MATH 33B : DIFFERENTIAL EQUATIONS APRIL 2013

Example: solving exact ODEs of the form

$$P(x,y) + Q(x,y)\frac{dy}{dx} = 0.$$

Show that $\boxed{\frac{dy}{dx} = -\frac{(ax+by)}{bx+cy}}$ is exact and hence find the particular solution that satisfies $y(0)=c$.

First, rewrite the ODE as:

$$(ax+by) + (bx+cy)\frac{dy}{dx} = 0 \quad ①$$

where $P(x,y) = ax+by$ ②

& $Q(x,y) = bx+cy.$ ③

To show the ODE is exact, we want to determine whether

$$\boxed{\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}.}$$

From ②, $\frac{\partial P}{\partial y} = \frac{\partial}{\partial y}(ax+by) = b$

From ③, $\frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}(bx+cy) = b$

$\Rightarrow \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} = b \Rightarrow$ ODE ① is exact.

We now look for the function $f(x,y)$ that allows us to write ① in the form $\boxed{\frac{d}{dx}\left[f(x,y)\right] = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dx} = 0}$ ④

From ④, it's easy to see that:

$$\frac{\partial f}{\partial x} = P(x,y) = ax+by \quad (5) \quad \text{and} \quad \frac{\partial f}{\partial y} = Q(x,y) = bx+cy. \quad (6)$$

Integrating ⑤ wrt x:

$$f(x,y) = \int ax+by \, dx = \frac{ax^2}{2} + bxy + k_1(y) \quad (7)$$

Integrating ⑥ wrt y,

$$f(x,y) = \int bx+cy \, dy = bxy + \frac{cy^2}{2} + k_2(x) \quad (8)$$

Comparing ⑦ & ⑧,

$$k_1(y) = \frac{cy^2}{2} \quad \text{and} \quad k_2(x) = \frac{ax^2}{2}$$

Using $k_1(y) = \frac{cy^2}{2}$ in ⑦ gives

$$f(x,y) = \frac{ax^2}{2} + bxy + \frac{cy^2}{2} \quad (9)$$

From ④, we know that the ODE we started with is given by:

$$\frac{d}{dx}\left[f(x,y)\right] = 0 \quad (10)$$ where $f(x,y)$ is given by ⑨.

Thus, integrating ⑩ wrt x:

general soln.

$$f(x,y) = k \quad \text{which is equal to:} \quad \boxed{\frac{ax^2}{2} + bxy + \frac{cy^2}{2} = k.} \quad (11)$$

Using the I.C. $y(0)=c$ in (11),

$$f(0,c) = c\cdot\frac{c^2}{2} = k \Rightarrow k = \frac{c^3}{2}$$

Back in (11),

$$\frac{ax^2}{2} + bxy + \frac{cy^2}{2} = \frac{c^3}{2}$$

Multiplying through by 2:

$$\boxed{ax^2 + 2bxy + cy^2 = c^3 \qquad (12)}$$

Eq. (12) is the particular soln to the IVP in implicit form.