Homework

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1 Chapter 9

1.1 Problem 7

1.1.1 Question

Suppose that f is a real-valued function defined in an open set $E \subset \mathbb{R}^n$, and that the partial derivatives $D_1 f, \ldots, D_n f$ are bounded in E. Prove that f is continuous in E.

Hint: Proceed as in the proof of Theorem 9.21.

1.1.2 Answer

Proof. This is very similar to Theorem 9.21. If we repeat Rudin's argument from that theorem we get

$$f(x+h) - f(x) = \sum_{j=1}^{n} [f(x+v_j) - f(x+v_{j-1})]$$

given $x + h \in E$ and $f(x+v_j) - f(x+v_{j-1}) = h_j D_j f(x + v_{j-1} + \theta_j h_j e_j)$ with $\theta \in (0,1)$.

From the problem we have that there exists some $M > 0$ such that $|D_j f(x)| \leq M$ for any $j = 1, \ldots, n$ and any $x \in E$. Thus

$$|f(x+v_j) - f(x+v_{j-1})| \leq M|h_j|$$

for any j. The triangle inequality gives us

$$|f(x+h) - f(x)| \leq \sum_{j=1}^{n} M|h_j| = M \sum_{j=1}^{n} |h_j|.$$

$$h \to 0 \Rightarrow \sum_{j=1}^{n} |j_j| \to 0$$

and

$$|f(x+h) - f(x)| \to 0$$

demonstrating that f is continuous at x as desired. □

1.2 Problem 9

1.2.1 Question

If f is a differentiable mapping of a *connected* open set $E \subset \mathbb{R}^n$ into $\mathbb{R}^m$, and if $f'(x) = 0$ for every $x \in E$, prove that f is constant in E.

1.2.2 Answer

I will first demonstrate that f is constant on any convex open neighborhood in E. Then I will show that the connectedness of E suffices to extend this result to all of E.

Proof. Theorem 9.19 states that if f maps a convex open set $E \subset \mathbb{R}^n$ into $\mathbb{R}^m$, f is differentiable in E and there is a real number M such that

$$||f'(x)|| \leq M$$

for every $x \in E$ then

$$|f(b) - f(a)| \leq M|b - a|$$

for all a, b in E.

So by this theorem f is constant on each convex subset of E. □

Now it remains only to demonstrate that this result extends to connected regions.

Proof. Open balls are convex sets. Thus, the subset of E on which f takes some value c is open if it is nonempty since for any point in this set there exists some open ball surrounding it in E (by openness of E) and on that ball the function takes value c (by convexity of balls). Similarly the subset of E where f takes any value $d \neq c$ is open.

Now note that the intersection of these sets must be empty since the function f is well defined. Also, one of these sets must me nonempty.

But, as the space E is connected one of these sets must be empty for any choice of constant c. Otherwise we could write E as their disjoint union. We therefore observe that f is constant valued over all E since, by well definition of f there exists some c for which the set of all points on which f evaluates to c is nonempty (just take $c = f(x)$ for some arbitrary $x \in E$). $\square$

1.3 Problem 10

1.3.1 Question

If f is a real function defined in a convex open set $E \subset \mathbb{R}^n$, such that $(D_1 f)(x) = 0$ for every $x \in E$, prove that $f(x)$ depends only on $x_2, \ldots, x_n$.

Show that the convexity of E can be replaced by a weaker condition, but that some condition is required. For example, if $n = 2$ and E is shaped like a horseshoe the statement may be false.

1.3.2 Answer

Proof. If we restrict our function f to only take points which have form $(x, \overline{x_2}, \overline{x_3}, \ldots, \overline{x_n}) \in E$ for arbitrary x and fixed other variables we obtain a function $g : \mathbb{R}^n \to \mathbb{R}^m$ which depends only on the value of x_1. We can make this into a function in one variable say $g'(x_1) = g(x_1, \overline{x_2}, \overline{x_3}, \ldots, \overline{x_n}))$. The partial derivative with respect to x_1 of this restriction g is still just 0 and furthermore the derivative of g' is 0 as well.

Note now that since the intersection of two convex sets is convex, the subset of E (a convex set) which intersects the convex subset of $\mathbb{R}^n$ given by $x_2 = \overline{x_2}, x_3 = \overline{x_3}, \ldots, x_n = \overline{x_n}$ is itself convex. This is the set on which g' is defined. Thus, we can apply Theorem 9.19 as in the above proof. Doing so we realize that g is a constant valued function with respect to x_1. Thus, we can surmise that the original function f did not depend on the value of x_1, only on the values of $x_2, x_3, \ldots, x_n$. $\square$

In the proof above the only time we used convexity of E was to show that the set on which g' was defined was convex, in order to apply Theorem 9.19. We can weaken this condition however, admitting all sets E such that all subsets say $K = E \cap \{(x_1, x_2, \ldots, x_n) | x_2 = \overline{x_2}, x_3 = \overline{x_3}, \ldots, x_n = \overline{x_n}\}$ are connected. Then, we can apply our result from the previous exercise in a similar manner to how we apply 9.19 in the above proof. The rest of the proof for this more general result follows similarly.