Q1 (25 pts). Find the general solutions.
(a).

$$x'' + 5x' - 14x = 0.$$

$\lambda^2 + 5\lambda - 14 = 0$

$(\lambda + 7)(\lambda - 2) = 0$

$\lambda = 2, -7$

$x = C_1 e^{2t} + C_2 e^{-7t}$

(b).

$$x'' + 10x' + 29x = 0.$$

$\lambda^2 + 10\lambda + 29 = 0$

$(\lambda + 5)^2 = -4$

$\lambda = -5 \pm 2i$

$x = e^{-5t}(C_1 \cos 2t + C_2 \sin 2t)$

(c).

$$x_1' = 12x_1 + 14x_2$$
$$x_2' = -7x_1 - 9x_2$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} 12 & 14 \\ -7 & -9 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$p(\lambda) = \det(A - \lambda I) = \det \begin{pmatrix} 12-\lambda & 14 \\ -7 & -9-\lambda \end{pmatrix}$$

$$= (12-\lambda)(-9-\lambda) + 98$$

$$= \lambda^2 - 3\lambda - 10$$
$$= (\lambda - 5)(\lambda + 2)$$

$$\lambda = -2, 5$$

$\lambda = -2$ $\quad (A - \lambda I)\vec{v}_1 = 0 \Rightarrow \begin{pmatrix} 14 & 14 \\ -7 & -7 \end{pmatrix} \vec{v}_1 = 0$

$$\vec{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$\lambda = 5$ $\quad (A - \lambda I)\vec{v}_2 = 0 \Rightarrow \begin{pmatrix} 7 & 14 \\ -7 & -14 \end{pmatrix} \vec{v}_2 = 0$

$$\vec{v}_2 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

$$\vec{x}(t) = c_1 e^{-2t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 e^{5t} \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

Q2 (25 pts). A mass weighing 1 kg stretches a spring 0.4 m. The system is placed in a viscous medium that provides a damping constant μ. The mass is pushed upward, contracting the spring a distance of 1 m from the mass-spring equilibrium, and then set in motion with a downward velocity of 7 m/s. The gravitational constant is 10 m/s^2.

(a). Take the downward direction to be positive. Find out the differential equation for the displacement. And write down the initial value.

$$k = \frac{mg}{x} = \frac{1 \cdot 10}{0.4} = 25 \ kg/s^2$$

$$m x'' = -kx - \mu x'$$

$$\begin{cases} x'' + 25x + \mu x' = 0 \\ x(0) = -1 \\ x'(0) = 7 \end{cases}$$

(b). Find the value of the damping constant μ for which the system is critically damped.

$\cancel{\lambda^2 + \lambda}$ $\quad \lambda^2 + \mu\lambda + 25 = 0$

$$\mu^2 - 4 \cdot 25 = 0$$

$$\mu = 10 \quad kg/s$$

(c). Let $\mu = 8$ kg/s. Solve the differential equation in (a). Find the amplitude, frequency and phase of the resulting motion.

$$x'' + 8x' + 25x = 0$$

$$\lambda^2 + 8\lambda + 25 = 0$$

$$(\lambda + 4)^2 = -9$$

$$\lambda = -4 \pm 3i$$

$$x(t) = e^{-4t}\left(C_1 \cos 3t + C_2 \sin 3t\right)$$

$$x(0) = -1 \Rightarrow \quad -1 = C_1$$

$$x'(t) = -4e^{-4t}\left(C_1 \cos 3t + C_2 \sin 3t\right) + e^{-4t}\left(-3C_1 \sin 3t + 3C_2 \cos 3t\right)$$

$$x'(0) = 7 \Rightarrow \quad 7 = -4C_1 + 3C_2$$

$$\Rightarrow C_2 = 1$$

$$x(t) = e^{-4t}\left(-\cos 3t + \sin 3t\right)$$

$$= e^{-4t} \cdot \sqrt{2}\cos\left(3t - \frac{3\pi}{4}\right)$$

Amplitude $= \sqrt{2}e^{-4t}$

Frequency $= 3$

phase $= \frac{3\pi}{4}$.

Q3 (25 pts).

(a). Find the general solution to the homogeneous equation,

$$y'' + 12y' + 36y = 0.$$

$$\lambda^2 + 12\lambda + 36 = 0$$

$$(\lambda + 6)^2 = 0$$

$$\lambda = -6$$

$$y(t) = C_1 e^{-6t} + C_2 t e^{-6t}$$

(b). Specify two linearly independent solutions from (a). Show that the Wronskian of these two solutions is always nonzero.

$$Y_1 = e^{-6t}, \quad Y_2 = t e^{-6t}$$

$$Y_1' = -6e^{-6t}, \quad Y_2' = e^{-6t} - 6t e^{-6t}$$

$$W = \det\begin{pmatrix} Y_1 & Y_2 \\ Y_1' & Y_2' \end{pmatrix} = \det\begin{pmatrix} e^{-6t} & te^{-6t} \\ -6e^{-6t} & e^{-6t} - 6te^{-6t} \end{pmatrix}$$

$$= e^{-12t} \det\begin{pmatrix} 1 & t \\ -6 & 1-6t \end{pmatrix} = e^{-12t}$$

$$W \neq 0.$$

(c). Solve the inhomogeneous equation

$$y'' + 12y' + 36y = 6te^{-6t} - 2e^{-6t} + t^2$$

by the method of undetermined coefficients.

$$y_p = t^2 (At + B) e^{-6t} + Ct^2 + Dt + E$$

$$y_p' = \left((3At^2 + 2Bt) - (6At^3 + 6Bt^2) \right) e^{-6t} + 2Ct + D$$

$$y_p'' = \left(6At + 2B - 18At^2 - 12Bt \right) e^{-6t}$$

$$- 6 (3At^2 + 2Bt - 6At^3 - 6Bt^2) e^{-6t} + 2C$$

te^{-6t} term: $\quad (6A - 12B - 12B) + 12(2B) = 6$

$$A = 1$$

e^{-6t} term: $\quad 2B = -2 \Rightarrow B = -1$

t^2 term: $\quad 36C = 1 \Rightarrow C = \frac{1}{36}$

t term: $\quad 12 \cdot 2C + 36D = 0 \Rightarrow D = -\frac{1}{54}$

constant term: $\quad 2C + 12D + 36E = 0 \Rightarrow E = \frac{1}{216}$

$$\Rightarrow y_p = t^2 (t-1) e^{-6t} + \frac{1}{36} t^2 - \frac{1}{54} t + \frac{1}{216}$$

the general soln is

$$y(t) = C_1 e^{-6t} + C_2 t e^{-6t}$$

$$+ t^2 (t-1) e^{-6t} + \frac{1}{36} t^2 - \frac{1}{54} t + \frac{1}{216}$$

Q4 (25 pts).

(a). Show that $y_1(x) = x^{-1/2}\sin x$ and $y_2(x) = x^{-1/2}\cos x$ are both solutions to the equation

$$x^2y'' + xy' + \left(x^2 - \frac{1}{4}\right)y = 0.$$

$$Y_1' = -\frac{1}{2}X^{-3/2}\sin X + X^{-\frac{1}{2}}\cos X$$

$$Y_1'' = \frac{3}{4}X^{-5/2}\sin X - \frac{1}{2}X^{-3/2}\cos X - \frac{1}{2}X^{-3/2}\cos X - X^{-\frac{1}{2}}\sin X$$

$$X^2Y_1'' + XY_1' + (X^2 - \tfrac{1}{4})Y_1$$

$$= \frac{3}{4}X^{-\frac{1}{2}}\sin X - X^{\frac{1}{2}}\cos X - X^{3/2}\sin X$$
$$-\frac{1}{2}X^{-1/2}\sin X + X^{\frac{1}{2}}\cos X + (X^2 - \tfrac{1}{4})\cdot X^{-\frac{1}{2}}\sin X$$

$$= 0.$$

$\Rightarrow Y_1$ is a soln.

$$Y_2' = -\frac{1}{2}X^{-3/2}\cos X - X^{-1/2}\sin X$$

$$Y_2'' = \frac{3}{4}X^{-5/2}\cos X + X^{-3/2}\sin X - X^{-1/2}\cos X$$

$$X^2Y_2'' + XY_2' + (X^2 - \tfrac{1}{4})Y_2 = 0$$

$\Rightarrow Y_2$ is a soln.

(b). Find the general solution of

$$x^2y'' + xy' + \left(x^2 - \frac{1}{4}\right)y = 3\sqrt{x}\sin x.$$

$$Y_p = V_1 Y_1 + V_2 Y_2$$

with $Y_1 = x^{-\frac{1}{2}}\sin x$. $Y_2 = x^{-\frac{1}{2}}\cos x$

V_1' and V_2' solves

$$\begin{cases} V_1' Y_1 + V_2' Y_2 = 0 \\ V_1' Y_1' + V_2' Y_2' = 3x^{\frac{1}{2}} \cdot x^{-2} \sin x \end{cases}$$

$$\Rightarrow \begin{cases} V_1'(x^{-\frac{1}{2}}\sin x) + V_2'(x^{-\frac{1}{2}}\cos x) = 0 \\ V_1'\left(-\frac{1}{2}x^{-3/2}\sin x + x^{-\frac{1}{2}}\cos x\right) + V_2'\left(-\frac{1}{2}x^{-3/2}\cos x - x^{-\frac{1}{2}}\sin x\right) = 3x^{-3/2}\sin x \end{cases}$$

$$\Rightarrow \begin{cases} V_1'\sin x + V_2'\cos x = 0 \\ V_1'\cos x - V_2'\sin x = 3x^{-1}\sin x \end{cases}$$

$$V_1' = \frac{3\sin x\cos x}{x} \Rightarrow V_1 = \int \frac{3\sin x\cos x}{x}\,dx$$

$$V_2' = -\frac{3\sin^2 x}{x} \Rightarrow V_2 = \int -\frac{3\sin^2 x}{x}\,dx$$

The general soln is

$$Y(t) = C_1 Y_1 + C_2 Y_2 + Y_p$$

$$= C_1 x^{-\frac{1}{2}}\sin x + C_2 x^{-\frac{1}{2}}\cos x$$

$$+ \int \frac{3\sin x\cos x}{x}\,dx \cdot x^{-\frac{1}{2}}\sin x - \int \frac{3\sin^2 x}{x}\,dx \cdot x^{-\frac{1}{2}}\cos x$$