Homework

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1 Problem 1

1.1 Question

Let $\{E_k\}_{k=1}^{\infty}$ be a countable disjoint collection of measurable sets. Prove that for any set A.

$$m^*\left(A \cap \bigcup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} m^*(A \cap E_k).$$

1.2 Answer

Proof. We will begin by assuming that both sides of our (alleged) equality are finite. By countable subadditivity we have

$$m^*\left(A \cap \bigcup_{k=1}^{\infty} E_k\right) \leq \sum_{k=1}^{\infty} m^*(A \cap E_k).$$

In order to prove the reverse inequality we will demonstrate in particular that given $\epsilon > 0$

$$m^*\left(A \cap \bigcup_{k=1}^{\infty} E_k\right) + \epsilon \geq \sum_{k=1}^{\infty} m^*(A \cap E_k).$$

Recall now that Proposition 6 (Page 36) of the book establishes the finite case of this proposition. In particular we have

$$m^*\left(A \cap \bigcup_{k=1}^{n} E_k\right) = \sum_{k=1}^{n} m^*(A \cap E_k).$$

Furthermore, given any $\epsilon > 0$ there exists some $n \in \mathbb{N}$ such that

$$\sum_{k=1}^{\infty} m^*(A \cap E_k) \leq \sum_{k=1}^{n} m^*(A \cap E_k) + \epsilon.$$

Taking these together we see that for any $\epsilon > 0$

$$\begin{aligned}\sum_{k=1}^{\infty} m^*(A \cap E_k) &\leq \sum_{k=1}^{n} m^*(A \cap E_k) + \epsilon \\ &= m^*\left(A \cap \bigcup_{k=1}^{n} E_k\right) + \epsilon \leq m^*\left(A \cap \bigcup_{k=1}^{\infty} E_k\right) + \epsilon\end{aligned}$$

$$\Rightarrow m^*\left(A \cap \bigcup_{k=1}^{\infty} E_k\right) + \epsilon \geq \sum_{k=1}^{\infty} m^*(A \cap E_k)$$

as desired.

Lastly we must consider the nonfinite cases. If either $m^*\left(A \cap \bigcup_{k=1}^{\infty} E_k\right)$ or $\sum_{k=1}^{\infty} m^*(A \cap E_k)$ is nonfinite then so is the other. In particular,

$$m^*\left(A \cap \bigcup_{k=1}^{\infty} E_k\right) = \infty \Rightarrow \sum_{k=1}^{\infty} m^*(A \cap E_k) = \infty$$

by subadditivity and

$$\sum_{k=1}^{\infty} m^*(A \cap E_k) = \infty \Rightarrow m^*\left(A \cap \bigcup_{k=1}^{\infty} E_k\right) = \infty$$

since

$$\sum_{k=1}^{\infty} m^*(A \cap E_k) = \infty$$

implies that given any N there is some corresponding m such that

$$\sum_{k=1}^{m} m^*(A \cap E_k) = N = m^*\left(A \cap \bigcup_{k=1}^{m} E_k\right)$$

so

$$m^*\left(A \cap \bigcup_{k=1}^{\infty} E_k\right) = \infty$$

as desired. $\square$

2 Problem 3

2.1 Question

For $\epsilon > 0$ and $n \in \mathbb{N}$, let $\mathcal{D}_{\epsilon,N}$ be the set of real numbers $x \in [0,1]$ with the following property: for every integer $q \geq N$ and for every integer $0 \leq p \leq q$:

$$\left| x - \frac{p}{q} \right| \geq \frac{1}{q^{2+\epsilon}}.$$

The number $x \in [0,1]$ is called a *Diophantine number* if $x \in \mathcal{D}_{\epsilon,N}$, for some ϵ, N. A number that is not Diophantine is called *Liouville*. Liouville numbers are well-approximable by rational number, and Diophantine numbers are not.

Prove that for every $\epsilon > 0$, the set

$$\bigcup_{N=1}^{\infty} \mathcal{D}_{\epsilon,N}$$

has Lebesgue measure 1 in $[0,1]$ (that is, its complement has measure 0). Conclude that for almost every $x \in [0,1]$ and for every $\epsilon > 0$, there exists a $C > 0$ such that

$$\left| x - \frac{p}{q} \right| \geq \frac{C}{q^{2+\epsilon}},$$

for all $p, q \in \mathbb{N}$. In particular, the set of Diophantine numbers has full measure (though this statement is even stronger than that).

2.2 Answer

Proof. Fixing ϵ we have that $x \notin \mathcal{K}_\epsilon = [0,1] \cap \bigcup_{N=1}^{\infty} \mathcal{D}_{\epsilon,N}$ if and only if for each $q, p \in \mathbb{N}$ with $p \leq q$

$$\left| x - \frac{p}{q} \right| < \frac{1}{q^{2+\epsilon}}.$$

Thus we can cover the complement $\mathcal{K}_\epsilon^C \cap [0,1]$ by a finite union of open balls defined as follows for some n:

$$\bigcup_{k=1}^{n} B_{\frac{1}{n^{2+\epsilon}}} \frac{k}{n}$$

So, in particular $\mathcal{K}_\epsilon^C \cap [0,1]$ has outer measure less than or equal to

$$n\frac{1}{n^{2+\epsilon}} = \frac{1}{n^{1+\epsilon}}$$

for some choice of n. However, since we can choose n to be arbitrarily large, this quantity can be as small as desired, and the outer measure of $\mathcal{K}_\epsilon^C \cap [0,1] = 0$ as desired.

Now we may conclude that for almost every $x \in [0,1]$ and for every $\epsilon > 0$, there exists a $C > 0$ such that

$$\left| x - \frac{p}{q} \right| \geq \frac{C}{q^{2+\epsilon}},$$

for all $p, q \in \mathbb{N}$. □

3 Problem 5

3.1 Question

Let E be a measurable set, and let N be the nonmeasurable set constructed in class. Prove that if $E \subset N$, then $m(E) = 0$.

3.2 Answer

Proof. Suppose towards a contradiction that there exists some measurable $E \subset N$ with nonzero measure. Then consider

$$F = \bigcup_{q \in [0,1] \cap \mathbb{Q}} E + q$$

However,

$$0 < m(E) \leq 1 \quad \text{and} \quad 0 < m(F) \leq 2$$

by monotonicity of measure, assumption. However this is a contradiction to the countable additivity of measure for measurable sets. □