1 Problem

We have two bucket each full, and containing 500 gal water. Pure water enters the first bucket at a rate of 5 gal / sec, mixes perfectly, and flows into the second bucket at a rate of 5 gal / sec. Mixed water leaves the second bucket at a rate of 5 gal / sec. Initially the first bucket has 100 lbs salt and the second has pure water.

2 General Solution

As usual for a mixing problem we want to write

$$\text{Rate} = \text{Rate In} - \text{Rate Out}$$

For the first bucket this is just

$$x_0'(t) = 0 - 5\frac{gal}{sec} \cdot \frac{x_0(t)lbs}{500gal}$$

The second is

$$x_1'(t) = 5\frac{gal}{sec} \cdot \frac{x_0(t)lbs}{500gal} - 5\frac{gal}{sec} \cdot \frac{x_1(t)lbs}{500gal}$$

So we have the matrix system

$$\begin{pmatrix} -\frac{1}{100} & 0 \\ \frac{1}{100} & -\frac{1}{100} \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \end{pmatrix} = \begin{pmatrix} x_0' \\ x_1' \end{pmatrix}$$

Now that we have a system we need to compute the eigenvalues. Write the Characteristic equation

$$|A - \lambda I| = \begin{vmatrix} -\frac{1}{100} - \lambda & 0 \\ \frac{1}{100} & -\frac{1}{100} - \lambda \end{vmatrix} = \left(-\frac{1}{100} - \lambda\right)^2 = 0$$

So, we have one real eigenvalue $\lambda = -1/100$ with multiplicity 2. Now to find a corresponding eigenvector we just compute

$$\begin{pmatrix} -\frac{1}{100} & 0 \\ \frac{1}{100} & -\frac{1}{100} \end{pmatrix} \begin{pmatrix} v_0 \\ v_1 \end{pmatrix} = -\frac{1}{100} \begin{pmatrix} v_0 \\ v_1 \end{pmatrix}$$

So, $v_0 = 0$. Let's choose our eigenvector to be

$$v = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Now we want to find a 'generalized eigenvector' w satisfying

$$(A - \lambda I)w = v$$

Computing, we must have

$$\begin{pmatrix} -\frac{1}{100} - (-\frac{1}{100}) & 0 \\ \frac{1}{100} & -\frac{1}{100} - (-\frac{1}{100}) \end{pmatrix} \begin{pmatrix} w_0 \\ w_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

or equivalently,

$$\frac{1}{100} w_0 = 1$$

So let's choose

$$w = \begin{pmatrix} 100 \\ 0 \end{pmatrix}$$

Now we can write the general solution using our formula as

$$x(t) = e^{\lambda t}\left((C_1 + C_2 t)v + C_2 w\right) = e^{-t/100}\left((C_1 + C_2 t)\begin{pmatrix} 0 \\ 1 \end{pmatrix} + C_2 \begin{pmatrix} 100 \\ 0 \end{pmatrix}\right)$$

3 Particular Solution / Analysis

We can split the general solution back into two equations:

$$x_0(t) = 100C_2e^{-t/100} \quad x_1(t) = (C_1 + C_2t)e^{-t/100}$$

Recall that we had as initial condition $x_0(0) = 100, x_1(0) = 0$. The first gives us

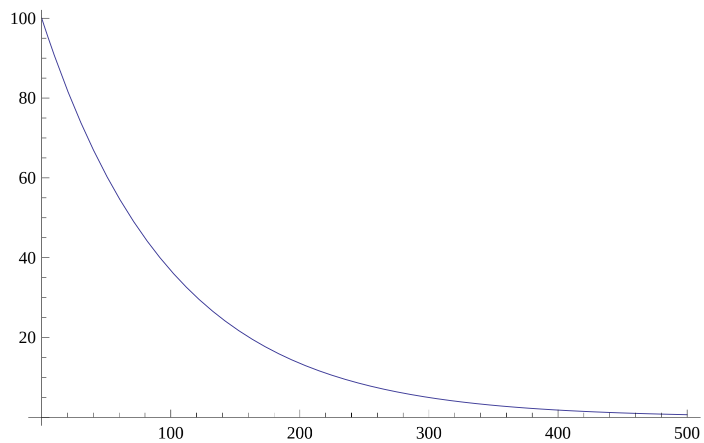
$$x_0(0) = 100C_2e^{0/100} = 100 \Rightarrow C_2 = 1$$

and from the second we have

$$x_1(0) = (C_1 + C_2 \cdot 0)e^{0/100} = C_1 = 0$$

So the particular solution is:

$$x_0(t) = 100e^{-t/100}$$



$$x_1(t) = te^{-t/100}$$