1. (a) Find the general solution of the system $\mathbf{y}' = A\mathbf{y}$, where

$$A = \begin{pmatrix} -1 & -2 \\ 4 & 3 \end{pmatrix}$$

The characteristic polynomial is given by

$$\lambda^2 - 2\lambda + 5$$

The eigenvalues are $\lambda = 1 + 2i$ and $\bar{\lambda} = 1 - 2i$. The eigenvector, $\mathbf{w}$, can be found from $(A - \lambda I)\mathbf{w} = 0$. One such eigenvector is $\mathbf{w} = \begin{pmatrix} -1 \\ 1+i \end{pmatrix}$. We can write the general solution as

$$\mathbf{y}(t) = C_1 e^t \begin{pmatrix} -\cos 2t \\ \cos 2t - \sin 2t \end{pmatrix} + C_2 e^t \begin{pmatrix} -\sin 2t \\ \sin 2t + \cos 2t \end{pmatrix}$$

(b) Find the solution to the initial value problem for the equation in (a) with

$$\mathbf{y}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

We plug in the initial condition to find the values of C_1 and C_2,

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} = C_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} + C_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

$C_1 = 0$ and $C_2 = 1$.

$$\mathbf{y}(t) = e^t \begin{pmatrix} -\sin 2t \\ \sin 2t + \cos 2t \end{pmatrix}$$

2. Consider a mass, spring, damper system with spring constant, $k = 1$, damping coefficient, $c = 3$, and mass, $m = 10$. The motion of the system is given by $my'' + cy' + ky = 0$.

(a) Solve the system for the motion of the mass with $y(0) = 5$ and $y'(0) = -v_o$.

The characteristic equation is given by

$$m\lambda^2 + c\lambda + k = 10\lambda^2 + 3\lambda + 1 = 0$$

The roots are given by $\lambda = -3/20 \pm i\sqrt{31}/20$, and the solution is given by

$$y(t) = e^{\frac{-3t}{20}} \left(C_1 \cos \frac{\sqrt{31}t}{20} + C_2 \sin \frac{\sqrt{31}t}{20} \right).$$

We can plug in the initial conditions to get the values of $C_1 = 5$ and $C_2 = \left(\frac{3}{4} - v_o\right)\frac{20}{\sqrt{31}}$. The solution is then given by

$$y(t) = e^{\frac{-3t}{20}}\left(5\cos\frac{\sqrt{31}t}{20} + \left(\frac{3}{4} - v_o\right)\frac{20}{\sqrt{31}}\sin\frac{\sqrt{31}t}{20}\right).$$

(b) Is this system over-damped, under-damped, or critically damped?

The roots of the characteristic equation are complex, so the system is underdamped.

(c) Sketch what the solution would look like on a y-t plot.