Homework

Frederick Robinson

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1 Problem 3

1.1 Question

Let f be a real-valued function of two variables (x, y) that is defined on the square $Q = \{(x, y) \mid 0 \le x \le 1, 0 \le y \le 1\}$ and is a measurable function of x for each fixed value of y. For each $(x, y) \in Q$, suppose that the partial derivatives $\partial f/\partial y$ exist. Suppose that there is a function g that is integrable over $[0, 1]$ and such that

$$\left|\frac{\partial f}{\partial y}(x, y)\right| \le g(x), \quad \forall (x, y) \in Q.$$

Prove that

$$\frac{\partial}{\partial y}\left[\int_0^1 f(x, y)dx\right] = \int_0^1 \frac{\partial f}{\partial y}(x, y)dx,$$

for all $y \in [0, 1]$.

1.2 Answer

Proof. Define the derivative at a point as the limit of a sequence of derivatives

$$\frac{\partial f}{\partial y}(x, y) = \lim_{n\to\infty} \frac{\partial f}{\partial y}(x, y - 2^{-n})$$

this sequence is dominated by g. We may therefore conclude by the dominated convergence theorem that

$$\frac{\partial}{\partial y}\left[\int_0^1 f(x, y)dx\right] = \int_0^1 \frac{\partial f}{\partial y}(x, y)dx,$$

as desired. □