Homework 4

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1 Question 1

1.1 Question

Let A be a commutative ring and let M and N be A-modules. Prove that $M \otimes_A N$ and $N \otimes_A M$ are (naturally) isomorphic.

1.2 Answer

Recall that $M \times N$ and $N \times M$ are isomorphic as A-modules say via $\varphi : N \times M \to M \times N$. By definition, the tensor product $M \otimes N$, is an abelian group, together with a bilinear map $\otimes : M \times N \to M \otimes N$ such that, for every abelian group Z, and bilinear map f, there exists unique $\bar{f}$ such that the diagram commutes. I claim that the same group $M \otimes N$ together with the bilinear map $\varphi^{-1} \circ \otimes$ is the tensor product $N \otimes M$. This is equivalent to saying that $M \otimes N \cong N \otimes M$.

$$\begin{tikzcd}
N \times M \arrow[r, bend right, "\varphi^{-1}"] \arrow[drr, "g"'] & M \times N \arrow[l, bend right, "\varphi"'] \arrow[r, "\otimes"] \arrow[dr, "f"] & M \otimes N \arrow[d, dotted, "\bar{f}"] \\
 & & Z
\end{tikzcd}$$

Proof. Let Z an abelian group, with $g : N \times M \to Z$ bilinear. Then defining $f = g \circ \varphi$, there exists a unique $\bar{f}$ which makes the diagram commute. By definition of isomorphism, this is also the unique map which makes $g = \bar{f} \circ (\varphi^{-1} \circ \otimes)$. $\square$

2 Question 2

2.1 Question

Prove that $M \otimes (N \oplus P) \cong M \otimes N \oplus M \otimes P$.

2.2 Answer

We follow the exposition in Dummit and Foote p373.

Proof. Define a map $\varphi : M \times (N \oplus P) \to (M \otimes N) \oplus (M \otimes P)$ by $(m, (n, p)) \mapsto (m \otimes n, m \otimes p)$. This is well defined, as we just treat n, p as members of $N \oplus P$ after inclusion. We can easily check bilinearity:

$$\varphi(a_1m_1 + a_2m_2, (n, p)) = ((a_1m_1 + a_2m_2) \otimes n, (a_1m_1 + a_2m_2) \otimes p)$$

$$= ((a_1m_1) \otimes n, (a_1m_1) \otimes p) + ((a_2m_2) \otimes n, (a_2m_2) \otimes p)$$
$$= a_1(m_1 \otimes n, m_1 \otimes p) + a_2(m_2 \otimes n, m_2 \otimes p)$$
$$= a_1\varphi(m_1, (n, p)) + a_2\varphi(m_2, (n, p))$$

also we check

$$\varphi(m, a_1(n_1, p_1) + a_2(n_2, p_2))$$
$$= (m \otimes (a_1n1 + a_2n2), m \otimes (a_1p1 + a_2p_2))$$
$$= (m \otimes a_1n_1, m \otimes a_1p_1) + (m \times a_2n_2, m \otimes a_2p_2)$$
$$= (a_1m \otimes n_1, a_1m \otimes p_1) + (a_1m \otimes n_2, a_2m \otimes p_2)$$
$$= a_1\varphi(m, (n_1, p_1)) + a_2\varphi(m, (n_2, p_2)).$$

Since φ is bilinear it induces a homomorphism of A-modules $X : M \otimes (N \oplus P) \to (M \otimes N) \oplus (M \otimes P)$.

Now we go the other way. Consider maps $f : M \times N \to M \otimes (N \oplus P)$ and $g : M \times P \to M \otimes (N \oplus P)$ defined as $f : (m, n) \mapsto m \otimes (n, 0)$ and $g : (m, p) \mapsto m \otimes (0, p)$. These are both bilinear, and hence induce maps $G : M \otimes N \to M \otimes (N \oplus P)$ and $(M \otimes P) \to M \otimes (N \oplus P)$. By the definition of direct sum, F and G together induce yet another map $Y : (M \otimes N) \oplus (M \otimes P) \to M \otimes (N \oplus P)$ which takes $Y((m_1 \otimes n, m_2 \otimes p)) = m_1 \otimes (n, 0) + m_2 \otimes (0, p)$. Therefore, $X \circ Y = Y \circ X = 1$, and we have proven the claimed isomorphism. $\square$

3 Question 3

3.1 Question

Prove that $A^m \otimes A^n \cong A^{mn}$.

3.2 Answer

Proof. We use induction on n. By the first, and next questions we have $A \otimes_A A^m \cong A^m \cong A^m \otimes_A A$. So, suppose (induction) that $A^m \otimes_A A^{n-1} \cong A^{m(n-1)}$. By the second exercise we conclude that $A^m \otimes_A A^n \cong A^m \otimes_A (A^{n-1} \oplus A) \cong (A^m \otimes_A A^{n-1}) \oplus (A^m \otimes_A A) \cong A^{m(n-1)} \oplus A^m \cong A^{mn}$. $\square$

4 Question 4

4.1 Question

Prove that $A \otimes_A M \cong M$.

4.2 Answer

Recall that the tensor product is just the object, map $\otimes$ that is universal for the following diagram

$$\begin{tikzcd} A \times M \arrow[r, "\otimes"] \arrow[dr, "f"'] & A \otimes M \arrow[d, dotted, "\bar{f}"] \\ & Z \end{tikzcd}$$

for f a bilinear map, Z an arbitrary abelian group. However, It is easy to see that this is just M. By bilinearity $(a, m) = (a \cdot 1, m) = (1, am)$, so, setting $A \otimes M = M$ there clearly exists an arrow $\bar{f}$ that makes the diagram commute. Moreover, $\bar{f}$ must also be unique, since were it not then there would exist $\bar{f} \neq \bar{g}$ which make the diagram commute. However, we can produce $f \neq g$ maps from $A \times M \to Z$ from these, a contradiction.

5 Question 5

5.1 Question

Prove that if I is an ideal of A, then $(A/I) \otimes_A M \cong M/IM$.

5.2 Answer

Proof. Define a map $\varphi : M \to (A/I) \oplus_A M$ defined as $\varphi : m \mapsto 1 \otimes m$. $IM \subseteq Ker(\varphi)$ since $\varphi(am) = 1 \oplus am = a \otimes m = 0 \otimes m = 0$. Hence, φ induces a homomorphism $X : M/IM \to (A/I) \otimes M$. This homomorphism is surjective by properties of homomorphism, the fact that $1 \otimes m$ generates $(A/I) \otimes_A M$.

Now, let $\psi : (A/I) \times M \to M/IM$ be $\psi : (a(modI), m) \mapsto am(modIM)$. Being bilinear, this induces a group homomorphism $Y : (A/I) \otimes M \to M/IM$ which sends $a(mdoI) \otimes m \mapsto am(modIM)$. $Y \circ X = 1$ with X a bijection. Hence, we have the desired isomorphism. $\square$

6 Question 6

6.1 Question

Write out a "bare hands" proof that if $M \to^{\psi} N \to P \to^{\phi} 0$ is exact, then so is $M \otimes_A Q \to^{1\otimes\psi} N \otimes_A Q \to^{1\otimes\phi} P \otimes_A Q \to 0$ for any A-module Q.

6.2 Answer

We follow the proof from Dummit and Foote p399.

Proof. Let the first sequence be exact. The map $\phi \otimes 1$ is surjective since $p \otimes q = \phi(n) \otimes q$ for all simple tensors, by surjectivity of ϕ. Now, take a general tensor $\sum m_i \otimes q_i$. We have $(\phi \otimes 1) \circ (\psi \otimes 1)(\sum m_i \otimes q_i) = \sum \phi \circ \psi(m_i) \otimes q_i = \sum 0 \otimes q_i = 0$. Hence, $Im(\psi \otimes 1) \subseteq Ker(\phi \otimes 1)$. Therefore, there exists a projection $\pi : (N \otimes_A Q)/Im(\psi \otimes 1) \to P \otimes_A Q$.

Now define $\gamma : P \times Q \to (N \otimes_A Q)/Im(\psi \otimes 1)$ by $\gamma((p, q)) = n \otimes q$ for $n \in N$ with $\phi(n) = p$. This map being bilinear it induces a homomorphism $\Gamma : P \otimes_A Q \to (N \otimes_A Q)/Im(\psi \otimes 1)$ by $\Gamma(p \otimes q) = n \otimes q$. Now $\Gamma \circ \pi(n \otimes q) = \Gamma(\phi(n) \otimes q) = n \otimes q$ so it is the identity. Since $\pi \circ \Gamma$ is also the identity we have $Im(\psi \otimes 1) = Ker(\phi \otimes 1)$. $\square$

7 Question 7

7.1 Question

Prove that a short exact sequence $0 \to M \to N \to P \to 0$ for A-modules is split if and only if the sequence exhibits an isomorphism $N \cong M \oplus P$; that is, if and only if there is a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r] \arrow[d] & N \arrow[r] \arrow[d] & P \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M \arrow[r] & M \oplus P \arrow[r] & P \arrow[r] & 0
\end{tikzcd}$$

where the left and right downward arrows are the identity maps, and the maps on the bottom line are the obvious inclusion and projection.

7.2 Answer

First we will show ($\Leftarrow$) that if the short exact sequence is split, then there exists such a commutative diagram.

Proof. By the definition of projective and injective modules there exist additional maps such that the following diagram commutes.

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & M \oplus P \arrow[r] \arrow[dl] & P \arrow[r] \arrow[d] \arrow[dl] & 0 \arrow[d] \\
0 \arrow[r] & M \arrow[r] & N \arrow[r] \arrow[ul] & P \arrow[r] \arrow[ul] & 0
\end{tikzcd}$$

or equivalently

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r, shift right] \arrow[dr, bend right] & M \oplus P \arrow[l, shift right, bend right] \arrow[r, shift right] \arrow[d, dotted, "g"'] & P \arrow[r] \arrow[l, shift right, bend right] \arrow[dl] & 0 \\
& & N \arrow[ul] \arrow[ur, bend right] & &
\end{tikzcd}$$

Thus, by definition of the coproduct there exists a unique map g (above), such that the diagram commutes. Applying the five lemma, we see that this homomorphism is in fact an isomorphism, as desired. ☐

Conversely ($\Rightarrow$), suppose that such an isomorphism of short exact sequences exists. We will demonstrate that it splits

Proof. So $0 \to M \to N \to P \to 0$. Let's call the map $p : N \to P$. Then, if we call the isomorphism $\varphi : N \to M \oplus P$, and the inclusion map $i : P \to M \oplus P$ we have a map from P to N such that its composition with p is the identity, namely $\varphi^{-1} \circ i$.

$$\begin{tikzcd}
M \oplus P \arrow[r] & P \arrow[l, bend right, "i"'] \arrow[d, "\cong"'] \\
N \arrow[u, "\varphi"'] \arrow[r, "p"'] & P
\end{tikzcd}$$

□

8 Question 8

8.1 Question

Prove the uniqueness assertion in the structure theorem for finitely generated modules over a PID.

8.2 Answer

We follow the proof of Dummit and Foote p466.

Throughout, let R be a PID and p a prime in R. F is a the field $R/(p)$.

Now, by way of a lemma we show that $M = R^r \Rightarrow M/pM \cong F^r$.

Proof. There is a map from R^r to $(R/(p))^r$ defined by modding out by (p). This is a surjective homomorphism of R-modules whose kernel is just those elements all of whose coordinates are in (p). Since these elements are just pR^r we have $R^r/pR^r \cong (R/(p))^r$ as claimed. □

As a second lemma I claim that if $M = R/(a_1) \oplus R/(a_2) \oplus \cdots \oplus R/(a_k)$ for each a_i divisible by p then $M/pM \cong F^k$.

Proof. First we will show that if $M = R/(a)$ where a is a nonzero element of R then

$$M/pM \cong \begin{cases} F & \text{if } p \text{ divides } a \text{ in } R \\ 0 & \text{if } p \text{ does not divide } a \text{ in } R. \end{cases}$$

The claim follows from this proposition, as well as the existence part of the structure theorem.

The above claim follows form the isomorphism theorems. Note first that $p(R/(a))$ is the image of the ideal (p) in the quotient $R/(a)$, hence is $(p) + (a)/(a)$. The idea $(p) + (a)$ is generated by a greatest common divisor of p and a, hence is (p) if p divides a and is $R = (1)$ otherwise. Hence $pM = (p)/(a)$ if p divides a and is $R/(a) = M$ otherwise. If p divides a then $M/pM - (R/(a))/((p)/(a)) \cong R/(p)$, and if p does not divide a then $M/pM = M/M = 0$. □

Now we will show that two finitely generated R-modules M and L are isomorphic if and only if they have the same list of invariant factors.

The forward ($\Rightarrow$) direction of this proof is clear.

So, suppose M and L are isomorphic. We will show that they have the same free rank and list of invariant factors.

Proof. Any isomorphism between M and L must map the torsion in one to the torsion in the other. So, $\text{Tor}(L) \cong \text{Tor}(M)$. So, $R^m \cong M/\text{Tor}(M) \cong L/\text{Tor}(L) \cong R^l$. Where m is the free rank of M and l of L. Let p be any nonzero prime from R. Then, as $R^m \cong R^l$ we have an isomorphism $R^m/pR^m \cong R^l/pR^l$. By the lemma this implies that $F^m \cong F^l$ where F is the field R/pR. Hence we have an isomorphism of an m-dimensional vector space over F with an l dimensional vector space over F, so that $m = l$ and M and L have the same free rank.

Now it remains only to show that M and L have the same lists of invariant factors. We need only work with the isomorphic torsion modules, so assume that both M and L are torsion R-modules.

First we'll demonstrate the they have the same elementary divisors. However, it suffices to show that for any prime p the elementary divisors which are a power of p are the same for both modules. If $L \cong M$ then the direct sum of the cclic factors whose elementary divisors are a power of p in L is isomorphic to the same direct sum in M.

We use induction on the power of p in the annihilator of L. If this power is 0, then both L and M are 0 and we are done. Otherwise L (and by the isomorphism M) have nontrivial elementary divisors. Suppose the elementary divisors of L are given by

$$\underbrace{p, p, \ldots, p}_{mtimes}, p^{\alpha_1}, p^{\alpha_2}, \ldots, p^{\alpha_t}$$

where $2 \leq \alpha_1 \leq \alpha_2 \leq \ldots \leq \alpha_s$. L is the direct sum of cyclic modules with generators $x_1, x_2, \ldots, x_m, x_{m+1}, \ldots, x_{m+s}$, say, whose annihilators are $(p), (p), \ldots, (p), (p^{\alpha_1}), \ldots, (p^{\alpha_s})$, respectively. Then the submodules pM_1 has elementary divisors

$$p^{\alpha_1 - 1}, p^{\alpha_2 - 1}, \ldots, p^{\alpha_s - 1}$$

since pL is the direct sum of the cyclic modules with generators $px_1, px_2, \ldots, px_m, px_{m+1}, \ldots, px_{m+s}$ whose annihilators are $(1), (1), \ldots, (1), (p^{\alpha_1 - 1}), \ldots, (p^{\alpha_s - 1})$, respectively. Similarly if the elementary divisors of M are given by

$$\underbrace{p, p, \ldots, p}_{ntimes}, p^{\beta_1}, p^{\beta_2}, \ldots, p^{\beta_t}$$

where $2 \leq \beta_1 \leq \beta_2 \leq \ldots \leq \beta_t$, then pM has elementary divisors

$$p^{\beta_1 - 1}, p^{\beta_2 - 1}, \ldots, p^{\beta_t - 1}$$

Since $M \cong L$, also $pL \cong pM$ and the power of p in the annihilator of pL is one less than the power of p in the annihilator of M. By induction, the elementary divisors for pL are the same as the elementary divisors for pM, i.e., $s = t$ and $\alpha_1 - 1 = \beta_i - 1$ for $i = 1, 2, \ldots, s$, hence $\alpha_i = \beta_i$ for $i = 1, 2, \ldots, s$. Finally since also $L/pL \cong M/pM$ we see from the second lemma that $F^{m+s} \cong F^{n+t}$, which shows that $m + s = n + t$ hence $m = n$ since we have already seen $s = t$. □

9 Question 9

9.1 Question

Complete the sketch proof of the existence of Jordan normal form.

9.2 Answer

Let T be a linear transformation on a finite dimensional vector space V over the field F, with minimal polynomial of T splitting completely over F. I claim that T can be put in Jordan normal form.

Proof. x acts on $v \in V$ by $x \cdot v = Tv$. For any $p(x) = a_n x^n + a_{n-1} x^{n-1} + ... + a_1 x + a_0 \in F[x]$, we define the action of $p(x)$ on $v \in V$ by $p(x)v = (a_n T^n + a_{n-1} T^{n-1} + ... + a_1 T + a_0)(v) = a_n T^n(v) + a_{n-1} T^{n-1}(v) + ... + a_1 T(v) + a_0 v$. This action makes V into an $F[x]$ module.

As V is a finite dimensional vector space over F, it is finitely generated as an $F[x]$-module. Since free modules over $F[x]$ have infinite dimension the $F[x]$ module is torsion. By the structure theorem for finitely generated modules over a PID, $V \cong F[x]/a_1(x) \oplus F[x]/a_2(x) \oplus \ldots \oplus F[x]/a_m(x)$ where $a_m(x)$ is the minimal polynomial for T.

By assumption, the minimal polynomial of T splits completely in F, so V is the direct sum of finitely many $F[x]$-module of the form $F[x]/(x - \gamma)^q$ for some $\gamma \in F$ and $q \geq 1$. Now consider the elements $(x - \gamma)^{q-1}, (x - \gamma)^{q-2}, ..., (x - \gamma), 1 \in F[x]/(x - \gamma)^q$. This set of elements is a basis for $F[x]/(x - \gamma)^q$ as a F vector space.

□

10 Question 10

10.1 Question

Give examples of matrices M over $\mathbb{Q}$ whose characteristic polynomial and minimal polynomial are respectively:

1. $(X-1)^5$ and $(X-1)^4$.
2. $(X-1)^2(X-2)^3$ and $(X-1)(X-2)$.
3. $(X-1)^2(X-2)^3$ and $(X-1)(X-2)^2$.
4. $(X-1)^2(X-2)^3$ and $(X-1)(X-2)^3$.

10.2 Answer

We exploit our knowledge of Jordan Canonical form to find the following matrices (Jordan Blocks highlighted).

1.

$$\left(\begin{array}{cccc|c} 1 & 1 & & & \\ & 1 & 1 & & \\ & & 1 & 1 & \\ & & & 1 & \\ \hline & & & & 1 \end{array}\right)$$

2.

$$\left(\begin{array}{c|c|c|c|c} 1 & & & & \\ \hline & 1 & & & \\ \hline & & 2 & & \\ \hline & & & 2 & \\ \hline & & & & 2 \end{array}\right)$$

3.

$$\left(\begin{array}{c|c|cc|c} 1 & & & & \\ \hline & 1 & & & \\ \hline & & 2 & 1 & \\ & & & 2 & \\ \hline & & & & 2 \end{array}\right)$$

4.

$$\left(\begin{array}{cc|ccc} 1 & & & & \\ & 1 & & & \\ \hline & & 2 & 1 & \\ & & & 2 & 1 \\ & & & & 2 \end{array}\right)$$