Math 33B: Differential Equations

Homework 3: Existence and uniqueness of IVPs & Autonomous equations

Due on: Fri., Apr. 26, 2013 - 9:00 AM *Instructor: Aliki M.*

Please include your *name*, *UID* and *discussion section* on the submitted homework.

Problem 1

Which of the following initial-value-problems (IVPs) is guaranteed to have a unique solution?

(i) $\dfrac{dy}{dx} = 1 + y^2 \quad \text{with} \quad y(0) = 1$

(ii) $\dfrac{dy}{dx} = \sqrt{y} \quad \text{with} \quad y(2) = 0$

Problem 2

Show that the IVP given by

$$\frac{dy}{dx} = 3y^{2/3} \quad \text{with} \quad y(0) = 0,$$

has two solutions given by $y(x) = 0$ and $y(x) = x^3$. Explain why the fact that there exists more than one solution does not contradict the uniqueness theorem.

Problem 3

Determine the initial conditions $y(a) = b$ for which a unique solution is guaranteed to the IVP given by,

$$\frac{dy}{dx} = \sqrt{y - x} \quad \text{with} \quad y(a) = b.$$

Problem 4

Consider the ODE,

$$x\frac{dy}{dx} - y = x^2 \cos x.$$

(i) Find the general solution and sketch several integral curves.

(ii) Show that there exists no solution to the IVP with $y(0) = -3$ and explain why this does not contradict the hypotheses stated in the existence and uniqueness theorems.

Problem 5

Consider the ODE,

$$\frac{dy}{dx} + 2y = 1.$$

(i) The ODE is autonomous. Find the critical point and determine stability.

(ii) Sketch some solutions (including the equilibrium solution).

(iii) Find the general solution $y(x)$ by separating variables.

Problem 6

Suppose the population of cod fish is described by the logistic equation which is modified to include the rate at which fish are caught. The latter is given by the term hy. This process is described by the following ODE

$$\frac{dy}{dt} = r(1 - y/K)y - hy;$$

where r, K and h are positive constants and $y(t)$ describes the species population. Note that the rate at which fish are caught depends on y.

(i) Show that if $h < r$, there are two critical points, $y = y_1$ and $y = y_2$. Find those points and plot the corresponding equilibrium solutions.

(ii) Show that one of the critical points is unstable while the other one is stable. Sketch some non-equilibrium solutions.

(iii) Let Y denote the sustainable yield rate. This represents the rate at which fish can be caught indefinitely without the species going extinct.

Mathematically, this is given by the product of h and the stable equilibrium solution. Plot Y against h and determine the value of h such that Y is maximized. The corresponding value is known as the maximum sustainable yield.