MATH 33B: DIFFERENTIAL EQUATIONS   April 2013

Examples: Applying existence & uniqueness theorems

Ex. 1: Determine whether $\boxed{\frac{dy}{dx} = x\ln(y)}$ is guaranteed to have a unique solution passing through the point $y(1)=e$

The ODE is expressed in the form $\frac{dy}{dx} = f(x,y)$ where

$f(x,y) = x\ln(y)$

The function $f(x,y)$ is continuous everywhere except where $y \leq 0$. There are no discontinuities that involve x.
$\Rightarrow$ by the existence thm $\exists$ at least one solution inside $-\infty < x < \infty$.

$\therefore$ To address the uniqueness of the solution, we determine $\frac{\partial f}{\partial y}$.

$$\frac{\partial f}{\partial y} \Rightarrow \frac{\partial}{\partial y}\left(x\ln y\right) = \frac{x}{y}.$$

$\frac{\partial f}{\partial y}$ is discont. where $y=0$

$\Rightarrow$ by the uniqueness thm, the solution will be unique as long as we avoid $y=0$.

The initial condition: the problem asks if there will be a unique solution passing through the pt $y(1)=e$

We have shown above that the values we want to avoid are $y \leq 0$ for existence and $y=0$ for uniqueness.

Since, however, $e > 0$ we are guaranteed by the hypotheses of the theorems that a unique soln. exists, somewhere within $-\infty < x < \infty$.

Ex. 2: Using the existence & uniqueness theorems, determine whether the following IVP has a unique solution.

$$\frac{dy}{dt} = y^{4/5} \quad \text{with } y(0) = 0$$

Here, $f(t,y) = y^{4/5}$ and $\dfrac{\partial f}{\partial y} = \dfrac{4}{5} y^{-1/5}$

While $f(t,y)$ is continuous at any point (hence by the existence thm, at least one solution exists), $\dfrac{\partial f}{\partial y}$ is discontinuous at $y = 0$.

Since for the initial condition $y(0) = 0$, $\dfrac{\partial f}{\partial y}$ is discontinuous we cannot conclude whether there exists a unique solution to the IVP.

Let us proceed to solve for the soln $y(t)$ of the IVP.

Sep variables, $\displaystyle\int y^{-4/5}\, dy = \int dt$

$$5y^{1/5} = t + k.$$

$$y(t) = \left(\frac{t+k}{5}\right)^5 = \frac{(t+k)^5}{3125} \quad \text{general soln.}$$

Applying the I.C: $\quad y(0) = \dfrac{k^5}{3125} = 0 \quad \Rightarrow k = 0$

Therefore, the particular soln is:

$$\boxed{y(t) = \frac{t^5}{3125}.} \quad (*)$$

Now (*) obviously passes through (0,0). However the IVP has another soln that satisfies both the ODE and I.C and that's given by $\boxed{y=0}$. This second solution is known as the equilibrium solution to the autonomous ODE. (see notes from LECTURE 8 + LECTURE 9).

$\therefore$ There are two solutions satisfying the IVP: $y=0$ (for all values of t!)

$$y = \frac{t^5}{3125}$$