Math 33B: Differential Equations

Review 1: Plotting direction fields

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1 Introduction

Definition of tangent lines: if $y = f(x)$ defines a function of x whose derivative, $\frac{dy}{dx}$ exists on an interval $\mathbb{I} : a < x < b$ (where, a and b are constants), then $\frac{dy}{dx}$ will give the slope of the graph of this function at each point whose x coordinate is in $\mathbb{I}$.

Direction fields give a geometric view of the differential equation (DE) and help us visualize the solutions to the DE and their late-time behavior.

2 Falling object example

Consider a particle of mass m kg falling under the action of gravity. It is assumed that only two forces act on the subject: air resistance (F_1) and gravity (F_2), as shown in Fig. 1.

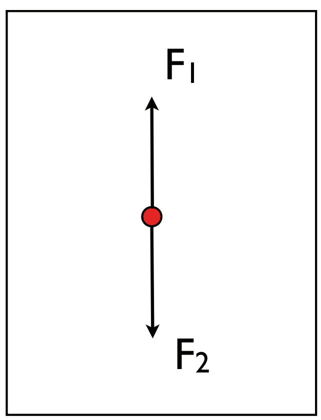

Figure 1: Falling object of constant mass m showing two forces acting on it: F_1 denotes the air resistance force and F_2 represents gravity.

The total force, F acting on the object is given by the sum of the two forces,

$$F = (-F_1) + F_2; \tag{1}$$

note we choose the upward direction to be negative, $-F_1$ while the downward direction is positive, F_2. By *Newton's second law*, the total force acting on the object is directly proportional to the acceleration, a produced by it,

$$F = ma. \tag{2}$$

We take the air resistance force to be proportional to the velocity of the object, v, which is in turn taken to be a function of time, t such that $v = v(t)$. Using c as the constant of proportionality, the air resistance may be modelled by,

$$F_1 = cv(t), \tag{3}$$

while the gravitational force is given by,

$$F_2 = mg, \tag{4}$$

and g is the gravitational acceleration which will be approximated here to be equal to $10\,\mathrm{m/s^2}$; these forces are shown in Fig. 2.

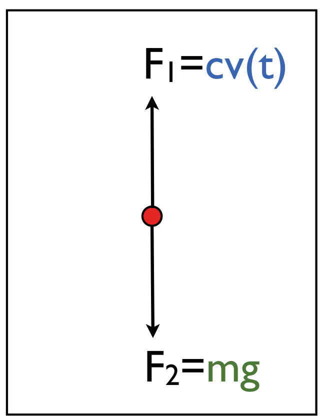

Figure 2: Labelled force balance for a falling object.

Equations (1)-(4) may be combined to give,

$$F = -cv(t) + mg = ma. \tag{5}$$

Now, since acceleration represents the rate of change of velocity with time

$$a = \frac{dv(t)}{dt}, \tag{6}$$

then, Eq. (5) may be expressed as a differential equation for $v(t)$,

$$\frac{dv(t)}{dt} = \frac{mg - cv(t)}{m}. \tag{7}$$

Let us assume that the object has a mass of 2 kg and $c = 4$ kg/s; then, Eq. (8) becomes

$$\frac{dv(t)}{dt} = 10 - 2v(t). \tag{8}$$

Note that the dependence of t in Eq. (8) only shows up *implicitly* through $v(t)$ [see notes in Section 3.3]. Now, any solution to this ordinary differential equation (ODE) will have $10-2v$ as the gradient at a particular value of the velocity, v. We illustrate this by choosing a set of values for v and simply drawing arrows (or line segments) at each v - along the t-axis - with slope $10-2v$. Suppose that at some time t, the velocity becomes 5 m/s; plugging in $v = 5$ in Eq. (8), gives $\frac{dv}{dt} = 0$. This tells us that if the velocity of the object happens to reach $v = 5$ at any point then its acceleration will be equal to 0 (i.e. moving at constant velocity). This is illustrated by drawing arrows along the $v = 5$ line with a slope equal to 0; i.e. the arrows are *horizontal*. The direction field corresponding to the zero-slope arrows is shown in Fig. 3 for $0 \le t \le 10$ and $0 \le v \le 10$.

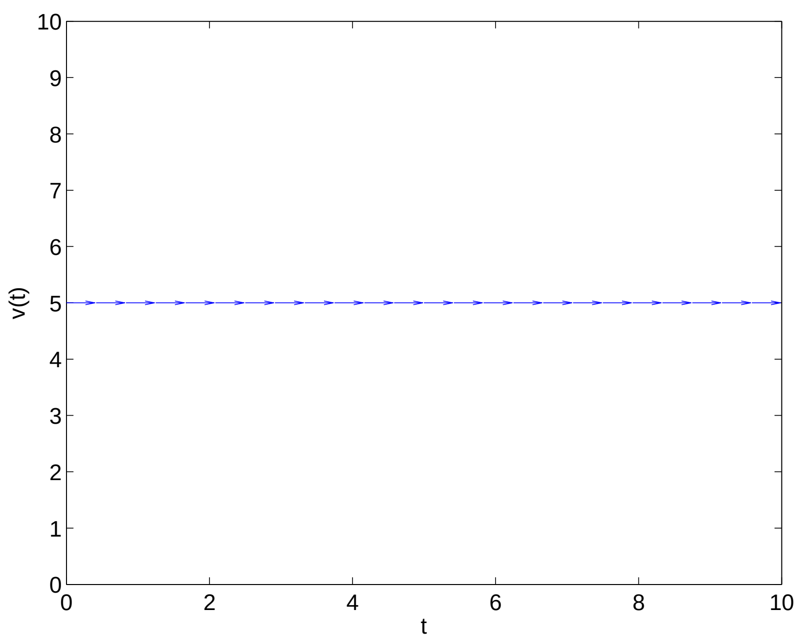

Figure 3: Direction arrows corresponding to the ODE given by Eq. (8) for $v = 5$ m/s.

Let us now assume that at some time t, the velocity becomes $v = 6$ m/s. Plugging in $v = 6$

in Eq. (8), gives a negative gradient, $\frac{dv}{dt} = -2$. Remember that $\frac{dv}{dt}$ gives a measure of the acceleration of the object and therefore a negative value would mean that at the particular velocity (here, $v = 6$ m/s), the velocity of the object must be decreasing. Similarly, a positive gradient would imply that the velocity of the object is increasing. In Fig. 4 the direction field corresponding to $v = 6$ is added to the existing arrow field shown in Fig. 3.

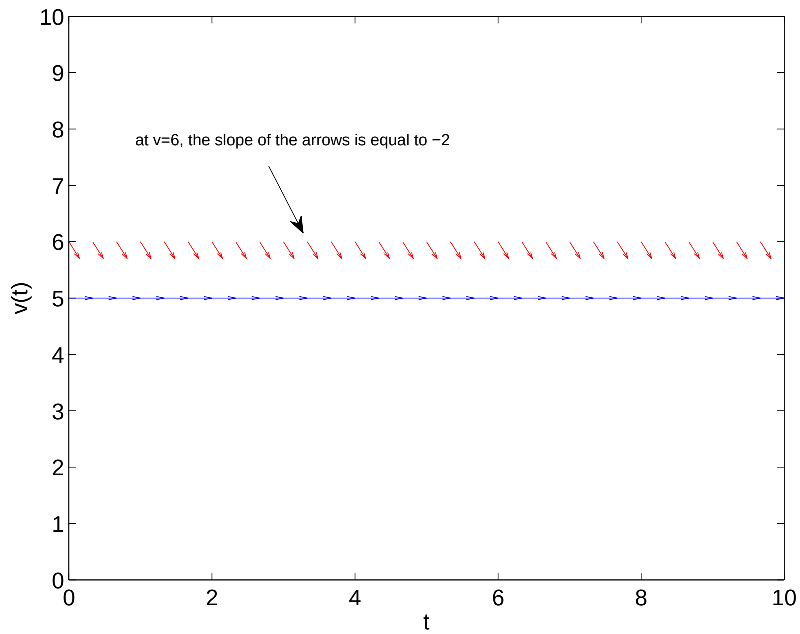

Figure 4: Direction field corresponding to the ODE given by Eq. (8) for $v = 5$ m/s (blue arrows) and $v = 6$ m/s (red arrows).

Finally, for $v = 4$, the value of the gradient is equal to $\frac{dv}{dt} = 2$. This gives a positive slope corresponding to the direction field given by the green arrows in Fig. 5

Adding more arrows for various values of v, allows us to visualize the direction field within the entire interval of interest which, in turn, helps us understand how the solutions to the ODE behave. Figure 6 shows various arrows, generated for different values of v. Note that both direction field plots in panels (a) and (b) are equivalent; arrowheads are not always included in direction fields (your textbook for example chooses to omit them). Technically speaking, the plot in panel (b) is called a 'slope field' as no *direction* is indicated by the line segments.

Now, let us go back to how a direction field helps us visualize the ODE solutions. We know that the arrows in the direction field are tangents to the actual solutions to the differential

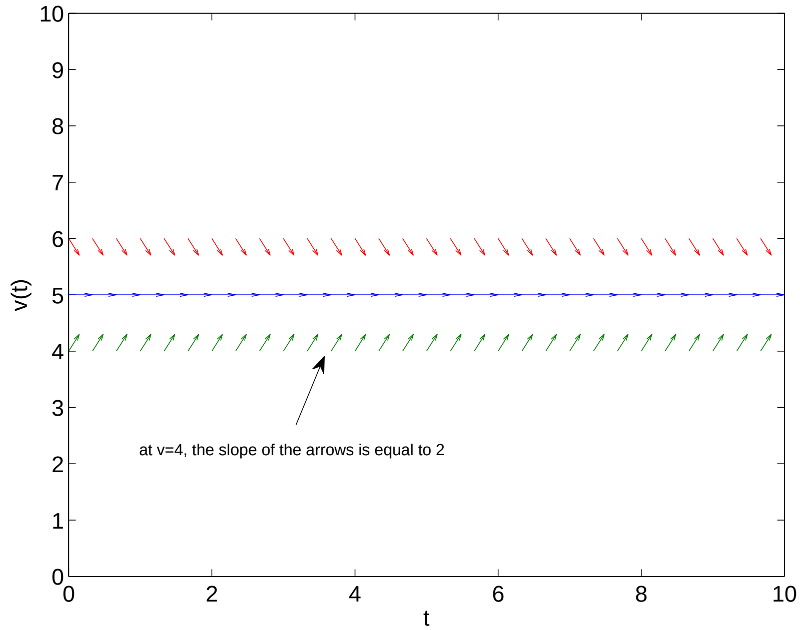

Figure 5: Direction field corresponding to the ODE given by Eq. (8) for $v = 5$ m/s (blue arrows), $v = 6$ m/s (red arrows) and $v = 4$ m/s (green arrows).

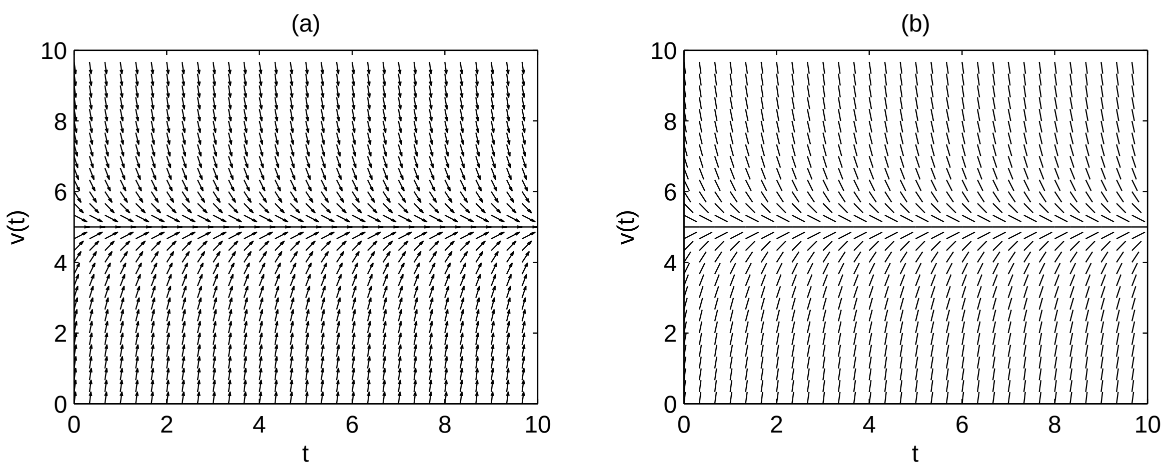

Figure 6: Direction field for the ODE given by Eq. (8). The slopes of the line segments are identical in both panels; the arrows in panel (a) indicate slope **and** direction while in panel (b) the line segments merely indicate slope.

equations and can therefore be used to sketch the solutions. Suppose we are given that at $t = 0$, the velocity is $v = 4$ (this is an initial condition). We know that the solution there should be increasing (because the gradient is positive); we therefore follow the direction of the arrows and, as the gradient becomes less and less positive, the solution becomes less and less steep. Three solutions (integral curves) are shown in Fig. 7 where the initial conditions are as follows: for the **red** curve, $v(0) = 6$, for the **black** curve, $v(0) = 5$ and for the **green** curve, $v(0) = 4$. From Fig. 7 it is evident that all solutions (regardless of the initial condition) approach $v = 5$ as t increases.

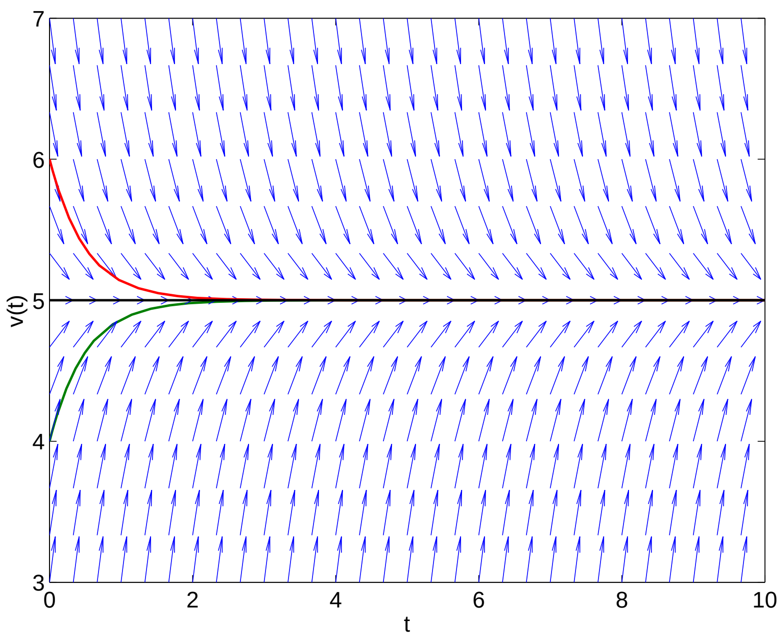

Figure 7: Solutions to the differential equation (8) shown by black curves and the associated direction field.

3 The method of isoclines

In the previous example in Section 2, it was fairly simple to plug in values in the differential equation to calculate the gradient and hence the slope of the arrows. Now, take a differential equation of the form $\frac{dy}{dx} = f(x, y)$, where x is the *independent* variable and y is a function of x, i.e. $y = y(x)$. Assuming that $f(x, y)$ is a complicated function of x and y, then, the method we have just used to obtain the direction field will prove to be somewhat lengthy. The method of *isoclines* gives us a quick way to obtain the direction field corresponding to a particular ODE. An isocline is a set of points (a line or curve) where *all* the direction field arrows that lie on it have the same slope.

Algebraically, we obtain the isoclines by solving the following equation

$$\frac{dy}{dx} = f(x, y) = c, \tag{9}$$

where c is a constant [please note that this c is in no way associated with the proportionality constant in Eq. (3)].

We will use the following ODE as an example to illustrate how to use the method of isoclines to draw the direction field corresponding to a DE,

$$\frac{dy}{dx} = -\frac{x}{y}. \tag{10}$$

3.1 Method

1. Set the differential equation of the form $\frac{dy}{dx} = f(x, y)$ to a constant c.

   Here,

   $$\frac{dy}{dx} = -\frac{x}{y} = c, \tag{11}$$

   which rearranged gives,

   $$y = -\frac{x}{c} \quad \text{or} \quad cy = -x. \tag{12}$$

2. Draw several isoclines by choosing different c values; e.g. in Eq. (12) for $c = 1$ the isocline is given by the straight line $y = -x$ [see Fig. 8(a)].

3. On each isocline, draw line segments (or arrows) along the line each having a slope c. For example, in Fig. 8(b), the isocline is generated using $c = 1$ (from Step 2) and therefore the line segments drawn **on** the line are of slope 1.* In Fig. 8(c), we draw another isocline corresponding to $c = -1$ which yields the equation of the new isocline as $y = x$. The line segments drawn on this isocline need to have a slope of $c = -1$. We continue to plot isoclines to find the regions of equal slope, gradually generating the direction field corresponding to the differential equation. Figure 8(d) shows the direction field in $-5 \leq x \leq 5$ and $-5 \leq y \leq 5$ (note that only a few isoclines are shown here).

*note that in this case, the isocline and the line segments are perpendicular to each other because the product of their slopes is equal to -1.

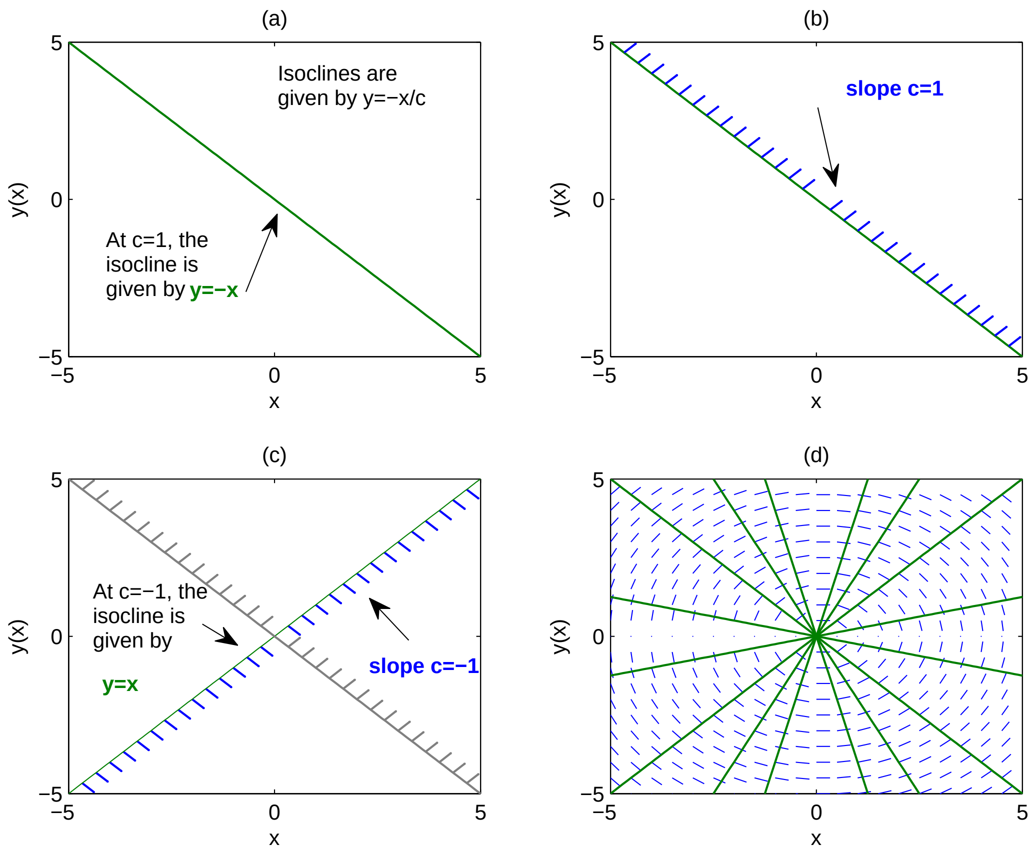

Figure 8: (a) Isocline corresponding to $c = 1$ in Eq. (12) with equation $y = -x$. (b) Isocline [from panel (a)] and line segments of slope $c = 1$. (c) Additional isocline for $c = -1$ (equation of line: $y = x$) and line segments (blue) with slope $c = -1$. (d) Direction field to ODE (10) and several isoclines at different values of c.

3.2 Visualizing the solution

Now that we have the direction field, we may choose a particular point in (x, y) and start drawing the solution keeping in mind that it should be tangent to the line segments/arrows. We can probably see that the solution forms a circle; see Fig. 9 for a solution to the ODE (10) superimposed on the direction field.

As discussed in lectures, for a particular solution to be valid it needs to be continuous **and** satisfy the initial condition. The derivative is discontinuous at $y = 0$ [see Eq. (10)]. The solution shown in Fig. 9 is actually made up of two semi-circles; the first one (**red**) is in the positive y-axis and was generated using $(1, 2)$ as the initial condition while the second one (**black**) was generated using $(1, -2)$. Which semi-circle is to be accepted as the solution will ultimately depend on the initial conditions.

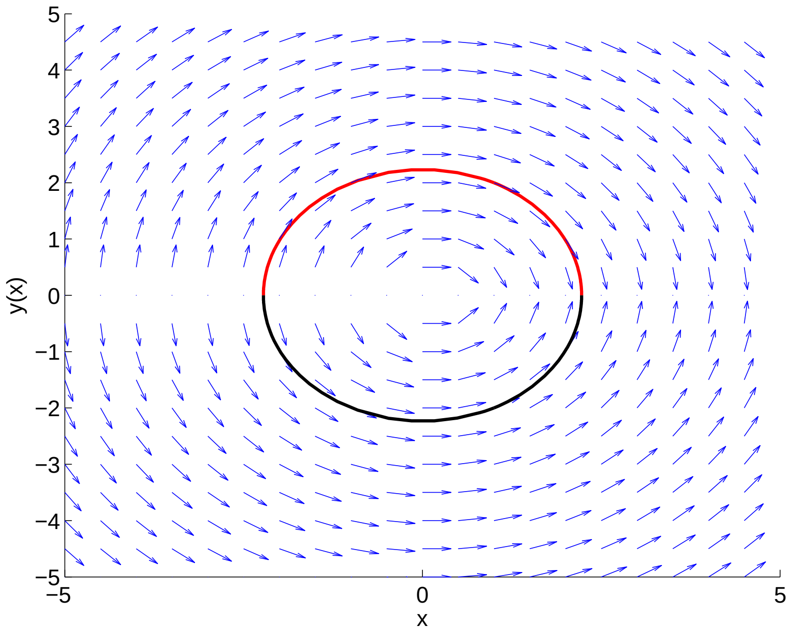

Figure 9: Direction field to ODE (10) and the corresponding solution (red curve) generated using $(1, 2)$ as the initial condition and a second solution (black curve) generated using $(1, -2)$ as the initial condition.

3.3 General notes

If we were to use the *method of isoclines* for the falling object example in Section 2, then the isoclines to the ODE (8) would all be *horizontal*. In fact, any ODE of the form,

$$\frac{dy}{dx} = f(y) \quad \text{(note } \textit{no explicit } x\text{-dependence in } f\text{)} \tag{13}$$

has horizontal isoclines. The slope of the line segments on the isoclines will of course depend on the corresponding value of c.

Similarly any ODE of the form,

$$\frac{dy}{dx} = f(x) \quad \text{(note } \textit{no explicit } y\text{-dependence in } f\text{)} \tag{14}$$

has vertical isoclines.

The shape of the isoclines corresponding to differential equations that take the more general form,

$$\frac{dy}{dx} = f(x, y) \tag{15}$$

depend on the actual form of the ODE.