Homework 4

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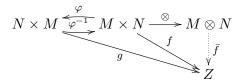
1 Question 1

1.1 Question

Let A be a commutative ring and let M and N be A-modules. Prove that $M \otimes_A N$ and $N \otimes_A M$ are (naturally) isomorphic.

1.2 Answer

Recall that $M \times N$ and $N \times M$ are isomorphic as A-modules say via $\varphi : N \times M \to M \times N$. By definition, the tensor product $M \otimes N$, is an abelian group, together with a bilinear map $\otimes : M \times N \to M \otimes N$ such that, for every abelian group Z, and bilinear map f, there exists unique \bar{f} such that the diagram commutes. I claim that the same group $M \otimes N$ together with the bilinear map $\varphi^{-1} \circ \otimes$ is the tensor product $N \otimes M$. This is equivalent to saying that $M \otimes N \cong N \otimes M$.



Proof. Let Z an abelian group, with $g: N \times M \to Z$ bilinear. Then defining $f = g \circ \varphi$, there exists a unique \bar{f} which makes the diagram commute. By definition of isomorphism, this is also the unique map which makes $g = \bar{f} \circ (\varphi^{-1} \circ \otimes)$.

2 Question 2

2.1 Question

Prove that $M \otimes (N \oplus P) \cong M \otimes N \oplus M \otimes P$.

2.2 Answer

We follow the exposition in Dummit and Foote p373.

Proof. Define a map $\varphi: M \times (N \oplus P) \to (M \otimes N) \oplus (M \otimes P)$ by $(m, (n, p)) \mapsto (m \otimes n, m \otimes p)$. This is well defined, as we just treat n, p as members of $N \oplus P$ after inclusion. We can easily check bilinearity:

$$\varphi(a_{1}m_{1} + a_{2}m_{2}, (n, p)) = ((a_{1}m_{1} + a_{2}m_{2}) \otimes n, (a_{1}m_{1} + a_{2}m_{2}) \otimes p)$$

$$= ((a_{1}m_{1}) \otimes n, (a_{1}m_{1}) \otimes p) + ((a_{2}m_{2}) \otimes n, (a_{2}m_{2}) \otimes p)$$

$$= a_{1}(m_{1} \otimes n, m_{1} \otimes p) + a_{2}(m_{2} \otimes n, m_{2} \otimes p)$$

$$= a_{1}\varphi(m_{1}, (n, p)) + a_{2}\varphi(m_{2}, (n, p))$$
also we check
$$\varphi(m, a_{1}(n_{1}, p_{1}) + a_{2}(n_{2}, p_{2}))$$

$$= (m \otimes (a_{1}n_{1} + a_{2}n_{2}), m \otimes (a_{1}p_{1} + a_{2}p_{2}))$$

$$= (m \otimes a_{1}n_{1}, m \otimes a_{1}p_{1}) + (m \times a_{2}n_{2}, m \otimes a_{2}p_{2})$$

$$= (a_{1}m \otimes n_{1}, a_{1}m \otimes p_{1}) + (a_{1}m \otimes n_{2}, a_{2}m \otimes p_{2})$$

$$= a_{1}\varphi(m, (n_{1}, p_{1})) + a_{2}\varphi(m, (n_{2}, p_{2})).$$

Since φ is bilinear it induces a homomorphism of A-modules $X:M\otimes (N\oplus P)\to (M\otimes N)\oplus (M\otimes P).$

Now we go the other way. Consider maps $f: M \times N \to M \otimes (N \oplus P)$ and $g: M \times P \to M \otimes (N \oplus P)$ defined as $f: (m,n) \mapsto m \otimes (n,0)$ and $g: (m,p) \mapsto m \otimes (0,p)$. These are both bilinear, and hence induce maps $G: M \otimes N \to M \otimes (N \oplus P)$ and $(M \otimes P) \to M \otimes (N \oplus P)$. By the definition of direct sum, F and G together induce yet another map $Y: (M \otimes N) \oplus (M \otimes P) \to M \otimes (N \oplus P)$ which takes $Y((m_1 \otimes n, m_2 \otimes p)) = m_1 \otimes (n,0) + m_2 \otimes (0,p)$. Therefore, $X \circ Y = Y \circ X = 1$, and we have proven the claimed isomorphism.

3 Question 3

3.1 Question

Prove that $A^m \otimes A^n \cong A^{mn}$.

3.2 Answer

Proof. We use induction on n. By the first, and next questions we have $A \otimes_A A^m \cong A^m \cong A^m \otimes_A A$. So, suppose (induction) that $A^m \otimes_A A^{n-1} \cong A^{m(n-1)}$. By the second exercise we conclude that $A^m \otimes_A A^n \cong A^m \otimes_A (A^{n-1} \oplus A) \cong (A^m \otimes_A A^{n-1}) \oplus (A^m \otimes_A A) \cong A^{m(n-1)} \oplus A^m \cong A^{mn}$. \square

4 Question 4

4.1 Question

Prove that $A \otimes_A M \cong M$.

4.2 Answer

Recall that the tensor product is just the object, map \otimes that is universal for the following diagram

$$A \times M \xrightarrow{\otimes} A \otimes M$$

$$f \qquad \qquad \downarrow \bar{f}$$

$$Z$$

for f a bilinear map, Z an arbitrary abelian group. However, It is easy to see that this is just M. By bilinearity $(a,m)=(a\cdot 1,m)=(1,am)$, so, setting $A\otimes M=M$ there clearly exists an arrow \bar{f} that makes the diagram commute. Moreover, \bar{f} must also be unique, since were it not then there would exist $\bar{f}\neq \bar{g}$ which make the diagram commute. However, we can produce $f\neq g$ maps from $A\times M\to Z$ from these, a contradiction.

5 Question 5

5.1 Question

Prove that if I is an ideal of A, then $(A/I) \otimes_A M \cong M/IM$.

5.2 Answer

Proof. Define a map $\varphi: M \to (A/I) \oplus_A M$ defined as $\varphi: m \mapsto 1 \otimes m$. $IM \subseteq Ker(\varphi)$ since $\varphi(am) = 1 \oplus am = a \otimes m = 0 \otimes m = 0$. Hence, φ induces a homomorphism $X: M/IM \to (A/I) \otimes M$. This homomorphism is surjective by properties of homomorphism, the fact that $1 \otimes m$ generates $(A/I) \otimes_A M$.

Now, let $\psi: (A/I) \times M \to M/IM$ be $\psi: (a(modI), m) \mapsto am(modIM)$. Being bilinear, this induces a group homomorphism $Y: (A/I) \otimes M \to M/IM$ which sends $a(mdoI) \otimes m \mapsto am(modIM)$. $Y \circ X = 1$ with X a bijection. Hence, we have the desired isomorphism.

6 Question 6

6.1 Question

Write out a "bare hands" proof that if $M \to^{\psi} N \to P \to^{\phi} 0$ is exact, then so is $M \otimes_A Q \to^{1\otimes\psi} N \otimes_A Q \to^{1\otimes\phi} P \otimes_A Q \to 0$ for any A-module Q.

6.2 Answer

We follow the proof from Dummit and Foote p399.

Proof. Let the first sequence be exact. The map $\phi \otimes 1$ is surjective since $p \otimes q = \phi(n) \otimes q$ for all simple tensors, by surjectivity of ϕ . Now, take a general tensor $\sum m_i \otimes q_i$. We have $(\phi \otimes 1) \circ (\psi \otimes 1)(\sum m_i \otimes q_i) = \sum \phi \circ \psi(m_i) \otimes q_i = \sum 0 \otimes q_i = 0$. Hence, $Im(\psi \otimes 1) \subseteq Ker(\phi \otimes 1)$. Therefore, there exists a projection $\pi : (N \otimes_A Q)/Im(\psi \otimes 1) \to P \otimes_A Q$.

Now define $\gamma: P \times Q \to (N \otimes_A Q)/Im(\psi \otimes 1)$ by $\gamma((p,q)) = n \otimes q$ for $n \in N$ with $\phi(n) = p$. This map being bilinear it induces a homomorphism $\Gamma: P \otimes_A Q \to (N \otimes_A Q)/Im(\psi \otimes 1)$ by $\Gamma(p \otimes q) = n \otimes q$. Now $\Gamma \circ \pi(n \otimes q) = \Gamma(\phi(n) \otimes q) = n \otimes q$ so it is the identity. Since $\pi \circ \Gamma$ is also the identity we have $Im(\psi \otimes 1) = Ker(\phi \otimes 1)$.

7 Question 7

7.1 Question

Prove that a short exact sequence $0 \to M \to N \to P \to 0$ for A-modules is split if and only if the sequence exhibits an isomorphism $N \cong M \oplus P$; that

is, if and only if there is a commutative diagram

$$0 \longrightarrow M \longrightarrow N \longrightarrow P \longrightarrow 0$$

$$\downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow$$

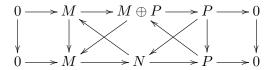
$$0 \longrightarrow M \longrightarrow M \oplus P \longrightarrow P \longrightarrow 0$$

where the left and right downward arrows are the identity maps, and the maps on the bottom line are the obvious inclusion and projection.

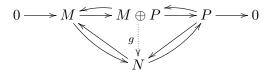
7.2 Answer

First we will show (\Leftarrow) that if the short exact sequence is split, then there exists such a commutative diagram.

Proof. By the definition of projective and injective modules there exist additional maps such that the following diagram commutes.



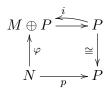
or equivalently



Thus, by definition of the coproduct there exists a unique map g (above), such that the diagram commutes. Applying the five lemma, we see that this homomorphism is in fact an isomorphism, as desired.

Conversely (\Rightarrow) , suppose that such an isomorphism of short exact sequences exists. We will demonstrate that it splits

Proof. So $0 \to M \to N \to P \to 0$. Let's call the map $p: N \to P$. Then, if we call the isomorphism $\varphi: N \to M \oplus P$, and the inclusion map $i: P \to M \oplus P$ we have a map from P to N such that its composition with p is the identity, namely $\varphi^{-1} \circ i$.



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8 Question 8

8.1 Question

Prove the uniqueness assertion in the structure theorem for finitely generated modules over a PID.

8.2 Answer

We follow the proof of Dummit and Foote p466.

Throughout, let R be a PID and p a prime in R. F is a the field R/(p). Now, by way of a lemma we show that $M = R^r \Rightarrow M/pM \cong F^r$.

Proof. There is a map from R^r to $(R/(p))^r$ defined by modding out by (p). This is a surjective homomorphism of R-modules whose kernel is just those elements all of whose coordinates are in (p). Since these elements are just pR^r we have $R^r/pR^r \cong (R/(p))^r$ as claimed.

As a second lemma I claim that if $M = R/(a_1) \oplus R/(a_2) \oplus \cdots \oplus R/(a_k)$ for each a_i divisible by p then $M/pM \cong F^k$.

Proof. First we will show that if M = R/(a) where a is a nonzero element of R then

$$M/pM \cong \left\{ \begin{array}{ll} F & \text{if } p \text{ divides } a \text{ in } R \\ 0 & \text{if } p \text{ does not divide } a \text{ in } R. \end{array} \right.$$

The claim follows from this proposition, as well as the existence part of the structure theorem.

The above claim follows form the isomorphism theorems. Note first that p(R/(a)) is the image of the ideal (p) in the quotient R/(a), hence is (p) + (a)/(a). The idea (p) + (a) is generated by a greatest common divisor of p and a, hence is (p) if p divides a and is R = (1) otherwise. Hence pM = (p)/(a) if p divides a and is R/(a) = M otherwise. If p divides a then $M/pM - (R/(a))/((p)/(a)) \cong R/(p)$, and if p does not divide p then p the

Now we will show that two finitely generated R-modules M and L are isomorphic if and only if they have the same list of invariant factors.

The forward (\Rightarrow) direction of this proof is clear.

So, suppose M and L are isomorphic. We will show that they have the same free rank and list of invariant factors.

Proof. Any isomorphism between M and L must map the torsion in one to the torsion in the other. So, $Tor(L) \cong Tor(M)$. So, $R^m \cong M/Tor(M) \cong L/Tor(L) \cong R^l$. Where m is the free rank of M and l of L. Let p be any nonzero prime from R. Then, as $R^m \cong R^l$ we have an isomorphism $R^m/pR^m \cong R^l/pR^l$. By the lemma this implies that $F^m \cong F^l$ where F is the field R/pR. Hence we have an isomorphism of an m-dimensional vector space over F with an l dimensional vector space over F, so that m = l and M and L have the same free rank.

Now it remains only to show that M and L have the same lists of invariant factors. We need only work with the isomorphic torsion modules, so assume that both M and L are torsion R-modules.

First we'll demonstrate the they have the same elementary divisors. However, it suffices to show that for any prime p the elementary divisors which are a power of p are the same for both modules. If $L \cong M$ then the direct sum of the cclic factors whose elementary divisors are a power of p in L is isomorphic to the same direct sum in M.

We use induction on the power of p in the annihilator of L. If this power is 0, then both L and M are 0 and we are done. Otherwise L (and by the isomorphism M) have nontrivial elementary divisors. Suppose the elementary divisors of L are given by

$$\underbrace{p, p, \dots, p}_{mtimes}, p^{\alpha_1}, p^{\alpha_2}, \dots, p^{\alpha_t}$$

where $2 \leq \alpha_1 \leq \alpha_2 \leq \ldots \leq \alpha_s$. L is the direct sum of cyclic modules with generators $x_1, x_2, \ldots, x_m, x_{m+1}, \ldots, x_{m+s}$, say, whose annihilators are $(p), (p), \ldots, (p), (p^{\alpha_1}), \ldots, (p^{\alpha_s})$, respectively. Then the submodules pM_1 has elementary divisors

$$p^{\alpha_1-1}, p^{\alpha_2-1}, \dots, p^{\alpha_s-1}$$

since pL is the direct sum of the cyclic modules with generators $px_1, px_2, \ldots, px_m, px_{m+1}, \ldots, px_{m+s}$ whose annihilators are $(1), (1), \ldots, (1), (p^{\alpha_1-1}), \ldots, (p^{\alpha_s-1})$, respectively. Similarly if the elementary divisors of M are given by

$$\underbrace{p, p, \dots, p}_{ntimes}, p^{\beta_1}, p^{\beta_2}, \dots, p^{\beta_t}$$

where $2 \leq \beta_1 \leq \beta_2 \leq \ldots \leq \beta_t$, then pM has elementary divisors

$$p^{\beta_1-1}, p^{\beta_2-1}, \dots, p^{\beta_t-1}$$

Since $M \cong L$, also $pL \cong pM$ and the power of p in the annihilator of pL is one less than the power of p in the annihilator of M. By induction,

the elementary divisors for pL are the same as the elementary divisors for pM, i.e., s=t and $\alpha_1-1=\beta_i-1$ for $i=1,2,\ldots,s$, hence $\alpha_i=\beta_i$ for $i=1,2,\ldots,s$. Finally since also $L/pL\cong M/pM$ we see from the second lemma that $F^{m+s}\cong F^{n+t}$, which shows that m+s=n+t hence m=n since we have already seen s=t.

9 Question 9

9.1 Question

Complete the sketch proof of the existence of Jordan normal form.

9.2 Answer

Let T be a linear transformation on a finite dimensional vector space V over the field F, with minimal polynomial of T splitting completely over F. I claim that T can be put in Jordan normal form.

Proof. x acts on $v \in V$ by $x \cdot v = Tv$. For any $p(x) = a_n x^n + a_{n-1} x^{n-1} + ... + a_1 x + a_0 \in F[x]$, we define the action of p(x) on $v \in V$ by $p(x)v = (a_n T^n + a_{n-1} T^{n-1} + ... + a_1 T + a_0)(v) = a_n T^n(v) + a_{n-1} T^{n-1}(v) + ... + a_1 T(v) + a_0 v$. This action makes V into an F[x] module.

As V is a finite dimensional vector space over F, it is finitely generated as an F[x]-module. Since free modules over F[x] have infinite dimension the F[x] module is torsion. By the structure theorem for finitely generated modules over a PID, $V \cong F[x]/a_1(x) \oplus F[x]/a_2(x) \oplus \ldots \oplus F[x]/a_m(x)$ where $a_m(x)$ is the minimal polynomial for T.

By assumption, the minimal polynomial of T splits completely in F, so V is the direct sum of finitely many F[x]-module of the form $F[x]/(x-\gamma)^q$ for some $\gamma \in F$ and $q \geq 1$. Now consider the elements $(x-\gamma)^{q-1}, (x-\gamma)^{q-2}, ..., (x-\gamma), 1 \in F[x]/(x-\gamma)^q$. This set of elements is a basis for $F[x]/(x-\gamma)^q$ as a F vector space.

10 Question 10

10.1 Question

Give examples of matrices M over \mathbb{Q} whose characteristic polynomial and minimal polynomial are respectively:

1. $(X-1)^5$ and $(X-1)^4$.

2. $(X-1)^2(X-2)^3$ and (X-1)(X-2).

3. $(X-1)^2(X-2)^3$ and $(X-1)(X-2)^2$.

4. $(X-1)^2(X-2)^3$ and $(X-1)(X-2)^3$.

10.2 Answer

We exploit our knowledge of Jordan Canonical form to find the following matrices (Jordan Blocks highlighted).

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4.

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& 1 & & & \\
& & 2 & 1 & \\
& & & 2 & 1 \\
& & & & 2
\end{pmatrix}$$