Homework 6

Frederick Robinson

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1 Question 1

1.1 Question

Prove that induction is transitive.

1.2 Answer

For $H \leq K \leq G$ we want $Ind_K^G(Ind_H^K V) = Ind_H^G V$.

Proof. This is equivalent to showing that

$$FG \otimes_{FK} (FK \otimes_{FH} V) \cong FG \otimes_{FH} V.$$

But this is clear.

$$\begin{aligned} FG \otimes_{FK} (FK \otimes_{FH} V) &\cong (FG \otimes_{FK} FK) \otimes_{FH} V \\ &\cong FG \otimes_{FH} V. \end{aligned}$$

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2 Question 2

2.1 Question

Use your earlier computation of the character table of S_5 to compute the character table of A_5.

2.2 Answer

Recall that the character table of S_5 is

	1 e	10 (12)	20 (123)	30 (1234)	24 (12345)	15 (12)(34)	20 (12)(345)
χ_1	1	1	1	1	1	1	1
χ_2	1	-1	1	-1	1	1	-1
χ_3	4	2	1	0	-1	0	-1
χ_4	4	-2	1	0	-1	0	1
χ_5	5	-1	-1	1	0	1	-1
χ_6	5	1	-1	-1	0	1	1
χ_7	6	0	0	0	1	-2	0

First we need to get the conjugacy classes. The identity is still in its own conjugacy class. We use the orbit-stabilizer theorem as in class and observe that the conjugacy classes in S_5 made up of even permutations correspond to a single conjugacy class if they commute with some odd permutation. If they do not, then they break up into two conjugacy classes of the same size.

The product of two disjoint 2-cycles commutes with a transposition, as does a 3-cycle. These are therefore still a single conjugacy class. The conjugacy class of 5-cycles however splits, as the centralizer of a given 5-cycle is just the cyclic subgroup generated by this cycle. However, in A_5 this centralizer contains a different number of elements.

Now, we note that there are 4 and 5 dimensional representations by the restriction argument from class. Then, since

$$\begin{aligned} 60 &= 1 + 16 + 25 + \sum n_i^2 \\ 18 &= \sum n_i^2 \end{aligned}$$

the remaining two must have dimension 3.

	1 e	20 (123)	12 (12345)	12 (12354)	15 (12)(34)
χ_1	1	1	1	1	1
χ_2	3	a	b	c	d
χ_3	4	1	-1	-1	0
χ_4	3	e	f	g	h
χ_5	5	-1	0	0	1

Now we use orthogonality of columns. First, use it wit the column corresponding to (123). We get $|A_5 : \langle (132) \rangle| = 1^2 + 1^2 + (-1)^2 + a^2 + e^2$, so $a = e = 0$. Now, using it with the column corresponding to $(12)(34)$ to get $|A_5 : \langle(12)(34)\rangle| = 4 = 1^2 + (-1)^2 + d^2 + h^2$, we see that d, h are each one of ± 1. Since the sum $\chi_2 + \chi_4$ is χ_7 from the character table of S_5 we must have $d = h = -1$.

	1 e	20 (123)	12 (12345)	12 (12354)	15 (12)(34)
χ_1	1	1	1	1	1
χ_2	3	0	b	c	-1
χ_3	4	1	-1	-1	0
χ_4	3	0	f	g	-1
χ_5	5	-1	0	0	1

Finally, we may apply row orthogonality to χ_1, χ_4 to obtain

$$3 + 12f + 12g - 15 = 0 \Rightarrow f + g = 1.$$

Now, since $\langle \chi_4, \chi_4 \rangle = 1$ we have

$$60 = 9 + 12(1-g)^2 + 12g^2 - 15 \Rightarrow g = \frac{1}{2}\left(1 \pm \sqrt{5}\right).$$

Using the fact that the sum $\chi_2 + \chi_4$ is χ_7 from the character table of S_5 again we now have

	1 e	20 (123)	12 (12345)	12 (12354)	15 (12)(34)
χ_1	1	1	1	1	1
χ_2	3	0	$\frac{1}{2}\left(1 \pm \sqrt{5}\right)$	$\frac{1}{2}\left(1 \mp \sqrt{5}\right)$	-1
χ_3	4	1	-1	-1	0
χ_4	3	0	$\frac{1}{2}\left(1 \mp \sqrt{5}\right)$	$\frac{1}{2}\left(1 \pm \sqrt{5}\right)$	-1
χ_5	5	-1	0	0	1

Now we can just make an arbitrary choice of sign, since making the other choice just amounts to a relabeling.

	1 e	20 (123)	12 (12345)	12 (12354)	15 (12)(34)
χ_1	1	1	1	1	1
χ_2	3	0	$\frac{1+\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$	-1
χ_3	4	1	-1	-1	0
χ_4	3	0	$\frac{1-\sqrt{5}}{2}$	$\frac{1+\sqrt{5}}{2}$	-1
χ_5	5	-1	0	0	1

3 Question 3

3.1 Question

Let H be a subgroup of G and let V be a representation of G. Prove that $\mathrm{Ind}_H^G(\mathrm{Res}_H V) \cong V \otimes \mathrm{Ind}_H^G 1$, where 1 is the trivial 1-dimensional representation.

3.2 Answer

Proof. We denote by $W = Fw$ the FH module affording the trivial representation on H. Manipulating the statement we want to prove a bit, we have.

$$Ind_H^G(Res_H V) = FG \otimes_{FH} (Res_H V),$$

and

$$\begin{aligned} V \otimes_F Ind_H^G 1 &= V \otimes_F (FG \otimes_{FH} W) \\ &\cong (FG \otimes_{FH} W) \otimes_F V \\ &\cong FG \otimes_{FH} (W \otimes_F V). \end{aligned}$$

So, it suffices to show that

$$W \otimes_F V \cong Res_H V.$$

I claim that $\varphi : W \otimes_F V \to Res_H V$ defined by $aw \otimes v \mapsto av$ for $a \in F$ is an isomorphism of FH modules. First, we verify that it is a homomorphism

$$\begin{aligned} \varphi(g \cdot (aw \otimes v)) &= \varphi(haw \otimes hv) \\ &= \varphi(aw \otimes hv) \\ &= a(hv) \\ &= h \cdot av \\ &= h\varphi(aw \otimes v) \end{aligned}$$

Also, it is surjective since the kernel is 0. Let $\sum a_i w \otimes v_i$ be in the kernel. Then,

$$\begin{aligned} \sum a_i w \otimes v_i &= \sum w a_i \otimes v_i \\ &= \sum w \otimes a_i v_i \\ &= w \otimes \sum a_i v_i \end{aligned}$$

and as $\sum a_i w \otimes v_i$ is in the kernel, we clearly have $\varphi(\sum a_i w \otimes v_i) = 0 = \sum a_i v_i$. Thus, our original element was $w \otimes 0 = 0$, and the map is injective. Clearly φ is surjective, so it is an isomorphism of FH modules, as claimed. $\square$

4 Question 4

4.1 Question

Prove that $\mathrm{Ind}_H^G 1$ is isomorphic to the permutation representation on the set G/H.

4.2 Answer

Proof. Denote by $V = Fv$ denote the FH module affording the identity representation on H. Let $\{g_i\}$ denote representatives of cosets of H in G. The representation on this set is given by left multiplication on this set. Given some $g_i, g \in G$ we have

$$gg_i \in g_j H$$

for some g_j in the representative set. Thus, fixing some $a \in F$ we have

$$\begin{aligned} g(g_i \otimes av) &= gg_i \otimes av \\ &= g_j h \otimes av \\ &= g \otimes ah_j v \\ &= g_j \otimes av \end{aligned}$$

So, we see that $FG \otimes_{FH} V$ has the same action as does the permutation representation $\square$

5 Question 5

5.1 Question

Fill in the details of the sketch of the construction of the character table of $\mathrm{GL}_2(\mathbb{F}_p)$ given in lectures, as follows.

1. Compute the conjugacy classes in $\mathrm{GL}_2(\mathbb{F})$ and their orders
2. Compute the characters of $I(\chi_1, \chi_2)$, and decompose it as a sum of irreducible characters.

3. Follow the approach in Lang to find the characters of some irreducible $(p-1)$-dimensional representations.

4. Show that you have computed the character table of $\mathrm{GL}_2(\mathbb{F}_p)$.

5.2 Answer

1. To compute the conjugacy classes in $\mathrm{GL}_2(\mathbb{F})$ we first observe that if the matrix has eigenvalues in $\mathbb{F}_p$ then it is conjugate to a matrix of one of the following 3 forms (Jordan Canonical Form)

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \quad \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \quad \begin{pmatrix} a & 1 \\ 0 & b \end{pmatrix}.$$

   One of these representatives is uniquely determined by the eigenvalues of a matrix.

   We need to deal with the case where the matrix has no eigenvalues in $\mathbb{F}_p$. By Rational Canonical Form, any such matrix is conjugate to one of the form

$$\begin{pmatrix} 0 & -b \\ 1 & -a \end{pmatrix}$$

   for a, b coefficients of the characteristic polynomial.

   Now that we have identified a choice of representative for every conjugacy class in the group we must count the number of distinct conjugacy classes of each form, and how many elements are in each of theses.

   For the first form there are $\frac{1}{2}(p-1)(p-2)$ such classes, since the order of the eigenvalues does not matter, yet they must be distinct. To compute the number of elements in a class of this form, first observe that the stabilizer is the set of diagonal matrices. There are $(p-1)^2$ such matrices, so there are

$$|G|/|Z(g)| = \frac{(p+1)p(p-1)^2}{(p-1)^2} = p(p+1)$$

   elements in a fixed such conjugacy class, by the orbit-stabilizer theorem.

   There are $p-1$ conjugacy classes of the one-eigenvalue form, as they are determined by some nonzero choice of eigenvalue. These commute with every element of the group, so the classes contain only one member each.

The non-diagonalizable, one eigenvalue form also has $p-1$ separate conjugacy classes, defined by the choice of (again nonzero) eigenvalue. These are stabilized by matrices of the form

$$\begin{pmatrix} x & y \\ 0 & x \end{pmatrix} \quad x \neq 0,$$

so again by the orbit-stabilizer theorem, one such conjugacy class contains

$$|G|/|Z(g)| = \frac{(p+1)p(p-1)^2}{(p-1)p} = (p+1)(p-1)$$

elements.

For the last type, there are $p(p-1)$ choices of eigenvalue, but the order doesn't matter so there are a total of $\frac{1}{2}p(p-1)$ choices of conjugacy class. By computing the number of elements of $\mathrm{GL}_2(\mathbb{F}_p)$, subtracting the contribution from elements of the first three forms, then dividing, we find that each of these conjugacy classes has

$$p(p-1)$$

elements.

2. The easy representations to see are the one dimensional ones. Just take any homomorphism $\varphi : \mathbb{F}_p^* \to \mathbb{C}^*$, and then compose it with the determinant. These representations are given by specifying which pth root of unity we take the generator of $\mathbb{F}_p$ to, so there are $p-1$ of them.

   The characters for these representations are

	Diagonalizable 1 Eigenvalue	1 Eigenvalue	2 Eigenvalues	No Eigenvalues
$\varphi \circ det$	$\varphi(a)^2$	$\varphi(a)^2$	$\varphi(ab)$	$\varphi(b)$

6 Question 6

6.1 Question

Define the *kernel* of a character χ to be the set of $g \in G$ such that $\chi(1) = \chi(g)$. Show that the kernel of χ is a normal subgroup of G, and that every normal subgroup of G is the intersection of the kernels of some set of irreducible characters of G.

6.2 Answer

First we show that the kernel is a normal subgroup.

Proof. Fixing some character χ we have

$$ker\chi = \{g \in G \mid \chi(1) = \chi(g)\}$$

so, assuming $x \in ker\chi$, $\chi(g^{-1}xg) = \chi(x) = \chi(1)$, since characters are class functions. Therefore, $g^{-1}xg \in ker\chi$ and $ker\chi$ is normal, by definition. □

Now, we show that every normal subgroup of G is the intersection of the kernels of a set of irreducible characters of G.

Proof. First we will show that if χ is a character of G which is a linear combination of some irreducible characters of G,

$$\chi = a_1\chi_1 + a_2\chi_2 + \cdots + a_n\chi_n$$

then $ker\chi = \bigcap\{\ker \chi_i\}$

Observe that $\chi(g) = \sum_i a_i\chi_i(g) = \sum_i a_i\chi_i(1) = \chi(1)$ as $\chi_i(g) \leq \chi_i(1)$. Therefore, $ker\chi \subseteq \bigcap\{\ker \chi_i\}$. The reverse inclusion is obvious, and we get the desired $ker\chi = \bigcap\{\ker \chi_i\}$.

Now fix a normal subgroup $N \trianglelefteq G$ and denoting by $A(G)$ the group algebra of G, define $\rho : G \to A(G/N)$ as

$$\rho_{gh}(f)(k) = f(kghN) = \rho_h(f)(kgN) = (\rho_g(\rho_h(f)))(k).$$

Clearly, this gives us a representation with $N \subseteq ker\rho$.

Now define a function $\delta_N : G/N \to \mathbb{C}$ by

$$\delta_N(g) = \begin{cases} 1 & \text{if } g \in N \\ 0 & \text{otherwise} \end{cases}$$

Observe now that if $g \in \ker \rho$ we have $\rho_g(\delta_N) = \delta_N$, and $1 = \delta_N(N) = \delta_N(gN)$. Hence either $N = gN$, or $g \in N$. Furthermore, as $g \in N, \ker \chi_\rho = N$. Therefore, $N = ker\chi = \bigcap ker\chi_i$, for χ_i irreducible, and we are done, by our earlier work. □

7 Question 7

7.1 Question

Using the character tables you have computed, determine all normal subgroups of D_{2n}, Q_8, S_4, A_5, and S_5.

7.2 Answer

We employ the result of the previous exercise throughout. Since every normal subgroup is the intersection of the kernels of some set of irreducible characters, we can enumerate all of the normal subgroups by computing the kernels of the irreducible representations from the character tables, and looking at the intersections of these.

1. First we have D_{2n}, n odd.

	1	r	$\cdots$	$r^{\frac{n-1}{2}}$	s
χ_1	1	1	$\cdots$	1	1
χ_2	1	1	$\cdots$	1	-1
χ_3	2	$2\cos\left(\frac{2\pi}{n}\right)$	$\cdots$	$2\cos\left((n-1)\pi/n\right)$	0
$\cdot$	$\cdot$	$\cdot$	$\cdots$	$\cdot$	$\cdot$
$\cdot$	$\cdot$	$\cdot$	$\cdots$	$\cdot$	$\cdot$
$\chi_{(n+3)/2}$	2	$2\cos\left(2(\frac{n+3}{12}-2)\pi/n\right)$	$\cdots$	$\cdot$	0

So the only proper, nontrivial normal subgroups is $\langle r \rangle$.

Now the even case

	1	r	r^2	$\cdots$	$r^{n/2}$	s	rs
χ_1	1	1	1	$\cdots$	1	1	1
χ_2	1	-1	1	$\cdots$	± 1	1	-1
χ_3	1	1	1	$\cdots$	± 1	1	-1
χ_4	1	-1	1	$\cdots$	± 1	1	-1
χ_5	2	$2\cos\left(2\pi/n\right)$	$\cdot$	$\cdots$	$2\cos\left(n\pi/n\right)$		
$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdots$	$\cdot$	$\cdot$	$\cdot$
$\chi_{n/2+3}$	2	$2\cos 2(n/2-1)\pi/n$	$\cdot$	$\cdots$	$\cdot$	0	0

so the only proper, nontrivial normal subgroups are $\langle r \rangle, \langle r^2, s \rangle, \langle r^2, rs \rangle$.

2. The character table for Q_8 is

	1	-1	i	j	k
χ_1	1	1	1	1	1
χ_2	1	1	-1	1	-1
χ_3	1	1	1	-1	-1
χ_4	1	1	-1	-1	1
χ_5	2	-2	0	0	0

So all subgroups of Q_8 are normal.

3. The character table of S_4 is

	1 e	6 (12)	8 (123)	6 (1234)	3 (12)(34)
χ_1	1	1	1	1	1
χ_2	1	-1	1	-1	1
χ_3	2	0	-1	0	2
χ_4	3	1	0	-1	-1
χ_5	3	-1	0	1	-1

So, the normal subgroups are $1, A_4, V, S_4$.

4. The character table of A_5 is

	1	20	12	12	15
χ_1	1	1	1	1	1
χ_2	3	0	$\frac{1+\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$	-1
χ_3	4	1	-1	-1	0
χ_4	3	0	$\frac{1-\sqrt{5}}{2}$	$\frac{1+\sqrt{5}}{2}$	-1
χ_5	5	-1	0	0	1

So there are no nontrivial, proper normal subgroups.

5. The character table of S_5 is

	1 e	10 (12)	20 (123)	30 (1234)	24 (12345)	15 (12)(34)	20 (12)(345)
χ_1	1	1	1	1	1	1	1
χ_2	1	-1	1	-1	1	1	-1
χ_3	4	2	1	0	-1	0	-1
χ_4	4	-2	1	0	-1	0	1
χ_5	5	-1	-1	1	0	1	-1
χ_6	5	1	-1	-1	0	1	1
χ_7	6	0	0	0	1	-2	0

and we observe that the normal subgroups are just S_5, A_5 and the trivial one.