Homework 4: Worked-out solutions   April 2013

Problem 1

The Wronskian of 2 functions $f(x), g(x)$ is defined as

$$W(x) = \begin{vmatrix} f(x) & g(x) \\ f'(x) & g'(x) \end{vmatrix} = f g' - f' g$$

where $f' = \frac{df}{dx}$ and $g' = \frac{dg}{dx}$.

(i) $f(x) = e^{2x}$; $g(x) = e^{-\frac{3x}{2}}$

$f'(x) = 2e^{2x}$ and $g'(x) = -\frac{3}{2} e^{-\frac{3}{2}x}$

the Wronskian is:

$$W(x) = \begin{vmatrix} e^{2x} & e^{-\frac{3x}{2}} \\ 2e^{2x} & -\frac{3}{2} e^{-\frac{3x}{2}} \end{vmatrix} = -\frac{3}{2} e^{x/2} - 2e^{x/2}$$

$$= -3\tfrac{1}{2} e^{x/2}$$

$$\boxed{W(x) = -\frac{7}{2} e^{x/2}}$$

(ii) $f(x) = e^{-2x}$; $g(x) = x e^{-2x}$.

$f'(x) = -2e^{-2x}$

$g'(x) = x(-2e^{-2x}) + e^{-2x}$ (using the product rule)

HW 4-2

$$\therefore W(x) = \begin{vmatrix} e^{-2x} & xe^{-2x} \\ -2e^{-2x} & [-2xe^{-2x} + e^{-2x}] \end{vmatrix}$$

$$= e^{-2x}(-2xe^{-2x} + e^{-2x}) + 2xe^{-4x}.$$

$$= \cancel{-2xe^{-4x}} + e^{-4x} + \cancel{2xe^{-4x}}$$

$$\boxed{W(x) = e^{-4x}}$$

(iii) $f(x) = x$; $g(x) = xe^{x}$.

$f'(x) = 1$; $g'(x) = xe^{x} + e^{x}$.

$$W(x) = \begin{vmatrix} x & xe^{x} \\ 1 & xe^{x} + e^{x} \end{vmatrix} = xe^{x}(x+1) - xe^{x}.$$

$$= x^2e^{x} + \cancel{xe^{x}} - \cancel{xe^{x}}.$$

$$\boxed{W(x) = x^2e^{x}.}$$

Problem 2

$y_1(t) = t^2$ $y_2(t) = t^{-1}$.

ODE: $t^2y'' - 2y = 0$. ① , $t > 0$.

For a function $y = y(t)$ to be a soln to ①, it must satisfy the ODE.

I.e. $t^2y_1'' - 2y_1$ should be equal to 0 if $y_1(t)$ is a soln.

HW4-3

$y_1' = 2t$ and $y_1'' = 2$.

Sub $y_1 = t^2$ and $y_1'' = 2$ in ①:

$t^2 \cdot 2 - 2t^2 = 2t^2 - 2t^2 = 0 \Rightarrow y_1 = t^2$ is a sol'n.

Similarly, we can show that $y_2 = t^{-1}$ is a soln.

$y_2' = \frac{-1}{t^2}$; $y_2'' = \frac{2}{t^3}$.

In ①, $t^2 y_2'' - 2y_2 = t^2 \cdot \left(\frac{2}{t^3}\right) - 2\left(\frac{1}{t}\right)$

$= \frac{2}{t} - \frac{2}{t} = 0 \Rightarrow y_2 = t^{-1}$ is a soln.

The G.S is $y(t) = c_1 t^2 + c_2 t^{-1}$ ② (for any c_1, c_2).

Diff. ② to get y', y'':

$y' = 2c_1 t - c_2 t^{-2}$

$y'' = 2c_1 + 2c_2 t^{-3}$. ③

Sub. ② & ③ in ①.

$t^2 y'' - 2y = t^2(2c_1 + 2c_2 t^{-3}) - 2(c_1 t^2 + c_2 t^{-1})$

$= 2\cancel{t^2} c_1 + \cancel{\frac{2c_2}{t}} - 2\cancel{t^2} c_1 - \cancel{\frac{2c_2}{t}}$

$= 0 \Rightarrow y = c_1 t^2 + c_2 t^{-1}$ is a soln to ①.

HW4 - 4

Problem 3

$y_1(x) = \cos 2x \qquad y_2(x) = \sin 2x.$

ODE: $y'' + 4y = 0$. (1)

Sub. y_1 and y_1'' in (1) to check if $y_1 = \cos 2x$ is a soln.

$y_1 = \cos 2x$ (2) $\rightarrow y_1' = -2\sin 2x \Rightarrow y_1'' = -4\cos 2x$.

$\Rightarrow y_1'' + 4y' = -4\cos 2x + 4\cdot\cos 2x = 0 \Rightarrow y_1 = \cos 2x$ is a soln.

$y_2 = \sin 2x$ (3) $\rightarrow y_2' = 2\cos 2x \rightarrow y_2'' = -4\sin 2x$.

In (1), $y_2'' + 4y_2 = -4\sin 2x + 4\sin 2x = 0 \Rightarrow y_2 = \sin 2x$ is a so'n.

For two solns y_1, y_2 to constitute a fundamental set of solns to a linear, second order, homog. ODE, the be linearly independent.

To check for linear independence, determine the Wronskian, $W(x)$, of the two fcts.

if $W(x) = 0$ for all x then y_1, y_2 are linearly dependent

if $W(x) \neq 0$ for all x, then y_1, y_2 are linearly independent

$$\therefore W(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix} = 2\cos^2 2x + 2\sin^2 2x.$$

$$= 2(\cos^2 2x + \sin^2 2x)$$

$W(x) = 2 \neq 0$ for all x

HW4 5

Thus, y_1, y_2 are linearly independent and constitute a fundamental set of solutions.

Problem 4

$$\boxed{x^2y'' - 4xy' + (x^2+6)y = 0 \quad (1) \qquad y(0)=0, \; y'(0)=0.}$$ IVP

Show that $y=0$ is a soln.

if $y=0$, obviously its derivs are also 0 so all 3 terms on LHS in (1) will be 0 thus satisfying the RHS.

The function $y(x)=0$ (note: it's 0 for all x) also satisfies the 2 initial conditions.

$\Rightarrow$ $y=0$ is a soln.

Show that $y = x^2 \sin x$ is a soln.

Sub $y = x^2\sin x$ and its derivs in (1) to check if the soln satisfies the ODE.

$$y' = x^2 \cdot \cos x + \sin x\,(2x)$$

$$y'' = x^2(-\sin x) + 2x\cos x + 2x\cos x + 2\sin x.$$

Sub y, y', y'' in (1)

$$x^2\underbrace{\left[-x^2\sin x + 4x\cos x + 2\sin x\right]}_{=y''} - 4x\underbrace{\left(x^2\cos x + 2x\sin x\right)}_{=y'} + \left(x^2+6\right)x^2\sin x$$

$$= \cancel{-x^4\sin x} + \cancel{4x^3\cos x} + 2x^2\sin x - \cancel{4x^3\cos x} - 8x^2\sin x + \cancel{x^4\sin x} + 6x^2\sin x$$

HW 4-6

$= 2x^2 \sin x - 8x^2 \sin x + 6x^2 \sin x$

$= 0.$

So, $x^2 \sin x$ satisfies the ODE. We now need to check if it satisfies the ICs:

$y(0) = 0 \cdot 0 = 0$ ✓

$y'(0) = 0 \cdot 1 + 0 \cdot 0 = 0$ ✓

$\Rightarrow$ $y = x^2 \sin x$ is a soln to the IVP

Rewrite the ODE in S.F:

$$y'' \underbrace{\boxed{-\frac{4}{x}}}_{=p(x)} y^2 + \underbrace{\boxed{\left(\frac{x^2+6}{x^2}\right)}}_{=q(x)} y = \underbrace{\boxed{0}}_{=f(x)}$$

$p(x) = \frac{-4}{x}$, $q(x) = 1 + \frac{6}{x^2}$, $f(x) = 0$

The Existence & Uniqueness thm for homog., second order linear ODEs states that as long as $p(x)$, $q(x)$ and $f(x)$ are continuous on some interval containing the point $x = x_0$ (at which the ICs are defined) then, there exists one (and only one) solution.

Clearly $p(x)$ & $q(x)$ are discont. at $x = 0$. The initial conditions are given at $x = 0$ $[y(\underline{\underline{0}}), y'(\underline{\underline{0}})]$ so any interval around $x = 0$ will include the discontinuity.

$\Rightarrow$ the hypotheses of the thm are not satisfied and therefore a unique soln is not guaranteed.

HW4-7

Problem 5

$$y'' - 3y' + 2y = 0. \quad ① \qquad y(0) = 1, \; y'(0) = 0$$

Write down the char. eqn corresponding to ①:

$$m^2 - 3m + 2 = 0 \quad ②$$ (this is b/c we assume that the sdn takes the form $y = e^{mx}$).

Eq. ② is $(m-2)(m-1) = 0$ ③

From ③ it's easy to see that the roots of the eqn are:

$m_1 = 2$ and $m_2 = 1$

These roots are real so the G.S. takes the form:

$$y(x) = c_1 e^{m_1 x} + c_2 e^{m_2 x}$$

$$\boxed{y(x) = c_1 e^{2x} + c_2 e^{x}}$$ ④ gen. sdn.

To find the particular sdn, apply ICs.

$y(0) = c_1 + c_2 = 1$ ⑤

To apply the second I.C. i.e. $y'(0) = 0$, diff ④ wrt x first:

$$y'(x) = 2c_1 e^{2x} + c_2 e^{-x}$$

Then apply I.C: $y'(0) = 2c_1 + c_2 = 0 \Rightarrow c_1 = \frac{-c_2}{2}$ ⑥

Using ⑥ in ⑤:

$c_1 + c_2 = 1 \Rightarrow -\frac{c_2}{2} + c_2 = 1 \Rightarrow c_2 = 2$

From ⑥, $c_1 = \frac{-c_2}{2} = -1$.

In ④, $y(x) = -e^{2x} + 2e^{x}$.

The partic. soln is $\boxed{y(x) = 2e^{x} - e^{2x}}$

Problem 6.

(i) $y'' + 4ky' - 12k^2 y = 0$. ①

The char. eqn is $m^2 + 4km - 12k^2 = 0$ ②.

Factorizing ②, $(m+6k)(m-2k) = 0$

The roots are $m_1 = -6k$ and $m_2 = 2k$. (real + distinct)

For 2 real roots, the G.S. is $\boxed{y(x) = c_1 e^{-6kx} + c_2 e^{2kx}}$.

(ii) $y'' + 8y = 0$.

The char. eqn is $m^2 + 8 = 0$ ①

The roots of ① are $m = \pm\sqrt{8}i = \pm 2\sqrt{2}i$. (complex + distinct)

$\Rightarrow$ Complex roots take the form $m = a \pm bi$

HW4-9

Here, $a=0$ and $b=2\sqrt{2}$

For complex roots the G.S is:

$y(x)=c_1y_1+c_2y_2$ (2)

where $y_1=e^{ax}(\cos bx+i\sin bx)$

$y_2=e^{ax}(\cos bx-i\sin bx)$

since $a=0$, $b=2\sqrt{2}$ y_1, y_2 reduce to: $\left.\begin{aligned} y_1 &= \cos(2\sqrt{2}x)+i\sin(2\sqrt{2}x) \\ y_2 &: \cos(2\sqrt{2}x)-i\sin(2\sqrt{2}x). \end{aligned}\right\}$ (3)

Sub. (3) in (2),

The G.S is: $y(x)=c_1[\cos(2\sqrt{2}x)+i\sin(2\sqrt{2}x)]+c_2[\cos(2\sqrt{2}x)-i\sin(2\sqrt{2}x)]$ (4)

Since we are looking for real-valued functions it is convenient to express (4) as:

$$\boxed{y(x)=k_1\cos(2\sqrt{2}x)+k_2\sin(2\sqrt{2}x)}.$$

where $k_1=c_1+c_2$ & $k_2=(c_1-c_2)i$.

(iii) $y''-2ay'+a^2y=0$ (1)

The char. eqn is $m^2-2am+a^2=0$

Factorizing, $(m-a)^2=0 \Rightarrow m=a$ (repeated root).

From the char. eqn, we obtain only one soln $\rightarrow y_1=e^{ax}$.

The G.S. requires a pair of fundamental solns. We usually obtain both from the char. eqn but in this case we only have one.

We need to look for the second one.

Given a known soln. (here, we know $y_1 = e^{ax}$) we can find the second one y_2 by assuming that y_1 & y_2 are two linearly indep. solns that satisfy ①.

Choose $y_2 = v(x)\cdot y_1(x)$ ②. [b/c. $\frac{y_2}{y_1} = v(x)$ = nonconstant fct of x.]

If ② satisfies the ODE ①, then by diff. y_2 to get y_2', y_2'' and sub. back in ①:

$$\underbrace{(vy_1'' + 2v'y_1' + y_1v'')}_{=y_2''} \mp 2a\underbrace{(vy_1' + y_1v')}_{=y_2'} + a^2\underbrace{vy_1}_{y_2} = 0.$$

Factorizing v, v', v''

$$v(\underbrace{y_1'' \mp 2ay_1' + a^2y_1}_{=0}) + v'(2y_1' \mp 2ay_1) + y_1v'' = 0. \quad ③$$

If y_1 is a soln to $y'' + 2ay' + a^2y = 0$ then $y_1'' + 2ay_1' + a^2y_1 = 0$

③ becomes: $v'(2y_1' \mp 2ay_1) = -y_1v''$. ④.

but $y_1 = e^{ax}$ and $y_1' = ae^{ax}$.

Sub. in ④, $v'(\cancelto{0}{2(ae^{ax}) \mp 2ae^{ax}}) = -e^{ax}v''$.

$v'' = 0$ ⑤ (since $e^{ax} \neq 0$).

Note that ⑤ is a result of:

1. choosing $y_2 = v(x) y_1(x)$ to satisfy the ODE ①.
2. sub. y_2, y_2', y_2'' in ①
3. simplifying (keeping in mind that y_1 is a soln to ① + $y_1 = e^{-ax}$).

If the ODE is linear with constant coef. and homog. and the characteristic eqn gives only one root then, by choosing $y_2 = v(x) \cdot y_1(x)$, we always end up with $v'' = 0$.

From ⑤, integ. twice wrt x:

$$v = k_1 x + k_2 . \quad ⑥$$

for $V(x)$ to be a nonconstant function of x, we need $c_1 \neq 0$.

The simplest form $V(x)$ can take is $V(x) = x$ ⑦ $(k_1 = 1, k_2 = 0)$.

from ②, the second soln is $y_2(x) = \underbrace{x}_{=V(x)} \cdot y_1(x)$

$$y_2(x) = x e^{ax}.$$

The G.S is $y(x) = c_1 y_1 + c_2 y_2$

$$\boxed{y(x) = e^{ax}(c_1 + c_2 x)} \quad ⑧$$

Note that if we had chosen to arbitrarily set $k_1 = 1, k_2 = 0$ in ⑥ we w still arrive at the same G.S. ⑧.

⊛ if you find that the characteristic eqn gives you only one root then you can immediately write-down that the second soln is $y_2 = x \cdot y_1(x)$. - no need to include the proof that $V(x) = x$.

HW 4-12

Problem 7

$y'' - 2y' + 10y = 0$ where $y = y(t)$.

Constant coef. linear ODE so look for solns of the form $y = e^{m_1 t}$.

The char. eqn is

$m^2 - 2m + 10 = 0$

Find roots: $m = \dfrac{2 \pm \sqrt{[4 - 40]}}{2}$

$m = \dfrac{2 \pm 6i}{2} = 1 \pm 6i$

$m_1 = 1 + 6i$ and $m_2 = 1 - 6i$.

The G.S is: $y(t) = e^{at}(k_1 \cos bt + k_2 \sin bt)$ where $a = 1$, $b = 6$.

$$\boxed{y(t) = e^{t}(k_1 \cos 3t + k_2 \sin 3t)}$$

note that if $k_1 = k_2 = 0$ then $y(t) = 0$ and as $t \to \infty$, $y \to 0$.

All the nonzero solns oscillate with an increasing amplitude that never ceases to grow.