M1 - 1

MATH 33B: Midterm Exam 1 — April 2013

QUESTION 1

$$(x^2 - y^2) + xy\frac{dy}{dx} = 0 \quad (1) \quad ; \quad y(1) = 1$$

Use $y(x) = v(x) \cdot x$. (2)

Diff (2) wrt x

$$\frac{dy}{dx} = v + x\frac{dv}{dx} \quad (3)$$

Sub (2) & (3) in (1)

$$(x^2 - x^2v^2) + x^2 v \underbrace{\left(v + x\frac{dv}{dx}\right)}_{=\frac{dy}{dx}} = 0$$

$$x^2 - \cancel{x^2v^2} + \cancel{x^2v^2} + x^3 v\frac{dv}{dx} = 0$$

$$x^2 = -x^3 v\frac{dv}{dx} \quad (4)$$

Separating variables & integrating

$$\int -v\,dv = \int \frac{1}{x}\,dx \quad \Rightarrow \quad -\frac{v^2}{2} = \ln x + C$$

$$\boxed{v^2 = K - 2\ln x} \quad \text{where } K = -2C$$

implicit gen. soln for $v(x)$

M1-2

Back in y variable space

$$\frac{y^2}{x^2} = k - 2\ln x$$

$\Rightarrow y^2 = x^2(k - 2\ln x)$ implicit gen. soln. for $y(x)$.

Apply I.C : $y(\phi) = 1$

$1 = (k - 0) \Rightarrow k = 1$

$\therefore \; y^2 = x^2(1 - 2\ln x)$

$\therefore \; y = \pm\sqrt{x^2(1 - 2\ln x)}$

we choose the +ve part of the soln as this satisfies $y(1)=1$

$$\boxed{y(x) = x\sqrt{1 - 2\ln x}}$$

QUESTION 2

$$(x - xy^2) + (6y - x^2y)\frac{dy}{dx} = 0. \quad (1)$$

Where $P(x,y) = x - xy^2$ & $Q(x,y) = 6y - x^2y$.

For exactness, we need $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$.

$\dfrac{\partial P}{\partial y} = -2xy$ and $\dfrac{\partial Q}{\partial x} = -2xy$

$\Rightarrow \dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x} = -2xy$

$\Rightarrow$ (1) is exact.

M1-3

Since ① is exact, it may be expressed as:

$$\frac{d}{dx}\left[f(x,y)\right] = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dx} = 0 \quad (*)$$

where $\frac{\partial f}{\partial x} = x - xy^2$ ② & $\frac{\partial f}{\partial y} = 6y - x^2y$. ③

Find $f(x,y)$

From ②, we integ. wrt x (keeping y const.)

$$f(x,y) = \int x - xy^2\,dx = \frac{x^2}{2} - \frac{x^2y^2}{2} + k_1(y) \quad ④$$

And integ. ③ wrt y (keeping x-constant)

$$f(x,y) = \int 6y - x^2y\,dy = 3y^2 - \frac{x^2y^2}{2} + k_2(x) \quad ⑤$$

By comparing ④ & ⑤: $k_1(y) = 3y^2$ & $k_2(x) = \frac{x^2}{2}$

Using ④, $\boxed{f(x,y) = \frac{x^2}{2} - \frac{x^2y^2}{2} + 3y^2}$

Having found $f(x,y)$, going back to (*),

$$\frac{d}{dx}\left[\frac{x^2}{2} - \frac{x^2y^2}{2} + 3y^2\right] = 0 \quad ⑥$$

Integ. ⑥ wrt x: $\frac{x^2}{2} - \frac{x^2y^2}{2} + 3y^2 = C$. ⑦

Apply I.C.: $2 - 8 + 12 = C \Rightarrow C = 6$

M1-4

In (7) $\frac{x^2}{2} - \frac{x^2y^2}{2} + 3y^2 = 6$

$$y^2\left(3 - \frac{x^2}{2}\right) = 6 - \frac{x^2}{2}$$

$$y^2(6-x^2) = (12-x^2)$$

$$y^2 = \frac{12-x^2}{6-x^2} \Rightarrow y = \pm\sqrt{\frac{(12-x^2)}{(6-x^2)}} \quad (8)$$

Choose the negative soln in (8) to satisfy $y(2) = -2$

$$\therefore \boxed{y = -\left[\frac{(12-x^2)}{(6-x^2)}\right]^{1/2}}$$

QUESTION 3

$$\frac{d}{dx}\left(\ln[\sec x + \tan x]\right) = \frac{1}{\sec x + \tan x} \cdot (\sec x \tan x + \sec^2 x)$$

$$= \frac{\sec x\cancel{(\sec x + \tan x)}}{\cancel{(\sec x + \tan x)}} \quad //.$$

Divide ODE by $\cos x$ to put in S.F.

$$\frac{dy}{dx} + \sec x \cdot y = \sec x - \frac{\sin x}{\cos x}.$$

$$\frac{dy}{dx} + \underbrace{\boxed{\sec x}}_{} y = \boxed{\sec x - \tan x} \quad (1)$$

$\sec x = P(x)$, $\sec x - \tan x = f(x)$

Find an I.F, $\mu(x) = e^{\int P(x)\,dx}$

$$e^{\int p(x)dx} = e^{\int \sec x\,dx} = e^{\ln[\sec x + \tan x]}$$

← using the result we were asked to show.

M1-5

So, $\mu(x) = \sec x + \tan x$. (2)

Multiply (1) by (2)

(3) $(\sec x + \tan x)\frac{dy}{dx} + \sec x(\sec x + \tan x)y = (\sec x + \tan x)(\sec x - \tan x)$

Use: $\frac{d}{dx}\left[(\sec x + \tan x)\cdot y\right] = (\sec x + \tan x)\frac{dy}{dx} + y\cdot(\sec x \tan x + \sec^2 x)$ ✓ (4)

Reduce LHS of (3) using (4)

$$\frac{d}{dx}\left[(\sec x + \tan x)y\right] = \sec^2 x - \tan^2 x. \quad (5)$$

By making use of the identity $\tan^2 x + 1 = \sec^2 x$
$\Rightarrow \sec^2 x - \tan^2 x = 1$,

we rewrite (5) as:

$$\frac{d}{dx}\left[(\sec x + \tan x)y\right] = 1$$

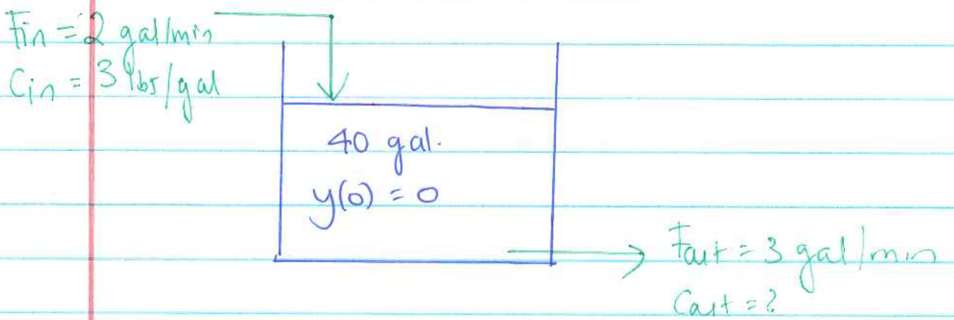
Integ. wrt x.,

$$(\sec x + \tan x)y = x + k.$$

$$\boxed{y = \frac{x+k}{\sec x + \tan x}.}$$ general solution.

M1-6

QUESTION 4



(a) $\frac{dy}{dt}$ = Rate in − Rate out

$$\frac{dy}{dt} = F_{in} C_{in} - \frac{F_{out} \cdot y}{V(t)} \quad \text{where } V(t) = \overset{=40}{V_0} + (F_{in} - F_{out})t$$

$$\Rightarrow \boxed{\frac{dy}{dt} = 6 - \frac{3y}{40-t}, \quad y(0) = 0}$$

[You may solve using I.F or by sep. variables].

Rewrite in S.F. $\quad \frac{dy}{dt} + \underbrace{\boxed{\frac{3}{40-t}}}_{}\overset{}{}y = 6. \quad$ ① $\quad$ ($=p(t)$)

The I.F is $\mu(t) = e^{\int \frac{3}{40-t} dt} = e^{-3\ln(40-t)} = \frac{1}{(40-t)^3}$. ②

Multiply ① by ②

$$\frac{1}{(40-t)^3}\frac{dy}{dt} + \frac{3}{(40-t)^4} y = \frac{6}{(40-t)^3} \quad ③$$

which is equivalent to: $\frac{d}{dt}\left[\frac{1}{(40-t)^3} \cdot y\right] = \frac{6}{(40-t)^3}$. ④

M1-7

Check that the LHS of ④ is equal to the LHS of ③.

$$\frac{d}{dt}\left[\frac{1}{(40-t)^3}\cdot y\right] = \frac{1}{(40-t)^3}\frac{dy}{dt} + y\left[\frac{-3}{(40-t)^4}\cdot(-1)\right]$$ ✓

Integ. ④ wrt t:

$$\frac{y}{(40-t)^3} = \int \frac{6}{(40-t)^3}\,dt = \frac{6}{(40-t)^2}\cdot\left(\frac{-1}{2}\right)(-1) + k$$

$$y = 3(40-t) + k(40-t)^3 \quad ⑤$$

Apply I.C. $y(0) = 120 + k(40)^3 = 0$

$$\therefore k = \frac{-120}{40^3}.$$

In ⑤, $\boxed{y(t) = 3(40-t) - \frac{120}{40^3}(40-t)^3}$ ⑥

(b) When the vol. is reduced to 20 gal:

$V(t) = 40 - t = 20 \Rightarrow t = 20$ mins.

Plugging in $t=20$ in ⑥:

$$y(20) = 3(20) - \frac{120}{40^3}\cdot(20)^3$$

$$= 60 - 120\left(\frac{1}{2}\right)^3$$

$$= 60 - \frac{120}{8}$$

$$= 60 - 15$$

$$\boxed{y(20) = 45 \text{ lbs.}}$$

M1-8

Question 5

$$\frac{dy}{dt} = \underbrace{-ay(b-y)(c-y)}_{=f(y)} \quad , \; y(0) = y_0.$$

(a) The eqm pts occur at $f(y) = 0$

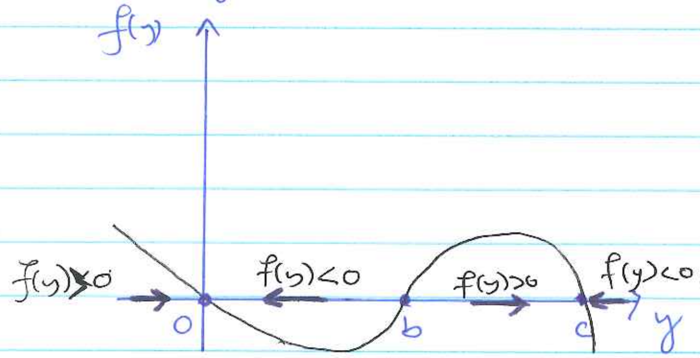
$\therefore \; -ay(b-y)(c-y) = 0$

$$\boxed{y=0, \; y=b, \; y=c}$$

Plot $f(y)$ vs y

if $\frac{dy}{dt} = f(y) < 0 \Rightarrow$ solns decrease

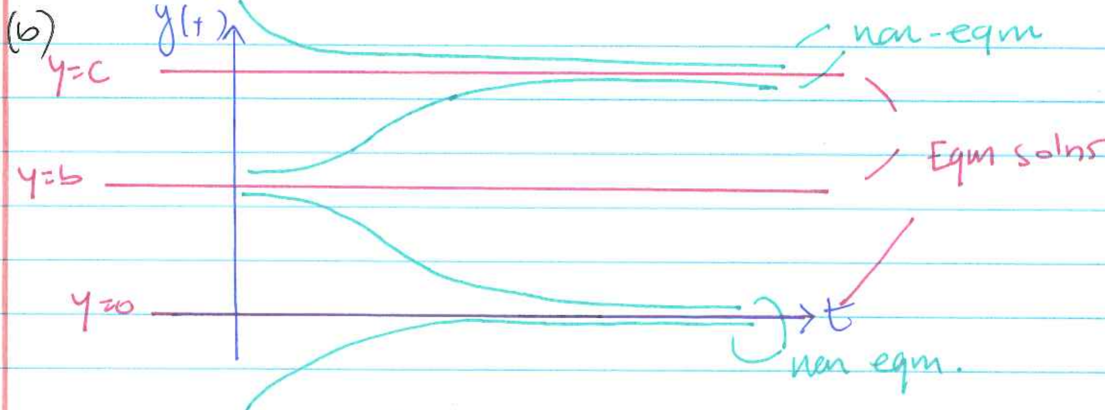
if $\frac{dy}{dt} = f(y) > 0 \Rightarrow$ solns increase.



The c.p.

$y=0$	is	$\rightarrow \bullet \leftarrow$	stable
$y=b$	is	$\leftarrow \bullet \rightarrow$	unstable
$y=c$	is	$\rightarrow \bullet \leftarrow$	stable

(b)



M1-9

$y=0$ would represent extinction. If $y_0<b$ the soln approach $y=0$ so we need y_0 to be initially above b such that $y=c$ is approached.

$\rightarrow \boxed{y_0>b}$