May / AGP

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10 June 2010

Aguilar, Gitler and Prieto 5.3

1 Question 11

Prove that the canonical inclusion $j : Y \to Z_f$ is a homotopy equivalence satisfying $j \circ f \simeq k$, where $k : X \to Z_f$ is the canonical inclusion induced by $x \mapsto (x, 1)$.

1.1 Answer

2 Question 19

Verify the details of the proof of the exactness property. In particular, show that $C_p \simeq \Sigma X^+$ and that up to precisely this homotopy equivalence, the map $\Sigma A^+ \to C_p$ corresponds to the inclusion.

2.1 Answer

3 Question 21

Construct an alternative proof of the relative exactness axiom (5.3.16) using the long exact homotopy sequence (4.3.41) of the quasifibration SP $Z_i \to$ SPC_i that is induced by the identification map $Z_i \to C_i$, where $i : A \to X$ is the inclusion.

3.1 Answer

4 Question 22

Assume that X is contractible. Prove that

$$H_q(X, A) \cong H_{q-1}(A)$$

if $q > 1$, and

$$H_q(X, A) \cong \tilde{H}_0(A).$$

4.1 Answer

5 Question 23

Take $A \subset B \subset X$ and assume that the inclusion $A \hookrightarrow B$ is a homotopy equivalence. Prove that the inclusion of pairs $(X, A) \hookrightarrow (X, B)$ induces an isomorphism

$$H_q(X, A) \to H_q(X, B)$$

for all q.

5.1 Answer

6 Question 25

Prove that if $f : (X, A) \to (Y, B)$ is a weak homotopy equivalence of pairs of topological spaces, then

$$f_* : H_q(X, A) \to H_q(Y, B)$$

is an isomorphism for all q. This is the so-called *weak homotopy equivalence axiom.* [*Hint:* See 5.1.35]

6.1 Answer

Recall that 5.1.35 states

If $\varphi : (\tilde{X}, \tilde{A}) \to (X, A)$ and $\gamma : (\tilde{Y}, \tilde{B}) \to (Y, B)$ are CW-approximations and $f : (X, A) \to (Y, B)$ is continuous, then there exists a map that is unique up to homotopy, say $\tilde{f} : (\tilde{X}, \tilde{A}) \to (\tilde{Y}, \tilde{B})$, such that the diagram

$$\begin{tikzcd}
(\tilde{X}, \tilde{A}) \arrow[r, "\varphi"] \arrow[d, "\tilde{f}"'] & (X, A) \arrow[d, "f"] \\
(\tilde{Y}, \tilde{B}) \arrow[r, "\gamma"] & (Y, B)
\end{tikzcd}$$

commutes up to homotopy, namely, $f \circ \varphi \simeq \gamma \circ \tilde{f}$ (by means of a homotopy of pairs).

7 Question 27

Prove that the excision axiom for excisive triads is equivalent to the following axiom. Suppose that (X, A) is a pair of spaces and that $U \subset A$ satisfies $\overline{U} \subset \mathring{A}$. Then the inclusion $i : (U - A, A - U) \to (X, A)$ induces an isomorphism $H_n(X - A, A - U) \cong H_n(X, A)$ for each $n \geq 0$.

7.1 Answer

8 Question 32

Let $(X, A) = \prod(X_\lambda, A_\lambda)$. Prove that for all q.

$$H_q(X, A) \cong \bigoplus_\lambda H_q(X_\lambda, A_\lambda).$$

This is the so-called *additivity axiom* for homology.

8.1 Answer