Homework

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1 Chapter 7

1.1 Problem 9

1.1.1 Question

Let $\{f_n\}$ be a sequence of continuous functions which converges uniformly to a function f on a set E. Prove that

$$\lim_{n\to\infty} f_n(x_n) = f(x)$$

for every sequence of points $x_n \in E$ such that $x_n \to x$, and $x \in E$. Is the converse of this true?

1.1.2 Answer

Proof. We need to show that, given ϵ there exists some N such that $n > N$ implies that $|f_n(x_n) - f(x)| < \epsilon$. So fix some ϵ. Since we know that each f_n is continuous and that the sequence of $\{f_n\}$ converges uniformly we must have f continuous by Theorem 7.12.

By continuity of f there is some δ such that $f(B_\delta x) \subset B_{\epsilon/2} f(x)$. Also, by convergence of x_n to x we know that there exists some M_1 for which $n > M_1$ gives $x \in B_\delta x$. Further, since $\{f_n\}$ converges uniformly to f there is some M_2 such that $n > M_2$ gives $|f_n(x) - f(x)| < \epsilon/2$ for any x.

Taking $N = \max(M_1, M_2)$ we get, by the triangle inequality each $f_n(x_n)$ has at most $d(f_n(x_n), f(x)) = \epsilon$. So, we have established

$$\lim_{n\to\infty} f_n(x_n) = f(x)$$

as desired. □

The converse is not true. Consider the following counterexample:

$$f_n(x) = e^{x/n}.$$

As has been proven previously this sequence of continuous functions converges pointwise to $f(x) = 1$ although it does not converge uniformly. Moreover, it is easy to see that given any sequence of points $x_n \in E$ such that $x_n \to x$, and $x \in E$ we get

$$\lim_{n\to\infty} f_n(x_n) = 1.$$

By convergence of x_n to x we know that there exists N with the property that each $n \geq N$ has $d(x_n, x) < \epsilon$. Next observe that each f_n is strictly increasing. So, since $f_n(x)$ converges to 1 pointwise in particular we have

$$\lim_{n\to\infty} f_n(x - \epsilon) = 1 \quad \text{and} \quad \lim_{n\to\infty} f_n(x + \epsilon) = 1.$$

That is, there exist M_1 and M_2 such that $m > M_1$ implies $d(f_m(x-\epsilon), 1) < \epsilon$ and $d(f_m(x + \epsilon), 1) < \epsilon$. Since we have increasing functions it must be that for $M = \max(M_1, M_2)$ then $m > M$ implies $d(f_m(y), 1) < \epsilon$ for all $m > M$ and given $y \in B_\epsilon x$.

Thus, we conclude that for each $n > \max(M, N)$ we have $d(f_n(x_n), 1) < \epsilon$, that is

$$\lim_{n\to\infty} f_n(x_n) = 1$$

as claimed.

1.2 Problem 12

1.2.1 Question

Suppose that g and $f_n (n = 1, 2, 3, \ldots)$ are defined on $(0, \infty)$, are Riemann-integrable on $[t, T]$ whenever $0 < t < T < \infty$, $|f_n| \leq g$, $f_n \to f$ uniformly on ever compact subset of $(0, \infty)$, and

$$\int_0^\infty g(x)dx < \infty$$

Prove that

$$\lim_{n\to\infty} \int_0^\infty f_n(x)dx = \int_0^\infty f(x)dx.$$

(See Exercises 7 and 8 of Chap. 6 for the relevant definitions.)

This is a rather weak form of Lebesgue's dominated convergence theorem (Theorem 11.32). Even in the context of the Riemann integral, uniform convergence can be replaced by pointwise convergence if it is assumed that $f \in \mathcal{R}$. (See the articles by F. Cunningham in *Math. Mag.*, vol. 40, 1967, pp. 179-186, and by H. Kestelman in *Amer. Math. Monthly*, vol. 77, 1970, pp. 182-187.)

1.2.2 Answer

We know that on any $[t, T]$ the function f is Riemann-integrable since each f_n is Riemann-integrable on every such interval, and $\{f_n\}$ converges uniformly to f. In particular we have on any $[t, T]$

$$\lim_{n\to\infty} \int_t^T f_n(x)dx = \int_t^T f(x)dx$$

by Theorem 2.16.

Also, we know that for a given x we have $|f(x)| \leq g(x)$ since each f_n has $|f_n| \leq g$. Thence, for any t, T we have

$$\int_t^T g(x)dx \geq \int_t^T |f(x)|dx,$$

and since

$$\lim_{t\to 0} \int_t^T g(x)dx = \int_0^T g(x)dx$$

exists so must

$$\int_0^T f(x)dx$$

as g is greater or equal in absolute value to f. Furthermore,

$$\lim_{T\to\infty} \int_0^T g(x)dx$$

exists which implies that

$$\int_0^\infty f(x)dx$$

does as well.

We have established the existence of the integral and the fact that it is given by

$$\lim_{n\to\infty}\int_0^\infty f_n(x)dx = \int_0^\infty f(x)dx$$

just follows from the previously mentioned fact that

$$\lim_{n\to\infty}\int_t^T f_n(x)dx = \int_t^T f(x)dx$$

on any interval $[t, T]$ since the improper integral is just defined as a limit of such integrals. That is, since each term of the limit is equal, so must be the limit itself.

2 Chapter 8

2.1 Problem 6

2.1.1 Question

Suppose $f(x)f(y) = f(x+y)$ for all real x and y.

1. Assuming that f is differentiable and not zero, prove that

   $$f(x) = e^{cx}$$

   where c is constant.

2. Prove the same thing, assuming only that f is continuous.

2.1.2 Answer

1. Differentiability implies continuity. See Part 2.

2. It follows by induction that $f(x)^n = f(x\cdot n)$ for any $n \in \mathbb{N}$. Thus, we observe in particular that given $q \in \mathbb{N}$ we have

   $$f\left(\frac{1}{q}\right)^q = f(1) \Rightarrow q\log f\left(\frac{1}{q}\right) = \log f(1).$$

   If we call $f(1) = e^c$ we get for any $q \in \mathbb{N}$

   $$\log f\left(\frac{1}{q}\right) = \frac{1}{q}\log e^c = \frac{c}{q}$$

which implies in particular that for any such $1/q$

$$f\left(\frac{1}{q}\right) = e^{c/q}.$$

More generally we see that for all positive rational number $p/q \in \mathbb{Q}^+$ we have

$$f\left(\frac{p}{q}\right) = f\left(\underbrace{\frac{1}{q} + \cdots + \frac{1}{q}}_{p \text{ times}}\right) = f\left(\frac{1}{p}\right)^p = \left(e^{c/q}\right)^p = e^{c(p/q)}.$$

Since we know that f is continuous and for a dense subset of $\mathbb{R}^+$ it is equivalent to e^{cx} then it must in fact be equal to e^{cx} on all of $\mathbb{R}^+$ by a previous proof. Moreover we have

$$f(-x)f(x) = f(0) = 1$$

So for $x > 0$ we just get

$$f(-x) = (e^{cx})^{-1} = e^{-cx}$$

and $f(x) = e^{cx}$ on $\mathbb{R}^-$ as well. Finally, we must prove that $f(0) = 1$ and that $f(1) \neq 0$.

The first is easy to see, for $f(0)f(x) = f(x) \Rightarrow f(0) = 0$ or $f(0) = 1$. However, if the former is the case then $f(x) = 0$ for all x, a contradiction.

The latter too is fairly clear, since $f(1) = 0$ would imply $f(x) = 0$ for each x by an argument identical to the one used above to establish $f(x) = e^{cx}$.