Homework

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1 Problem 3

1.1 Question

A set of real numbers is said to be a G_δ *set* if it is the countable intersection of open sets

Show that for any bounded set $E \subset \mathbb{R}$, there is a G_δ set for which

$$E \subseteq G \quad \text{and } m^*(G) = m^*(E).$$

1.2 Answer

Proof. By the definition of outer measure for any $\epsilon = 1/n$ there exists some open cover of E say G_n such that

$$m^*(E) < \sum_{I \in G_n} l(I) \leq m^*(E) + \epsilon$$

If we let $G = \bigcap_{i=1}^{\infty} G_i$ I claim that G is G_δ and moreover that it has $E \subseteq G$ and $m^*(G) = m^*(E)$.

It is clear that G is G_δ since it is defined as the intersection of countably many open covers. Moreover, since each G_n has E as a subset (by definition of cover) their intersection must also have $E \subseteq G$.

Finally, we must verify that $m^*(G) = m^*(E)$. Since $E \subseteq G$ we have by a proof in class that $m^*(G) \geq m^*(E)$. By definition of G we have that $G \subseteq G_n$ for all n. So, $m^*(G) \leq m^*(G_n)$. Therefore, since given δ there is some G_n with $m^*(G_n) - m^*(E) < \delta$ it follows that $m^*(G) \leq m^*(E)$. Hence $m^*(E) = m^*(G)$ as desired. $\square$

2 Problem 6

2.1 Question

Let $E \subset \mathbb{R}$ be a measureable set with $m(E) > 0$. Prove that for any $\alpha < 1$ there is an open interval $I \subset \mathbb{R}$ such that $m(E \cap I) > \alpha m(I)$.

2.2 Answer

Proof. First consider the case where $m(E) < \infty$. By definition of outer measure, for any $\epsilon > 0$ there exists some open cover of E say $\{A_k\}$ such that

$$m(E) \leq \sum_{k=1}^{\infty} l(A_k) < \epsilon + m(E)$$

Moreover, we can reduce any such open cover to a open cover by disjoint open sets $\{A'_k\}$ by taking $A'_1 = A_1$ and $A'_i = A_i \setminus \bigcup_{j=1}^{i-1} A_j$. Finally we can take each A'_i to be an interval since all open sets in $\mathbb{R}$ are countable unions of open intervals and countable collections of countable things are countable.

Suppose towards a contradiction that there exists $\alpha < 1$ such that $m(E \cap I) \leq \alpha m(I)$ for all open intervals I. Since each A'_i is an interval they are all measurable. Hence $E \cap A'_i$ is measurable for all i, and by the countable additivity of measurable sets we have

$$m(E) = \sum_{i=1}^{\infty} m(E \cap A'_i) \leq \alpha \sum_{i=1}^{\infty} m(A'_i) < \alpha(\epsilon + m(E))$$

Since $\epsilon > 0$ was arbitrary and $\alpha < 0$ this is a contradiction.

In the case where $m(E) = \infty$ consider the set $E'_x = E \cap (-x, x)$ for $x \in \mathbb{R}^+$. For any x this set is measurable since intervals are always measurable and the intersection of two measurable sets is measurable. Moreover, there must be some x such that $m(E'_x) \neq 0$ since the measure of E is nonzero. Fixing one such E'_x we can apply the proof from the first case to demonstrate that there exists some interval such that $m(E \cap I) > \alpha m(I)$ for any $\alpha < 1$. □

3 Problem 7

3.1 Question

Let $E \subset \mathbb{R}$ be a measurable set with $m(E) > 0$. Prove that the set

$$E - E = \{x - y : x, y \in E\}$$

contains an interval.

Hint: take $\alpha > 3/4$ and let I be as in the previous exercise. Then $E - E$ contains $(-\frac{1}{2}m(I), \frac{1}{2}m(I))$.

3.2 Answer

Proof. By the previous exercise there exists some interval $I = (a, b) \subset \mathbb{R}$ such that $m(I \cap E) > 3/4m(I)$. I claim that $(-\frac{1}{2}m(I), \frac{1}{2}m(I)) \subset E - E$. Assume towards a contradiction that there is some $x \in (-\frac{1}{2}m(I), \frac{1}{2}m(I))$ such that $x \notin E - E$. Then for any $y \in E \cap I$ we have $y - x \notin E \cap I$ and $y + x \notin E \cap I$.

Since the measure is invariant under translation we know $m(I \cap E + x) = m(I \cap E) > 3/4m(I)$. These are disjoint sets, each of which is measurable, so $m(I \cap E + x) + m(I \cap E) = m((I \cap E + x) \cup (I \cap E)) > 3/2m(I) = 3/2(b - a)$. Yet $(I \cap E + x) \cup (I \cap E) \subset (a, b + x)$ and $m(a, b + x) = b + x - a < 3/2(b - a)$. Contradiction. $\square$