Math 33b, Winter 2013, Tonći Antunović - Homework 8

From the textbook solve the problems:

Section 9.2: 2, 8, 18, 24, 34, 40, 42, 44, 46, 48, 50, 52, 54, 56 and 58.

And also the problems below:

Problem 1. Find the solution of the initial value problem $y' = Ay$, $y(0) = y_0$ where

$$A = \begin{pmatrix} -3 & -2 \\ 2 & 2 \end{pmatrix}, \quad y_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Solution: The characteristic polynomial is

$$p(\lambda) = \det \begin{pmatrix} \lambda+3 & 2 \\ -2 & \lambda-2 \end{pmatrix} = \lambda^2 + \lambda - 2.$$

The zeros of this polynomial are $\lambda_1 = 1$ and $\lambda_2 = -2$ so these are eigenvalues. The eigenvectors for the eigenvalue $\lambda_1 = 1$ are the non-zero vectors in the nullspace of

$$A - I = \begin{pmatrix} -4 & -2 \\ 2 & 1 \end{pmatrix},$$

which are multiples of the vector $(1, -2)^T$. The eigenvectors for the eigenvalue $\lambda_2 = -2$ are the non-zero vectors in the nullspace of

$$A + 2I = \begin{pmatrix} -1 & -2 \\ 2 & 4 \end{pmatrix},$$

which are multiples of the vector $(2, -1)^T$. The general solution of the equation $y' = Ay$ is

$$y = C_1 e^t \begin{pmatrix} 1 \\ -2 \end{pmatrix} + C_2 e^{-2t} \begin{pmatrix} 2 \\ -1 \end{pmatrix}.$$

Then

$$y(0) = C_1 \begin{pmatrix} 1 \\ -2 \end{pmatrix} + C_2 \begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} C_1 + 2C_2 \\ -2C_1 - C_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

so we need $C_1 + 2C_2 = 1$ and $-2C_1 - C_2 = 1$, which gives $C_1 = -1$ and $C_2 = 1$. Then the solution is

$$y = e^t \begin{pmatrix} -1 \\ 2 \end{pmatrix} + e^{-2t} \begin{pmatrix} 2 \\ -1 \end{pmatrix}.$$

Problem 2. Find the solution of the initial value problem $y' = Ay$, $y(0) = y_0$ where

$$A = \begin{pmatrix} -1 & -1 \\ 1 & -3 \end{pmatrix}, \qquad y_0 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

Solution: The characteristic polynomial is

$$p(\lambda) = \det \begin{pmatrix} \lambda+1 & 1 \\ -1 & \lambda+3 \end{pmatrix} = \lambda^2 + 4\lambda + 4,$$

so there is only one eigenvalue $\lambda_1 = -2$. The eigenvectors are non-zeros vectors in the nullspace of

$$A + 2I = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$$

which are multiples of vector $(1,1)^T$. Then

$$e^{-2t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \text{and} \quad e^{-2t}(v_2 + t\begin{pmatrix} 1 \\ 1 \end{pmatrix})$$

form a fundamental solution set, where $(A+2I)v_2 = (1,1)^T$. We can take $v_2 = (2,1)^T$ so

$$e^{-2t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \text{and} \quad e^{-2t}(\begin{pmatrix} 2 \\ 1 \end{pmatrix} + t\begin{pmatrix} 1 \\ 1 \end{pmatrix})$$

form a fundamental solution set, and the general solution is given as

$$y = C_1 e^{-2t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 e^{-2t}(\begin{pmatrix} 2 \\ 1 \end{pmatrix} + t\begin{pmatrix} 1 \\ 1 \end{pmatrix}).$$

Since

$$y(0) = C_1\begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} C_1 + 2C_2 \\ C_1 + C_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

we get $C_1 = 4$ and $C_2 = -1$ so the solution is

$$y = 4e^{-2t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} - e^{-2t}(\begin{pmatrix} 2 \\ 1 \end{pmatrix} + t\begin{pmatrix} 1 \\ 1 \end{pmatrix}) = e^{-2t}(\begin{pmatrix} 2 \\ 3 \end{pmatrix} + t\begin{pmatrix} 1 \\ 1 \end{pmatrix})$$

Problem 3. Find the solution of the initial value problem $y' = Ay$, $y(0) = y_0$ where

$$A = \begin{pmatrix} 3 & 1 \\ -17 & -5 \end{pmatrix}, \qquad y_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Solution: The characteristic polynomial is

$$p(\lambda) = \det(\lambda I - A) = \det \begin{pmatrix} \lambda - 3 & -1 \\ 17 & \lambda + 5 \end{pmatrix} = \lambda^2 + 2\lambda + 2,$$

and the zeros are $-1+i$ and $-1-i$. For $\lambda_1 = -1+i$ we set

$$(A - (-1+i)I)v = 0 \quad \Rightarrow \quad \begin{pmatrix} 4-i & 1 \\ -17 & -4-i \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

One possible choice is $z_1 = 1$ and $z_2 = -4+i$ and we obtain a complex solution

$$y = e^{(-1+i)t} \begin{pmatrix} 1 \\ -4+i \end{pmatrix} = e^{-t}(\cos t + i \sin t) \left(\begin{pmatrix} 1 \\ -4 \end{pmatrix} + i \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right)$$

$$= e^{-t} \left(\cos t \begin{pmatrix} 1 \\ -4 \end{pmatrix} - \sin t \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) + ie^{-t} \left(\cos t \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \sin t \begin{pmatrix} 1 \\ -4 \end{pmatrix} \right).$$

The general solution is now obtained by taking linear combinations of the real and the imaginary part.

$$y = C_1 e^{-t} \left(\cos t \begin{pmatrix} 1 \\ -4 \end{pmatrix} - \sin t \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) + C_2 e^{-t} \left(\cos t \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \sin t \begin{pmatrix} 1 \\ -4 \end{pmatrix} \right)$$

Then we have

$$y(0) = C_1 \begin{pmatrix} 1 \\ -4 \end{pmatrix} + C_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

which gives $C_1 = 1$ and $C_2 = 4$ so the solution is

$$y = e^{-t} \left(\cos t \begin{pmatrix} 1 \\ -4 \end{pmatrix} - \sin t \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) + 4e^{-t} \left(\cos t \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \sin t \begin{pmatrix} 1 \\ -4 \end{pmatrix} \right)$$

$$= e^{-t} \begin{pmatrix} \cos t + 4 \sin t \\ -5 \sin t \end{pmatrix}$$

Problem 4. Find all pair of real valued functions (x_1, x_2) which satisfy

$$x_1' = 5x_1 - 2x_2, \quad x_2' = 2x_1.$$

Solution: Writting the system in the vector form we get

$$y = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad y' = Ay,$$

where

$$A = \begin{pmatrix} 5 & -2 \\ 2 & 0 \end{pmatrix}.$$

The characteristic polynomial is

$$p(\lambda) = \det \begin{pmatrix} \lambda - 5 & 2 \\ -2 & \lambda \end{pmatrix} = \lambda^2 - 5\lambda + 4.$$

The eigenvalues are zeros of the characteristic polynomial $\lambda_1 = 1$ and $\lambda_2 = 4$. The eigenvectors for $\lambda_1 = 1$ are non-zero vectors in the nullspace of the matrix

$$A - I = \begin{pmatrix} 4 & -2 \\ 2 & -1 \end{pmatrix}$$

which are multiples of the vector $(1, 2)^T$. The eigenvectors for $\lambda_2 = 4$ are non-zero vectors in the nullspace of the matrix

$$A - 4I = \begin{pmatrix} 1 & -2 \\ 2 & -4 \end{pmatrix}$$

which are multiples of the vector $(2, 1)^T$. The general solution of $y' = Ay$ is then

$$y = C_1 e^t \begin{pmatrix} 1 \\ 2 \end{pmatrix} + C_2 e^{4t} \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

Therefore,

$$x_1 = C_1 e^t + 2C_2 e^{4t}, \quad x_2 = 2C_1 e^t + C_2 e^{4t}.$$

Problem 5. Write down the matrix A such that the differential equation $y' = Ay$ has the general solution

$$y = e^t \begin{pmatrix} 1 \\ 1 \end{pmatrix} + e^{-2t} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

Solution: From the form of the solution we see that the matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

has eigenvalues 1 and -2 and the corresponding eigenvectors

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

So we need

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

and

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = -2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -2 \\ -4 \end{pmatrix}.$$

This gives the equations

$$a + b = 1, \quad c + d = 1, \quad a + 2b = -2, \quad c + 2d = -4,$$

which after solving give $a = 4$, $b = -3$, $c = 6$ and $d = -5$ so

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 4 & -3 \\ 6 & -5 \end{pmatrix}.$$