Homework

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1 Problem 1

1.1 Question

Construct a measurable set E with $m(E) > 0$ such that for interval $I \subset \mathbb{R}$, we have $m(E \cap I) < m(I)$.

1.2 Answer

Fix some sequence of real numbers say $\{x_i\}$ with $x_i > 0$ and

$$\sum_{k=1}^{\infty} x_k = y < 1$$

Now, construct a set $\mathcal{C}_i$ by removing intervals (each of the same length) having a total length x_i from the middle of each interval in the set $\mathcal{C}_{i-1}$. We set $\mathcal{C}_0 = [0, 1]$. Define $\mathcal{C}$ by the limit of the sequence of $\mathcal{C}_i$.

The measure of $\mathcal{C}$ is clearly just $1 - y$. I claim that for any interval $I \subset \mathbb{R}$, we have $m(E \cap I) < m(I)$.

Proof. Clearly we may consider only intervals I which are subsets of the unit interval since if $I \cap [0, 1] \neq I$ the above property holds with no further verification necessary.

This established fix some $I = [a, b]$ for $0 \leq a \leq b \leq 1$. There is some subinterval of I say $[c, d]$ which is not present in E since for every $\mathcal{C}_i$ the longest interval completely contained in $\mathcal{C}_i$ has length strictly less than $1/2^i$. Thus $m(E \cap I) \leq (b - a) - (d - c) < b - a = m(I)$ and the property holds as desired. □

2 Problem 4

2.1 Question

A nonempty subset X of $\mathbb{R}$ is called *perfect* provided it is closed and each neighborhood of any point in X contains infinitely many points of X. 1) Prove that every perfect subset of $\mathbb{R}$ is uncountable. 2) Show that the middle-thirds Cantor set is perfect.

2.2 Answer

1. *Proof.* See Rudin Page 41

   Suppose X is countable, and denote the points of X by $x_1, x_2, x_3, \ldots$. No construct a sequence of neighborhoods as follows.

   Let V_1 be any neighborhood of x_1. If V_1 consists of all $y \in \mathbb{R}$ such that $|y - x_1| < r$, the closure $\bar{V_1}$ of V_1 is the set of all $y \in \mathbb{R}$ such that $|y - x_1| \leq r$.

   Suppose V_n has been constructed, so that $V_n \cap X$ is not empty. Since every point of X is a limit point of X, there is a neighborhood V_{n+1} such that $\bar{V}_{n+1} \subset V_n$, $x_n \notin \bar{V}_{n+1}$, and $V_{n+1} \cap X$ is nonempty. By this last property V_{n+1} satisfies our inductive hypothesis and we can continue the construction.

   Set $K_n - \bar{V}_n \cap X$. Since $\bar{V}_n$ is closed and bounded, $\bar{V}_n$ is compact. Since $x_n \notin K_{n+1}$, no point of X lies in $\bigcup_1^\infty K_n$. Since $K_n \subset X$, this implies that $\bigcup_1^\infty K_n$ is empty. But, each K_n is nonempty, and $K_n \supset K_{n+1}$, a contradiction since each K is compact.

   $\square$

2. *Proof.* The complement of the middle thirds cantor set is a countable union of open intervals. Thus, the complement is open and the Cantor set itself is closed.

   If we fix some point x in the Cantor set and some neighborhood of radius ϵ we notice that there are infinitely many "boundary points" within ϵ of x.

   Every time we remove an open interval from $B_\epsilon x$ we introduce at least one new boundary point into $B_\epsilon x$. Since these boundary points are all in the final Cantor set, and we remove infinitely many intervals from every neighborhood ϵ of x during the construction of the Cantor set we may conclude that the middle thirds Cantor set is perfect. $\square$