Hatcher

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Hatcher 4.1

1 Question 16

1.1 Question

Show that a map $f : X \to Y$ between connected CW complexes factors as a composition $X \to Z_n \to Y$ where the first map induces isomorphisms on π_i for $i \leq n$ and the second map induces isomorphisms on π_i for $i \geq n+1$.

1.2 Answer

Adapt Example 4.17 on page 354.

Construct Z_n as follows. Start with a copy of X, then add $(n+1)$-cells to it corresponding to generators of $\pi_{n+1}(Y)$. This space has a π_{n+1} that is too large but by adding $n+2$-cells we can get the appropriate group.

These $n+2$-cells (as well as the just-added $n+1$ cells) map to Y. by taking generators of π_{n+1} to generators in Y, and relations to relations. Repeating this process infinitely many times, we are done.

Hatcher 4.2

2 Question 1

2.1 Question

Use homotopy groups to show there is no retraction $\mathbb{R}P^n \to \mathbb{R}P^k$ if $n > k > 0$.

2.2 Answer

First we'll make some calculations of the homotopy groups of $\mathbb{R}P^n$.

Proof. Recall that we have the fibration

$$S^0 \to S^n \to \mathbb{R}P^n$$

for all n. So, we may write the long exact sequence of the fibration as

$$\cdots \to \pi_k(S^0) \to \pi_k(S^n) \to \pi_k(\mathbb{R}P^n) \to \pi_{k-1}(S^0) \to \cdots \to \pi_0(S^n) \to 0.$$

Clearly,

$$\pi_n(S^0) = \begin{cases} 0 & n \neq 0 \\ \mathbb{Z}/2\mathbb{Z} & n = 0 \end{cases}.$$

Thus, $k > 1$, $\pi_k(\mathbb{R}P^n)$ fits into the exact sequence

$$0 \to \pi_k(S^n) \to \pi_k(\mathbb{R}P^n) \to 0$$

and we have $\pi_k(S^n) \cong \pi_k(\mathbb{R}P^n)$ for $k > 1$. At the end of this long exact sequence we have

$$\cdots \to \pi_1(S^n) \to \pi_1(\mathbb{R}P^n) \to \pi_0(S^0) \to \pi_0(S^n) \to 0$$

$$\cdots \to \pi_1(S^n) \to \pi_1(\mathbb{R}P^n) \to \mathbb{Z}/2\mathbb{Z} \to \pi_0(S^n) \to 0.$$

Assuming that $n > 1$, this is just

$$0 \to \pi_1(\mathbb{R}P^n) \to \mathbb{Z}/2\mathbb{Z} \to 0$$

and $\pi_1(\mathbb{R}P^n) \cong \mathbb{Z}/2\mathbb{Z}$ for $n > 1$. Since $\mathbb{R}P^1 \cong S^1$ we have $\pi_1(\mathbb{R}P^1) \cong \pi_1(S^1) \cong \mathbb{Z}$ in the case of $n = 1$.

So, by substituting in the values of $\pi_k(S^n)$ where appropriate we have a table of homotopy groups that looks like

	π_1	π_2	π_3	π_4	π_5	π_6	$\cdots$
$\mathbb{R}P^1$	$\mathbb{Z}$	0	0	0	0	0	$\cdots$
$\mathbb{R}P^2$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/12\mathbb{Z}$	$\cdots$
$\mathbb{R}P^3$	$\mathbb{Z}/2\mathbb{Z}$	0	$\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/12\mathbb{Z}$	$\cdots$
$\mathbb{R}P^4$	$\mathbb{Z}/2\mathbb{Z}$	0	0	$\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\cdots$
$\mathbb{R}P^5$	$\mathbb{Z}/2\mathbb{Z}$	0	0	0	$\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\cdots$
$\mathbb{R}P^6$	$\mathbb{Z}/2\mathbb{Z}$	0	0	0	0	$\mathbb{Z}$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

$\square$

Now, it is easy to see that there is no retraction $\mathbb{R}P^n \to \mathbb{R}P^k$ if $n > k > 0$.

Proof. Suppose towards a contradiction that there is such a retraction $r : \mathbb{R}P^n \to \mathbb{R}P^k$. But then we should be able to factor the group $\pi_k(\mathbb{R}P^k)$ by

$$\pi_k(\mathbb{R}P^k) \xrightarrow{i_*} \pi_k(\mathbb{R}P^n) \xrightarrow{r_*} \pi_k(\mathbb{R}P^k)$$

first including, then retracting. This is a contradiction, since the middle group is either 0, or $\mathbb{Z}/2\mathbb{Z}$ while the outer one is $\mathbb{Z}$. $\square$

3 Question 2

3.1 Question

Show the action of $\pi_1(\mathbb{R}P^n)$ on $\pi_n(\mathbb{R}P^n) \cong \mathbb{Z}$ is trivial for n odd and non-trivial for n even.

3.2 Answer

Let's consider $S^n \subset \mathbb{R}^{n+1}$ the unit sphere. Then $\pi_n(S^n)$ is generated by the identity map on the sphere, and $-1 \in \pi_n(S^n)$ is generated by any negative determinant member of $SO(n+1)$ (which fixes the basepoint). Furthermore, $\mathbb{R}P^n$ is obtained by associating antipodal points in the sphere. That is, $x \sim y$ if $x = -Iy$ for I the identity matrix in $SO(n+1)$. Denote by p the map which sends $S^n \to \mathbb{R}P^n$ by this association.

Since $\pi_1(\mathbb{R}P^n) = \mathbb{Z}/2$ it suffices to check that the nontrivial element of $\pi_1(\mathbb{R}P^n$ sends the generator of $\pi_n(\mathbb{R}P^n)$ to itself with n odd and to -1 for n even. This nontrivial element has a representative whose preimage under p is a great circle in S^n going through the base point of our homotopy.

Assuming without loss of generality that this basepoint has coordinate $(1, 0, 0, \ldots)$, we see that acting by this element induces a (not basepoint preserving) rotation in our sphere, exchanging the basepoint, with the opposite pole. That is, it acts by the matrix

$$A = \begin{pmatrix} -1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & & -1 \end{pmatrix}.$$

(Note: there is nothing special about this particular matrix, it just needs to have -1 in the first entry, and determinant 1.)

Now, taking into account that anitpodal points are associated, we see the desired result. A pair, $(x, -Ix)$ is taken by this map to $A(x, -Ix) = (Ax, -Ax) = -A(-Ix, x)$. In the case that $n+1$ is odd, $-A$ has negative determinant, however if $n+1$ is even, $-A$ has positive determinant. Since

4 Question 6

4.1 Question

Show that the relative form of the Hurewicz Theorem in dimension n implies the absolute form in dimension $n-1$ by considering the pair (CX, X) where CX is the cone on X.

4.2 Answer

The Hurewicz Theorem states:

If (X, A) is an $(n-1)$-connected pair of path-connected spaces with $n \geq 2$ and $A \neq 0$ then $h' : \pi'_n(X, A, x_0) \to H_n(X, A)$ is an isomorphism and $H_i(X, A) = 0$ for $i < n$.

The absolute version is just the result of taking A to be the basepoint. Namely:

If X is an $(n-1)$-connected space with $n \geq 2$ then $h' : \pi'_n(X, x_0) \to H_n(X)$ is an isomorphism and $H_i(X) = 0$ for $i < n$.

Proof. Let $X \neq 0$ be an $(n-2)$-connected space for $n \geq 2$. Then, as the pair (CX, X) is $(n-1)$-connected, and each is path connected, $X \neq 0$, $h' : \pi'_n(CX, X, x_0) \to H_n(CX, X)$ is an isomorphism and $H_i(CX, X) = 0$ for $i < n$ by the relative Hurewicz Theorem.

However this implies that we have $H_i(X) = 0$ for $i < n-1$ by the long exact sequence of relative homology groups. the fact that $h' : \pi'_n(CX, X, x_0) \to H_n(CX, X)$ is an isomorphism implies that $h' : \pi'_n(X, x_0) \to H_n(X)$ is an isomorphism in one lower dimension, since $\pi_n(CX, X) \cong \pi_{n-1}(X)$, and similarly for homology groups. $\square$

5 Question 7

5.1 Question

Construct a CW complex X with prescribed homotopy groups $\pi_i(X)$ and prescribed actions of $\pi_1(X)$ on the $\pi_i(X)$'s.

5.2 Answer