MATH 33B: DIFFERENTIAL EQUATIONS APRIL 2013

Example: constant coefficients, second order linear ODEs.

Find the general solution to the following ODEs:

(i) $y'' - y' - 2y = 0$
(ii) $y'' + 2y' + 17y = 0$
(iii) $4y'' + 4y' + y = 0$.

In all the ODEs above the differential eqn is of the form:

$$\boxed{y'' + \boxed{p}\,y' + \boxed{q}\,y = \boxed{0}} \quad (1)$$

$= f(x)$

constants.

second order linear ODE with constant coefficients

A solution that satisfies Eq. (1) (for constant p, q and $f(x) = 0$), is the exponential fct, i.e. $y = e^{mx}$ (where m is to be found)

The starting point for solving the ODEs in (i), (ii) & (iii) is to write down the characteristic eqn for each ODE.
The soln for part (i) includes a preliminary step (Step 0) which may be skipped when solving such ODEs.

(i) $y'' - y' - 2y = 0$ (2)

Step 0: Try a soln, $y = e^{mx}$ (3).

Diff (3): $y' = me^{mx}$ (4) & $y'' = m^2 e^{mx}$ (5)

Sub. ③, ④ & ⑤ in ②:

$$m^2e^{mx} - me^{mx} - 2e^{mx} = 0$$

$$e^{mx}(m^2 - m - 2) = 0 \quad ⑥$$

Since $e^{mx} \neq 0$, ⑥ reduces to

$$\boxed{m^2 - m - 2 = 0}$$ ← this is known as the characteristic equation

Step 1: Write down the characteristic eqn.

$$m^2 - m - 2 = 0 \quad ⑦$$

Comparing this to the ODE ②,
- m^2 corresponds to the y'' term
- $-m$ corresponds to $-y'$
- -2 corresponds to $-2y$.

Step 2: Find roots of ⑦

from ⑦, $(m-2)(m+1) = 0 \Rightarrow \left.\begin{array}{l} m_1 = 2 \\ m_2 = -1 \end{array}\right\}$ real + distinct

Step 3: Construct general soln.

We first need a pair of fundamental solns. Since the soln we tried was $y = e^{mx}$ and we have two roots m_1, m_2, it makes sense to write the 2 solns as:

$$y_1(x) = e^{\overset{=m_1}{\boxed{2}}x} \quad \text{and} \quad y_2(x) = e^{\overset{=m_2}{\boxed{-1}}x}$$

The G.S. is a linear comb. of y_1 & y_2:

$$\boxed{y(x) = c_1e^{2x} + c_2e^{-x}}$$ where c_1, c_2 are constants.

(ii) $y'' + 2y' + 17y = 0.$ ①

Step 1: write down characteristic equation

$m^2 + 2m + 17 = 0$ ②

Step 2: find roots of ②

$m = \frac{-2 \pm \sqrt{[4 - 68]}}{2}$ } complex + distincts

$m_1 = -1 + 4i$ and $m_2 = -1 - 4i$

Step 3: Construct general solution

The G.S. is $y(x) = c_1 e^{m_1 x} + c_2 e^{m_2 x}$ ③

Now, $e^{(-1+4i)x} = e^{-x}(\cos 4x + i \sin 4x)$ ④

$e^{(-1-4i)x} = e^{-x}(\cos 4x - i \sin 4x)$ ⑤

Sub. ④ e ⑤ in ③

$y(x) = e^{-x}\left[\cos 4x\,(c_1 + c_2) + i \sin 4x\,(c_1 - c_2)\right]$

or $\boxed{y(x) = e^{-x}(k_1 \cos 4x + k_2 \sin 4x).}$ A G.S. where complex numbers don't explicitly appear is preferred since we're looking for real-valued functions

(iii) $4y'' + 4y' + y = 0.$ ①

Step 1: Write down char. eqn

$4m^2 + 4m + 1 = 0$ ②

Step 2: Find roots of ②

$$(2m+1)^2 = 0$$

$$m = -\frac{1}{2} \quad \text{(repeated root).}$$

Step 3: Construct G.S.

In this case we can only obtain one fundamental soln using the characteristic eqn. and that is

$$y_1 = e^{-\frac{1}{2}x}. \quad ③$$

The second soln takes the form $\boxed{y_2 = x y_1}$ ⊛

Note: You may either remember the result given by ⊛ (which is true when we only have a single, repeated root) or derive it using the method below:

If one of the fund. solutions is known, then the second one can be found by assuming that y_2 takes the form:

$$y_2 = v(x) y_1(x) \quad ④$$ (the reason behind this idea, is that we need y_2 to be a nonconstant multiple of y_1)

Diff. ④ & sub. in ①

$$y_2' = v y_1' + y_1 v' \quad \& \quad y_2'' = v y_1'' + 2 y_1' v' + y_1 v''$$

$$4\underbrace{(v y_1'' + 2 y_1' v' + y_1 v'')}_{= y_2''} + 4\underbrace{(v y_1' + y_1 v')}_{= y_2'} - \underbrace{v y_1}_{= y_2} = 0 \quad ⑤$$

Factorize v, v' & v'' in ⑤

$$v(4y_1'' + 4y_1' - y_1) + v'(8y_1' + 4y_1) + 4y_1 v'' = 0. \quad ⑥$$

In (6), the coef. of v i.e. $4y_1'' + 4y_1' - y_1$ must be zero. if y_1 satisfies the ODE: $4y'' + 4y' - y = 0$.

So (6) reduces to:

$$v'(8y_1' + 4y_1) + 4y_1 v'' = 0 \quad (7)$$

Let us now diff. y_1 wrt x:

$$y_1 = e^{-\frac{1}{2}x} \longrightarrow y_1' = -\frac{1}{2}e^{-\frac{1}{2}x}.$$

Sub y_1 & y_1' in (7) gives:

$$v'\left(\cancel{-4e^{-\frac{1}{2}x}} + \cancel{4e^{-\frac{1}{2}x}}\right) + 4e^{-\frac{1}{2}x} v'' = 0$$

$$\therefore v'' = 0 \quad (8) \quad \text{since } 4e^{-\frac{1}{2}x} \neq 0.$$

<u>Integ. (8) wrt x twice</u>

$v = k_1 x + k_2$ (for any nonzero k_1 & any k_2. We choose $k_2 = 0$, $k_1 = 1$ to obtain the simplest form of $v(x)$ — remember this must be a nonconstant fct of x!)

$\Rightarrow v(x) = x$

and $y_2 = v(x)y_1(x) \Rightarrow y_2 = xe^{-\frac{1}{2}x}$.

$\Rightarrow$ the G.S. is

$$\boxed{y(x) = c_1 e^{-\frac{x}{2}} + c_2 x e^{-\frac{x}{2}}}$$