PPI-1

MATH 33B: Worked out Practice Problems I April 2013

PROBLEM 1

$$\frac{dy}{dx} = \frac{3x^2}{3y^2-4} \quad (1), \quad y(1)=0.$$

Sep. variables & integrating,

$$\int (3y^2-4)\,dy = 3\int x^2\,dx.$$

$$y^3-4y = x^3 + C. \quad (1) \text{ general soln.}$$

Apply I.C

$$0 = 1 + C \Rightarrow C = -1$$

In (1), $y^3 - 4y = x^3 - 1$

$$\boxed{y^3-4y-x^3+1=0} \quad (2) \quad \text{particular soln.}$$

From (1), the derivative is defined everywhere where $3y^2-4 \neq 0$

at $3y^2-4=0$,

$y = \pm \frac{2}{\sqrt{3}}$ the tangent to the integral soln are completely vertical.

In order to define the interval of existence, we determine what x-values correspond to $y = \pm\frac{2}{\sqrt{3}}$ using (2):

$$\left(\pm\frac{2}{\sqrt{3}}\right)^3 - 4\left(\pm\frac{2}{\sqrt{3}}\right) = x^3 - 1$$

$$\pm\left(\frac{8}{3\sqrt{3}} - \frac{8}{\sqrt{3}}\right) = x^3 - 1$$

$$\mp\frac{16}{3\sqrt{3}} = x^3 - 1$$

(Notes: The coordinates $\left(\frac{-16}{3\sqrt{3}}, \frac{-2}{\sqrt{3}}\right)$ and $\left(\frac{16}{3\sqrt{3}}, \frac{2}{\sqrt{3}}\right)$ are the points where the soln has vertical tangents and hence the derivative is not defined)

$\Rightarrow$ The soln is defined within all values of x where $y(x)$ is differentiable:

$$\frac{-16}{3\sqrt{3}} < x^3 - 1 < \frac{16}{3\sqrt{3}}$$

or $$\boxed{|x^3 - 1| < \frac{16}{3\sqrt{3}}}$$

PROBLEM 2

$$\frac{dy}{dx} = \frac{x(x^2+1)}{4y^3} \quad (1) \quad ; \quad y(0) = -1/\sqrt{2}$$

(i) Sep. variables and integrating

$$4\int y^3\, dy = \int x^3 + x\, dx$$

$$y^4 = \frac{x^4}{4} + \frac{x^2}{2} + C \quad (2) \quad \text{gen. soln.}$$

Apply I.C

$$\frac{1}{4} = 0 + c \Rightarrow c = \frac{1}{4}$$

PPI-3

Back in ②,

$$y^4 = \frac{x^4}{4} + \frac{x^2}{2} + \frac{1}{4}$$

which is

$$y = \pm\left(\frac{x^4+2x^2+1}{\sqrt{2}}\right)^{1/4} = \pm\left[\frac{(x^2+1)^2}{\sqrt{2}}\right]^{1/4}$$

We know that $y(0) = -\frac{1}{\sqrt{2}}$, so we choose the negative part of the soln to be the one that satisfies the IVP:

$$\boxed{y(x) = -\frac{\sqrt{x^2+1}}{\sqrt{2}}} \quad ③$$

particular soln.

(ii) We determine the interval in which the soln the same way as in problem 1.

From the ODE ①, $\frac{dy}{dx}$ is undefined if $y=0$.

The soln ③ tells us that $y(x)$ can never be zero (b/c $x^2 \not< 0$) which means that the soln is well-defined for all values of x:

$$\boxed{-\infty < x < \infty}$$

PROBLEM 3

$$\frac{dy}{dx} = 2y(1-y) \quad ①$$

This is an autonomous eqn b/c the RHS of ① is only a function of the dependent variable, y.

PPI-4

Critical pts occur at $\frac{dy}{dx} = f(y) = 0$.

Where $f(y) = 2y(1-y)$.

$\Rightarrow 2y(1-y) = 0$ $\quad$ $y=0,\ y=1$ are the critical pts.

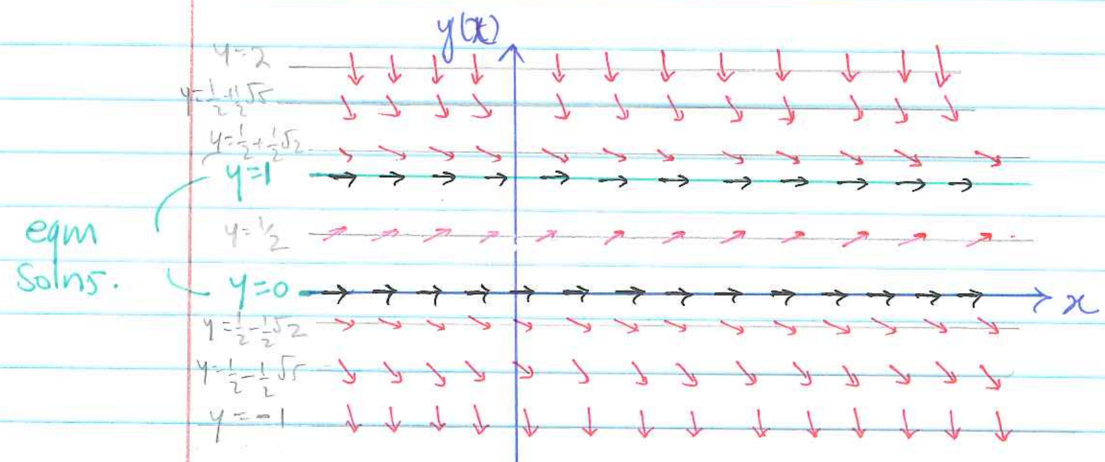
The corresponding eqm solns are $\left.\begin{array}{l} y=0 \\ y=1 \end{array}\right\}$ for all values of x.

Direction field

Using the method of isoclines, set $f(y)=k$ where k is a constant.

Then the equation of the isocline is given by:

$$y = \frac{1}{2} \pm \frac{1}{2}\sqrt{1-2k}$$



ISOCLINES

$k=0$, $y=1$ & $y=0$
$\Rightarrow$ eqm solns have arrows of slope 0, not suprising.

$k=\frac{1}{2}$, $y=\frac{1}{2}$

$k=-\frac{1}{2}$, $y=\frac{1}{2}\pm\frac{1}{2}\sqrt{2}$

$k=-2$, $y=\frac{1}{2}\pm\frac{1}{2}\sqrt{5}$

$k=-4$, $y=\frac{1}{2}\pm\frac{3}{2}$

Notes: 1) The direction field arrows have a −ve slope above $y=1$ and below $y=0$.

2) The arrows have a positive slope within $0<y<1$ and their slope can't exceed $k=\frac{1}{2}$ (this corresponds to the dir. field at $y=\frac{1}{2}$)

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PROBLEM 4

$$\frac{dy}{dx} + 2xy = xy^2. \quad (1)$$

Use $y = \frac{1}{u}$ (2) where $u = u(x)$

then, $\frac{dy}{dx} = \frac{-1}{u^2} \cdot \frac{du}{dx}$. (3)

Sub. (2), & (3) in (1)

$$\frac{-1}{u^2}\frac{du}{dx} + 2x\left(\frac{1}{u}\right) = \frac{x}{u^2}.$$

Multiply both sides by $-u^2$:

$$\boxed{\frac{du}{dx} - 2xu = -x.} \quad (4)$$

Rearrange (4): $\frac{du}{dx} = 2xu - x = x(2u-1)$ separable.

Sep. variables & integrating:

$$\int \frac{1}{2u-1}\,du = \int x\,dx.$$

$$\frac{1}{2}\ln(2u-1) = \frac{x^2}{2} + \ln k.$$

$$\ln\left(\frac{(2u-1)^{1/2}}{k}\right) = \frac{x^2}{2} \Rightarrow (2u-1)^{1/2} = ke^{x^2/2}$$

$$u = \frac{k^2e^{x^2}+1}{2}$$

$\therefore u(x) = \frac{1}{2}(Ce^{x^2}+1)$ where $C = k^2$.

PPI-6

Back in y-variable:

$$u(x) = \frac{1}{2}\left(ce^{x^2}+1\right) \quad \text{and} \quad u = \frac{1}{y}.$$

$$\therefore \quad \frac{1}{y} = \frac{1}{2}\left(Ce^{x^2}+1\right) \Rightarrow y(x) = \frac{2}{Ce^{x^2}+1} \quad ⑤ \quad \text{gen. solution.}$$

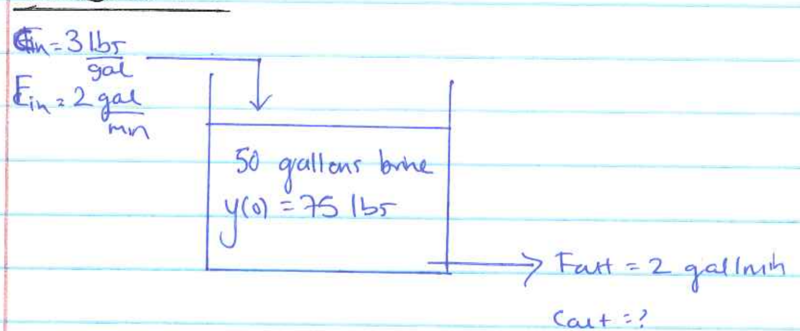
Applying I.C: $y(0)=1$

In ⑤, $\quad 1 = \dfrac{2}{c+1} \Rightarrow c = 1.$

$$\Rightarrow \quad \boxed{y(x) = \frac{2}{e^{x^2}+1}}$$

Note: The original ODE also has separable variables but ④ was much simpler to solve.

PROBLEM 5



(1) $\dfrac{dy}{dt} = F_{in}C_{in} - F_{out} \cdot \dfrac{y(t)}{V}$ where $V = V_0 = 50$ (constant)

$$\Rightarrow \quad \boxed{\frac{dy}{dt} = 6 - \frac{y}{25}, \quad y(0) = 75} \quad ①$$

PPI -7

(ii) At what t is $y = 125$ lbs?

To solve for this, we need to solve the IVP given by ①

The ODE is separable: $\frac{dy}{dt} = \frac{150-y}{25}$

Sep. variables & integrating,

$$25\int \frac{1}{150-y}\,dy = \int dt.$$

$$-25\ln(150-y) = t + \ln k.$$

$$\ln\left[\frac{(150-y)^{-25}}{k}\right] = t \Rightarrow (150-y)^{-25} = ke^{t}$$

$$y = 150 - k^{-1/25} e^{-1/25 t}$$

$$\Rightarrow y(t) = 150 - ce^{-1/25 t} \quad ② \quad \text{where } c = k^{-1/25}$$

Apply I.C.

$$y(0) = 75 \Rightarrow y(t=0) = 150 - c = 75 \Rightarrow c = 75$$

In ②, $\boxed{y(t) = 75(2 - e^{-1/25 t})}$ particular soln.

at $t = t_1$, $y(t_1) = 125$ lbs:

$$y(t_1) = 75(2 - e^{-1/25 t_1}) = 125$$

$$2 - e^{-1/25 t_1} = \frac{125}{75} = \frac{5}{3}$$

$$e^{-1/25 t_1} = 1/3 \Rightarrow t_1 = -25\ln(1/3)$$

PPI-8

$t_1 = 25 \ln 3$ mins (note: in exams you may leave your answer in this form since you won't have a calculator)

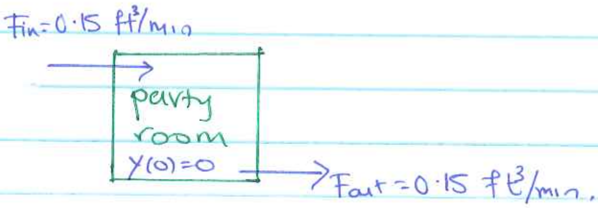
$$\therefore \boxed{t_1 \approx 27.47 \text{ mins.}}$$

(iii) As $t \to \infty$, $e^{-\frac{1}{25}t} \to 0$ and $y(t_\infty) \to 150$

$\therefore$ After a long time, there will be 150 lbs of salt dissolved.

PROBLEM 6

(i)



The total "rate in" of CO is given by: $F_{in}\left[\frac{ft^3 \text{ air}}{\text{min}}\right] \times 6\% \frac{[ft^3 \text{ smoke}]}{[ft^3 \text{ air}]}$

The total "rate out" of CO is: $F_{out} \cdot \left[\frac{ft^3 \text{ air}}{\text{min}}\right] \times \frac{y}{V} \frac{ft^3 \text{ smoke}}{[ft^3 \text{ air}]}$

The vol. of air in the party room is constant $= 1800$ ft³. air.

$$\therefore \frac{dy}{dt} = 0.15 \cdot \left(\frac{6}{100}\right) - 0.15 \cdot \frac{y}{1800}.$$

$$\boxed{\frac{dy}{dt} = \frac{9}{1000} - \frac{0.15y}{1800} \qquad y(0) = 0.}$$

Solving the ODE using IF (alternatively, use sep. of variables)

Put ODE in SF. $\quad \frac{dy}{dt} + \frac{0.15}{1800} y = \frac{9}{1000} \quad (1)$

PPI-9

The I·F is: $\mu(t) = e^{\frac{0.15t}{1800}}$. ②

<u>Multiply ① by ②:</u>

$$e^{\frac{0.15t}{1800}} \frac{dy}{dt} + \frac{0.15}{1800} e^{\frac{0.15t}{1800}} y = \frac{9}{1000} e^{\frac{0.15t}{1800}}.$$

<u>Reduce LHS:</u>

$$\frac{d}{dt}\left[e^{\frac{0.15t}{1800}} y\right] = \frac{9}{1000} e^{\frac{0.15t}{1800}}.$$

Check: $\frac{d}{dt}\left[e^{\frac{0.15t}{1800}} y\right] = e^{\frac{0.15t}{1800}} \frac{dy}{dt} + y \cdot e^{\frac{0.15t}{1800}} \cdot \frac{0.15}{1800}$ ✓

$$\Rightarrow e^{\frac{0.15t}{1800}} y = \frac{9}{1000} \cdot \frac{1800}{0.15} e^{0.15t/1800} + k.$$

$$y = 108 + k e^{-\frac{0.15t}{1800}} \quad (1000 \times 2)$$

$$y(t) = 108 + k e^{-\frac{t}{12000}}$$

<u>Apply I.C:</u> $y(0) = 108 + k = 0 \Rightarrow k = -108.$

The particular soln is: $\boxed{y(t) = 108\left(1 - e^{-t/12000}\right)}$ ③

(ii) The conc. of CO is given by the amt of smoke per the amount of air in the room (the concept is similar to the salt concentr. as we've seen in mixing problems in class & HW)

$\therefore$ conc. CO $= \frac{y}{V} = \frac{y(t)}{1800}$ (= conc. of CO at any time t)

$C_{danger} = 0.00018$ (stated in problem)

$\rightarrow C_{danger} = \frac{y_{danger}}{1800} = 0.00018$ ④

From ④, we obtain that a prudent person should leave the party before y reaches ydanger:

$$y_{danger} = (1800)\cdot(0.00018).$$

⇒ Back in ③,

$$y(t_{danger}) = 108\left(1 - e^{-t/12000}\right) = 1800\,(0.00018)$$

$$1 - \frac{(1800)(0.00018)}{108} = e^{-t_{dan.}/12000}.$$

$$t_{danger} = -12000 \ln\left[1 - \frac{(1800)(0.00018)}{108}\right]$$

(again, you can leave your answer like this in an exam)

$$\therefore \boxed{t_{danger} = 36 \text{ mins}}.$$

A prudent person should leave no later than 36 mins after the smoking starts. (i.e. that's a pretty short party time!)

PROBLEM 7

$$a\frac{dy}{dx} + by = ke^{-\lambda x}$$

(i) In SF. $$\frac{dy}{dx} + \frac{b}{a}y = \frac{k}{a}e^{-\lambda x}. \quad ①$$

Linear, first order ODE ⇒ use I.F.

$$\mu(x) = e^{\int \frac{b}{a}dx} = e^{\frac{bx}{a}}$$

PPI-11

Multiply ① by $\mu(x)$:

$$e^{\frac{b}{a}x}\frac{dy}{dx} + \frac{b}{a}e^{\frac{b}{a}x}y = \frac{k}{a}e^{(\frac{b}{a}-\lambda)x}. \quad ②$$

Reduce LHS: $\frac{d}{dx}\left[e^{\frac{b}{a}x}y\right] = e^{\frac{b}{a}x}\frac{dy}{dx} + e^{\frac{b}{a}x}\cdot\frac{b}{a}y.$

⇒ ② becomes:

$$\frac{d}{dx}\left[e^{\frac{b}{a}x}\cdot y\right] = \frac{k}{a}e^{(\frac{b}{a}-\lambda)x}.$$

Integ. wrt x:

$$e^{\frac{b}{a}x}\cdot y = \frac{k}{a}\int e^{(\frac{b}{a}-\lambda)x}\,dx.$$

$$y = \frac{k}{a}e^{-\frac{b}{a}x}\left[e^{(\frac{b}{a}-\lambda)x}\frac{1}{\frac{b}{a}-\lambda}\right] + Ce^{-\frac{b}{a}x}$$

$$y(x) = \frac{k}{\cancel{a}}e^{-\lambda x}\left(\frac{\cancel{a}}{b-\lambda a}\right) + Ce^{-\frac{b}{a}x}.$$

$$\boxed{y(x) = \left(\frac{k}{b-\lambda a}\right)e^{-\lambda x} + Ce^{-\frac{b}{a}x}.} \quad ③$$

gen. solution

(ii) If $\lambda = 0$, then ③ becomes:

$$y(x) = \frac{k}{b} + Ce^{-\frac{b}{a}x}$$

As $x \to \infty$, $e^{-\frac{b}{a}x} \to 0$ and $y \to \frac{k}{b}$

(iii) If $\lambda > 0$, as $x \to \infty$, $e^{-\lambda x} \to 0$.

Also, as $x \to \infty$, $e^{-\frac{b}{a}x} \to 0$ (since a, b are positive constants).

$\Rightarrow$ ③ becomes:

$$\boxed{y = 0 \quad \text{as } x \to \infty.}$$

PROBLEM 8

$$\left(ye^{2xy} + x\right) + axe^{2xy}\frac{dy}{dx} = 0 \quad ①$$

Comparing ① to the form: $P(x,y) + Q(x,y)\frac{dy}{dx} = 0$.

$P(x,y) = ye^{2xy} + x$ and $Q(x,y) = axe^{2xy}$.

To show that ~~eqn~~ ① is exact, we require that

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}.$$

$$\Rightarrow \frac{\partial}{\partial y}\left[ye^{2xy} + x\right] = y \cdot e^{2xy} \cdot 2x + e^{2xy}.$$

$$\frac{\partial P}{\partial y} = e^{2xy}(2xy + 1) \quad ②$$

$$\frac{\partial}{\partial x}\left(axe^{2xy}\right) = a\left(xe^{2xy} \cdot 2y + e^{2xy}\right)$$

$$\frac{\partial Q}{\partial x} = e^{2xy}(2axy + a) \quad ③$$

PPI-13

Equating ② & ③

$$e^{2xy}\left(2xy+1\right) = e^{2xy}\left(2axy+a\right)$$

$\Rightarrow$ $a=1$.

With $a=1$, the ODE becomes:

$$\left(ye^{2xy}+x\right) + xe^{2xy}\frac{dy}{dx} = 0. \quad ④$$

Since ④ is exact, it may be expressed as:

$$\frac{d}{dx}\left[f(x,y)\right] = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} = 0 \quad (*)$$

where $\frac{\partial f}{\partial x} = ye^{2xy}+x$ ⑤ & $\frac{\partial f}{\partial y} = xe^{2xy}$ ⑥.

Integ. ⑤ wrt x (keeping y constant)

$$f(x,y) = \int ye^{2xy}+x\,dx = y\cdot\frac{1}{2y}e^{2xy} + \frac{x^2}{2} + k_1(y) \quad ⑦$$

Integ. ⑥ wrt y (keeping x-constant)

$$f(x,y) = \int xe^{2xy}\,dy = xe^{2xy}\cdot\frac{1}{2x} + k_2(x) \quad ⑧$$

Compare ⑦ & ⑧

from ⑦, $f(x,y) = \frac{e^{2xy}}{2} + \frac{x^2}{2} + k_1(y)$

from ⑧, $f(x,y) = \frac{e^{2xy}}{2} + k_2(x) + 0$

This implies that $k_1(y) = 0$

$$k_2(x) = \frac{x^2}{2}.$$

<u>Sub. $k_2(x) = \frac{x^2}{2}$ in ⑧,</u>

$$f(x,y) = \frac{e^{2xy}}{2} + \frac{x^2}{2}$$

From ⊛ the ODE is now,

$$\frac{d}{dx}\left[\frac{e^{2xy}}{2} + \frac{x^2}{2}\right] = 0.$$

<u>Integ. wrt x:</u>

$$\frac{e^{2xy} + x^2}{2} = C$$

or $\boxed{e^{2xy} + x^2 = C_1}$ where $C_1 = 2C$.

general solution

<u>PROBLEM 9</u>

$$\frac{dy}{dt} = ay - y^3 = f(y) \quad ①$$

(i) For critical pt, $f(y) = 0$

$\therefore f(y) = ay - y^3 = y(a - y^2) = 0 \Rightarrow y = 0$ & $y = \pm\sqrt{a}$

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if $a>0$, then $y=0$, $y=\sqrt{a}$, $y=-\sqrt{a}$ (real solns)

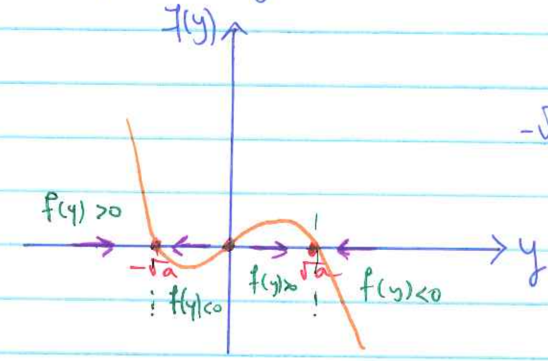
and the corresponding eqm solns are: $\left.\begin{array}{l} y=0 \\ y=\pm\sqrt{a} \end{array}\right\}$ for all values of t.

if $a<0$, then $y=0$ is the only critical point.

if $a=0$, again, $y=0$ is the only critical point

Determine the stability

plot $f(y)$ vs y. $a>0$



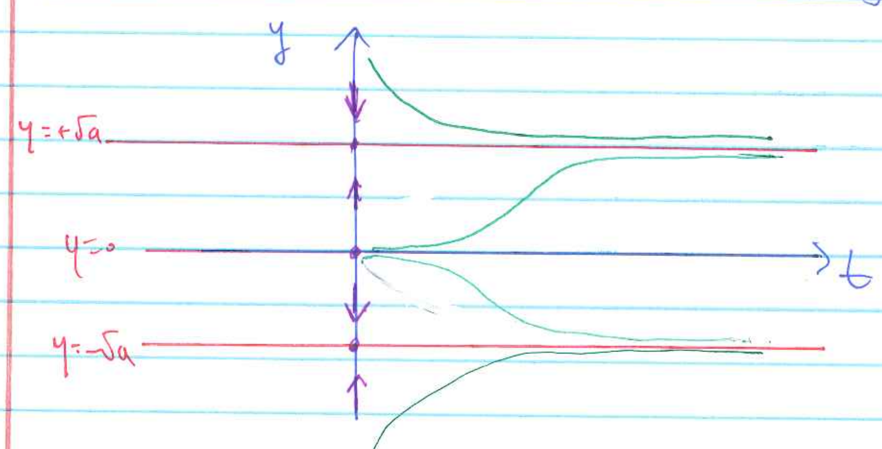
$y<-\sqrt{a}$; $f(y)>0$: solns increase

$-\sqrt{a}<y<0$; $f(y)<0$: solns decrease

$0<y<\sqrt{a}$; $f(y)>0$: solns increase

$y>\sqrt{a}$; $f(y)<0$: solns decrease

(ii) Plot eqm & non-eqm solutions in t-y plane



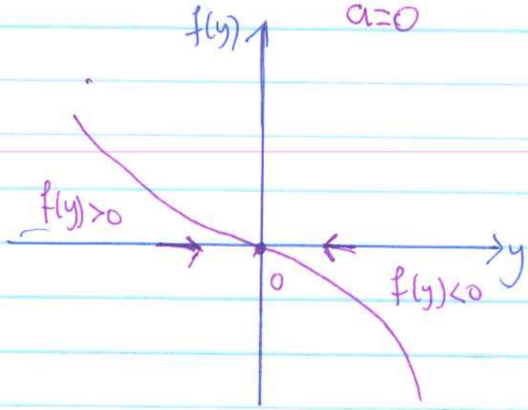
$y=0$ $\leftarrow\bullet\rightarrow$

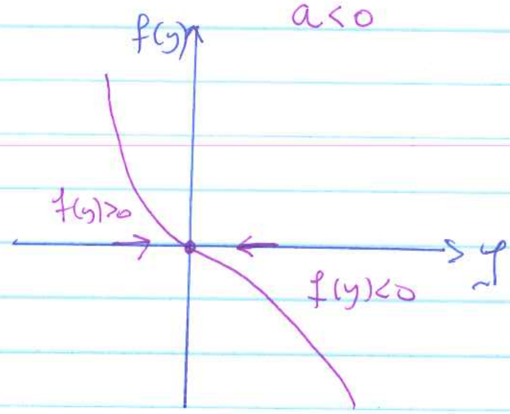
asymptotically unstable

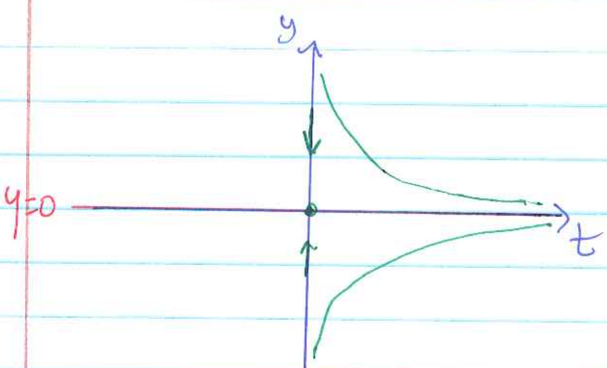
$y=\pm\sqrt{a}$ $\rightarrow\bullet\leftarrow$

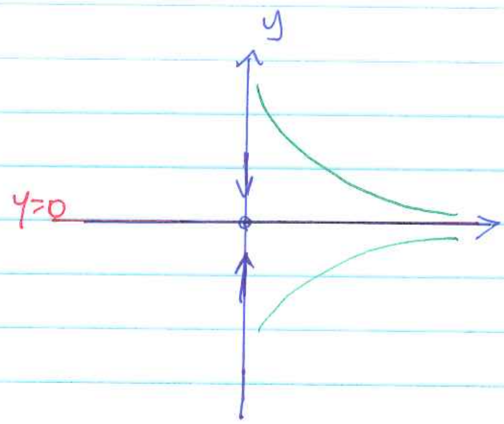
both asymptotically stable

Following the same analysis for $a=0$, $a<0$









For both cases $a<0$, $a=0$ the eqm soln $y=0$ $\rightarrow \bullet \leftarrow$ is asymptotically stable.

PROBLEM 10

$$\frac{dy}{dt} = ay(1-y) \quad ; \quad y(0)=y_0.$$

(i) Critical points occur at $f(y)=ay(1-y)=0$

$\therefore$ $ay(1-y)=0$ $\quad$ $y=0$ and $y=1$ are critical pts

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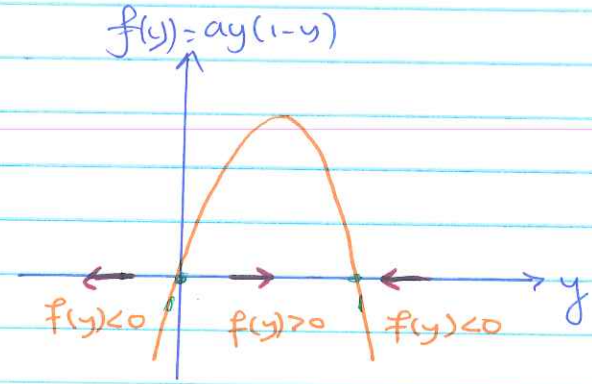
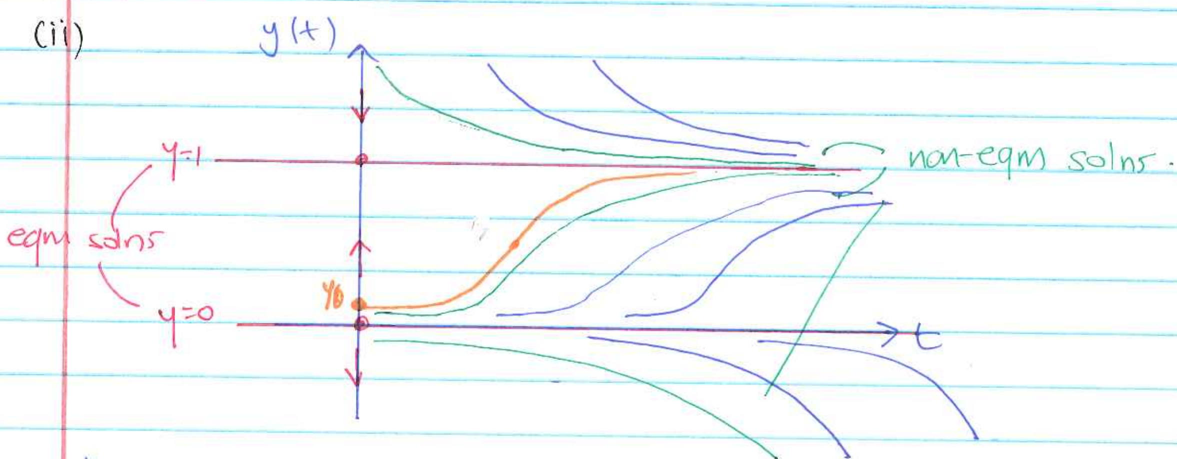
<u>To determine their stability, sketch $f(y)$ vs y:</u>



The c.p. $y=0$ $\leftarrow \bullet \rightarrow$ is <u>asymptotically</u> unstable while the c.p. $y=1$ $\rightarrow \bullet \leftarrow$ is <u>asymptotically stable</u>.

(ii)



the green curves are non-eqm solns

the blue curves are non-eqm solns which have been translated to the right

the orange curve shows the particular soln passing through $y(t=0)=y_0$.

(iii) As $t \to \infty$, <u>all</u> solutions approach $y=1$ which means that the disease has spread to the entire population (note that having $y_0<0$ makes no sense physically so the solns shown below $y=0$ only exist mathematically and have zero physical significance)