Homework

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Contents

1 Chapter 10 Section 1

1.1 Problem 3

1.1.1 Question

Assume that $rm = 0$ for some $r \in R$ and some $m \in M$ wiht $m \neq 0$. Prove that r does not have a left inverse (i.e., there is not $s \in R$ such that $sr = 1$).

1.1.2 Answer

Proof. Assume towards a contradiction that there exists some $s \in R$ such that $sr = 1$. Then we have

$$0 = s(rm) = (sr)m = m$$

Contradiction. □

1.2 Problem 5

1.2.1 Question

For any left ideal I of R define

$$IM = \{\sum_{\text{finite}} a_i m_i \mid a_i \in I, m_i \in M\}$$

to be the collection of all finite sums of elements of the form am where $a \in I$ and $m \in M$. Prove that IM is a submodule of M.

1.2.2 Answer

Proof. First we note that the set IM must be nonempty for since I is nonempty it contains some element say x and M is nonempty it contains some m. Thus $xm \in IM$

Now we must show that

$$m + rn \in IM \; \forall m, n \in IM, \forall r \in R$$

Let $m, n \in IM$, $r \in R$. Then we have

$$m + rn = \sum_{\text{finite}} a_i m_i + r \sum_{\text{finite}} a_i n_i$$

but r distributes over the sum and any product $ra_i \in I$. Thus we may just rewrite as

$$m + rn = \sum_{\text{finite}} a_i m_i + \sum_{\text{finite}} b_i n_i$$

for $a_i, b_i \in I$. This is just a finite sum of elements of the form am where $a \in I$ and $m \in M$ though since two finite sets taken together is just a finite set.

Hence IM is a submodules of M as claimed. □

1.3 Problem 8

1.3.1 Question

An element m of the R-module M is called a *torsion element* if $rm = 0$ for some nonzero element $r \in R$. The set of torsion elements is denoted

$$\text{Tor}(M) = \{m \in M \mid rm = 0 \text{ for some nonzero } r \in R\}.$$

1. Prove that if R is an integral domain then $\text{Tor}(M)$ is a submodule of M (called the *torsion* submodule of M).

2. Give an example of a ring R and an R-module M such that $\text{Tor}(M)$ is not a submodule. [Consider the torsion elements in the R-module R.]

3. If R has zero divisors show that every nonzero R-module has nonzero torsion elements.

1.3.2 Answer

1. *Proof.* $\text{Tor}(M)$ is nonempty since in particular it contains 0. (R is an integral domain $\Rightarrow 1 \in R \Rightarrow 1 \cdot 0 = 0 \Rightarrow 0 \in \text{Tor}(M)$).

   Moreover, given $m, n \in \text{Tor}(M)$, $x \in R$ say that $x'm = 0 = x''n$. We therefore have $x'x''(m + xn) = x'x''m + xx'x''n = x''(x'm) + xx'(x''n) = 0$. So we have shown the submodule criterion and $\text{Tor}(M)$ must be a submodule as claimed. □

2. Consider the ring given by the set of all 2×2 matrices with entries in $\mathbb{R}$ and the module given the ring operating on elements from same. and say

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

We have $A, B \in \mathrm{Tor}(M)$ since $AB = BA = 0$ yet since $A + B = I$ we know that there is no nonzero $r \in R$ such that $r(A + B) = 0$. Thus $\mathrm{Tor}(M)$ is not a submodule.

3. Let $ab = 0$ for $a \neq 0$ and $b \neq 0$. Now fix some $m \neq 0 \in M$. If $bm = 0$ then m is a nonzero torsion element and we're done. If not then $a(bm) = (ab)m = 0m = 0$ so bm is a nonzero torsion element.

1.4 Problem 18

1.4.1 Question

Let $F = \mathbb{R}$, let $V = \mathbb{R}^2$ and let T be the linear transformation from V to V which is rotation clockwise about the origin by $\pi/2$ radians. Show that V and 0 are the only $F[x]$-submodules for this T.

1.4.2 Answer

I will begin by establishing some facts that are true in general and will be used in this and the next 2 problems.

We know from the book that $F[x]$-submodules are subspaces in this context, and moreover that $F[x]$-submodules are precisely those subspaces which are mapped into themselves by T. We will exploit these observations and treat the next problems linear-algebraically.

Moreover, we note that in general a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ takes $0 \to 0$ and $\mathbb{R}^2 \to \mathbb{R}^2$. Thus these are both $F[x]$-submodules since they are subspaces.

Furthermore, if $\vec{v}$ is a vector such that $T(\vec{v}) = c\vec{v}$ then $\text{Span}(\vec{v})$ is an $F[x]$-submodule as any element of $\text{Span}(\vec{v})$ is just $c\vec{v}$ (and every such vector is an element of $\text{Span}(\vec{v})$) and by linearity we get $T(c\vec{v}) = cT(\vec{v})$. Conversely if $\vec{v}$ is a vector such that $T(\vec{v}) \neq c\vec{v}$ then $\text{Span}(\vec{v})$ is not an $F[x]$-submodule.

Proof. We consider subspaces spanned by one vector. Thus, assume towards a contradiction that there exists some nonzero $\vec{v} \in \mathbb{R}^2$ such that $T(\vec{v}) = c\vec{v}$. However this imples

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = c \begin{bmatrix} a \\ b \end{bmatrix} \Rightarrow \begin{bmatrix} b \\ -a \end{bmatrix} = c \begin{bmatrix} a \\ b \end{bmatrix}$$

So $b/a = -a/b = c$. Contradiction. $\square$

1.5 Problem 19

1.5.1 Question

Let $F = \mathbb{R}$, let $V = \mathbb{R}^2$ and let T be the linear transformation from V to V which is projection onto the y-axis. Show that V, 0, the x-axis and the y-axis are the only $F[x]$-submodules for this T.

1.5.2 Answer

Proof. We consider subspaces spanned by one vector. (See previous problem for justification) We seek $\vec{v} = (a, b) \neq \vec{0}$ such that,

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = c \begin{bmatrix} a \\ b \end{bmatrix} \Rightarrow \begin{bmatrix} b \\ 0 \end{bmatrix} = c \begin{bmatrix} a \\ b \end{bmatrix}$$

So $b = ca$ and $cb = 0$ for some c. This is only the case for vectors of the form $(a, 0)$ or $(0, b)$. Hence the x-axis and the y-axis are $F[x]$-submodules as claimed. $\square$

1.6 Problem 20

1.6.1 Question

Let $F = \mathbb{R}$, let $V = \mathbb{R}^2$ and let T be the linear transformation from V to V which is rotation clockwise about the origin by π radians. Show that *every* subspace of V is an $F[x]$-submodules for this T.

Proof. We consider subspaces spanned by one vector. (See 1.4.2 for justification) We seek $\vec{v} = (a, b) \neq \vec{0}$ such that,

$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = c \begin{bmatrix} a \\ b \end{bmatrix} \Rightarrow \begin{bmatrix} -a \\ -b \end{bmatrix} = c \begin{bmatrix} a \\ b \end{bmatrix}$$

So $-a = ca$ and $-b = cb$ for some c. This is the case for all a, b however (take $c = -1$) so all one dimensional subspaces are $F[x]$-submodules as claimed. $\square$

1.6.2 Answer

2 Chapter 10 Section 2

2.1 Problem 1

2.1.1 Question

Use the submodule criterion to show that kernels and images of R-module homomorphisms are submodules.

2.1.2 Answer

Proof. Say $\varphi : M \to M$ is a homomorphism of R-modules. Then take $x, y \in \ker(\varphi)$ and $r \in R$. $\varphi(x) = \varphi(y) = 0$ by definition. Thus we have $\varphi(x + ry) = \varphi(x) + r\varphi(y) = 0 + r0 = 0$. Therefore the kernel of φ is a submodule as claimed. $\square$

Proof. Let $x, y \in \mathrm{Img}(\varphi)$ and $r \in R$. Then there exist corresponding x', y' such that $\varphi(x') = x$ and $\varphi(y') = y$. So, we get $\varphi(x' + ry') = \varphi(x') + r\varphi(y') = x + ry \in \mathrm{Img}(\varphi)$ and the image of φ is a submodule as claimed. $\square$

2.2 Problem 3

2.2.1 Question

Give an explicit example of a map from one R-module to another which is a group homomorphism but not an R-module homomorphism.

2.2.2 Answer

Consider the matrix module over $\mathrm{GL}_2(\mathbb{R})$ with same as the ring. Moreover let

$$\varphi(x) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} x.$$

That such an mapping is a group homomorphism has been proven previously. However, $\varphi(rx) \neq r\varphi(x)$ since in particular we have

$$\varphi[\begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}\begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}] = \begin{bmatrix} 249 & 290 \\ 565 & 658 \end{bmatrix} \neq \begin{bmatrix} 353 & 410 \\ 477 & 554 \end{bmatrix} = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}\varphi[\begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}]$$

2.3 Problem 7

2.3.1 Question

Let z be a fixed element of the center of R. Prove that the map $m \mapsto zm$ is an R-module homomorphism from M to itself. Show that for a commutative ring R the map from R to $\operatorname{End}_R(M)$ given by $r \mapsto rI$ is a ring homomorphism (where I is the identity endomorphism).

2.3.2 Answer

Proof. We fix $s \in R$ and $x, y \in M$. No we verify $\varphi(x + y) = r(x + y) = rx + ry = \varphi(x) + \varphi(y)$ and $\varphi(sx) = rsx = srx = s\varphi(x)$ as desired. □

Proof. The mapping $\varphi(x) : R \to \operatorname{End}_R(M)$ given by $r \mapsto rI$ is a well defined by the above proof. Now let $x, y \in R$ and note that $\varphi(x+y) = (x+y)I = xI+yI = \varphi(x)+\varphi(y)$ and $\varphi(xy) = (xy)I = (yI)x = x\varphi(y)$ by commutativity. □

2.4 Problem 8

2.4.1 Question

Let $\varphi : M \to N$ be an R-module homomorphism. Prove that $\varphi(\operatorname{Tor}(M)) \subseteq \operatorname{Tor}(N)$ (cf. Exercise 8 in Section 1).

2.4.2 Answer

Let $m \in \mathrm{Tor}(N)$. Then by definition there exists some nonzero $r \in R$ such that $rm = 0$. Hence $0 = \varphi(0) = \varphi(rm) = r\varphi(m)$ and $\varphi(m) \in \mathrm{Tor}(N)$. This implies that $\varphi(\mathrm{Tor}(M)) \subseteq \mathrm{Tor}(N)$ as claimed.

2.5 Problem 10

2.5.1 Question

Let R be a commutative ring. Prove that $\mathrm{Hom}_R(R, R)$ and R are isomorphic as rings.

2.5.2 Answer

I claim that the mapping $\varphi : \mathrm{Hom}_R(R, R) \to R$ defined by $\varphi(x) = \psi(1)$ for ψ a homomorphism is an isomorphism of rings.

Proof. We first show that it is a homomorphism of rings. Towards this let x, y in $\mathrm{Hom}_R(R, R)$. Now observe that $\varphi(x+y) = (x+y)(1) = x(1)+y(1) = \varphi(x) + \varphi(y)$ and that furthermore $\varphi(xy) = (xy)(1) = x(1)y(1) = \varphi(x)\varphi(y)$.

Now it remains to show that this homomorphism is bijective. First we will show injectivity. Assume that $\varphi(x) = \varphi(y) \Rightarrow x(1) = y(1)$ but since for any $z \in R$ we have $1z = z$ we just get $x(z) = x(z1) = zx(1) = zy(1) = y(z)$ and therefore $x = y$ by definition.

Now surjectivity. Assume towards a contradiction that there exists $r \in R$ such that for no x do we have $\varphi(x) = r$. This is a contradiction since we can define a homomorphism by $\psi(x) = rx$ (see a previous exercise) and this homomorphism surely has $\psi(1) = r$. Contradiction. $\square$