Math 33b, Winter 2013, Tonći Antunović - Homework 6

From the textbook solve the problems:
Section 4.5: 18, 20, 22, 26, 28, 32, 34, 36, 38, 42
Section 4.6: 2, 8, 14

And also the problems below:

Problem 1. Find the general solution of the equation

$$y'' + 2y' + y = e^t - \sin t.$$

Solution: General solution to the homogeneous equation $y''+2y'+y=0$ is $y = C_1e^{-t} + C_2te^{-t}$. First we find a particular solution for $y''+2y'+y=e^t$ and we search it in the form $y_{p,1} = ae^t$ and we get $ae^t + 2ae^t + ae^t = e^t$ so $a = 1/4$ and $y_{p,1} = e^t/4$. Then search for particular solution $y_{p,2}$ for $y''+2y'+y = -\sin t$ and search for it in the form $y_{p,2} = b\sin t + c\cos t$. We get $y'_{p,2} = b\cos t - c\sin t$ and $y''_{p,2} = -b\sin t - c\cos t$ so

$$-b\sin t - c\cos t + 2b\cos t - 2c\sin t + b\sin t + c\cos t = -\sin t,$$

which gives $-2c = -1$ and $2b = 0$, so $y_{p,2} = \frac{1}{2}\cos t$. The general solution is then

$$y = C_1e^{-t} + C_2te^{-t} + \frac{1}{4}e^t + \frac{1}{2}\cos t.$$

Problem 2. Find the solution of the initial value problem

$$y'' - 4y' + 4y = -e^{2t}, \quad y(0) = 0, \ \ y'(0) = 1.$$

Solution: The general solution to the homogeneous equation $y'' - 4y' + 4y = 0$ is $y = C_1e^{2t} + C_2te^{2t}$. Particular solution y_p can't be of the form ae^{2t} or ate^{2t} so we set $y_p = at^2e^{2t}$ which gives $y'_p = ae^{2t}(2t^2 + 2t)$ and $y''_p = ae^{2t}(4t^2 + 8t + 2)$. Then

$$ae^{2t}(4t^2 + 8t + 2 - 8t^2 - 8t + 4t^2) = -e^{2t} \quad \Rightarrow \quad a = -1/2$$

so $y_p = -\frac{1}{2}t^2e^{2t}$ and the general solution is

$$y = e^{2t}(C_1 + C_2t - \frac{1}{2}t^2).$$

To find C_1 and C_2 use initial conditions. Since $y' = e^{2t}(2C_1 + C_2 + (2C_2 - 1)t - t^2)$ we get equations

$$C_1 = 0, \quad 2C_1 + C_2 = 1,$$

which gives $C_1 = 0$ and $C_2 = 1$ and the solution is

$$y = e^{2t}(t - \frac{1}{2}t^2).$$

Problem 3. Find the solution of the initial value problem

$$y'' + 2y' - 3y = 13\sin(2t) - 2\sin t, \qquad y(0) = 0, \;\; y'(0) = 0.$$

Solution: The general solution to $y'' + 2y' - 3y = 0$ is $y = C_1e^{-3t} + C_2e^t$. The particular solution $y_{p,1}$ for $y'' + 2y' - 3y = 13\sin(2t)$ in the form $y_{p,1} = a\sin(2t) + b\cos(2t)$ we get $y'_{p,1} = 2a\cos(2t) - 2b\sin(2t)$ and $y''_{p,1} = -4a\sin(2t) - 4b\cos(2t)$ and so

$$-4a\sin(2t) - 4b\cos(2t) + 4a\cos(2t) - 4b\sin(2t) - 3a\sin(2t) - 3b\cos(2t) = 13\sin(2t),$$

so $-7a - 4b = 13$ and $4a - 7b = 0$ which gives $a = -7/5$ and $b = -4/5$ and

$$y_{p,1} = -\frac{7}{5}\sin(2t) - \frac{4}{5}\cos(2t).$$

For a particular solution for $y'' + 2y' - 3y = -2\sin t$ in the form $y_{p,2} = c\sin t + d\cos t$ we get $y'_{p,2} = c\cos t - d\sin t$ and $y''_{p,2} = -c\sin t - d\cos t$ and

$$-c\sin t - d\cos t + 2c\cos t - 2d\sin t - 3c\sin t - 3d\cos t = -2\sin t,$$

so we get $-4c - 2d = -2$ and $2c - 4d = 0$ and so $c = 2/5$ and $d = 1/5$. Therefore, $y_{p,2} = \frac{2}{5}\sin t + \frac{1}{5}\cos t$. The general solution is then

$$y = C_1e^{-3t} + C_2e^t - \frac{7}{5}\sin(2t) - \frac{4}{5}\cos(2t) + \frac{2}{5}\sin t + \frac{1}{5}\cos t.$$

Since

$$y = -3C_1e^{-3t} + C_2e^t + \frac{8}{5}\sin(2t) - \frac{14}{5}\cos(2t) - \frac{1}{5}\sin t + \frac{2}{5}\cos t.$$

the initial conditions give $C_1 + C_2 = 4/5 - 1/5 = 3/5$ and $-3C_1 + C_2 = 14/5 - 2/5 = 12/5$ and so $C_1 = -9/20$ and $C_2 = 21/20$ and the solution is

$$y = -\frac{9}{20}e^{-3t} + \frac{21}{20}e^t - \frac{7}{5}\sin(2t) - \frac{4}{5}\cos(2t) + \frac{2}{5}\sin t + \frac{1}{5}\cos t.$$

Problem 4. Write down a second order linear differential equation with constant coefficients whose general solution is $y = e^{-t}(C_1 \sin t + C_2 \cos t) + e^t$.

Solution: Let the equation be $y'' + py' + qy = f$. From the solution we see that $e^{-t}(C_1 \sin t + C_2 \cos t)$ is a general solution for the homogeneous equation $y'' + py' + qy = 0$. For this we need the characteristic roots to be $-1 + i$ and $-1 - i$, and the characteristic equation to be $(\lambda - (-1 + i))(\lambda - (-1 - i)) = 0$ which is the same as $(\lambda + 1)^2 + 1 = 0$ or $\lambda^2 + 2\lambda + 2 = 0$. Therefore the homogeneous equation should be $y'' + 2y' + 2y = 0$, so $p = 2$ and $q = 2$. We also need e^t to be a particular solution of $y'' + 2y' + 2y = f$, so $f(t) = e^t + 2e^t + 2e^t = 5e^t$ and so the equation is $y'' + 2y' + 2y = 5e^t$.

Problem 5. Consider the differential equation

$$y'' + py' + qy = f(t),$$

where p and q are constants. Let y_1, y_2 and y_3 be three solutions of this equation such that $y_1 - y_2$ and $y_1 - y_3$ are linearly independent. Show that the general solution of this equation is $y = (1 + C_1 + C_2)y_1 - C_1 y_2 - C_2 y_3$.

Solution: By linearity and using the computations from the class one can check that $y_1 - y_2$ and $y_1 - y_3$ are solutions to the homogeneous equation $y'' + py' + qy = 0$, and since they are independent the general solution of $y'' + py' + qy = 0$ is $y = C_1(y_1 - y_2) + C_2(y_1 - y_3)$. Since y_1 is a particular solution to $y'' + py' + qy = f(t)$ the general solution to this equation is

$$y = C_1(y_1 - y_2) + C_2(y_1 - y_3) + y_1 = (1 + C_1 + C_2)y_1 - C_1 y_2 - C_2 y_3.$$