Hatcher

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Hatcher 2.2

1 Question 4

1.1 Question

Construct a surjective map $S^n \to S^n$ of degree zero, for each $n \geq 1$.

1.2 Answer

For S^1 we can demonstrate such a map explicitly by

$$\varphi(\theta) = \begin{cases} 2\theta & \theta \leq \pi \\ -2\theta & \theta > \pi \end{cases}$$

where S^1 is coordinatized by $0 \leq \theta < 2\pi$ in the usual polar way.

To get such a map for S^n, consider S^n as $S^{n-1} \times [0, 1]$ modulo the association of $S^{n-1} \times 0$ and $S^{n-1} \times 1$ to points. Then, we have a map on S^n induced by one on S^{n-1}. The S^1 map induces the required on on S^2 and so on.

2 Question 9

2.1 Question

Compute the homology groups of the following 2-complexes:

1. The quotient of S^2 obtained by identifying the north and south poles to a point.
2. $S^1 \times (S^1 \vee S^1)$.

3. The space obtained from D^2 by first deleting the interiors of two disjoint subdisks in the interior of D^2 and then identifying all three resulting boundary circles together via homeomorphisms preserving clockwise orientations of these circles.

4. The quotient space of $S^1 \times S^1$ obtained by identifying points in the circle $S^1 \times \{x_0\}$ that differ by $2\pi/m$ rotation and identifying points in the circle $\{x_0\} \times S^1$ that differ by $2\pi/n$ rotation.

2.2 Answer

1. We can create a CW approximation with 1 0-cell, 1 1-cell, and 1 2-cell. Associate the 1-cell to the 0-cell, to form a loop, then associate the 2 cell to the 1 cell twice around its boundary, in one direction, then the other. (The 1-cell goes from the north pole to the south pole of S^2 before our association.)

   Now, observe that the chain complex is just

   $$\mathbb{Z} \overset{\partial_0}{\leftarrow} \mathbb{Z} \overset{\partial_1}{\leftarrow} \mathbb{Z} \leftarrow 0 \leftarrow \cdots$$

   with ∂_0 the 0 map, since the north and south poles are associated, and ∂_1 the zero map as well, as the boundary traverses the 1-cell in one direction then the other, and is therefore nullhomotopic.

   Therefore, we have

   $$H_n(X) = \begin{cases} \mathbb{Z} & n = 0, 1, 2 \\ 0 & \text{otherwise} \end{cases}$$

2. We write

   $$S^1 \times (S^1 \vee S^1) = S^1 \times S^1 \cup_{S^1} S^1 \times S^1) = T^2 \cup_{S_1} T^2.$$

   Now, we represent this as a CW complex with 1 0-cell, 3 1-simpleces, and 2 2-simpleces. So we have

   $$\mathbb{Z} \overset{\partial_0}{\leftarrow} \mathbb{Z}^3 \overset{\partial_1}{\leftarrow} \mathbb{Z}^2 \leftarrow 0 \leftarrow \cdots$$

   with $\partial_0 = 0, \partial_1(a) = \alpha - \beta - \alpha + \beta = 0, \partial_1(b) = \alpha - \gamma - \alpha + \gamma = 0$ if we call the 1 cells α, β, γ and the 2 cells a, b. Thus,

   $$H_n(X) = \begin{cases} \mathbb{Z} & n = 0 \\ \mathbb{Z}^3 & n = 1 \\ \mathbb{Z}^2 & n = 2 \\ 0 & \text{otherwise} \end{cases}.$$

3. Let's split the boundary circle into 2 1 cells, joined at 2 0-simpleces, to form circles. Finally, add two one cells: one from the outside to the first removed disk, and one from the first removed disk to the second. Now we have 2 0-cells 4 1-cells, and 1 2-cell. Call the 0-cells α and β, and the 1-cells a, b, c, d. The boundary of a is $\alpha - \beta$, the boundary of b is $\beta - \alpha$, the boundary of c is $\alpha - \alpha = 0$, and the boundary of d is $\beta - \alpha$. The boundary of the 2-cell is $a+b+c-b+d-b-a-d-a-c = -a-b$. Our chain complex is

$$\mathbb{Z}^2 \xleftarrow{\partial_0} \mathbb{Z}^4 \xleftarrow{\partial_1} \mathbb{Z} \leftarrow 0 \leftarrow \cdots$$

Where the ∂s are given as above. Therefore,

$$H_n(X) = \begin{cases} \mathbb{Z} & n = 0 \\ \mathbb{Z}^2 & n = 1 \\ 0 & \text{otherwise} \end{cases}.$$

4. To create a CW approximation of this space we use 1 0-cell, 2 1-cells, and 1 2-cell. We've got

$$\mathbb{Z} \xleftarrow{\partial_0} \mathbb{Z}^2 \xleftarrow{\partial_1} \mathbb{Z} \leftarrow 0 \leftarrow \cdots$$

and $\partial_1 = 0$. Therefore

$$H_n(X) = \begin{cases} \mathbb{Z} & n = 0, 2 \\ \mathbb{Z}^2 & n = 1 \\ 0 & \text{otherwise} \end{cases}.$$

3 Question 10

3.1 Question

Let X be the quotient space of S^2 under the identification $x \sim -x$ for x in the equator S^1. Compute the homology groups $H_i(X)$. Do the same for S^3 with antipodal points of the equatorial $S^2 \subset S^3$ identified.

3.2 Answer

For the first space we can approximate with 1 0-cell, 1 1-cell, and 2 2-cells. For

$$\mathbb{Z} \xleftarrow{\partial_0} \mathbb{Z} \xleftarrow{\partial_1} \mathbb{Z}^2 \leftarrow 0 \leftarrow \cdots$$

we have $\partial_0 = 0, \partial_1(a) = \alpha + \alpha = 2\alpha, \partial_1(b) = -\alpha - \alpha = -2\alpha$. Hence,

$$H_n(X) = \begin{cases} \mathbb{Z} & n = 2, 0 \\ \mathbb{Z}/2\mathbb{Z} & n = 1 \\ 0 & \text{otherwise} \end{cases}.$$

For the next space, we can get a decomposition with 2 3-cells 1 2-cell, 1 1-cell, and 1 0-cell. For

$$\mathbb{Z} \overset{\partial_0}{\leftarrow} \mathbb{Z} \overset{\partial_1}{\leftarrow} \mathbb{Z} \overset{\partial_2}{\leftarrow} \mathbb{Z}^2 \leftarrow 0 \leftarrow \cdots$$

we have $\partial_0 = \partial_2 = 0, \partial_1(a) = 2\alpha$ All of the maps from cells of dimension ≤ 2 are from $\mathbb{R}P^2$. The last map is just $\partial_2 = \pm d_2$ depending on which generator we evaluate on (where d_2 is the corresponding map from $\mathbb{R}P^n$). Hence,

$$H_n(X) = \begin{cases} \mathbb{Z} & n = 0 \\ \mathbb{Z}/2\mathbb{Z} & n = 1 \\ \mathbb{Z}^2 & n = 3 \\ 0 & \text{otherwise} \end{cases}.$$

4 Question 12

4.1 Question

Show that the quotient map $S^1 \times S^1 \to S^2$ collapsing the subspace $S^1 \vee S^1$ to a point is not nullhomotopic by showing that it induces an isomorphism on H_2. On the other hand, show via covering spaces that any map $S^2 \to S^1 \times S^1$ is nullhomotopic.

4.2 Answer

Let's approximate T^2 by one 0-cell, two 1-cells, and a 2-cell say x, a, b, f, and S^2 by a 0 cell, and a 2-cell say x', f'. The boundary maps for these spaces are $\partial_2(f) = a + b - a - b = 0, \partial_1(a) = \partial_1(b) = 0, \partial_2(f') = 0, \partial_1 = 0$.

We compute $H_2(T^2) = \langle f \rangle / 0 = \mathbb{Z}$. The isomorphism induced on H_2 is defined by sending $f \mapsto f', 0 \mapsto 0$, and is therefore nontrivial.

Any map $S^2 \to T^2$ lifts to a map on the universal cover of T^2, $\mathbb{R}^2$. Since $\mathbb{R}^2$ is contractible, the map is nullhomotopic in the cover, and therefore in the original space.

5 Question 19

5.1 Question

Compute $H_i(\mathbb{R}P^n/\mathbb{R}P^m)$ for $m < n$ by cellular homology, using the standard CW structure on $\mathbb{R}P^n$ with $\mathbb{R}P^m$ as its m-skeleton.

5.2 Answer

This computation is just as in Example 2.42 ($\mathbb{R}P^n$) except that all members of the chain complex of dimension $i, 0 < i \leq m$ are set to 0. The chain becomes

$$0 \xleftarrow{0} \overbrace{\underbrace{\mathbb{Z} \xleftarrow{0} 0 \xleftarrow{0} \cdots \xleftarrow{0} 0}_{\text{dimensions } 0 \text{ to } m} \xleftarrow{0} \mathbb{Z} \leftarrow \cdots \leftarrow \mathbb{Z}}^{\text{dimensions } 0 \text{ to } n} \xleftarrow{0 \text{ or } 2} 0$$

and we have homology groups

$$H_i(\mathbb{R}P^n/\mathbb{R}P^m) = \begin{cases} \mathbb{Z} & i = 0 \\ \mathbb{Z}/2\mathbb{Z} & m+1 = i, i \text{ odd} \\ \mathbb{Z} & m+1 = i, i \text{ even} \\ \mathbb{Z}/2\mathbb{Z} & m+1 < i < n \text{ odd} \\ \mathbb{Z} & i = n \text{ odd} \\ 0 & \text{otherwise} \end{cases}$$

6 Question 20

6.1 Question

For finite CW complexes X and Y, show that $\chi(X \times Y) = \chi(X)\chi(Y)$.

6.2 Answer

Let $X \times Y = A$, and denote by z_i the number of i-cells in Z.

$$\begin{aligned}
\chi(X \times Y) &= \chi(A) \\
&= \sum_i (-1)^i a_i \\
&= \sum_i (-1)^i \sum_{m+n=i} x_n y_m \\
&= \sum_n (-1)^n x_n \cdot \sum_m (-1)^m y_m \\
&= \chi(X) \cdot \chi(Y)
\end{aligned}$$

as desired.

7 Question 21

7.1 Question

If a finite CW complex X is the union of subcomplexes A and B, show that $\chi(X) = \chi(A) + \chi(B) - \chi(A \cap B)$.

7.2 Answer

If a, b, x have a_n, b_n, x_n n cells, we observe that $x_n = a_n + b_n - (a \cap b)_n$. Hence,

$$\begin{aligned}
\chi(X) &= \sum_n (-1)^n x_n \\
&= \sum_n (-1)^n a_n + \sum_n (-1)^n b_n - \sum_n (-1)^n (a \cap b)_n \\
&= \chi(A) + \chi(B) - \chi(A \cap B)
\end{aligned}$$

as desired.