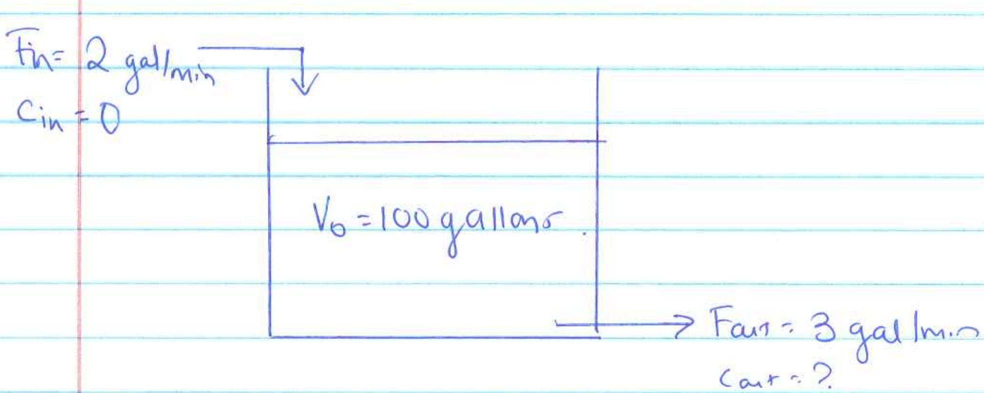
HW 2 - 1

Homework 2: Worked-out solutions

April 2013

PROBLEM 1



(i)

Let $y(t)$ denote the amount of salt in tank.

The liquid volume is changing (decreasing) with time as $F_{out} > F_{in}$:

This is given by: $V(t) = V_0 + (F_{in} - F_{out})t$ ①

initial volume

$\therefore V(t) = 100 - t$ ②

The ODE describing the process is

$$\frac{dy}{dt} = \text{Rate in} - \text{Rate out}$$

$$\frac{dy}{dt} = F_{in}C_{in} - \frac{F_{out}\,y}{V(t)}$$

$$\Rightarrow \frac{dy}{dt} = 0 - \frac{3y}{100-t} \quad ③$$

initial amt of salt in tank.

The IVP is given by ③ and $y(0) = 30$

HW2 -2

(ii) Eq. 3 is separable:

$$\int \frac{1}{3y}\,dy = -\int \frac{1}{100-t}\,dt.$$

$$\frac{1}{3}\ln y = -\ln(100-t)\cdot(-1) + \ln k.$$

$$\ln y^{1/3} = \ln(100-t) + \ln k.$$

rearranging, $\ln\left(\frac{y^{1/3}}{k}\right) = \ln(100-t)$

$$\therefore \quad y = [k(100-t)]^3 \quad (4) \quad \text{general s/n.}$$

Applying I.C

$$y(0) = k^3(100)^3 = 30 \quad \Rightarrow k^3 = \frac{30}{100^3} \quad (5)$$

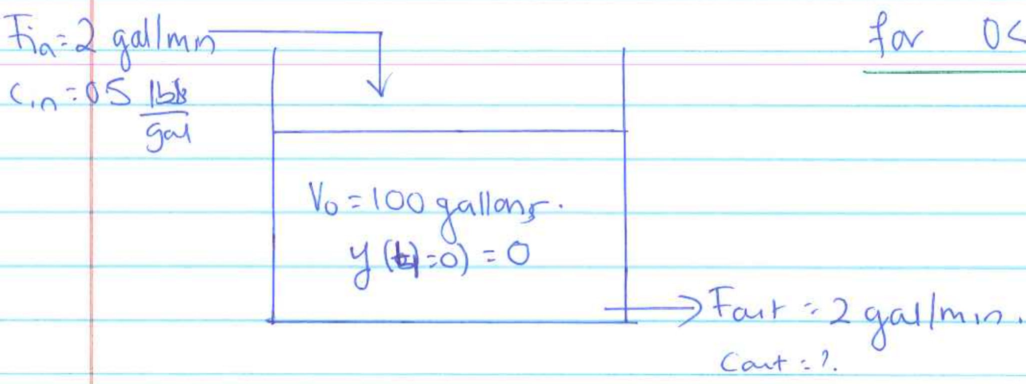
Sub 5 in 4

$$\boxed{y(t) = \frac{30}{100^3}\cdot(100-t)^3} \quad \text{particular solution}$$

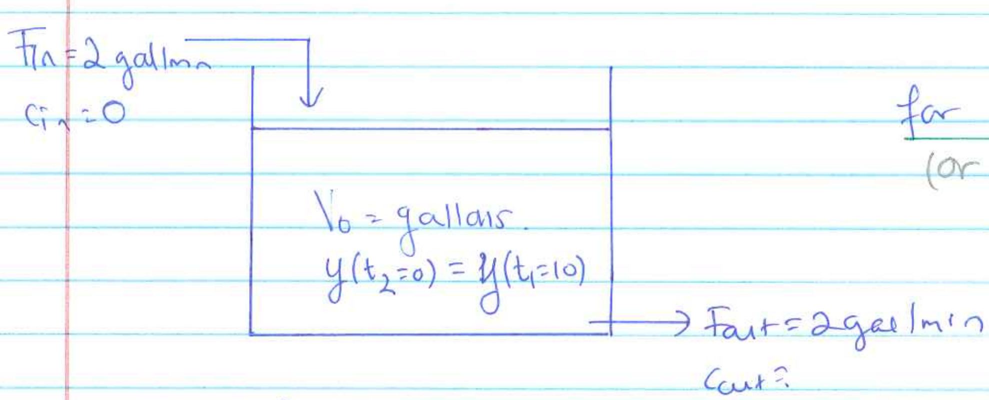
After 10 minutes, i.e. $y(10) = \frac{30}{100^3}(90)^3 = \underline{21.87 \text{ lbs}}$

HW2-3

Problem 2



for $0 < t_1 < 10$



for $0 < t_2 < \infty$.
(or $10 < t_1 < \infty$)

$y(t_2 = 10)$?

From process 1, the IVP is

$$\frac{dy}{dt_1} = 2 \times 0.5 - \frac{2y}{100} = 1 - \frac{y}{50}. \quad \text{with } y(0) = 0$$

Rewrite as $\frac{dy}{dt_1} = \frac{50 - y}{50}$ sep. variables. (alternatively you can use I.F.)

$$50 \int \frac{1}{50-y} \, dy = \int dt_1$$

$$-50 \ln(50-y) = t_1 + \ln k$$

$$(50-y)^{-50} = ke^{t_1}$$

HW2 - 4

$$50 - y = \left(ke^{t_1}\right)^{-1/50}$$

$$\boxed{y = 50 - ce^{-\frac{1}{50}t_1}} \quad (1) \qquad \text{where } c = k^{-1/50}$$

Apply I.C. : $y(0) = 0$

$y(0) = 50 - c = 0 \Rightarrow c = 50$

In (1), $y(t_1) = 50\left(1 - e^{\frac{-1}{50}t_1}\right)$

at $t_1 = 10$ the first process is stopped and the amt of salt in the tank is:

$y(10) = 50\left(1 - e^{-1/5}\right)$. (this is the I.C for process 2)

From process 2, the IVP is

$$\frac{dy}{dt_2} = 0 - \frac{2y}{100} \qquad \text{with } y(0) = 50\left(1 - e^{-1/5}\right)$$

Sep. variables

$$-\int \frac{50}{y}\, dy = \int dt_2$$

$$-50 \ln y = t_2 + \ln k_2$$

$$\ln \frac{y^{-50}}{k_2} = t_2$$

$$y = \left(k_2 e^{t_2}\right)^{-1/50}$$

$$y(t_2) = c_2 e^{-\frac{1}{50}t_2} \quad (2) \qquad \text{where } c_2 = k_2^{-1/50}.$$

HW 2-5

Apply I.C.

$$y(0) = c_2 = 50(1-e^{-1/5}).$$

In ②,

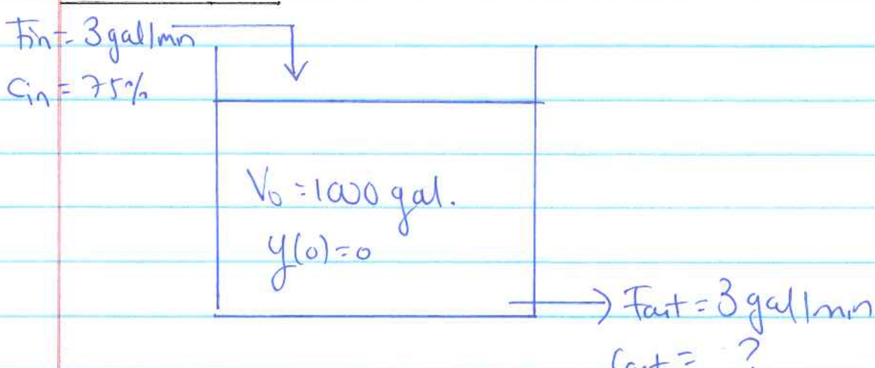
$$y(t_2) = 50(1-e^{-1/5})e^{-1/50\, t_2}.$$

After an additional 10 mins, when $t_2 = 10$,

$$y(10) = 50(1-e^{-1/5})e^{-1/5}.$$

$$y(10) \approx 7.42 \text{ lbs}.$$

Problem 3



$y(t)$ denotes the amt of gallons of alcohol in tank.

(i) The [illegible] describing the process is:

① $$\frac{[\text{gal alc}]}{[\text{min}]}\ \frac{dy}{dt} = F_{in}\frac{[\text{gal liq}]}{[\text{min}]} \times C_{in}\frac{[\text{gal alc.}]}{[\text{gal liq}]} - F_{out}\frac{[\text{gal liq}]}{[\text{min}]} \cdot \frac{y}{V}\frac{[\text{gal alc}]}{[\text{gal liq}]}$$

HW 2.6

In ①, I've indicated the units of the different quantities. The rate of change $\frac{dy}{dt}$ is in gallons of alc. per unit time and therefore all the terms on the RHS should have ... units.

The IVP is: $\frac{dy}{dt} = 3 \times \frac{75}{100} - \frac{3y}{1000}$ with $y(0) = 0$

no alcohol in tank initially.

(ii) $\Rightarrow \frac{dy}{dt} = \frac{9}{4} - \frac{3y}{1000}$ ②.

Rewrite as $\frac{dy}{dt} + \frac{3}{1000} y = \frac{9}{4}$. ③

The I.F. is $\mu(t) = e^{\frac{3}{1000}t}$ ④

Multiplying ③ by ④

$$e^{3/1000 t} \frac{dy}{dt} + e^{3/1000 t} \cdot \frac{3}{100} y = \frac{9}{4} e^{3/1000 t} \quad ⑤$$

Reduce as: $\frac{d}{dt}\left[e^{3/1000 t} \cdot y\right] = \frac{9}{4} e^{3/1000 t}$ ⑥

Check: $\frac{d}{dt}\left[e^{3/1000 t} \cdot y\right] = e^{3/1000 t} \frac{dy}{dt} + y \frac{3}{1000} e^{3/1000 t}$ ✓ equiv. to LHS of ⑤

Integ. ⑥ wrt t:

$$e^{3/1000 t} \cdot y = \frac{9}{4} \int e^{3/1000 t} \, dt.$$

$$\Rightarrow y(t) = \left[\frac{9}{4} \cdot \frac{1000}{3} e^{3/1000 t} + k\right] e^{-3/1000 t}.$$

HW2-7

Which simplifies to:

$$y(t) = 750 + ke^{-3/1000\,t}.$$

Apply I.C:

$$y(0) = 750 + k = 0 \Rightarrow k = -750.$$

$$\therefore \boxed{y(t) = 750\left(1 - e^{-\frac{3}{1000}t}\right)} \quad (7)$$ Particular soln.

(iii) For the mixture to be 50% alcohol then, the volume of alcohol in the tank needs to be $y(t) = 500$ (50% of total volume)

Use (7) to find the time t for which $y(t) = 500$

$$y(t) = 750\left(1 - e^{-3/1000\,t}\right) = 500.$$

$$e^{-3/1000\,t} = 1 - \frac{2}{3} = \frac{1}{3}$$

$$t = -\frac{1000}{3}\ln\frac{1}{3} = 366 \text{ mins.}$$

$t = 366$ mins.

Problem 4

For all ODEs in Problem 4, we compare the ODE to the form:

$$P(x,y) + Q(x,y)\frac{dy}{dx} = 0.$$

HW2-8

1. $(2x+y) + (x+2y)\dfrac{dy}{dx} = 0.$

For exactness, $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}.$

$$\left.\begin{aligned} \frac{\partial P}{\partial y} &= \frac{\partial}{\partial y}(2x+y) = 1 \\ \frac{\partial Q}{\partial x} &= \frac{\partial}{\partial x}(x+2y) = 1 \end{aligned}\right\} \quad \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} = 1 \Rightarrow \text{ODE is exact.}$$

The exact ODE is expressed as

$$\frac{d}{dx}\left[f(x,y)\right] = 0 \quad (*)$$

where $\dfrac{d}{dx}[f] = \dfrac{\partial f}{\partial x} + \dfrac{\partial f}{\partial y}\cdot\dfrac{dy}{dx}$

and $\dfrac{\partial f}{\partial x} = P(x,y)$ and $\dfrac{\partial f}{\partial y} = Q(x,y)$

$\Rightarrow \dfrac{\partial f}{\partial x} = 2x+y$ ① and $\dfrac{\partial f}{\partial y} = x+2y.$ ②

Integrate ① wrt x. (keeping y constant)

③ $f(x,y) = \int 2x+y\,dx = x^2+xy+k_1(y)$ ← fct of what we were keeping constant when integrating

Integrate ② wrt y (keeping x constant)

④ $f(x,y) = \int x+2y\,dy = xy+y^2+k_2(x)$

HW2 - 9

Comparing ③ & ④

$f(x,y) = x^2 + xy + k_1(y)$ ⎫ $k_1(y) = y^2$

$f(x,y) = k_2(x) + xy + y^2$ ⎭ $k_2(x) = x^2$

$\Rightarrow \boxed{f(x,y) = x^2 + xy + y^2}$ ⑤

Using ⊛ & ⑤ the ODE is:

$$\frac{d}{dx}\left[x^2 + xy + y^2\right] = 0 \qquad ⑥$$

we can now integ. ⑥ wrt x (note the LHS is the total deriv. wrt x so once we integrate ⑥ the RHS will simply be equal to a constant).

$\therefore \boxed{x^2 + xy + y^2 = C}$ implicit general soln.

2. $(x + 2y) + (2x + y)\frac{dy}{dx} = 0.$

Test for exactness: $\frac{\partial}{\partial y}(x + 2y) = 2$ ⎫ $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} = 2$

$\frac{\partial}{\partial x}(2x + y) = 2$ ⎭ ODE is exact.

Find $f(x,y)$ that satisfies the ODE $\frac{d}{dx}\left[f(x,y)\right] = 0$

since ODE is exact, $f(x,y)$ should satisfy

$$\frac{\partial f}{\partial x} = x + 2y \quad ① \qquad \text{and} \qquad \frac{\partial f}{\partial y} = 2x + y \quad ②$$

HW2 10

$\Rightarrow$ Integ. (1) wrt x:

$$f(x,y) = \int x + 2y \, dx = \frac{x^2}{2} + 2xy + k_1(y) \quad (3)$$

Integ. (2) wrt y:

$$f(x,y) = \int 2x + y \, dy = 2xy + \frac{y^2}{2} + k_2(x). \quad (4)$$

Compare (3) & (4)

$k_1(y) = \frac{y^2}{2}$ and $k_2(x) = \frac{x^2}{2}$.

Sub. $k_1(y) = \frac{y^2}{2}$ in (3)

$$f(x,y) = \frac{x^2}{2} + 2xy + \frac{y^2}{2}.$$

The ODE therefore is:

$$\frac{d}{dx}\left[\frac{x^2}{2} + 2xy + \frac{y^2}{2}\right] = 0 \quad (5)$$

To obtain the G.S. integ (5) wrt x

$$\frac{x^2}{2} + 2xy + \frac{y^2}{2} = C$$

or $\boxed{x^2 + 4xy + y^2 = C_2}$ where $C_2 = 2C$.

implicit gen. soln.

HW 2-11

3. $\left[2xy - x\sin(xy)\right] + \left[y^2 - y\sin(xy)\right]\frac{dy}{dx} = 0$

$\frac{\partial P}{\partial y} = 2x - x^2\cos xy$ $\qquad$ $\frac{\partial Q}{\partial x} = 0 - y^2\cos(xy)$

The ODE is not exact b/c $\frac{\partial P}{\partial y} \neq \frac{\partial Q}{\partial x}$.

4. $\left[y^2 - y\sin(xy)\right] + \left[2xy - x\sin(xy)\right]\frac{dy}{dx} = 0.$

$\frac{\partial P}{\partial y} = 2y - xy\cos(xy) - \sin(xy)$

$\frac{\partial Q}{\partial x} = 2y - xy\cos(xy) - \sin(xy)$

$\Rightarrow$ ODE is exact since $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$.

The ODE can be expressed as

$$\frac{d}{dx}\left[f(x,y)\right] = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dx} = 0 \qquad (1)$$

where $\frac{\partial f}{\partial x} = y^2 - y\sin(xy)$ (2) and $\frac{\partial f}{\partial y} = 2xy - x\sin(xy)$ (3)

Integ. (2) wrt x (keep y constant)

$$f(x,y) = \int y^2 - y\sin(xy)\,dx = y^2x + \frac{y\cos(xy)}{y} + k_1(y) \qquad (4)$$

Integ. ③ wrt y

$$f(x,y) = \int 2xy - x\sin(xy)\, dy$$

$$= xy^2 + \frac{x\cos(xy)}{x} + k_2(x). \quad ⑤$$

Compare ④ + ⑤

$$\left.\begin{aligned} f(x,y) &= xy^2 + \cos(xy) + k_1(y) \\ f(x,y) &= xy^2 + \cos(xy) + k_2(x) \end{aligned}\right\}$$ these must be equal therefore $k_1(y) = k_2(x) = 0$

$$\Rightarrow f(x,y) = xy^2 + \cos(xy).$$

From ①, the ODE is: $\frac{d}{dx}\left[xy^2 + \cos(xy)\right] = 0$

$\therefore$ integ. wrt x: $\boxed{xy^2 + \cos(xy) = C}$ implicit gen. soln.

Problem 5

$$\underbrace{\left(\frac{1}{x^2} + \frac{1}{y^2}\right)}_{= P(x,y)} + \underbrace{\left(\frac{ax+1}{y^3}\right)}_{= Q(x,y)} \frac{dy}{dx} = 0. \quad ①$$

For ① to be exact, $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$

HW 2-13

$$\frac{\partial P}{\partial y} = \frac{\partial}{\partial y}\left(\frac{1}{x^2} + \frac{1}{y^2}\right) = 0 - \frac{2}{y^3} \quad ①$$

$$\frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}\left(\frac{ax+1}{y^3}\right) = \frac{1}{y^3} \cdot a \quad ③$$

Equating ② + ③

$$-\frac{2}{y^3} = \frac{a}{y^3} \Rightarrow a = -2.$$

The exact ODE is: $\left(\frac{1}{x^2} + \frac{1}{y^2}\right) + \left(\frac{1-2x}{y^3}\right)\frac{dy}{dx} = 0.$

Which is equivalent to: $\frac{d}{dx}\left[f(x,y)\right] = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dx} = 0.$

Where $\frac{\partial f}{\partial x} = \frac{1}{x^2} + \frac{1}{y^2}$ ④ & $\frac{\partial f}{\partial y} = \frac{1}{y^3} - \frac{2x}{y^3}$ ⑤

Find $f(x,y)$

Integ. ④ wrt x.

$$f(x,y) = -\frac{1}{x} + \frac{x}{y^2} + k_1(y) \quad ⑥$$

Integ. ⑤ wrt y

$$f(x,y) = \frac{-1}{2y^2} - 2x\left(\frac{-1}{2y^2}\right) + k_2(x) \quad ⑦$$

HW 2-14

Comparing ⑥ & ⑦:

$$f(x,y) = \frac{-1}{x} + \frac{x}{y^2} + k_1(y)$$

$$f(x,y) = k_2(x) + \frac{\cancel{2}x}{\cancel{2}y^2} - \frac{1}{2y^2}.$$

$\Rightarrow k_1(y) = -\frac{1}{2y^2}$

$k_2(x) = -\frac{1}{x}.$

$$\rightarrow f(x,y) = \frac{-1}{x} + \frac{x}{y^2} - \frac{1}{2y^2}.$$

The ODE is: $\frac{d}{dx}\left[\frac{-1}{x} + \frac{x}{y^2} - \frac{1}{2y^2}\right] = 0$

So, integ. wrt x:

$$\frac{-1}{x} + \frac{x}{y^2} - \frac{1}{2y^2} = C$$

multiply by $2xy^2$:

$$-2y^2 + 2x^2 - x = \cancel{2C}\,xy^2$$

$\boxed{2x^2 - 2y^2 - x = c_1 xy^2}$ where $c_1 = 2c$.

gen. soln in implicit form.