Homework 2

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8 November 2010

1 Question 1

1.1 Question

Suppose that α is algebraic over a field k. Describe how to compute $1/\alpha$ as an element of $k[\alpha]$ (i.e. as a polynomial in α with coefficients in k), and illustrate this by computing $(1+\sqrt{2}+\sqrt{3})^{-1}$.

1.2 Answer

Observe that if α is a root of the irreducible polynomial

$$p(x) = p_n x^n + p_{n-1}x^{n-1} + \cdots + p_1 x + p_0$$

we can compute α^{-1} from

$$\alpha(p_n\alpha^{n-1} + p_{n-1}\alpha^{n-2} + \cdots + p_1) = -p_0$$

namely

$$\alpha^{-1} = \frac{-1}{p_0}(p_n\alpha^{n-1} + p_{n-1}\alpha^{n-2} + \cdots + p_1).$$

Since $p(x)$ is irreducible $p_0 \neq 0$. (Dummit and Foote p.516)

It is easy to verify that $\alpha = 1+\sqrt{2}+\sqrt{3}$ is a root of the irreducible polynomial

$$-8 + 16x - 4x^2 - 4x^3 + x^4 \in \mathbb{Q}[x]$$

Therefore, we compute as above

$$\alpha^{-1} = \frac{1}{8}(16 - 4\alpha - 4\alpha^2 + \alpha^3).$$

2 Question 2

2.1 Question

Suppose that K/k is an algebraic extension, and that $\alpha \in K$ has odd degree over k. Show that $k(\alpha) = k(\alpha^2)$.

2.2 Answer

Recall that $\alpha \in K$ has odd degree over k if and only if $[k(\alpha) : k]$ is odd.

Proof. It is clear by closure that $k(\alpha^2) \subseteq k(\alpha)$. Moreover, since $\alpha \in k(\alpha)$ we have $[k(\alpha) : k(\alpha^2)] = 1$ or $[k(\alpha) : k(\alpha^2)] = 2$. Thus, by multiplicativity of extension degree, together with the fact that $[k(\alpha) : k]$ is odd $[k(\alpha) : k(\alpha^2)] = 1$ and $k(\alpha) \subseteq k(\alpha^2) \Rightarrow k(\alpha) = k(\alpha^2)$. $\square$

3 Question 3

3.1 Question

Let $E = k(X)$ with X transcendental over k. If $E/F/k$ is a proper intermediate extension, show that E/F is algebraic.

3.2 Answer

Proof. Suppose towards a contradiction that E/F is transcendental. Then there are two cases. Either F/k is transcendental, or it is algebraic.

If F/k is transcendental, then E/k has transcendence degree at least 2, a contradiction.

If F/k is algebraic, there is some minimal nonempty, algebraically independent set U such that $F = k(U)$, since we assumed F to be a nontrivial intermediate extension. However, $U \cup X$ is a transcendence basis for E/k since, were X algebraic over U then it would be algebraic over k, and were any members of U algebraic with the addition of X, F/k would not be an algebraic extension.

But, this is a contradiction, since $|U \cup X| > 1$. $\square$

4 Question 4

4.1 Question

Let $K = \mathbb{F}_p(X, Y)$ and let $L = \mathbb{F}_p(X^{1/p}, Y^{1/p})$. What is the degree of the extension L/K? Show that there is no element $\alpha \in L$ for which $L = K(\alpha)$.

4.2 Answer

$$[L : K] = p^2$$

Proof. The extension $\mathbb{F}_p(X^{1/p})/\mathbb{F}_p(X)$ has order at most p since the polynomial

$$X^p - 1 = 0$$

has root $X^{1/p}$. Furthermore, if there is some polynomial of smaller degree with root $X^{1/p}$ then $\mathbb{F}_p(X^{1/p})$ would have size not a power of p for some prime. Hence, $[\mathbb{F}_p(X^{1/p}) : \mathbb{F}_p(X)] = p$. Similarly for Y.

Therefore, since $\mathbb{F}_p(X) \neq \mathbb{F}_p(X, Y)$

$$[L : K] = p^2$$

as claimed. □

Proof. Assume towards a contradiction that there exists such an α. However, $\mathbb{F}_p$ is perfect, so this implies that there exists an irreducible, separable polynomial of degree p^2 over $\mathbb{F}_p$. Contradiction. □

5 Question 5

5.1 Question

Find the degree of $\mathbb{Q}\left(\sqrt{2 + \sqrt{2}}\right)$. Is it a normal extension?

5.2 Answer

Denote $\alpha = \sqrt{2 + \sqrt{2}}$. $\mathbb{Q}(\alpha)$ has degree 4, since its minimal polynomial is $2 - 4x^2 + x^4$. The extension is normal. Note that the roots of the minimal polynomial are

$$x = -\sqrt{2 - \sqrt{2}} \quad x = \sqrt{2 - \sqrt{2}} \quad x = -\sqrt{2 + \sqrt{2}} \quad x = \sqrt{2 + \sqrt{2}}$$

and

$$2/\alpha - \alpha = -\sqrt{2-\sqrt{2}} \quad \alpha - 2/\alpha = \sqrt{2-\sqrt{2}}$$
$$-\alpha = -\sqrt{2+\sqrt{2}} \quad \alpha = \sqrt{2+\sqrt{2}}.$$

6 Question 6

6.1 Question

Suppose that K/k is an algebraic extension. A *normal closure* of K/k is an extension L/K such that

1. L/k is normal, and
2. no proper subfield of L that contains K is normal over k.

1. Show that K/k has a normal closure, and that any two normal closures are isomorphic.
2. If K/k is finite then any normal closure is also finite over k
3. If K/k is separable then so is the normal closure, and thus the normal closure is Galois over k.
4. Find a normal closure of $\mathbb{Q}(\sqrt[5]{3})/\mathbb{Q}$. What is its degree over $\mathbb{Q}$?

6.2 Answer

1. It suffices to show that there exists some field extension E/k such that E is normal, and $E \supseteq K$, since if there is such an extension the minimal such extension is just the intersection of all such.

   The field extension generated by the minimal polynomials of all $x \in K$, $x \notin k$ is such an extension, by definition.

   Suppose that E, L are both normal closures of K. Then I claim that $E = k(\alpha_i)_{i\in I} = L$ where $\{\alpha_i\}$ is the set of roots to the minimial polynomials of each $k \in K$. If there is some such $k \notin E, L$ then the field is not normal. Conversely, if there an element $x \in E, L$ such that $k \notin k(\alpha_i)_{i\in I}$ then there is a proper subfield of E, L satisfying the requisite properties, namely $k(\alpha_i)_{i\in I}$.

2. If K/k is finite, then $K = k(\alpha_1, \dots, \alpha_n)$ for some set of $\{\alpha_i\}$ algebraic, and the normal closure is contained within the normal field obtained by the following procedure:

   Compute the minimal polynomial of each α_i and adjoin all of its roots to k.

   Since each α_i is algebraic, there are only finitely many such roots for each of the finite set of α_i. Hence, the normal closure is contained within a finite extension and is itself finite.

3. Since the minimal polynomial of each $\alpha \in K$ is separable in K then the same holds true for the minimal polynomial of each $\beta \in L$, since when we construct L we only add elements which have the same minimal polynomial as elements α already in K.

4. The normal closure of $\mathbb{Q}(\sqrt[5]{3})/\mathbb{Q}$ is just

   $$\mathbb{Q}(x^5-3) = \mathbb{Q}\left(-(-3)^{1/5}, 3^{1/5}, (-1)^{2/5}3^{1/5}, -(-1)^{3/5}3^{1/5}, (-1)^{4/5}3^{1/5}\right),$$

   an extension of degree 5 since $x^5 - 3$ is the minimal polynomial for $\sqrt[5]{3}$.

7 Question 7

7.1 Question

Suppose that K/k is an algebraic extension and that $K = k(\alpha_i)_{i \in I}$ for some elements $\alpha_i \in K$. If each $k(\alpha_i)/k$ is separable, show that K/k is separable.

7.2 Answer

We must show that every finitely generated subextension

$$k(\alpha_1, \dots, \alpha_n)$$

is separable over k. Fix some such set $\{\alpha_1, \dots, \alpha_n\} \subset \{\alpha_i\}$ and consider the tower

$$k \subset k(\alpha_1) \subset \cdots \subset k(\alpha_1, \alpha_2, \dots, \alpha_n).$$

As every α_i is separable over k, each α_i is separable over $k(\alpha_1, \alpha_2, \alpha_{i-1})$ for $i \geq 2$. Thus, by the tower theorem the entire field $k(\alpha_1, \alpha_2, \dots, \alpha_n)$ is separable over k as desired.

(Lang p. 241)