MATH 33B: DIFFERENTIAL EQUATIONS MAY, 2013.

EXAMPLE: Method of variation of parameters: Homogeneous ODE.

find the general solution to:

$$x^2y'' + xy' - y = 0. \quad ①.$$

given that $y_1 = x$ is a solution.

Note: this is the example we covered in Lecture 15 (section I) on Monday, May 6th.

Eq. ① is a second order LINEAR ODE and it's homogeneous. So the general solution we seek takes the form:

$$y(x) = C_1 y_1 + C_2 y_2. \quad ②.$$

We already have y_1 (given) so we just need to find y_2.

The method we use to find the second solution once the first one is known is to assume that y_2 is given by:

$$y_2 = V(x) \cdot y_1(x). \quad ③$$

(the motivation here is that we need y_2 to be linearly independent to y_1).

Determine y_2

We determine y_2 by solving $y_2'' + \dfrac{y_2'}{x} - \dfrac{y_2}{x^2} = 0$ ④ (this is Eq. ① in standard form)

Given $y_1 = x$, from (3) we have $y_2 = v(x) \cdot x$. (5).

Take (5) & diff. wrt x.

$y_2' = v \cdot + x v'$.

$y_2'' = x v'' + 2v'$.

Sub. y_2, y_2', y_2'' in (4)

$$(xv''+2v') + \frac{1}{x}(v + xv') - \frac{1}{x^2} v \cdot x = 0.$$

Factorize v, v', v''

$$v\left(\frac{1}{x} - \frac{x}{x^2}\right) + v'\left(2 + \frac{x}{x}\right) + xv'' = 0.$$

$$\cancel{v\left(\frac{1}{x} - \frac{1}{x}\right)}^{0} + 3v' = -xv''.$$

$$\therefore \quad \frac{v''}{v'} = \frac{-3}{x}. \quad (6).$$

Eq. (6) is a second order ODE for $v(x)$. So, if we can integrate (6) wrt x twice, we'll have the required $v(x)$ that will help us solve for y_2.

Let's note here, that $v(x)$ has to be a nonconstant fct of x. Or else, y_1 and y_2 are linearly dependent and therefore do not form a fundamental pair of solutions.

Integrating both sides of (6) wrt x. (here, we are essentially separating variables)

$\ln v' = -3 \ln x$ (7)

Exponentiating both sides of 7

$$v' = x^{-3}.$$

and upon integrating once more:

$$v = \int x^{-3}\,dx = \frac{-1}{2x^2} \quad (8).$$

Eq. 8 gives us the functional form of $v(x)$

Now, from 3, $y_2 = v(x) \cdot y_1(x)$

$$y_2 = \frac{-1}{2x^2} \cdot x$$

$$y_2 = \frac{-1}{2x}.$$

(this is the second fundamental soln. in fact, fundamental solution, the coef., $\frac{-1}{2}$ is just a constant that can be absorbed in c_2)

$\Rightarrow$ The G.S. is:

$$y(x) = c_1 y_1 + c_2 y_2$$

$$\boxed{y(x) = c_1 x + \frac{c_2}{x}.}$$

($\frac{-1}{2}$ has been absorbed in c_2)

Notes: You will have noticed that when we were integrating to get $v(x)$ we completely ignored the constants of integration.

Let us now return to Eq. 6 and redo the integration with the constants:

Back to ⑥:

$$\frac{v''}{v'} = -\frac{3}{x}.$$

Integrating once gives: $\ln v' = -3\ln x + \ln k_1$.

$$v' = k_1 x^{-3}.$$

Integrating a second time,

$$v = \frac{-k_1}{2x^2} + k_2.$$

Now, we are looking for any nonconstant fct of x i.e $v(x)$ that makes y_2 satisfy the homogeneous ODE ①.

So we can arbitrarily set k_1 and k_2 to whatever we want, as long as $V(x)$ is nonconstant.
if $k_1 = 0$ then $v(x)$ becomes a constant so we need to stay away from that.

The simplest* form $v(x)$ can take is if $k_1 = 1$ and $k_2 = 0$.

$$\Rightarrow \quad V(x) = \frac{-1}{2x^2}$$ (as given by Eq. ⑧)

* actually, you can set $k_1 = -2$ so that $V(x) = \frac{1}{x^2}$. That would be the simplest form.

Alternative method of solution.

In Lecture 15, we have shown that for the general 2nd order linear ODE [i.e. for arbitrary $p(x)$, $q(x)$], the unknown function $v(x)$ takes the form:

$$\boxed{V(x) = \int \frac{1}{y_1^2} e^{-\int p(x)\,dx}\, dx} \quad ⑨$$

where y_1 is the known solution..

For this example the ODE in standard form is:

$$y'' + \frac{1}{x}y' - \frac{1}{x^2}y = 0 \quad (10) \quad \text{and } y_1 = x.$$

from (10), $p(x) = \frac{1}{x}$.

If you have (9) memorized, then you can use (9) to find $v(x)$:

So, since $p(x) = \frac{1}{x}$, $\quad e^{-\int \cdots dx} = e^{-\int \frac{1}{x} dx} = e^{-\ln x} = \frac{1}{x}$.

Sub. $e^{-\int p(x)\,dx} = \frac{1}{x}$ in (9):

$$V = \int \frac{1}{\underbrace{x^2}_{y_1^2}} \cdot \overbrace{\boxed{\frac{1}{x}}}^{e^{-\int p(x)\,dx}} dx = \int \frac{1}{x^3}\,dx$$

$$V = \frac{-1}{2x^2}$$

(again, no need for the constant of integration)

Once we have $v(x)$ we have y_2: $\quad y_2 = v(x)\cdot y_1$

$$y_2 = \frac{-1}{2x^2}\cdot x = \frac{-1}{2x}.$$

like before.

You are welcome to memorize (9) and use it if you prefer.