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Question

Prove that Spec $(A_1 \times A_2) =$ Spec $A_1 \coprod$ Spec A_2, and if Spec $A = X \coprod Y$, then $A = B \times C$ with $X =$ Spec $B, Y =$ Spec C.

Answer

Proof. If $\mathfrak{p} \in A_1 \times A_2$ is a prime ideal, it is of the form $a_1 \times a_2$ for $a_i \in$ Spec A_i or A_i, as $\pi_i(\mathfrak{p}) \subset A_i$ must be prime or A_i. In fact, precisely one of the a_i must be A_i, as for $(a, b) \in \mathfrak{p}$ we have $(a, b) = (a, 1)(1, b)$. If $(a, 1) \in \mathfrak{p}$, then $a_2 = A_2$, otherwise, $a_1 = A_1$. Finally, observe that $\mathfrak{p} \times A_2$ is a prime ideal for all prime $\mathfrak{p} \in A_1$, as if $(a_1, a_2)(a_3, a_4) \in \mathfrak{p} \times A_2$ either a_1 or a_3 is in $\mathfrak{p}$ by primeness of $\mathfrak{p}$. As both a_2, and a_4 are in A_2, this is sufficient for on of $(a_1, a_2), (a_3, a_4)$ to be in $\mathfrak{p} \times A_2$.

The set $X = \{(\mathfrak{p}, A_2) \mid \mathfrak{p} \in \text{Spec } A_1\}$ is both closed and open, since $V((0, A_2)) = X = D((A_1, 0))$. Similarly, for $Y = \{(A_1, \mathfrak{p} \mid \mathfrak{p} \in \text{Spec } A_2\}$ we have $V((A_a, 0)) = X = D((0, A_2))$. Thus, Spec $(A_1 \times A_2) = X \coprod Y$.

Finally, we verify that $A_1 \cong X$ via $\mathfrak{p} \mapsto (\mathfrak{p}, A_2)$. This is a bijection by our identification of the prime ideals of $A_1 \times A_2$ above. Furthermore, it is continuous, with continuous inverse, since a closed set $V(\mathfrak{p}) \in A_1$ corresponds to the closed set $V((\mathfrak{p}, A_2)) \in A_1 \times A_2$. Similarly, $A_2 \cong Y$ via $\mathfrak{p} \mapsto (A_1, \mathfrak{p})$. □

Proof. For the opposite direction, assume Spec $A = X \coprod Y$. Thus, there exist I, J such that $V(I) = X, V(J) = Y$. Now observe that $I + J$ is the entire ring. Were it not, it would be contained in some maximal (and therefore prime) ideal, but $V(I) \cap V(J) = \emptyset$. Therefore, the Chinese Remainder Theorem applies, so $IJ = I \cap J$, and $f : A/IJ \to A/I \times A/J$ defined by $f(x + IJ) = (x + I, x + J)$ is an isomorphism.

Since $IJ = I \cap J \subseteq \mathfrak{p}$ for all $\mathfrak{p}$ prime $IJ \in$ nil A, and every $x \in IJ$ is nilpotent. Hence, Spec $A \cong$ Spec $A/IJ \cong$ Spec $(A/I \times A/J)$. □

Question

Prove that the Zariski topology is a topology on Spec A.

Answer

We must verify (i) that sets of the form $V(X)$ cover Spec A, (ii) 0, Spec $A = V(X)$ for some X (iii)$\{V(X) \mid X \in A\}$ is closed under finite union, arbitrary intersection.

Each prime ideal $\mathfrak{p}$ is contained in at least the set $V(\mathfrak{p})$ by definition.

$0 = V(A)$, Spec $A = V(0)$.

$V(\mathfrak{p}) \cup V(\mathfrak{q}) = \{a \in \text{Spec } A \mid \mathfrak{p} \subseteq a\} \cup \{a \in \text{Spec } A \mid \mathfrak{q} \subseteq a\} = \{a \in \text{Spec } A \mid \mathfrak{p} + \mathfrak{q} \subseteq a\} = V(\mathfrak{p} + \mathfrak{q})$. But the set of ideals is closed under this addition operation.

Furthermore, $\bigcap_{i \in I} V(J_i) = \bigcup_{i \in I}\{\mathfrak{p} \in \text{Spec } A \mid J_i \subseteq \mathfrak{p}\} = \{\mathfrak{p} \in \text{Spec } A \mid \bigcap_{i \in I} J_i \subseteq \mathfrak{p}\} = V(\bigcap_{i \in I} J_i)$.