Homework 8

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1 Question 1

1.1 Question

1. If $a, b \in A$, prove that $D(a) \cap D(b) = D(ab)$.
2. Prove that $a \in A^\times$ if and only if $D(a) =$Spec A.
3. Prove that if $a \in A$ and $u \in A^\times$ then $D(a) = D(au)$.

1.2 Answer

1. If an ideal I contains a, then clearly it also contains ab. Hence, $D(ab) \subseteq D(a)$. By a similar argument, $D(ab) \subseteq D(b) \Rightarrow D(ab) \subseteq D(a) \cap D(b)$. Now let $\mathfrak{p} \in D(a) \cap D(b)$ be a prime ideal not containing a or b. Clearly $\mathfrak{p}$ does not contain ab either, since $\mathfrak{p}$ is assumed to be prime and so by definition $\mathfrak{p} \in D(ab) \Rightarrow D(a) \cap D(b) \subseteq D(ab)$.
2. Given $a \in A^\times$, any $(p) \ni a$ also contains 1, and therefore $A = \mathfrak{p}$, but this can't be, as $\mathfrak{p}$ is assumed to be proper.

   Conversely, suppose that $D(a) = \text{Spec}A$. Then, a is contained in no prime ideal. In particular then a is contained in no maximal ideal. Hence, a generates the entire ring. Thus, it has an inverse.
3. It follows from parts 1 and 2 that $D(au) = D(a) \cap D(u) = D(a) \cap \text{Spec}A = D(a)$ as desired.

2 Question 2

2.1 Question

If I is an ideal of A, define $\text{rad}(I) := \{a \in A \mid a^n \in I \text{ for some } n \geq 1\}$.

1. Prove that $\text{rad}(I)$ is the preimage of $\text{nil}(A/I)$ under the surjection $A \to A/I$.
2. Prove that $\text{rad}(I) = \bigcap_{\mathfrak{p} \supset I} \mathfrak{p}$ where (as indicated) the intersection is taken over the set of all prime ideals containing $\mathfrak{p}$.

2.2 Answer

1. Suppose $x \in \text{nil}(A/I)$. Then, for some $n \geq 1, (x+I)^n = I \Rightarrow x^n \in I$ as desired. For the reverse inclusion, let $a \in \text{rad}(I)$. Thus, $a^n \in I$, and $f(a)^n = f(a^n) = 0$, so $f(a) \in \text{nil}(A/I)$ as desired.
2. By the previous part this is equivalent to showing that
$$f^{-1}(\text{nil}(A/I)) = \bigcap_{\mathfrak{p} \supset I} \mathfrak{p}$$
but
$$\begin{aligned} f^{-1}(\text{nil}(A/I)) &= f^{-1}\Big(\bigcap_{\mathfrak{p} \subset A/I} \mathfrak{p}\Big) \\ &= \bigcap_{\mathfrak{p} \subset A/I} f^{-1}(\mathfrak{p}) \\ &= \bigcap_{\mathfrak{p} \supset I} \mathfrak{p} \end{aligned}$$

as desired.

3 Question 3

3.1 Question

If $a, b \in A$, prove that the following are equivalent:

1. $D(b) \subset D(a)$
2. $b^n \in aA$ for some $n \geq 1$
3. The natural map $A \to A_b$ factors through the natural map $A \to A_a$.

[Some hints: for (1) $\Rightarrow$ (2), consider the statement on complements, and use exercise 2 part 2. To see that (2) $\Rightarrow$ (3), recall that the natural map $A \to A_a$ is initial among all maps $A \to B$ in which a becomes invertible. To see that (3) $\Rightarrow$ (1), recall that the image of Spec A_a in Spec A under the natural map is $D(a)$ (and similarly with b in place of a).]

3.2 Answer

First we'll show $1 \Rightarrow 2$

Proof. Suppose $D(b) \subset D(a)$. Then, $V(a) \subset V(b)$, or equivalently, $\{\mathfrak{p} \mid a \in \mathfrak{p}\} \subset \{\mathfrak{p} \mid b \in \mathfrak{p}\}$. Thus, by exercise 2 part 2 $\mathrm{rad}(bA) \subset \mathrm{rad}(aA)$. As b is in $\mathrm{rad}(bA)$ trivially, it is also in $\mathrm{rad}(aA)$. But this is what we wanted to show. □

Now we'll show $2 \Rightarrow 3$

Proof. By the hint it suffices to show that 2 implies that a has inverse in $A_b = A[x]/(bx - 1)$. Fix n such that $b^n \in aA$. So, $b^n = ay$ for some $y \in A$. Hence, $b^n x^n = 1 = ayx^n$ and a has inverse in A_b as claimed. □

Finally, $3 \Rightarrow 2$

Proof. Recall from class that the natural map $A \to A_p$ induces a map of spectrums, Spec $A_p \overset{f^{-1}}{\to}$ Spec A with image $D(p)$. So, assume that $f : A \to A_b$ factors as the composition of $g : A \to A_a$ and $h : A_a \to A_b$. Then, $f^{-1}(A_b) = D(b) = g^{-1} \circ h^{-1}(A_b)$. Since $h^{-1}(A_b) \subseteq A_a$, we have $g^{-1} \circ h^{-1}(A_b) \subseteq h^{-1}(A_a)$. That is $D(b) \subseteq D(a)$, as desired. □

4 Question 4

4.1 Question

Let $a \in A$, write $A_a := A[x]/(ax - 1)$, and let $f : A \to A_a$ the natural map, inducing a map f^{-1} : Spec $A_a \to$Spec A. In class we proved that f^{-1} induces a bijection between Spec A_a and $D(a)$. The goal of this exercise is to carefully prove that this bijection is a homeomorphism (when $D(a)$ is equipped with the topology induced from that of Spec A).

1. If $b \in A_a$, show that $a^n b = f(a')$ for some $a' \in aA$.
2. Prove that the bijection Spec $A_a \to D(a)$ induced by f^{-1} restrcts to a bijection between $D(b)$ (which is a subset of Spec A_a) and $D(a')$ (which is a subset of Spec A). [Hint: We showed in class that for any $f : A \to B$, and any $a \in A$, the preimage of $D(a)$ under f^{-1} is $D(f(a))$. Apply this with a replaced by a'. Then use the result of exercise 1 to show that $D(a') \subset D(a)$ and that $D(f(a')) = D(b)$.]
3. Using (2), conclude that f^{-1} induces a homeomorphism between Spec A_a and $D(a)$.

4.2 Answer

1. Fix some $b \in A_a$. We can write b as a polynomial in x with coefficients $b_i \in A$: $b = \sum_{i=0}^{n} b_i x^i$. Now, multiplying through by a^{n+1} we just get $a^{n+1}b = \sum_{i=0}^{n} b_i a^{n-i+1}$. Seen as a member of aA, this is precisely the desired a'.

2. Choosing $a^n b = f(a')$ as above, we have by a proof in class, $(f^{-1})^{-1}D(a') = D(f(a')) = D(a^n b)$, and by question 1, $D(a^n b) = D(b)$, since a^n is a unit in A_a. Since this is a restriction of a bijection, it is itself a bijection.

3. The previous part demonstrates that f^{-1} restricts to a bijection between an open basis for Spec A_a, and $D(a)$. Thus, it is continuous with continuous inverse.

Question 4*

Question

Keeping the notation of exercise 2, try to write a proof that Spec $A_a \to D(a)$ is a homeomorphism using closed sets rather than distinguished open sets.

Answer

5 Question 5

5.1 Question

1. Prove that if $U \subset V$ is an inclusion of open subsets of Spec A, and if $f \in \mathcal{O}(U)$, then $f|_V$ (the restriction of f to V) is an element of $\mathcal{O}(V)$. [Hint: Exercise 3 may be helpful.]

2. Prove that if $U_{i_{i \in I}}$ is any open cover of Spec A, and that if $f_i \in \mathcal{O}(U_i)$ are such that f_i and f_j coincide on $U_i \cap U_j$ for all $i, j \in I$, then the function f defined on $\bigcup_i U_i$ via

$$f(\mathfrak{p}) = f_i(\mathfrak{p}) \text{ if } \mathfrak{p} \in U_i$$

is well-defined, and is an element of $\mathcal{O}(\bigcup U_i)$.

[These are the *sheaf properties* of $\mathcal{O}$.]

5.2 Answer

1. There exists $a \in A$, $D(a) \subseteq U$ such that $f|_{D(a)}$ is given $x \in A_a$. Given a b with $D(b) \subseteq V$, $D(b) \subseteq D(a)$ we would like an element $y \in A_b$ with $f|_{D(b)}$ given by y. If we write x and y as polynomials in $1/a$ and $1/b$, and sum by matching denominators we have $x = x'/a^n, y = y'/b^m$. By exercise 3, we can put $b^l = ac$ for some $c \in A$. Hence, $x'/a^n = x'c^n/a^nc^n = x'c^n/b^{nl}$. So, $y' = x'c^n$, $m = nl$ is just what we wanted to demonstrate.

2. For some $p \in U_i, U_j$ we have $f(p) = f_i(p), f_j(p)$, but these two values coincide by assumption, so it is well defined.

   Moreover, $f \in \mathcal{O}$, as given $p \in U$ there is some U_i with $p \in U_i$. Hence there exists $a \in A$ with $D(a) \subseteq U_i$ and $f_i|_{D(a)}$ is given by $x \in A_a$. But since $f = f_i$, this same element gives $f|_{D(a)}$, and $f \in \mathcal{O}(U)$.