the isoclines are all horizontal

Next, we choose values of c to plot various isoclines:

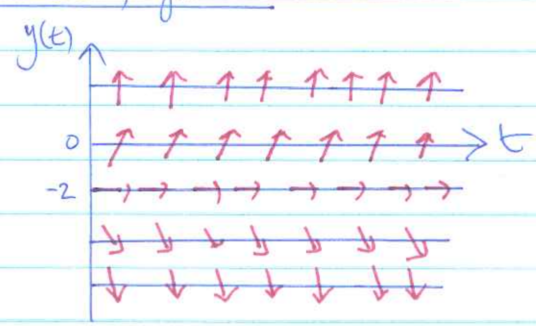
for $c=2$, $y=0$

for $c=0$, $y=-2$

for $c=-2$, $y=-4$



Adding more isoclines, gives:



HW1-5

The behavior of the solutions (i.e. of the integral curves) at late times (i.e. as $t \to \infty$) will depend on the initial condition, say $y(t_0) = y_0$. More specifically it will depend on the value of y_0 at $t = t_0$:

if $y_0 > -2$, then $y(t) \to \infty$ as $t \to \infty$.

$y_0 < -2$, then $y(t) \to -\infty$ as $t \to \infty$

$y_0 = -2$, then $y(t) \to -2$ as $t \to \infty$ (here, the sol'n is simply the line $y = -2$)

Three integral curves are shown on the matlab generated plot on the next page, corresponding to each range of initial values above. (again, you didn't need to plot these for the homework)

Extra notes.

The ODE $\frac{dy}{dt} = y + 2$ is a linear, first order ODE which can be solved through the integrating factor method.

Solving the ODE gives $y(t) = -2 + Ke^{t}$

now if the initial condition is given by $y(t = t_0) = -2$ then

$$y(t_0) = -2 + Ke^{t} = -2 \quad (4)$$

(4) can be satisfied by choosing $K = 0$. Furthermore, for $K = 0$, eq. (4) is satisfied for **any** value of t_0.
This makes sense by looking at the direction field since the isoclines are horizontal and therefore independent of t.

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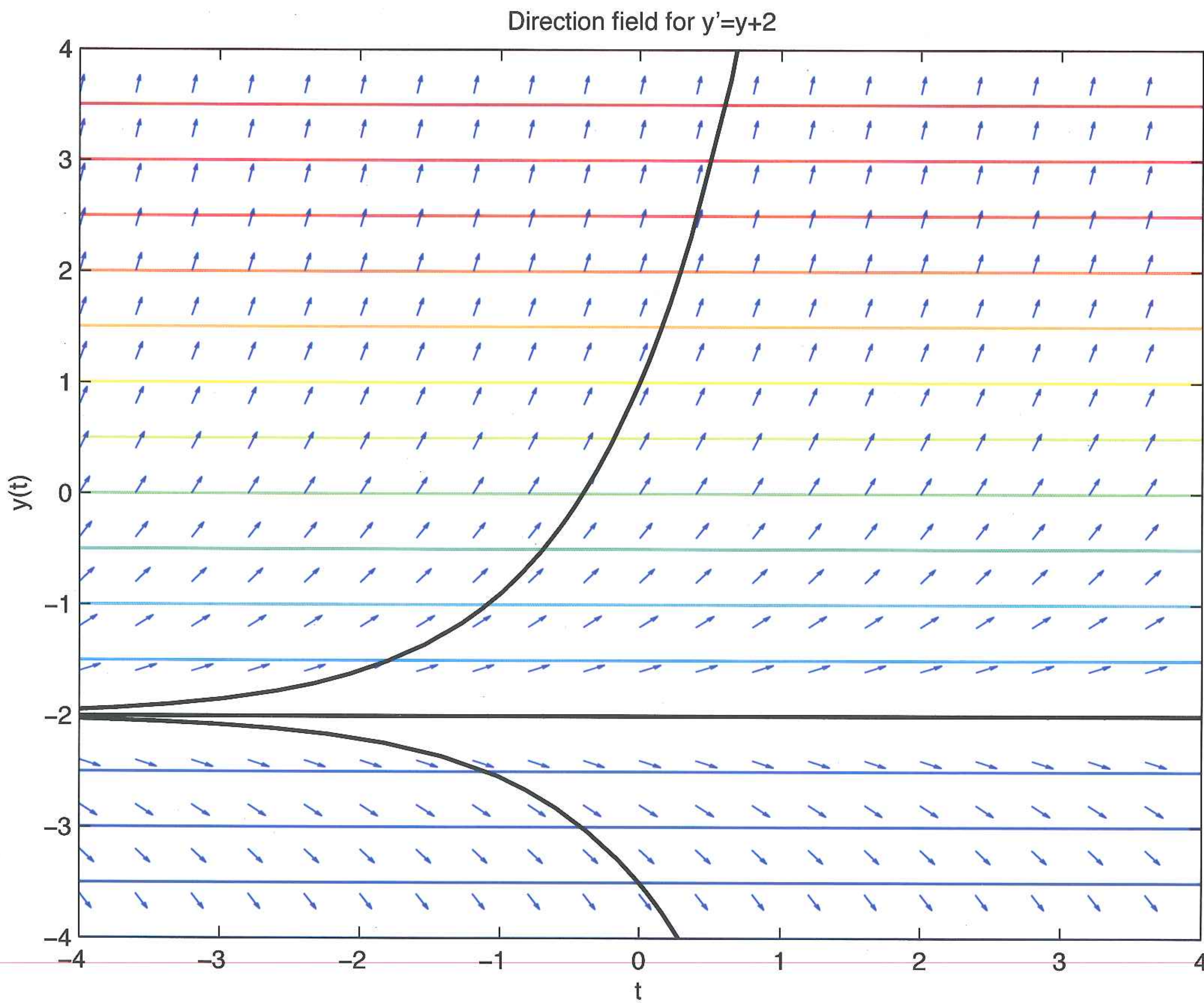


HW 1-7

Problem 3

$$\frac{dy}{dx} = \frac{x-y}{x+y} \quad (1)$$

Find equation for isoclines.

Set (1) equal to a constant, c

$$\frac{dy}{dx} = \frac{x-y}{x+y} = c \quad \Rightarrow \quad \frac{x-y}{x+y} = c \quad (2)$$

Rearranging (2),

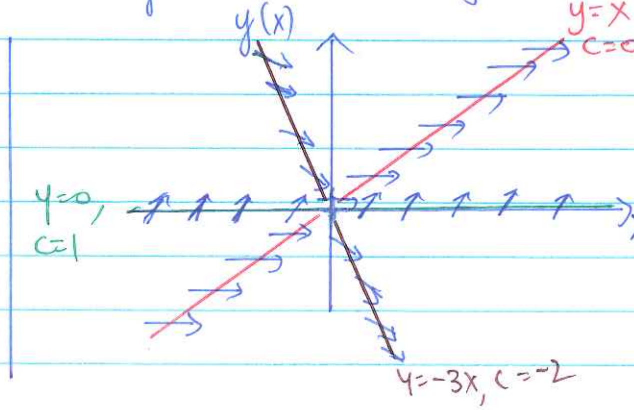
$$x - y = c(x+y)$$

$$y(c+1) = x(1-c) \quad \Rightarrow \quad \boxed{y = \left(\frac{1-c}{1+c}\right) \cdot x.} \quad (3)$$

Isocline eqn.

The first thing to notice about (1) is that the derivative (and hence the direction field) is not defined at $y=-x$

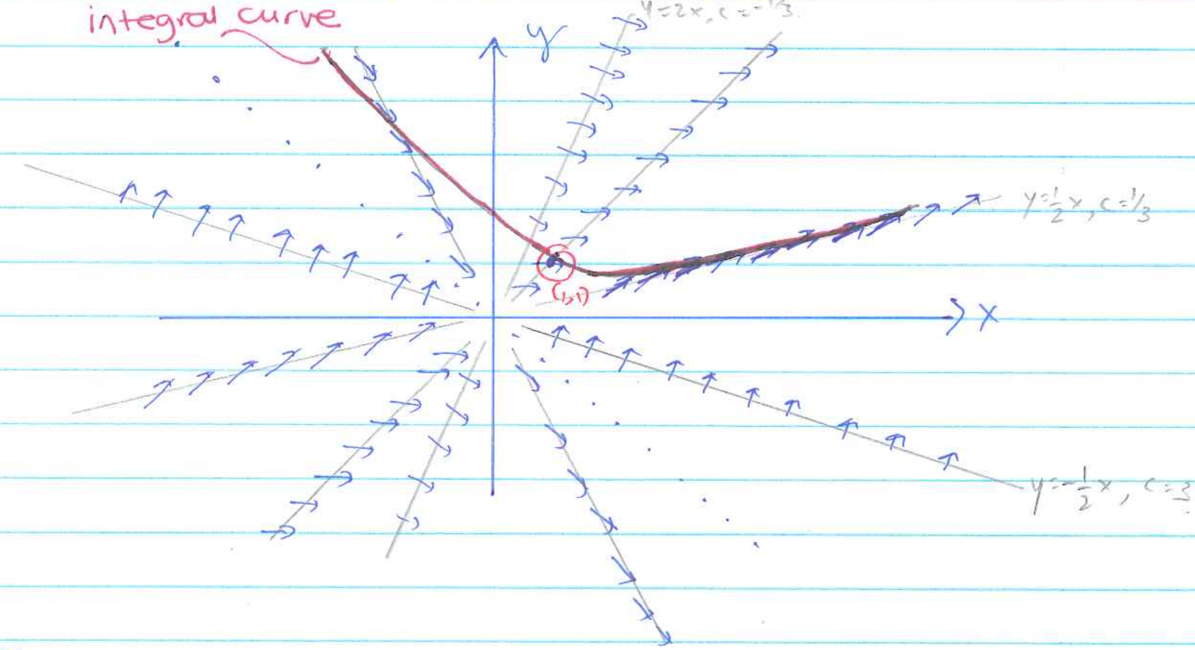
Just like in problems 1 and 2, we need to choose various values of c to plug into (3) to obtain the isoclines. Contrary to problems 1 and 2, the isoclines are neither horizontal nor vertical. In this case y is a linear function of x.



- For $c=0$, (3) gives $y=x$.
(on the red line, the arrows are of Slope $c=0$)
- For $c=1$, $y=0$
- For $c=-2$, $y=-3x$

HW 1-8

Plugging in more value of c in ③ allows us to plot more isoclines and direction arrows:



The isoclines shouldn't really pass through the origin because the direction field at (0,0) is undefined.

For all the pts that lie on $y=-x$, the ODE given by ① is undefined (the line $y=-x$ is shown by dots above and in the matlab generated plot next page)

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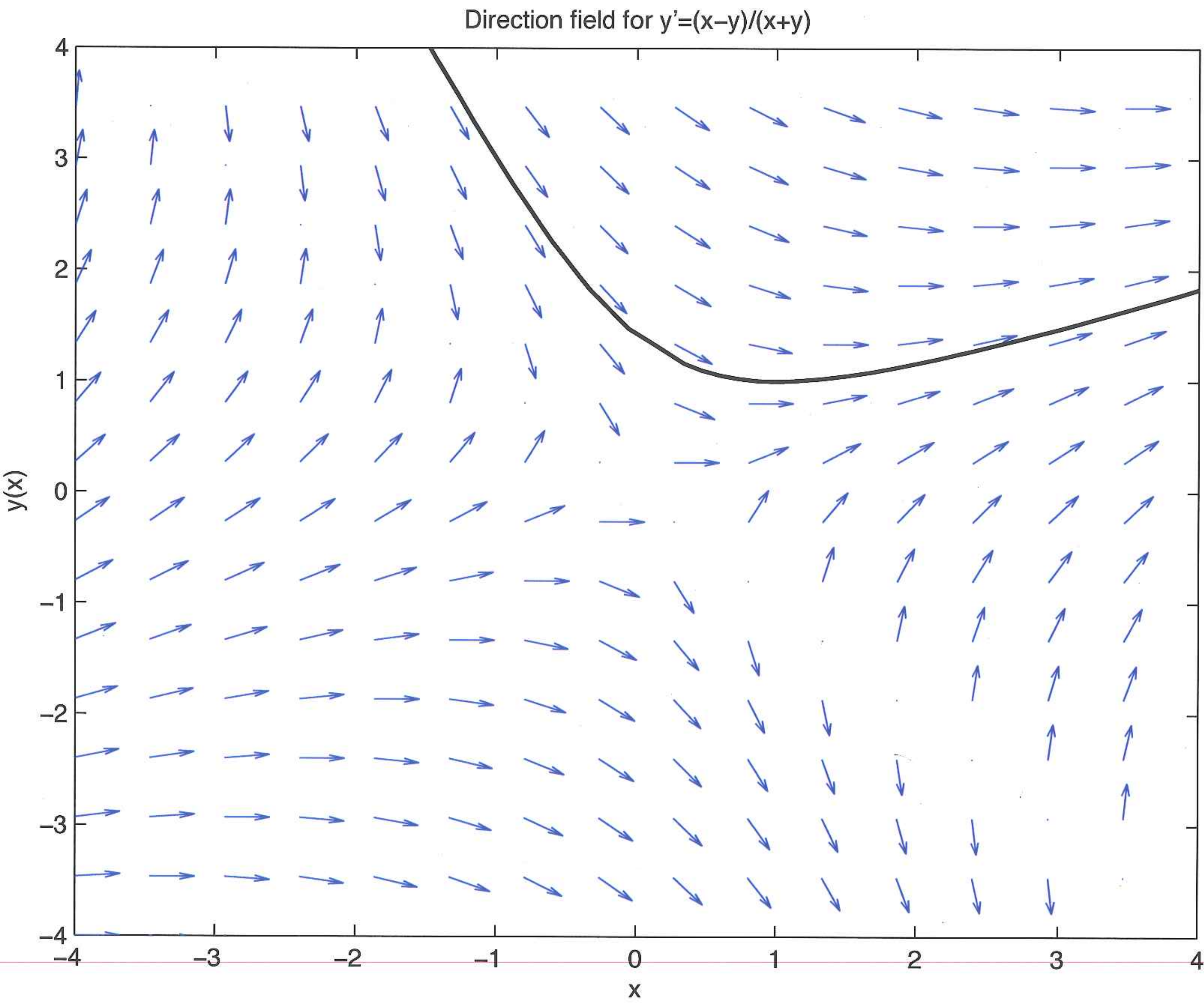

HW1-10

Problem 4

(i) $\frac{dy}{dx} = 1 + \frac{1}{y^2}$ with $y(1)=0$.

rewrite the ODE as:

$$\frac{dy}{dx} = \frac{y^2+1}{y^2}$$

Then, by separating variables,

$$\int \frac{y^2}{y^2+1}\, dy = \int 1\, dx. \quad (1)$$

For the integral on the LHS of (1), we use the following trig. substitution.

let $y = \tan u$ then $\frac{dy}{du} = \sec^2 u\, du$.

Also, $y^2+1 = \tan^2 u + 1$ (2)

but $\boxed{\tan^2 u + 1 = \sec^2 u}$ (3) $\Rightarrow$ (2) becomes $y^2+1 = \sec^2 u$.

(1) is expressed as:

$$\int \frac{\tan^2 u}{\cancel{\sec^2 u}} \cancel{\sec^2 u}\, du = \int 1\, dx$$

We are left with $\int \tan^2 u\, du = \int 1\, dx$. (4)

using identity (3) again, $\tan^2 u = \sec^2 u - 1$

HW1 - 11

So, (4) is:

$$\int (\sec^2 u - 1)\, du = \int 1 \, dx.$$

$$\tan u - u = x + k. \quad (5)$$

Back to the y variable,

recall $y = \tan u \Rightarrow u = \tan^{-1} y$.

In (5), $\boxed{y - \tan^{-1} y = x + k.}$ (6) general soln.

Applying the I.C, $y = 0$ at $x = 1$,

$$0 - \tan^{-1} 0 = 1 + k \Rightarrow k = -1$$

Sub $k = -1$ in (6)

$$\boxed{y - \tan^{-1} y = x - 1}$$ particular soln.

(ii) $\dfrac{dy}{dx} = \dfrac{x^2}{y(1+x^3)}$ $\quad y(0) = 1$

Separate the variables $\quad y\, dy = \dfrac{x^2}{1+x^3}\, dx$

Integrate both sides, $\quad \int y\, dy = \int \dfrac{x^2}{1+x^3}\, dx \quad (7)$

The numerator on the integrand on the RHS of ⑦ is the derivative of the denominator (off by a factor of 3) so we could either integrate immediately by observation or through a substitution.

let $u = 1 + x^3 \Rightarrow \dfrac{du}{dx} = 3x^2 \Rightarrow dx = \dfrac{du}{3x^2}$

Sub. u and du in the RHS of ⑦:

$$\int y \, dy = \int \frac{\cancel{x^2}}{u} \frac{du}{3\cancel{x^2}}$$

Integrating,

$$\frac{y^2}{2} = \frac{1}{3} \ln|u| + k.$$

Back in x-variable space,

$$\frac{y^2}{2} = \frac{1}{3} \ln|1+x^3| + k \quad ⑧$$

Applying I.C, $y(0)=1$

$$\frac{1}{2} = \frac{1}{3} \cancel{\ln|1|}^{0} + k \Rightarrow k = 1/2$$

Sub. $k = 1/2$ in ⑧

$$\frac{y^2}{2} = \frac{1}{3} \ln|1+x^3| + \frac{1}{2}$$

$$y^2 = \frac{2}{3} \ln|1+x^3| + 1$$

Expressing the soln in explicit form,

$$\boxed{y = \sqrt{\frac{2}{3}\ln(1+x^3) + 1}}$$

HW1-13

Problem 5

(i) $4xy + (x^2+1)\frac{dy}{dx} = 0.$

rewrite as

$$4xy = -(x^2+1)\frac{dy}{dx}$$

then separate the variables,

$$\frac{1}{y}dy = \frac{-4x}{x^2+1}dx. \quad ①$$

integrate both sides of ①,

$$\int \frac{1}{y}dy = -2\int \frac{2x}{x^2+1}dx.$$

$$\ln y = -2\ln|x^2+1| + \ln k.$$

since all the terms involve the natural log, you may express the constant of integration as $\ln k$ as well.

$$\ln y - \ln k = \ln(x^2+1)^{-2}$$

$$\ln\left|\frac{y}{k}\right| = \ln\left(x^2+1\right)^{-2}$$

exponentiating both sides,

$$\boxed{y = k(x^2+1)^{-2}}$$

(ii) $x\tan\frac{y}{x} + y - x\frac{dy}{dx} = 0. \quad ②$

Since ② is neither linear nor separable, we have no way of solving it using the techniques we've learned so far

HW1-14

By dividing the ODE through by x,

$$\tan\frac{y}{x} + \frac{y}{x} - \frac{dy}{dx} = 0. \quad (3).$$

The hint suggests a change of variables to transform the ODE to a separable one.

Starting from $y = v(x)\cdot x$, it's easy to see that

$$v = \frac{y}{x}. \quad (4)$$

also if $y = v\cdot x$ then $\frac{dy}{dx} = v\cdot 1 + x\frac{dv}{dx}$ (5)

(v is a fct of x so v·x is a product of fcts of x, hence we use the product rule to diff. wrt x)

Sub. (4) & (5) in (3),

$$\tan v + v - \left(v + x\frac{dv}{dx}\right) = 0$$

Simplifying, $\tan v - x\frac{dv}{dx} = 0$ this is a separable ODE in v(x) and x.

Sep. variables, $\frac{1}{\tan v}\,dv = \frac{1}{x}\,dx$

Integrating, $\int \frac{1}{\tan v}\,dv = \int \frac{1}{x}\,dx$

which is equivalent to. $\int \frac{\cos v}{\sin v}\,dv = \int \frac{1}{x}\,dx$

Since $\cos v$ is the derivative of $\sin v$, HW1 - 15

$$\ln|\sin v| = \ln x + \ln k.$$

$$\ln\left|\frac{\sin v}{k}\right| = \ln x.$$

$$\sin v = kx$$

Back to the y-variable, $v = y/x$,

$$\sin\left(\frac{y}{x}\right) = k \cdot x.$$ implicit gen. soln.

Taking the inverse sine fct on both sides,

$$\frac{y}{x} = \sin^{-1}(kx)$$

$$\boxed{y(x) = x\sin^{-1}(kx)}$$

Problem 6

(i) $(x^2+1)\dfrac{dy}{dx} + xy = x.$ (1) 1st order, linear ODE.

Firstly, we put (1) in standard form

$$\frac{dy}{dx} + \frac{x}{x^2+1}y = \frac{x}{x^2+1} \quad (2)$$

Where $p(x) = \dfrac{x}{x^2+1}$ and $f(x) = \dfrac{x}{x^2+1}$

HW1-16

Next, we find an I.F. using $\mu(x) = e^{\int p(x)\,dx}$.

$$\int p(x)\,dx = \int \frac{x}{x^2+1}\,dx = \frac{1}{2}\ln|x^2+1|$$

$\therefore$ the I.F is $\mu(x) = e^{\ln|x^2+1|^{1/2}} = (x^2+1)^{1/2}$.

Multiply (2) by $\mu(x)$

$$(x^2+1)^{1/2}\frac{dy}{dx} + \frac{x(x^2+1)^{1/2}}{(x^2+1)}y = \frac{x(x^2+1)^{1/2}}{(x^2+1)}$$

Simplifying,

$$(x^2+1)^{1/2}\frac{dy}{dx} + \frac{x}{(x^2+1)^{1/2}}y = \frac{x}{(x^2+1)^{1/2}} \qquad (3).$$

The LHS of (3) is reduced as $\frac{d}{dx}\left[\underbrace{(x^2+1)^{1/2}}_{=\mu(x)}\cdot y\right]$ (4)

we check that the reduced form is correct:

$$\frac{d}{dx}\left[(x^2+1)^{1/2}\cdot y\right] = (x^2+1)^{1/2}\frac{dy}{dx} + y\cdot\frac{1}{2}(x^2+1)^{-1/2}\cdot 2x$$

$$\frac{d}{dx}\left[(x^2+1)^{1/2}\cdot y\right] = \underbrace{(x^2+1)^{1/2}\frac{dy}{dx} + y\frac{x}{(x^2+1)^{1/2}}}$$

this is equivalent to the LHS of (3) so the reduced form is correct.

(3) becomes.

$$\frac{d}{dx}\left[(x^2+1)^{1/2}\cdot y\right] = \frac{x}{(x^2+1)^{1/2}} \qquad (5)$$

HW1-17

Eq. ⑤ may be integrated wrt x to give:

$$(x^2+1)^{1/2} \cdot y = \int \frac{x}{(x^2+1)^{1/2}}\, dx.$$

$$(x^2+1)^{1/2} \cdot y = (x^2+1)^{1/2} + k.$$

$$\boxed{y = 1 + k(x^2+1)^{-1/2}}$$

(ii) $\frac{dy}{dx} + (2\tan x)y = \sin x$ ⑥ $y(\pi/3) = 0.$

The ODE is already in S.F and $p(x) = 2\tan x$

$f(x) = \sin x.$

The I.F is $\mu(x) = e^{2\int \tan x\, dx}.$

$$2\int \tan x\, dx = 2\int \frac{\sin x}{\cos x}\, dx = -2\ln|\cos x|$$

$$\Rightarrow \mu = e^{\ln(\cos x)^{-2}} = (\cos x)^{-2}$$

Multiply ⑥ by I.F

$$\frac{1}{\cos^2 x}\frac{dy}{dx} + 2\tan x \cdot \cos^{-2} x\, y = \sin x \cdot \cos^{-2} x.$$

Simplifying, $\frac{1}{\cos^2 x}\frac{dy}{dx} + 2\frac{\sin x}{\cos x} \cdot \frac{1}{\cos^2 x}\, y = \frac{\sin x}{\cos^2 x}.$

$$\frac{1}{\cos^2 x}\frac{dy}{dx} + 2\frac{\sin x}{\cos^3 x}\, y = \frac{\sin x}{\cos^2 x}. \quad ⑦$$

HW1-18

The reduced form for the LHS of ⑦ is

$$\frac{d}{dx}\left[\frac{1}{\cos^2 x}\cdot y\right] \overset{\text{check}}{=} \frac{1}{\cos^2 x}\cdot\frac{dy}{dx} + y\cdot\frac{1}{\cos^3 x}(-\sin x)(-2)$$

$$= \frac{1}{\cos^2 x}\frac{dy}{dx} + \frac{2\sin x}{\cos^3 x}\, y$$

So ⑦ becomes,

$$\frac{d}{dx}\left[\frac{1}{\cos^2 x}\cdot y\right] = \frac{\sin x}{\cos^2 x}.$$

Integ. w.r.t. x:

$$\frac{1}{\cos^2 x}\cdot y = \int \frac{\sin x}{\cos^2 x}\,dx$$

$$\frac{1}{\cos^2 x}\, y = \frac{1}{\cos x} + k$$

$$\boxed{y(x) = \cos x + k\cos^2 x.}$$ gen. soln.

Applying the I.C., $y(\pi/3) = \cos\left(\frac{\pi}{3}\right) + k\cos^2\frac{\pi}{3} = 0$

$$0 = \frac{1}{2} + \frac{k}{4} \quad \Rightarrow k = -2$$

$$\boxed{y(x) = \cos x - 2\cos^2 x}$$