Math 33b, Winter 2013, Tonći Antunović - Homework 5

From the textbook solve the problems:
  Section 4.3: 12, 14, 20, 22, 28, 34, 36;
  Section 4.4: 8, 10, 12, 14, 18;
  Section 4.5: 2, 4, 6, 8

And also the problems below:

Problem 1. Solve the initial value problem

$$9y'' - 6y' + y = 0, \quad y(0) = 3, \ \ y'(0) = 2.$$

Solution: The characteristic equation is $9\lambda^2 - 6\lambda + 1 = 0$ which has only one solution $\lambda = 1/3$, so the general solution is to the equation alone is

$$y = C_1 e^{t/3} + C_2 t e^{t/3}.$$

Since $y' = \frac{C_1}{3} e^{t/3} + C_2 e^{t/3} + \frac{C_2}{3} t e^{t/3}$ the initial conditions give $C_1 = 3$ and $\frac{C_1}{3} + C_2 = 2$ which gives $C_1 = 3$ and $C_2 = 1$. So the solution is

$$y = 3e^{t/3} + te^{t/3}.$$

Problem 2. Solve the initial value problem

$$y'' = -16y, \quad y(0) = 1/2, \ \ y'(0) = 2.$$

Solution: The equation $y'' + 16y = 0$ has the characteristic equation $\lambda^2 + 16 = 0$ which has solutions $\lambda_1 = 4i$ and $\lambda_2 = -4i$ so the general solution is to the equation alone is

$$y = C_1 \cos(4t) + C_2 \sin(4t).$$

Since $y' = -4C_1 \sin(4t) + 4C_2 \cos(4t)$ the initial conditions give $C_1 = 1/2$ and $4C_2 = 2$ which gives

$$y = \frac{1}{2} \cos(4t) + \frac{1}{2} \sin(4t).$$

Problem 3. An object is moving on a spring without any resistance. The amplitude is $A = 2$, the phase $\phi = \pi/4$ and period is 4. Find the initial position and the initial velocity of the object (at time $t = 0$).

Solution: The solution can be written in the form $y = A\cos(w_0 t - \phi)$, where A is the amplitude, ϕ is the phase and w_0 is the natural frequency. Since period is $2\pi/w_0$ we get $2\pi/w_0 = 4$ and $w_0 = \pi/2$ and we can write

$$y(t) = 2\cos(\pi t/2 - \pi/4).$$

Since $y'(t) = -\pi\sin(\pi t/2 - \pi/4)$ the initial position is $y(0) = 2\cos(-\pi/4) = \sqrt{2}$ and the initial velocity is $y'(0) = -\pi\sin(-\pi/4) = \pi/\sqrt{2}$.

Problem 4. An object is attached to a spring and is moving in viscous liquid. Assume that the system is overdamped. We pull the object away from the equilibrium position and release it from rest. Show that after it's released, the object will never reach the equilibrium position.

Solution: Since the system is overdamped the solution is of the form

$$y = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t},$$

where λ_1 and λ_2 are necessarily negative characteristic roots. Assume without loss of generality that $\lambda_1 < \lambda_2 < 0$. We have $y = C_1\lambda_1 e^{\lambda_1 t} + C_2\lambda_2 e^{\lambda_2 t}$ and since the initial velocity is zero we have

$$C_1\lambda_1 + C_2\lambda_2 = 0, \quad \Rightarrow \quad C_2 = -\frac{\lambda_1}{\lambda_2}C_1,$$

so we can write the solution as

$$y = C_1\Big(e^{\lambda_1 t} - \frac{\lambda_1}{\lambda_2}e^{\lambda_2 t}\Big).$$

When the object passes through the equilibrium position the value of y is equal to zero, which can happen only if

$$e^{\lambda_1 t} - \frac{\lambda_1}{\lambda_2}e^{\lambda_2 t} = 0 \quad \Rightarrow \quad e^{(\lambda_1 - \lambda_2)t} = \frac{\lambda_1}{\lambda_2}.$$

However, this can't happen for any value $t > 0$ since $\lambda_1 - \lambda_2 < 0$ so $e^{(\lambda_1-\lambda_2)t} < 1$ for any $t > 0$ and on the other hand by the assumption $\lambda_1 < \lambda_2 < 0$ it holds that $\frac{\lambda_1}{\lambda_2} > 1$.