MATH33B: DIFFERENTIAL EQUATIONS

May 1st, 2013

Example: constant coefficient, non-homogeneous ODEs

$$y'' + py' + qy = \boxed{f(x)}$$ forcing term.

CASE 1: $f(x)$ is an exponential function.

Note: these are the examples we looked at during Lecture 13 on Wednesday, May 1st & Friday, May 3rd.

Example 1

$$y'' - 3y' - 4y = 3e^{2x} \quad (1)$$

(1) is a constant coefficient, nonhomogeneous, second-order ODE with $f(x) = 3e^{2x}$.

The G.S. to (1) takes the form: $\boxed{y(x) = y_h + y_p}$ (2) (proof in L-13)

y_h is the soln to the HOMOGENEOUS ODE (i.e. when $f(x)=0$)

y_p is a particular soln to the NONHOMOGENEOUS ODE (i.e. Eq (1)).

Step 1: Find y_h

(1) reduces to $y'' - 3y' - 4y = 0$ (3) when $f(x) = 0$.

To find y_h, we solve (3).

The characteristic eqn for (3) is $m^2 - 3m - 4 = 0$

And the roots are: $(m-4)(m+1)=0$

$m=4$ or $m=-1$ roots are real + distinct

$\therefore \boxed{y_h = c_1 e^{4x} + c_2 e^{-x}}$ ④

Step 2: Find y_p

We use the method of undetermined coefficients (M.U.C). Having y_h isn't really needed to apply the method, but we'll see in Example 2 that it's useful to have it.

The M.U.C requires a little bit of 'guessing'. Educated guessing perhaps.

The term $f(x)$ in ① is $f(x)=3e^{2x}$.

For y_p we guess a similar form to $f(x)$. In the case of exponential $f(x)$ we want the exponent i.e. $2x$ to be the same in $f(x)$ and our guess for y_p.

$\Rightarrow$ Guess $y_p = Ae^{2x}$ ⑤

this is the undetermined coefficient

If y_p is a soln to the nonhomogeneous ODE ① then it must satisfy it:

$$y_p'' - 3y_p' - 4y_p = 3e^{2x}. \quad ⑥$$

We take ⑤, diff. wrt x to obtain y_p', y_p''

$y_p' = 2Ae^{2x}$ and $y_p'' = 4Ae^{2x}$

In ⑥, $\underbrace{4Ae^{2x}}_{=y_p''} - 3\underbrace{(2Ae^{2x})}_{=y_p'} - 4\underbrace{Ae^{2x}}_{=y_p} = 3e^{2x}$

Simplifying,

$$e^{2x}\left(\cancel{4A} - 6A - \cancel{4A}\right) = 3e^{2x}.$$

Comparing coef. of e^{2x} on both sides

$$-6A = 3 \Rightarrow \boxed{A = -\frac{1}{2}}$$

Now, that we have A, we go back to 5:

$\boxed{y_p = -\frac{1}{2}e^{2x}}$ this is a particular soln that satisfies the nonhomog. eqn. 1.

Using 2, we construct the G.S to 1

$$\boxed{y(x) = c_1 e^{4x} + c_2 e^{-x} - \frac{1}{2}e^{2x}.}$$

Note: As for homogeneous eqns, we will need 2 initial conditions here to solve for c_1 & c_2.

Example 2

We will look at a similar ODE in this example:

$$y'' - 3y' - 4y = 4e^{4x}. \quad (7)$$

The G.S is given by $y(x) = y_h + y_p$.

Step 1: Find y_h

We already have this from example 1 (note that the LHS of 7 is the same as the LHS of 1).

$\therefore\ y_h = c_1 e^{4x} + c_2 e^{-x}$. ⑧

Step 2: Find y_p

For the particular soln, we guess a soln for y_p by looking at the $f(x)$ term.

The $f(x)$ term here is $f(x) = 4e^{4x}$.

So, our guess will be $y_p = Ae^{4x}$ ⑨ ← note we want to match the exponent in $f(x)$ for our guess of y_p.

We proceed as in example 1:

Take ⑨, diff. wrt x to obtain y_p', y_p''

$y_p' = 4Ae^{4x}$ and $y_p'' = 16Ae^{4x}$.

In ⑦, $y_p'' - 3y_p' - 4y_p = 4e^{4x}$ ← Again, if y_p is a soln to ⑦ it must satisfy it.

$\Rightarrow\ 16Ae^{4x} - 3(4Ae^{4x}) - 4Ae^{4x} = 4e^{4x}$.

Simplifying, $e^{4x}\underbrace{(16A - 12A - 4A)}_{=0} = 4e^{4x}$

The LHS vanishes (and so do all the A's) so we can't determine the value of A.
⇒ This implies our initial guess, $y_p = Ae^{4x}$, wasn't good enough.

This is where having the homogeneous soln, y_h, becomes useful.
If we compare y_h to our guess for y_p, we have:

$y_h = \boxed{c_1 e^{4x}} + c_2 e^{-x}$ and $y_p = \boxed{Ae^{4x}}$

Our guess, $y_p = Ae^{4x}$, matches one of the fundamental solutions.

→ Essentially, c_1e^{4x} is a <u>constant multiple</u> of Ae^{4x} (since both c_1 & A are constants).

→ This means that the 2 solns are <u>linearly dependent</u>

In view of this, we choose to multiply our initial guess by x:

$$y_p = Axe^{4x}$$ <u>, where A is still to be determined</u>

And try again:

$$y_p' = A(4xe^{4x} + e^{4x}) \quad \text{and} \quad y_p'' = A(16xe^{4x} + 8e^{4x})$$

In 7, $$A(16xe^{4x} + 8e^{4x}) - 3A(4xe^{4x} + e^{4x}) - 4Axe^{4x} = 4e^{4x}.$$

<u>Simplifying,</u>

$$xe^{4x}(\cancelto{0}{16A - 12A - 4A}) + e^{4x}(8A - 3A) = 4e^{4x}$$

<u>Comparing coef. of e^{4x} on both sides.</u>

$$5A = 4 \Rightarrow A = 4/5.$$

And back into our second guess, ∴ Particular soln is

$$\boxed{y_p = \frac{4}{5}xe^{4x}.}$$

<u>The G.S to 7 is:</u>

$$\boxed{y(x) = \underbrace{\boxed{c_1e^{4x} + c_2e^{-x}}}_{= y_h} + \underbrace{\boxed{\frac{4}{5}xe^{4x}}}_{= y_p}}$$

Notes:

- As a general rule if y_p matches any of the 2 solutions in y_h then we multiply our initial guess by x.

- Bear in mind, that you may need to multiply by higher powers of x.

- It's useful to have the homogeneous soln, y_h, but it is not required when trying to find y_p. Having y_h will allow you to 'guess' the correct form of y_p.

- The M.U.C works well but only for a limited class of functions: The RHS of the nonhomogeneous eqn, $f(x)$ must be:

  (i) an exponential. (covered in this example + Lecture 13)

  (ii) trigonometric fct ($a\sin bx$, $a\cos bx$) } in Lecture 14.

  (iii) polynomial

MATH 33B: DIFFERENTIAL EQUATIONS - May 2013

Example: constant coefficient, non-homogeneous ODEs:

$$y'' + py' + qy = f(x)$$

CASE 2: $f(x)$ is a sine and/or cosine term.

Example

$$y'' - 3y' - 4y = 2\sin x. \quad (1)$$

For nonhomogeneous linear ODEs, the general solution takes the form $y(x) = y_h + y_p$. (2)

Step 1: Find y_h (the soln to the homogeneous ODE)

$$y'' - 3y' - 4y = 0$$

char. eqn is $m^2 - 3m - 4 = 0$

$(m-4)(m+1) = 0 \Rightarrow m_1 = 4, \; m_2 = -1$ } roots are real + distinct.

$\boxed{y_h = c_1e^{4x} + c_2e^{-x}}$ (3)

Step 2: Find y_p (a particular soln to (1))

We need to guess an appropriate form for y_p.

The form of $f(x)$ should be our first hint: $f(x) = 2\sin x$.

Ignoring the constant, the functional form of $f(x)$ is $\sin x$.

Our first guess is to have $\boxed{y_p = A\sin x}$ ④ (where A is to be determined).

This should be a good guess unless the chosen form of y_p contains a term that we already have in the homogeneous solution. In this case it doesn't.

As we've seen in lecture and as discussed in the notes (Review 9), the form of y_p given by ④ won't help us determine A.

The correct form to use is $\boxed{y_p = A\sin x + B\cos x.}$ ⑤.

Even if f(x) doesn't include a cosine term, we need to include it because it will appear once we diff. sinx once. When we diff. cos x we go back to $-\sin x$ (which is a constant multiple of the original function) and so we are good with just the two functions: sinx & cosx.

More formally, we need y_p to include all the linearly dependent derivatives of the chosen fct.

Once we've settled on what form y_p should take, we need to determine the undetermined coefficients: A & B.

Determine A & B.

Take ⑤ diff. wrt x: $y_p' = A\cos x - B\sin x$ ⑥

$$y_p'' = -A\sin x - B\cos x. \quad ⑦.$$

Sub. ⑤, ⑥ & ⑦ in ①

$$(-A\sin x - B\cos x) - 3(A\cos x - B\sin x) - 4(A\sin x + B\cos x) = 2\sin x. \quad ⑧$$

Eq. 8 needs to be satisfied if y_p is to be a soln to 1.

Rewrite 8 as:

$\sin x(-5A+3B) + \cos x(-5B-3A) = 2\sin x + 0\cos x.$

Compare coef. of $\sin x$ & $\cos x$ on both sides

$-5A+3B = 2$ (9)

$-5B-3A=0$ (10) $\Rightarrow -5B = 3A$

$B = -\frac{3}{5}A$

In (9), $-5A + 3\left(-\frac{3A}{5}\right) = 2$

$-5A - \frac{9A}{5} = 2$

$-\frac{34}{5}A = 2 \Rightarrow A = -5/17.$

$\therefore B = -\frac{3}{5}\left(\frac{-5}{17}\right) = 3/17.$

The desired particular soln is $y_p = \frac{-5}{17}\sin x + \frac{3}{17}\cos x.$

The G.S. to 1 is : $\boxed{y(x) = c_1e^{4x} + c_2e^{-x} - \frac{5}{17}\sin x + \frac{3}{17}\cos x.}$

MATH 33B: DIFFERENTIAL EQUATIONS MAY. 2013.

Example: Constant coefficient, nonhomogeneous ODEs

$$y'' + py' + qy = f(x)$$

CASE 3: $f(x)$ is a polynomial

Consider: $y'' - 3y' - 4y = x^2$. ①

where $f(x) = x^2$.

Just like in previous examples, the general soln to ① is made up of the homog soln y_h and a particular soln, y_p.

y_h satisfies $y'' - 3y' - 4y = 0$

and y_p satisfies ①.

$\Rightarrow \boxed{y(x) = y_h + y_p}$ ← general soln to ①.

Step 1: Get y_h

$y_h = c_1 e^{4x} + c_2 e^{-x}$ ② (see previous examples on CCLE)

Step 2: Get y_p

As mentioned earlier, y_p is a particular solution that satisfies the nonhomogeneous eqn ①.

The RHS of ①, $f(x) = x^2$, helps us specify the form y_p should take. In this case we have a polynomial of order 2.

The form of y_p therefore, should definitely include Ax^2

Now, as mentioned in class, notes and in example 2 (where $f(x) = 2\sin x$) the method of undetermined coefficients is guaranteed to work if y_p:

(i) does not contain a term that belongs in y_h

(ii) there exist a finite number of linearly dependent derivatives.

In this case, the linearly dependent derivatives of x^2 (ignoring coefficients) are:

$$x \quad , 1$$

Which means that the appropriate form of y_p should be

$$\boxed{y_p = Ax^2 + Bx + C} \quad (3)$$ (A, B, C are the undetermined coefficients)

Once we have the form of y_p, we want to check whether any terms in y_p match any of the fundamental solutions in y_h.

Since $y_h = c_1 e^{4x} + c_2 e^{-x}$, we are good to continue with (3) as our best guess for y_p:

Determine A, B & C

Take (3) and diff. wrt x: $\quad y_p' = 2Ax + B \quad (4)$

$$y_p'' = 2A \quad (5)$$

Substitute (3), (4) & (5) in (1)

$$2A - 3(2Ax + B) - 4(Ax^2 + Bx + C) = x^2$$

Collect like terms together

$$\therefore -4Ax^2 + x(-6A - 4B) + 2A - 3B - 4C = x^2.$$

Compare coefs of $x^2, x,$

of x^2: $-4A = 1 \Rightarrow A = -1/4$

of x^1: $-6A - 4B = 0$

$-6\left(\frac{-1}{4}\right) - 4B = 0 \Rightarrow B = 3/8$

of x^0: $2A - 3B - 4C = 0$

$2\left(\frac{-1}{4}\right) - 3\left(\frac{3}{8}\right) - 4C = 0 \Rightarrow C = \frac{-13}{32}$

$$\Rightarrow \boxed{y_p = -\frac{x^2}{4} + \frac{3x}{8} - \frac{13}{32}} \quad (6)$$

The gen. soln is: $y(x) = y_h + y_p$

$$\boxed{y(x) = c_1 e^{4x} + c_2 e^{-x} - \frac{x^2}{4} + \frac{3x}{8} - \frac{13}{32}} \quad (7)$$

Notes

(i) Don't confuse y_p (a particular soln) with the particular soln to the nonhomogeneous ODE

(ii) Eq. (7) is the general solution to (1) since c_1 and c_2 represent any two constants.

(iii) To find the particular soln to (1), we need two initial conditions, say, $y(x_0) = y_0$, $y'(x_0) = y'_0$ to determine c_1, c_2.

(iv) Finally, there are infinite other y_p solutions that would satisfy (1). We've just managed to find one using a relatively simple method.