

Homework

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1 Chapter 13 Section 1

1.1 Problem 1

1.1.1 Question

Show that $p(x) = x^3 + 9x + 6$ is irreducible in $\mathbb{Q}[x]$. Let θ be a root of $p(x)$. Find the inverse of $1 + \theta$ in $\mathbb{Q}(\theta)$

1.1.2 Answer

I claim that $p(x) = x^3 + 9x + 6$ is irreducible in $\mathbb{Q}[x]$.

Proof. Recall that Eisenstein's Criterion states that

If p is a prime in \mathbb{Z} and $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$ with $n \geq 1$ such that p divides a_i for all $0 \leq i \leq n-1$ but p^2 does not divide a_0 then $f(x)$ is irreducible in both $\mathbb{Z}[x]$ and $\mathbb{Q}[x]$

So, we can verify that $p = 3$ satisfies this criterion, and therefore $p(x)$ is irreducible over $\mathbb{Q}[x]$ as desired. \square

Now, we want to find polynomials $a(x)$ and $b(x)$ in $\mathbb{Q}[x]$ such that

$$a(x)(x+1) + b(x)(x^3 + 9x + 6) = 1$$

we can check that if we take $a(x) = (x^2 - x + 10)/4$ and $b(x) = -1/4$ we get

$$\begin{aligned} -\frac{1}{4}(-x^3 + x^2 - 10x - x^2 + x - 10) - \frac{1}{4}(x^3 + 9x + 6) &= -\frac{1}{4}(-4) \\ &= 1 \end{aligned}$$

as desired. So,

$$\theta^{-1} = \frac{1}{4}(\theta^2 - \theta + 10).$$

1.2 Problem 3

1.2.1 Question

Show that $x^3 + x + 1$ is irreducible over \mathbb{F}_2 and let θ be a root. Compute the powers of θ in $\mathbb{F}_2(\theta)$.

1.2.2 Answer

I claim that $f(x) = x^3 + x + 1$ is irreducible over \mathbb{F}_2 .

Proof. If $x^3 + x + 1$ were reducible, at least one of its factors must be an order 1 polynomial. Therefore, if f is reducible it has a root. However, we can check that f has no roots. In particular $f(0) = 1 \neq 0$ and $f(1) = 1 \neq 0$. Therefore, f is not reducible over \mathbb{F}_2 . \square

Since we know that $\theta^3 + \theta + 1 = 0$ in $\mathbb{F}_2(\theta)$ we have the relation $\theta^3 = -\theta - 1 = \theta + 1$. Now we can compute

<i>order</i>	<i>power</i>	<i>order</i>	<i>power</i>
0	1	4	$\theta^2 + \theta$
1	θ	5	$\theta^2 + \theta + 1$
2	θ^2	6	$\theta^2 + 1$
3	$\theta + 1$	7	1

Since we have in general $\theta^k = \theta^{m \cdot 7} \theta^{k \bmod 7}$ for some $m \in \mathbb{N} \cup \{0\}$ we can just write

$$\begin{aligned} \theta^k &= \theta^{m \cdot 7} \theta^{k \bmod 7} \\ &= (\theta^7)^m \cdot \theta^{k \bmod 7} \\ &= 1^m \cdot \theta^{k \bmod 7} \\ &= \theta^{k \bmod 7} \end{aligned}$$

So the above table completely describes all powers of θ .

2 Chapter 13 Section 2

2.1 Problem 3

2.1.1 Question

Determine the minimal polynomial over \mathbb{Q} for the element $1 + i$.

2.1.2 Answer

The minimal polynomial over \mathbb{Q} for $1 + i$ is just

$$f(x) = x^2 - 2x + 2$$

Proof. f is monic and a member of $\mathbb{Q}[x]$ so it remains only to verify that it is irreducible and that it has $1 + i$ as a root. First we'll check that $1 + i$ is a root

$$\begin{aligned} (1 + i)^2 - 2(1 + i) + 2 &= 2i - 2(1 + i) + 2 \\ &= 0. \end{aligned}$$

Now we must verify that f is irreducible, however this is clear, by Eisenstein's criterion if we take $p = 2$. \square

2.2 Problem 4

2.2.1 Question

Determine the degree over \mathbb{Q} of $2 + \sqrt{3}$ and of $1 + \sqrt[3]{2} + \sqrt[3]{4}$

2.2.2 Answer

The degree of $2 + \sqrt{3}$ over \mathbb{Q} is 2.

Proof. We can verify that the minimal polynomial of $2 + \sqrt{3}$ in \mathbb{Q} is

$$f(x) = x^2 - 4x + 1.$$

First we observe that

$$\begin{aligned} (2 + \sqrt{3})^2 - 4(2 + \sqrt{3}) + 1 &= (7 + 4\sqrt{3}) - (8 + 4\sqrt{3}) + 1 \\ &= 0 \end{aligned}$$

so, $2 + \sqrt{3}$ is a root of f . So, if f were reducible, one of its factors would necessarily be $x - (2 + \sqrt{3})$. Since $2 + \sqrt{3}$ is not in \mathbb{Q} f must be irreducible. Thus, by Proposition 11 (Page 521), the degree of $2 + \sqrt{3}$ is 2. \square

The degree of $1 + \sqrt[3]{2} + \sqrt[3]{4}$ over \mathbb{Q} is 3.

Proof. I claim that the minimal polynomial of $1 + \sqrt[3]{2} + \sqrt[3]{4}$ is

$$f(x) = x^3 - 3x^2 - 3x - 1$$

First check that

$$\begin{aligned} (1 + \sqrt[3]{2} + \sqrt[3]{4})^3 - 3(1 + \sqrt[3]{2} + \sqrt[3]{4})^2 - 3(1 + \sqrt[3]{2} + \sqrt[3]{4}) - 1 \\ = (19 + 152^{1/3} + 122^{2/3}) - 3(5 + 42^{1/3} + 32^{2/3}) - 3(1 + \sqrt[3]{2} + \sqrt[3]{4}) - 1 \\ = 0 \end{aligned}$$

Now, observe that, were f reducible, then $1 + \sqrt[3]{2} + \sqrt[3]{4}$ would be a root of some polynomial with degree 2 or less. In particular that would imply that some nontrivial linear combination of $(1 + \sqrt[3]{2} + \sqrt[3]{4})^2 = 5 + 4\sqrt[3]{2} + 3\sqrt[3]{4}$, $1 + \sqrt[3]{2} + \sqrt[3]{4}$, and 1 with rational coefficients is zero. However, such linear combinations are of form $(4a + b)\sqrt[3]{2} + (3a + b)\sqrt[3]{4} + (5a + b + c)$. However, as $\sqrt[3]{2}, \sqrt[3]{4}$ are not members of \mathbb{Q} such linear combinations must have $4a + b = 3a + b = 5a + b + c = 0$. There are no such linear combinations. So, f must be irreducible. We can thus conclude that $1 + \sqrt[3]{2} + \sqrt[3]{4}$ is of degree 3 over \mathbb{Q} . \square

2.3 Problem 5

2.3.1 Question

Let $F = \mathbb{Q}(i)$. Prove that $x^3 - 2$ and $x^3 - 3$ are irreducible over F .

2.3.2 Answer

Proof. First observe that since each of these polynomials has degree 3, they are both reducible if and only if they have roots. Furthermore, for any root of either of these equations, the field extension generated by that root over \mathbb{Q} has degree 3 since both equations are irreducible in \mathbb{Q} by Eisenstein. This is the degree of the minimal field extension which contains the roots. Thus, neither polynomial can have roots in F which is of degree 2, and consequently neither is reducible over F . \square

2.4 Problem 7

2.4.1 Question

Prove that $\mathbb{Q}(\sqrt{2} + \sqrt{3}) = \mathbb{Q}(\sqrt{2}, \sqrt{3})$ [one inclusion is obvious, for the other consider $(\sqrt{2} + \sqrt{3})^2$, etc.] Conclude that $[\mathbb{Q}(\sqrt{2} + \sqrt{3}) : \mathbb{Q}] = 4$. Find an irreducible polynomial satisfied by $\sqrt{2} + \sqrt{3}$

2.4.2 Answer

Proof. Clearly $\mathbb{Q}(\sqrt{2} + \sqrt{3}) \subseteq \mathbb{Q}(\sqrt{2}, \sqrt{3})$. To show the reverse inclusion it suffices to note that

$$\frac{(\sqrt{2} + \sqrt{3})^3 - 9(\sqrt{2} + \sqrt{3})}{2} = \sqrt{2} \quad \text{and} \quad \frac{11(\sqrt{2} + \sqrt{3}) - (\sqrt{2} + \sqrt{3})^3}{2} = \sqrt{3}.$$

Since we can write the generators of $\mathbb{Q}(\sqrt{2}, \sqrt{3})$ as members of $\mathbb{Q}(\sqrt{2} + \sqrt{3})$ we must have $\mathbb{Q}(\sqrt{2}, \sqrt{3}) \subseteq \mathbb{Q}(\sqrt{2} + \sqrt{3})$ as desired. \square

We must have $[\mathbb{Q}(\sqrt{2} + \sqrt{3}) : \mathbb{Q}] = 4$ since $[\mathbb{Q}(\sqrt{2}, \sqrt{3}) : \mathbb{Q}] = 4$ (minimal polynomials $x^2 - 2$ and $x^2 - 3$) and by the above proof $\mathbb{Q}(\sqrt{2} + \sqrt{3}) = \mathbb{Q}(\sqrt{2}, \sqrt{3})$.

I claim that the polynomial

$$f(x) = x^4 - 10x^2 + 1$$

is irreducible and has $\sqrt{2} + \sqrt{3}$ as a root.

Proof. First we check

$$\begin{aligned} (\sqrt{2} + \sqrt{3})^4 - 10(\sqrt{2} + \sqrt{3})^2 + 1 &= (49 + 20\sqrt{6}) - 10(5 + 2\sqrt{6}) + 1 \\ &= 0 \end{aligned}$$

This suffices also to show that f is irreducible since if it were reducible, then we would have $[\mathbb{Q}(\sqrt{2} + \sqrt{3}) : \mathbb{Q}] < 4$ a contradiction. \square

2.5 Problem 8

2.5.1 Question

Let F be a field of characteristic $\neq 2$. Let D_1 and D_2 be elements of F , neither of which is a square in F . Prove that $F(\sqrt{D_1}, \sqrt{D_2})$ is of degree 4 over F if $D_1 D_2$ is not a square in F and is of degree 2 over F otherwise. When $F(\sqrt{D_1}, \sqrt{D_2})$ is of degree 4 over F the field is called a *biquadratic extension* of F .

2.5.2 Answer

Proof. We have by Lemma 16 that $F(\sqrt{D_1}, \sqrt{D_2}) = (F(\sqrt{D_1}))(\sqrt{D_2})$. Since we know that D_1 is not a square in F , it follows that $\sqrt{D_1}$ is of degree 2 in F . Now there are two cases.

If $D_1 D_2$ is a square in F then either $D_1 = D_2 l^2$ or $D_2 = D_1 l^2$ for some $l \in F$. Hence, $\sqrt{D_2} \in F(\sqrt{D_1})$ and $F(\sqrt{D_1}, \sqrt{D_2}) = (F(\sqrt{D_1}))(\sqrt{D_2}) = F(\sqrt{D_1})$ which in turn implies $[F(\sqrt{D_1}, \sqrt{D_2}) : F] = 2$.

On the other hand, if $D_1 D_2$ is not a square in F then $\sqrt{D_2} \notin F(\sqrt{D_1})$. Since $\sqrt{D_2}$ is of degree 2 in F it is of at most degree 2 in $F(\sqrt{D_1})$. Thus, $\sqrt{D_2}$ is of degree exactly 2 in $F(\sqrt{D_1})$ and $[F(\sqrt{D_1}, \sqrt{D_2}) : F] = 4$ as desired. \square