Math 33B: Differential Equations

Review 7: Introduction to second order, linear ODEs

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1 Standard form

The second order, linear differential equation is given in standard form as,

$$y'' + p(x)y' + q(x)y = f(x) \tag{1}$$

where y' and y'' denote the first and second derivatives of y wrt x, respectively.

Just like first order linear ODEs (**Review 3**), we note that the linear ODE given by (1), has the following properties:

- the function $y(x)$ and its derivatives, $y'(x)$, $y''(x)$ are **not** present at any power higher than 1;
- there are **no** products of the function $y(x)$ and its derivatives, $y'(x)$, $y''(x)$.

The x-dependent functions $p(x)$, $q(x)$ and $f(x)$ need to be continuous but they can take any form: they can be constants, linear or nonlinear functions.

Terminology

If $f(x) = 0$ in Eq. (1), the ODE reduces to:

$$y'' + p(x)y' + q(x)y = 0. \tag{2}$$

Equation (2) is known as the homogeneous equation.

If $f(x) \neq 0$, the ODE is known as the nonhomogeneous equation.

For the rest of this review, we will be concerned with the *homogeneous* equation given by (2) where we will identify some important ideas on the structure of the general solutions to such ODEs.

2 Structure of the general solution

We start this section with an example. Consider the following ODE,

$$y'' - 4y = 0, \tag{3}$$

which is equivalent to (2) with $p(x) = 0$, $q(x) = -4$.

The differential equation (3) gives $y'' = 4y$ which tells us that we are looking for a solution $y(x)$ which, when differentiated twice (to obtain y''), should be equal to 4 times the original function.

An obvious solution is $y_1 = e^{2x}$ and so is any constant multiple of it, i.e. $y_1 = c_1 e^{2x}$. A perhaps not so obvious solution is $y_2 = e^{-2x}$ and, again, so is any constant multiple of it, i.e. $y_2 = c_2 e^{-2x}$. We can check that y_1 and y_2 are in fact solutions to (3) by differentiating twice and plugging back into the ODE:

e.g. take $y_1 = c_1 e^{2x}$ and, upon differentiating twice wrt x, we get $y_1'' = 4c_1 e^{2x}$.

Substituting in Eq. (3), we get:

$$y_1'' - 4y_1 = 4c_1 e^{2x} - 4c_1 e^{2x} = 0 \tag{4}$$

thus, $y_1 = c_1 e^{2x}$ is a solution. Similarly, we can show that $y_2 = c_2 e^{-2x}$ is also a solution.

In fact, any *linear combination* of y_1 and y_2, i.e.,

$$y(x) = c_1 y_1(x) + c_2 y_2(x)$$

can also be shown to satisfy the ODE. This leads us to state the following definition about the structure of the general solution to homogeneous, second order linear differential equations.

If $y_1(x)$ and $y_2(x)$ are two *linearly independent* solutions to a *linear*, *homogeneous* ODE then

$$y(x) = c_1 y_1(x) + c_2 y_2(x) \tag{5}$$

is also a solution for any pair of constants c_1 and c_2.

The pair of solutions $y_1(x)$ and $y_2(x)$ are known to be a fundamental set of solutions.

A few things to note about the statement in the box:

- It is *not* true for nonhomogeneous ODEs;
- It *is* true for a linear homogeneous ODE of *any* order, say of order N, but note that we should have a set of N fundamental solutions appearing in Eq. (5), i.e. $y(x) = c_1 y_1(x) + c_2 y_2(x) + \ldots + c_N y_N(x)$.

  So, for an ODE of order 2, we expect to have two solutions, for an ODE of order 3, we expect three and so on.

Initial conditions

We notice that the general solution contains two arbitrary constants c_1 and c_2. Therefore, a particular solution to a second order differential equation requires two initial conditions (to solve for the 2 constants). These are given by the value of the function y and its first order derivative y' at a particular point in x, say x_0:

$$y(x_0) = y_0 \quad \text{and} \quad y'(x_0) = y'_0, \tag{6}$$

where y_0 and y'_0 are constants.

Note that both conditions in Eq. (6) are specified at the **same** point x_0. If they are different, then the problem (consisting of the ODE and the initial conditions), is known as a *boundary value problem.* The method for solving boundary value problems as opposed to initial value problems is completely different and we don't deal with it in this course.

3 Wronskian determinant

Linear independent functions

For the pair of solutions $y_1(x)$ and $y_2(x)$ to constitute a *fundamental* set of solutions we require that the two solutions are linearly independent.

When referring to two functions being linearly independent, roughly speaking, it means that the two functions cannot be a constant multiple of each other which is easy to see in some cases but not so easy in others.

To check if two functions, $y_1(x)$ and $y_2(x)$ are linearly independent/dependent, we use the *Wronskian determinant*.

This is defined as follows:

$$W(x) = \begin{vmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{vmatrix} = y_1 y'_2 - y_2 y'_1; \tag{7}$$

where, if $W(x) = 0$ for all values of x on some interval $\mathcal{I}$, the two functions are said to be **linearly dependent**. Now, if $W(x) \neq 0$ at *any* point in x on $\mathcal{I}$, the functions are said to be **linearly independent**. This result is true for any arbitrary pair of functions.

However, if the functions $y_1(x)$ and $y_2(x)$ are also solutions to the homogeneous, linear second order ODE [i.e. Eq. (2)], then we claim the following result with respect to the Wronskian:

> If $y_1(x)$ and $y_2(x)$ are a *linearly independent* pair of fundamental solutions to Eq. (2), then their Wronskian is <u>never</u> zero on some interval $\mathcal{I}$. Then (and only then) their linear combination i.e. $y(x) = c_1 y_1(x) + c_2 y_2(x)$ form a general solution of the ODE (2). If the two solutions are *linearly dependent* then their Wronskian is <u>always</u> zero for all x on $\mathcal{I}$.

The proof for the statement in the box, i.e. that the Wronskian is either **never zero** or **always zero** is given by *Abel's theorem* which is discussed below. Please note that Abel's theorem is **not** examinable. It is included here for anyone who wants to know a little bit more about this.

Abel's theorem

Let us begin by assuming that $y_1(x)$ and $y_2(x)$ are two solutions to the homogeneous ODE:

$$y'' + p(x)y' + q(x)y = 0. \tag{8}$$

Their Wronskian is given by Eq. (7), i.e.

$$W(x) = y_1 y_2' - y_2 y_1'. \tag{9}$$

Now, differentiating $W(x)$ wrt x gives,

$$\begin{aligned} W'(x) &= y_1 y_2'' + y_2' y_1' - y_2 y_1'' - y_2' y_1' \\ &= y_1 y_2'' - y_2 y_1'' \end{aligned} \tag{10}$$

Next, since $y_1(x)$ and $y_2(x)$ are solutions to (8), we have:

$$y_1'' + p(x)y_1' + q(x)y_1 = 0 \tag{11}$$

for y_1 and

$$y_2'' + p(x)y_2' + q(x)y_2 = 0 \tag{12}$$

for y_2.

Now, multiply Eq. (11) by y_2:

$$y_2 y_1'' + p(x) y_2 y_1' + q(x) y_2 y_1 = 0, \tag{13}$$

and Eq. (12) by y_1:

$$y_1 y_2'' + p(x) y_1 y_2' + q(x) y_1 y_2 = 0. \tag{14}$$

Subtracting Eq. (13) from Eq. (14), we obtain:

$$(y_1 y_2'' - y_2 y_1'') + p(x)(y_1 y_2' - y_2 y_1') = 0. \tag{15}$$

Note that the terms in blue are equivalent to $W'(x)$ given by (10) and the terms in red are equivalent to $W(x)$ given by (9). So, Eq. (15) may be re-written as:

$$W'(x) + p(x) W(x) = 0. \tag{16}$$

Equation (16) is a first order ODE for $W(x)$ which may be integrated (by separating variables) to give the following general solution:

$$\boxed{W(x) = c e^{-\int p(x)\, dx}. \tag{17}}$$

Since the exponential function is never zero, we see from Eq. (17), that $W(x)$ is either *identically zero* when $c = 0$ and *never zero* if $c \neq 0$.

4 Existence & uniqueness of solutions

Consider the initial value problem:

$$\begin{aligned} &y'' + p(x) y' + q(x) y = f(x), \\ &y(x_0) = y_0, \quad y'(x_0) = y_0'. \end{aligned} \tag{18}$$

If $p(x)$, $q(x)$ and $f(x)$ are continuous on the interval $a < x < b$, containing the point $x = x_0$ then, there exists a **unique solution** $y = y(x)$ and this solution exists throughout the entire interval.

The theorem applies to second order linear ODEs of the form given by (1) and guarantees that the given initial value problem will always have a twice-differentiable solution (existence) and this solution will be unique (uniqueness) on any interval containing x_0 as long as $p(x)$, $q(x)$ and $f(x)$ are continuous on the same interval.

Of course, neither existence nor uniqueness of a solution is guaranteed at a discontinuity of $p(x)$, $q(x)$ *or* $f(x)$.