Homework

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21 February 2010

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1 Chapter 10 Section 3

1.1 Problem 4

1.1.1 Question

An R-module M is called a *torsion* module if for each $m \in M$ there is a nonzero element $r \in R$ such that $rm = 0$, where r may depend on m (i.e., $M = \text{Tor}(M)$ in the notation of Exercise 8 of Section 1). Prove that every finite abelian group is a torsion $\mathbb{Z}$-module. Give an example of an infinite abelian group that is a torsion $\mathbb{Z}$-module.

1.1.2 Answer

Proof. Let G be a finite abelian group and let $x \in G$ for some $x \neq 0$. I claim that there exists some $xn \in G$ such that $xn = 0$ for $n \neq 0$. Since the entire group is finite there exists some least xm such that $xm = x$ and $m \neq 1$. Thus $xm = x \Rightarrow xm - x = 0 \Rightarrow x(m-1) = 0$. So we have constructed a nonzero n which fulfills the desired criterion. □

I claim that the infinite direct sum of $\mathbb{Z}/i\mathbb{Z}$ for each $i \in \mathbb{Z}^+$ say X is a torsion $\mathbb{Z}$-module.

Proof. There are only finitely many nonzero components of a given element of X. Say that each nonzero element is in position k_j (that is, it is a member of the cyclic group of order $\mathbb{Z}/k_j\mathbb{Z}$). Take the product $k_1k_2\dots k_m$. The product of this number with our element is zero since in any cyclic group the product of a multiple of its order and any element is just zero. □

1.2 Problem 6

1.2.1 Question

Prove that if M is a finitely generated R-module that is generated by n elements that every quotient of M may be generated by n (or fewer) elements. Deduce that quotients of cyclic modules are cyclic.

1.2.2 Answer

Proof. Assume that M is generated by some set say $\{m_1, m_2, \ldots, m_n\}$. Then any element of M can be written in the form $m = r_1m_1 + r_2m_2 + \cdots + r_nm_n$.

The quotient module M/N must be finitely generated. In particular the set $\{M_1, M_2, \ldots, M_n\}$ generates M/N where each $M_i \in M/N$ has $m_i \in M_i$ as an arbitary element $x \in M$ can be written as $x = r_1m_1 + r_2m_2 + \cdots + r_nm_n$ so $x \in r_1M_1 + r_2M_2 + \cdots + r_nM_n$ □

A module is cyclic if and only if it is generated by one element. Thus, by the above proof if a module is cyclic it is generated by one element and its quotients are also generated by one element and are therefore cyclic too.

1.3 Problem 7

1.3.1 Question

Let N be a submodule of M. Prove that if both M/N and N are finitely generated then so is M.

1.3.2 Answer

Proof. We may divide the elements of M into equivalence classes where two elements belong to the same equivalence class if and only if they are brought to the same element of M/N by the natural quotient homomorphism. Moreover we know that any element of one of these equivalence classes say $l \in L$ can be written as $l = n + x$ for some $n \in N$ and x the same value for each $l \in L$.

Hence, since N is finitely generated, and M/N is finitely generated there is a finite collection of $m_1, m_2, \ldots, m_n$ which generate each member of the equivalence classes corresponding to the members of the quotient group which generate it. These are in particular the set of generators for N together with one element of each generator in the quotient module.

So $m_1, m_2, \ldots, m_n$ generate M as each element of the quotient module can be written as a product of the generating elements and thence any member of the corresponding equivalence class can be written as a product of members of the equivalence classes corresponding to the generating elements of the quotient module. □

1.4 Problem 9

1.4.1 Question

An R-module M is called *irreducible* if $M \neq 0$ and if 0 and M are the only submodules of M. Show that M is irreducible if and only if $M \neq 0$ and M is a cyclic module with any nonzero element as generator. Determine all the irreducible $\mathbb{Z}$-modules.

1.4.2 Answer

Throughout assume $M \neq 0$

We prove ($\Rightarrow$) that if any nonzero element of M generates it then M is irreducible.

Proof. Let $N \subset M$ be a submodule of M. Suppose that it is nonzero. Then it contains some nonzero element $n \in N$. By assumption n generates N. Hence, by closure under product $M = Rn \subset N$. Thus $N = M$ and every nonzero submodule of M is all of M. $\square$

We prove ($\Leftarrow$) that any nonzero element of M generates it if M is irreducible.

Proof. Let $x \in M$ be nonzero. Since x is nonzero we know that $\langle x \rangle \neq 0$ thus, by irreducibility of M it must be that $\langle x \rangle = M$ as claimed. $\square$

The irreducible $\mathbb{Z}$ modules are precisely $\mathbb{Z}/p\mathbb{Z}$ for p prime. R modules are just abelian groups in $\mathbb{Z}$ and irreducible modules are just those which have each element a generator by the above proof. Thus, by the fundamental theorem of cyclic groups each y $\mathbb{Z}/p\mathbb{Z}$ for p prime is a irreducible module.

1.5 Problem 18

1.5.1 Question

Let R be a Principal Ideal Domain and let M be an R-module that is annihilated by the nonzero, proper ideal (a). Let $a = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ be the unique factorization of a into distinct prime powers in R. Let M_i be the annihilator of $p_i^{\alpha_i}$ in M, i.e., M_i is the set of $\{m \in M \mid p_i^{\alpha_i} m = 0\}$ — called the *p_i-primary component of M*. Prove that

$$M = M_1 \oplus M_2 \oplus \cdots \oplus M_k.$$

1.5.2 Answer

We will employ results from 16 and 17 without proof.

Proof. Since $(a)M = 0$ we know that $M/(a)M = M$. Moreover the ideals $p_i^{\alpha_i}$ are each pairwise comaximal so we can employ the result of Exercise 17 to conclude that

$$M \cong M/(p_1^{\alpha_1})M \times \cdots \times M/(p_k^{\alpha_k})M.$$

Now we shall let $m = (m_1, \ldots, m_k)$ be an element of this cartesian product.

Suppose that $p_i^{\alpha_i} m = 0$. We then have $p_i^{\alpha_i} m_j = 0$ for all j. If $j \neq i$ then $p_j^{\alpha_j} m_j = 0$ as well since $(p_i^{\alpha_i})$ and $(p_j^{\alpha_j})$ are comaximal this implies that $m_j = 0$. Hence the $p_i^{\alpha_i}$-primary component of the cartesian product is $M/(p_i^{\alpha_i} M)$.

The isomorphism gives a direct product – and therefore direct sum since we have a finite number of factors – decomposition of M into its $p_i^{\alpha_i}$-primary components. By definition each such component is just M_i. Hence

$$M = M_1 \oplus M_2 \oplus \cdots \oplus M_k$$

as claimed. □

1.6 Problem 20

1.6.1 Question

Let I be a nonempty index set and for each $i \in I$ let M_i be an R-module. The *direct product* of the modules M_i is defined to be their direct product as abelian groups (cf. Exercise 15 in section 5.1) with the action of R componentwise multiplication. The *direct sum* of modules M_i is defined to be the restricted direct product of the abelian groups M_i (cf. Exercise 17 in Section 5.1) with the action or R componentwise multiplication. In other words, the direct sum of the M_i's is the subset of the direct product, $\prod_{i \in I} M_i$, which consists of elements $\prod_{i \in I} m_i$ such that only finitely many of the components m_i are nonzero; the action of R on the direct product or direct sum is given by $r\prod_{i\in I} m_i = \prod_{i\in I} rm_i$ (cf. Appendix I for the definition of Cartesian products of infinitely many sets). The direct sum will be denoted by $\oplus_{i\in I} M_i$.

1. Prove that the direct product of the M_i's is an R-module and the direct sum of the M_i's is a submodule of their direct product.

2. Show that if $R = \mathbb{Z}$, $I = \mathbb{Z}^+$ and M_i is the cyclic group of order i for each i, then the direct sum of the M_i's is not isomorphic to their direct product. [Look at torsion.]

1.6.2 Answer

1. First I prove that the direct product of the M_i's is an R-module.

   Proof. We have proven previously that the direct product of the abelian groups are abelian groups under componentwise addition. Thus in particular the direct product of each M_i is an abelian group under componentwise addition.

   The first property we must verify is that $(r+s)m = rm + sm$. This just follows from the same property in each M_i though. In particular note that if m_i is a component of some m we have $(r+s)m_i = rm_i + sm_i$. Hence since our operations are defined componentwise we have $(r+s)m = rm + sm$ as desired.

   Now we verify that $(rs)m = r(sm)$. Again, this fact follows from the corresponding property in each M_i. In particular note that if m_i

is a component of some m we have $(rs)m_i = r(sm_i)$. Hence since our operations are defined componentwise we have $(rs)m = r(sm)$ as desired.

Now we show that $r(m+n) = rm + rn$ for any m, n in the direct product. Just as in the previous two properties this just follows from the corresponding property in each M_i. In particular note that if m_i, n_i are corresponding components of some m, n we have $r(m_i + n_i) = rm_i + rn_i$. Hence since our operations are defined componentwise we have $r(m+n) = rm + rn$.

Finally we note that if $1 \in R$ then $1m = m$ for all m in the direct product since for each $m \in M_i$ we have $1m = m$. □

Now I will show that the direct sum is a submodule of the direct product using the submodule criterion.

Proof. It is readily apparent that the direct sum is a nonempty subset. Now we wish to show that for arbitrary m, n in the direct sum and $r \in R$ we have $m + rn$ in the direct sum.

By closure of each M_i we know that rn is in the direct sum. Any member of the direct sum has a finite number of nonzero components. Say that m has l and n has k. Then $m + rn$ has at most $l + k$ nonzero components. So we have demonstrated the submodule criterion and shown that the direct sum is a submodule of the direct product as desired. □

2. If $A \cong B$ are two modules over the same ring R and $A = \text{Tor}(A)$ then $B = \text{Tor}(B)$.

   Proof. Let $a \in A$. There exists some nonzero $r \in R$ such that $ra = 0$. If φ is the isomorphism then $\varphi(ra) = r\varphi(a) = 0$. Since isomorphisms are bijective there exists some $a \in A$ such that $\varphi(a) = b$ for each $b \in B$. Hence there exists a nonzero $r \in R$ such that $rb = 0$ for every $b \in B$ and $B = \text{Tor}(B)$ by definition. □

   Now I claim that the direct sum is a torsion module but that the direct product is not. For a proof of the first claim see 1.1.2

   Proof. There exists no $r \in \mathbb{Z}$ such that $r(1, 1, \ldots) = (0, 0, \ldots)$. In particular assume that there is such an r. But $r1 = r \neq 0 \in \mathbb{Z}/(r+1)\mathbb{Z}$. Contradiction. Hence the direct product is not a torsion module as claimed. □

1.7 Problem 22a

1.7.1 Question

Let R be a Principal Ideal Domain, let M be a torsion R-module (cf. Exercise 4) and let p be a prime in R (do not assume M is finitely generated, hence it need not have a nonzero annihilator — cf. Exercise 5). The *p-primary* *component* of M is the set of all elements of M that are annihilated by some positive power of p.

1. Prove that the p-primary component is a submodule. [See Exercise 13 in Section 1.]

1.7.2 Answer

Proof. We will use the submodule criterion.

The p-primary component is nonempty because in particular it contains 0. (0 is annihilated by any $r \in R$)

I claim that given m, n in the p-primary component $r \in R$, $m + rn$ is in the p-primary component. Since m, n in the p-primary component there are positive powers of p which annihilate them. Say $p^\alpha m = p^\beta n = 0$, then $p^{\alpha+\beta}(m + rn) = p^\beta(p^\alpha m) + rp^\beta(p^\alpha n) = 0 + 0 = 0$. Thus, the p-primary component is a submodule as claimed. $\square$