Math 33B: Differential Equations

Review 2: First order ODEs with separable variables

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1 General method

An ordinary differential equation (ODE) has **separable variables** if it is of the form,

$$f(y)\frac{dy}{dx} = g(x), \tag{1}$$

or if it can be expressed in the above form. We note that in (1), all functions of y are multiplied by the derivative, $\frac{dy}{dx}$ while all functions of x are separated from it:

$$\underbrace{f(y)\frac{dy}{dx}}_{\substack{\text{y-functions multiplied}\\ \text{by the derivative}}} \quad = \quad \underbrace{g(x)}_{\substack{\text{x-functions}\\ \text{are separated}}}.$$

Once the ODE is expressed in the form given by (1), then we may use the following steps to solve the differential equation:

1. **Separate the variables:**

$$f(y)dy = g(x)dx \tag{2}$$

2. **Integrate both sides*:**

$$\int f(y)\,dy = \int g(x)\,dx \tag{3}$$

3. **Solve for $y(x)$:**

   i.e. obtain an *explicit solution* (if possible).

*Note that in **Step 2**, we are essentially integrating **both** sides with respect to x. Equation (3) is equivalent to:

$$\int f(y)\,\frac{dy}{dx}\,dx = \int g(x)\,dx. \tag{4}$$

2 Worked-out examples

2.1 Example 1

Find the general solution to the following DE by separating the variables,

$$\frac{dy}{dx} = 6y^2x. \tag{5}$$

Solution:

1. Separate the variables,

$$\frac{1}{y^2}\,dy = 6x\,dx \tag{6}$$

2. Integrate both sides,

$$\begin{aligned} \int \frac{1}{y^2}\,dy &= 6\int x\,dx \\ -\frac{1}{y} &= 3x^2 + k \end{aligned} \tag{7}$$

3. Solve for $y(x)$

$$y(x) = -\frac{1}{3x^2 + k} \tag{8}$$

The general solution given by Eq. (8), represents a 1-parameter family of solutions. This means that depending on the value of k (the parameter), we obtain a different solution. With additional information, such as an initial condition (IC), we can solve for the value of k that satisfies the ODE and the IC; this yields a particular solution. The *interval of existence* of the solution to the ODE posed ultimately depends on the value of k which, in turn, depends on the IC.

2.2 Example 2

Find the general solution to the following DE by separating the variables,

$$\frac{dy}{dx} = \frac{2y}{x}. \tag{9}$$

Solution:

1. Separate the variables,

$$\frac{1}{y}\,dy = \frac{2}{x}\,dx \tag{10}$$

2. Integrate both sides,

$$\begin{aligned} \int \frac{1}{y}\,dy &= 2\int \frac{1}{x}\,dx \\ \ln|y| &= 2\ln|x| + k \end{aligned}$$

3. Solve for $y(x)$.

   Exponentiating both sides,

$$\begin{aligned} |y| &= x^2 e^k \\ \text{or} \quad y &= \pm e^k x^2 \end{aligned}$$

   Setting $c = \pm e^k$ gives,

$$y(x) = cx^2. \tag{11}$$

Equation (11) is the general solution to the ODE given by (9).

3 Physical applications

A number of physical applications are described by *separable differential equations*:

- <u>The growth of bacteria in biology</u>

  Suppose the number of bacteria in a yeast culture grows at a rate that is proportional to the number of bacteria present. Then, if we denote the number of bacteria by y, time by t and take the constant of proportionality to be c then we may mathematically describe this process by,

  $$\frac{dy}{dt} = cy(t).$$

- <u>Radioactive decay</u>

  The same form of the ODE used in the first example may be used to describe radioactive decay in a radioactive isotope

  $$\frac{dN}{dt} = -\lambda N(t),$$

  where N is the number of atoms in the isotope and $\lambda > 0$ is the decay constant.

- <u>Newton's law of cooling</u>

  It has been proven expermentally that the rate of change of the temperature of a body, T_b, which has been immersed in a medium of some constant temperature, T_m is proportional to the difference between the temperatures in the body and the medium. This is described by the following separable ODE

  $$\frac{dT_b}{dt} = -k(T_b - T_m),$$

  where $T_b = T_b(t)$ and $k > 0$ is a proportionality constant.

In general, many physical problems that deal with temperature, decomposition/growth and some simple chemical reactions take the form of the equations described above whose solutions involve the exponential function.