Homework

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1 Chapter 6

1.1 Problem 16

1.1.1 Question

For $1 < s < \infty$, define

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

(This is Riemann's zeta function, of great importance in the study of the prime numbers.) Prove that

1. $$\zeta(s) = s \int_1^{\infty} \frac{[x]}{x^{s+1}} dx$$

2. $$\zeta(s) = \frac{s}{s-1} - s \int_1^{\infty} \frac{x - [x]}{x^{s+1}} dx.$$

where $[x]$ denotes the greatest integer $\geq x$.

Prove that the integral in 2 converges for all $s > 0$.

Hint: To prove 1, compute the difference between the integral over $[1, N]$ and the Nth partial sum of the series that defines $\zeta(s)$.

1.1.2 Answer

1. *Proof.* We take the expression

$$s \int_1^\infty \frac{[x]}{x^{s+1}} dx$$

and express it as a sum of integrals on the intervals $(n, n+1)$ to get

$$s \left(\int_1^2 \frac{[x]}{x^{s+1}} dx + \int_2^3 \frac{[x]}{x^{s+1}} dx + \dots \right)$$

but since each such interval $[x]$ is the same, we just write

$$s \left(\int_1^2 \frac{1}{x^{s+1}} dx + \int_2^3 \frac{2}{x^{s+1}} dx + \dots \right) \tag{1}$$

Now we exploit the Fundamental Theorem of Calculus, computing

$$\int_n^{n+1} \frac{n}{x^{s+1}} dx = n \left[-\frac{x^{-s}}{s} \right]_n^{n+1} = n \left(-\frac{(n+1)^{-s}}{s} + \frac{n^{-s}}{s} \right)$$

So, the summation in Equation 1 can, more explicitly be written as

$$s \sum_{n=1}^\infty n \left(-\frac{(n+1)^{-s}}{s} + \frac{n^{-s}}{s} \right) = \sum_{n=1}^\infty \left(\frac{n}{n^s} - \frac{n}{(n+1)^s} \right)$$

However, grouping common denominators, we observe that the sum partially telescopes to yield more simply

$$\sum_{n=1}^\infty \frac{1}{n^s} = \zeta(s)$$

□

2. Having now proved Part 1 it suffices to show that

$$s \int_1^\infty \frac{[x]}{x^{s+1}} dx = \frac{s}{s-1} - s \int_1^\infty \frac{x - [x]}{x^{s+1}} dx.$$

Proof. By the Fundamental Theorem of Calculus we have

$$\int_1^\infty \frac{1}{x^s}dx = \frac{1}{s-1}$$

So

$$\begin{aligned}
\int_1^\infty \frac{x}{x^{s+1}}dx &= \frac{1}{s-1} \\
\Rightarrow s\int_1^\infty \frac{x}{x^{s+1}}dx &= \frac{s}{s-1} \\
\Rightarrow s\int_1^\infty \left(\frac{x-[x]}{x^{s+1}} + \frac{[x]}{x^{s+1}}\right)dx &= \frac{s}{s-1} \\
\Rightarrow s\int_1^\infty \left(\frac{x-[x]}{x^{s+1}} + \frac{[x]}{x^{s+1}}\right)dx &= \frac{s}{s-1} \\
\Rightarrow s\int_1^\infty \frac{[x]}{x^{s+1}}dx &= \frac{s}{s-1} - s\int_1^\infty \frac{x-[x]}{x^{s+1}}dx
\end{aligned}$$

as desired $\square$

It remains now to show that the integral in Part 2 converges.

Proof. Since for $x \in (1, \infty)$ we have

$$0 \leq \frac{x-[x]}{x^{s+1}} \leq \frac{1}{x^{s+1}}$$

we know that

$$\int_1^\infty \frac{x-[x]}{x^{s+1}}dx$$

converges if and only if

$$\int_1^\infty \frac{1}{x^{s+1}}dx$$

converges.

However,

$$\int_1^\infty \frac{1}{x^{s+1}}dx$$

converges by the integral test (Problem 8) since we have already shown that the sequence

$$\sum_{x=1}^{\infty} \frac{1}{x^{s+1}}$$

is convergent for $1 < s < \infty$. $\square$

2 Chapter 7

2.1 Problem 2

2.1.1 Question

If $\{f_n\}$ and $\{g_n\}$ converge uniformly on a set E, prove that $\{f_n + g_n\}$ converges uniformly on E. If, in addition, $\{f_n\}$ and $\{g_n\}$ are sequences of bounded functions, prove that $\{f_n g_n\}$ converges uniformly on E.

2.1.2 Answer

If $\{f_n\}$ and $\{g_n\}$ converge uniformly on a set E, then $\{f_n + g_n\}$ converges uniformly on E.

Proof. Let $\{f_n\}$ and $\{g_n\}$ converge uniformly on E to f and g respectively.

Then, by definition we have that for every $\epsilon > 0$ there is a corresponding N such that $n > N \Rightarrow |f_n(e) - f(e)| \leq \epsilon$ for any $e \in E$. And, moreover the same holds for $\{g_n\}$. That is, for any $\epsilon > 0$ there exists some M such that $m > M \Rightarrow |g_m(e) - g(e)| \leq \epsilon$ for any $e \in E$. But it is easy to see that these sums converge to $f + g$.

For, given ϵ, let $\delta = \frac{\epsilon}{2}$. Then take the maximum of N, M where N, M are selected so that $n > N \Rightarrow |f_n(e) - f(e)| \leq \delta$ and $m > M \Rightarrow |g_m(e) - g(e)| \leq \delta$ respectively. Say $\max\{N, M\} = K$.

Then for all $n > K$ we have $|f_n(e) - f(e)| + |g_n(e) - g(e)| \leq 2\delta = \epsilon \Rightarrow |f_n(e) - f(e) + g_n(e) - g(e)| \leq \epsilon \Rightarrow |(f_n(e) + g_n(e)) - (f(e) + g(e))| \leq \epsilon$. This however is just the definition for uniform convergence of $\{f_n + g_n\}$. $\square$

Now we wish to show that if $\{f_n\}$ and $\{g_n\}$ converge uniformly and are bounded on a set E then $\{f_n g_n\}$ converges uniformly on E.

Proof. We begin as above by noting that by definition we have that for every $\epsilon > 0$ there is a corresponding N such that $n > N \Rightarrow |f_n(e) - f(e)| \leq \epsilon$ for any $e \in E$. And, moreover the same holds for $\{g_n\}$. That is, for any $\epsilon > 0$ there exists some M such that $m > M \Rightarrow |g_m(e) - g(e)| \leq \epsilon$ for any $e \in E$.

Now recall the identity (Page 50)

$$f_n g_n - fg = (f_n - f)(g_n - g) + f(g_n - g) + g(f_n - f).$$

So, call the bounds on $\{f_n\}$ and $\{g_n\}$ F and G respectively. Now, fixing some $\epsilon > 0$ there must be values N, M, N', and M' so that $n > N \Rightarrow |f_n - f| \leq \sqrt{\epsilon/3}$, $n > M \Rightarrow |g_n - g| \leq \sqrt{\epsilon/3}$, $n > N' \Rightarrow |f_n - f| \leq \epsilon/(3G)$, and $n > M' \Rightarrow |g_n - g| \leq \epsilon/(3F)$. So taking $O = \max\{N, M, N', M'\}$ we see that $n > O \Rightarrow f_n g_n - fg = (f_n - f)(g_n - g) + f(g_n - g) + g(f_n - f) \leq \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon$. Thus we have shown that $\{f_n g_n\}$ converges uniformly as desired. □

2.2 Problem 3

2.2.1 Question

Construct sequences $\{f_n\}$, $\{g_n\}$ which converge uniformly on some set E, but such that $\{f_n g_n\}$ does not converge uniformly on E (of course, $\{f_n g_n\}$ must converge on E).

2.2.2 Answer

Let $\{f_n(x)\} = 1 + \frac{1}{n}$ and let $\{g_n(x)\}$ be the constant valued series with $\{g_n(x)\} = x$ for each n.

Clearly both of these series converge uniformly. The series $\{f_n\}$ converges (uniformly) to $f(x) = 1$. For any $\epsilon > 0$ take $1/\delta = N$ where $\delta \in \mathbb{Q}$ is some rational number expressible as $1/p$ $p \in \mathbb{N}$ which has $\delta < \epsilon$. We see that for each $n > N$ we have $|f_n(e) - 1| \leq \epsilon$ for any $e \in \mathbb{R}$. So $\{f_n\}$ converges uniformly.

Since the second sequence $\{g_n\}$ is constant valued it is uniformly convergent (to its value) for clearly $|g_n(x) - g(x)| = 0 < \epsilon$ for any $n \in \mathbb{N}$, $\epsilon > 0$ $\epsilon \in \mathbb{R}$ and $x \in \mathbb{R}$.

Now it remains only to show that the product $\{f_n g_n\} = x + \frac{x}{n}$ does not converge uniformly.

Proof. We see that this sequence converges pointwise to $h(x) = x$ as fixing x the sequence $x + \frac{x}{n}$ converges to x.

However this is not uniformly convergent, for suppose towards a contradiction that it is uniformly convergent. Then pick some $\epsilon > 0$. By assumption there is some N such that for each $n > N$ we have $|f_n(x) - x| \leq \epsilon$ so pick some $n_0 > N$. Now we see however that for $x = 2n\epsilon$ and so $|f_n(x) - x| = |x + \frac{x}{n} - x| = |\frac{x}{n}| = |2\epsilon| > \epsilon$. This is a contradiction. □