MATH 33B: DIFFERENTIAL EQUATIONS

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Example: Variation of parameters (nonhomogeneous ODE).

Find the G.S to $y'' + 4y = \tan 2x$. (1)

Eqn (1) is a linear second order ODE with the forcing term being $f(x) = \tan 2x$.

Since $f(x) = \tan 2x$ is not one of the 'simple' fcts we looked at for the method of undetermined coefficients i.e ae^{bx} or $a\sin bx / a\cos bx$ or polynomial of degree n, we need to use the method of variation of parameters

To use variation of parameters, we need to know the homogeneous solution, y_h.

This is given by $y_h = c_1 y_1 + c_2 y_2$ and y_h solves (1) when $f(x) = 0$.

Step 1: Find homogeneous solution

The homog. eqn is $y'' + 4y = 0$. (2)

This is a constant coefficient ODE so, we can use the characteristic eqn:

$$m^2 + 4 = 0 \implies m = \pm 2i \quad \text{(roots are complex)}$$

Since the roots are complex, the homog. soln is:

$$\boxed{y_h = c_1 \cos 2x + c_2 \sin 2x} . \quad (3)$$

Step 2: Construct desired particular solution.

Now that y_h is known, we seek a particular soln to the nonhomogeneous equation, ① which takes a similar form to y_h. The idea here is to vary the parameters (hence, the name of the method!) in y_h as follows:

Take Eq. ③ and replace c_1 with $v_1(x)$ and c_2 with $v_2(x)$. $v_1(x)$ & $v_2(x)$ are unknown and nonconstant functions of x.

$$\Rightarrow \quad \boxed{y_p = v_1(x)\cos 2x + v_2(x)\sin 2x.} \quad ④$$

Note that v_1 & v_2 need to be nonconstant or else y_p would be a constant multiple of y_h and hence a solution to the homogeneous problem and not the nonhomogeneous one.

Step 3: Find 2 independent eqns for v_1' & v_2'

This is the long step.

We determine y_p by solving:

$$y_p'' + 4y_p = \tan 2x. \quad ⑤.$$

We therefore need expressions for y_p'' & y_p to plug in to ⑤.

We already know y_p – it is given by ④.

Diff. y_p once: $y_p' = v_1'\cos 2x - 2v_1\sin 2x + v_2'\sin 2x + 2v_2\cos 2x.$ ⑥

To make our life easier, let's make the following essential assumption:

$$\boxed{v_1'\cos 2x + v_2'\sin 2x = 0} \quad ⑦$$

Using: Eq. (7) in (6). reduces (6) to:

$$y_p' = -2v_1 \sin 2x + 2v_2 \cos 2x. \quad (8)$$

Note that the new y_p' expression has no first order derivs of v_1 or v_2.

Diff (8) to get y_p''

$$y_p'' = -4v_1 \cos 2x - 2v_1' \sin 2x - 4v_2 \sin 2x + 2v_2' \cos 2x. \quad (9).$$

Now, sub. (9) & (4) in (5).

$$\left(-4v_1 \cos 2x - 2v_1' \sin 2x - 4v_2 \sin 2x + 2v_2' \cos 2x\right) + 4\left(v_1 \cos 2x + v_2 \sin 2x\right) = \tan 2x$$

Factorize v_1, v_2

$$v_1 \left(\cancel{-4\cos 2x + 4\cos 2x}\right)^{0} + v_2 \left(\cancel{-4\sin 2x + 4\sin 2x}\right)^{0} - 2v_1' \sin 2x + 2v_2' \cos 2x = \tan 2x$$

$$\Rightarrow \quad \boxed{2v_2' \cos 2x - 2v_1' \sin 2x = \tan 2x} \quad (10)$$

The 2 independent eqns we are looking for are given by (7) & (10):

$$\boxed{\begin{aligned} &v_1' \cos 2x + v_2' \sin 2x = 0. \\ &2v_2' \cos 2x - 2v_1' \sin 2x = \tan 2x. \end{aligned}}$$

Now, $v_2' = -\dfrac{v_1' \cos 2x}{\sin 2x}$

Sub. v_2' in (10):

$$2\left(\frac{-v_1'\cos 2x}{\sin 2x}\right)\cos 2x - 2v_1'\sin 2x = \tan 2x.$$

$$v_1'\left[\frac{-2\cos^2 2x - 2\sin^2 2x}{\sin 2x}\right] = \tan 2x.$$

$$\therefore\ v_1' = \frac{-\tan 2x \cdot \sin 2x}{2}.$$

$$\therefore\ \boxed{v_1' = \frac{-\sin^2 2x}{2\cos 2x}.} \quad (11)$$

Sub. v_1' in the expression for v_2':

$$v_2' = -\frac{\cancel{\cos} 2x}{\cancel{\sin 2x}}\left(\frac{-\sin^{\cancel{2}} 2x}{2\cancel{\cos} 2x}\right)$$

$$\boxed{v_2' = \frac{\sin 2x}{2}} \quad (12)$$

Step 4: Solve for v_1 & v_2.

Integrate (11): $v_1 = -\frac{1}{2}\int \frac{\sin^2 2x}{\cos 2x}\,dx. = -\frac{1}{2}\int \frac{1-\cos^2 2x}{\cos 2x}\,dx$

$$= -\frac{1}{2}\int \frac{1}{\cos 2x} - \cos 2x\,dx$$

$$v_1 = -\frac{1}{2}\int \sec 2x - \cos 2x\,dx. \quad (13)$$

Integrating sec 2x:

$$I = \int \sec 2x \, dx = \int \sec 2x \frac{(\sec 2x + \tan 2x)}{(\sec 2x + \tan 2x)} \, dx$$

$$I = \int \frac{\sec^2 2x + \sec 2x \tan 2x}{\sec 2x + \tan 2x} \, dx$$

note that the numerator is the derivative of the denomin. by a factor of a $\frac{1}{2}$, so,

$$I = \frac{1}{2} \ln(\sec 2x + \tan 2x)$$

Using this result in (13),

$$V_1 = -\frac{1}{2}\left[\frac{1}{2}\ln(\sec 2x + \tan 2x) - \frac{1}{2}\sin 2x\right]$$

$$\boxed{V_1 = \frac{1}{4}\left[\sin 2x - \ln(\sec 2x + \tan 2x)\right].}$$

Finally, integrating (12) wrt x.

$$V_2 = \frac{1}{2}\int \sin 2x \, dx = -\frac{1}{4}\cos 2x.$$

$$\boxed{V_2 = -\frac{1}{4}\cos 2x.}$$

Note: we don't really need to take into account the constants of integration because we are looking for a particular solution that satisfies the nonhomogeneous eqn.

The desired particular solution is:

$$y_p(x) = v_1 y_1 + v_2 y_2$$

$$y_p(x) = \frac{1}{4}\left[\sin 2x - \ln(\sec 2x + \tan 2x)\right] \cdot \cos 2x - \frac{1}{4}\cos 2x \cdot \sin 2x$$

Simplifying:

$$y_p(x) = \cancel{\frac{1}{4}\sin 2x \cos 2x} - \frac{1}{4}\ln(\sec 2x + \tan 2x)\cos 2x \cancel{- \frac{1}{4}\cos 2x \sin 2x}$$

$$\therefore y_p = -\frac{1}{4}\ln(\sec 2x + \tan 2x)\cos 2x.$$

The G.S. is $y(x) = y_h + y_p$

$$\boxed{y(x) = C_1 \cos 2x + C_2 \sin 2x - \frac{1}{4}\ln(\sec 2x + \tan 2x)\cos 2x} \quad (14)$$

Note: To solve for c_1, c_2 we need initial conditions. These would take the form $y(x_0) = y_0$, $y'(x_0) = y_0'$ } apply these in (14) to solve for c_1, c_2.

ALTERNATIVE METHOD OF SOLUTION.

In lecture 16, we derive the formulas for $v_1(x)$ & $v_2(x)$ for the general ODE: $y'' + p(x)y' + q(x)y = f(x)$

These are: $$v_1 = \int \frac{-y_2 \cdot f(x)}{W(x)}\,dx \quad (15) \qquad \& \qquad v_2 = \int \frac{y_1 f(x)}{W(x)}\,dx. \quad (16)$$

where $W(x)$ is the Wronskian of y_1, y_2.

Now, given that we have the homogeneous solution, y_1, y_2, and $W(x)$ are known functions of x. Also, $f(x)$ is the forcing term on the RHS of the ODE – also known.

So using (15) & (16), we can solve for v_1, v_2.

Let's find the Wronskian first.

The fundamental solns to the homog. ODE are $y_1 = \cos 2x$, $y_2 = \sin 2x$ } from Eq. (3)

$$\therefore W(x) = \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix} = 2\cos^2 2x + 2\sin^2 2x = 2.$$

In (15),

$$v_1 = -\int \frac{\sin 2x \cdot \tan 2x}{2}\, dx = -\int \sin 2x \cdot \frac{\sin 2x}{2\cos 2x}\, dx$$

$$\boxed{v_1 = -\frac{1}{2}\int \frac{\sin^2 2x}{\cos 2x}\, dx} \quad (17)$$

The same expression was derived using the variation of parameters method from first principles.

In (16) $$v_2 = \int \frac{\cos 2x \cdot \tan 2x}{2}\, dx = \int \cos 2x \cdot \frac{\sin 2x}{2\cos 2x}\, dx$$

$$\boxed{v_2 = \frac{1}{2}\int \sin 2x\, dx} \quad (18)$$

Carrying out the integrals in (17) & (18), will give us the same particular soln as the first method. You may use whatever method you prefer.