Homework

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1 Chapter 7

1.1 Problem 6

1.1.1 Question

Prove that the series

$$\sum_{n=1}^{\infty}(-1)^n \frac{x^2+n}{n^2}$$

converges uniformly in every bounded interval, but does not converge absolutely for any value of x.

1.1.2 Answer

The above series converges uniformly on every bounded interval $[a, b]$

Proof.

$$\sum_{n=1}^{\infty}(-1)^n \frac{x^2+n}{n^2} = \sum_{n=1}^{\infty}\left((-1)^n \frac{x^2}{n^2} + \frac{(-1)^n}{n}\right) = \sum_{n=1}^{\infty}(-1)^n \frac{x^2}{n^2} + \sum_{n=1}^{\infty}\frac{(-1)^n}{n}$$

Now we just observe that we know already the alternating harmonic series

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{n}$$

converges. So, taken as a series of functions this converges uniformly. (*Proof:* Weierstrass M-Test with itself.)

It is moreover easy to see that on a bounded interval $[a, b]$ the series

$$\left|(-1)^n \frac{x^2}{n^2}\right| \leq \frac{b^2}{n^2}$$

So this component converges uniformly by the Weierstrass M-Test.

Since each component taken individually converges uniformly it must be that the entire sequence converges uniformly, and we have shown that

$$\sum_{n=1}^{\infty} (-1)^n \frac{x^2 + n}{n^2}$$

converges uniformly, as desired. □

Now we will show that the above sum does not converge absolutely for any value of x.

Proof. If we fix a value of x the absolute value of the sum becomes

$$\sum_{n=1}^{\infty} \left(\frac{x^2}{n^2} + \frac{1}{n}\right)$$

but of course, for any value of x, n we have

$$\left(\frac{x^2}{n^2} + \frac{1}{n}\right) \geq \frac{1}{n}$$

since x^2 and n^2 are both positive.

Hence, the sum

$$\sum_{n=1}^{\infty} \left(\frac{x^2}{n^2} + \frac{1}{n}\right)$$

diverges by comparison with

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

□

1.2 Problem 7

1.2.1 Question

For $n = 1, 2, 3, \ldots$, x real, put

$$f_n(x) = \frac{x}{1 + nx^2}.$$

Show that $\{f_n\}$ converges uniformly to a function f, and that the equation

$$f'(x) = \lim_{n\to\infty} f_n'(x)$$

is correct if $x \neq 0$, but false if $x = 0$.

1.2.2 Answer

I claim that $\{f_n\}$ converges uniformly to $f(x) = 0$

Proof. First we establish that $\{f_n\}$ converges pointwise to 0. Fixing an x we see that

$$\left|\frac{x}{1 + nx^2}\right| \leq \left|\frac{x}{nx^2}\right| \leq \left|\frac{1}{x}\right| \frac{1}{n}$$

and so by comparison test we have pointwise convergence to 0.

Now, for a fixed n we observe that $\lim_{x\to\infty} f_n(x) = 0$. Since

$$\left|\frac{x}{1 + nx^2}\right| \leq \left|\frac{x}{nx^2}\right| = \frac{1}{n}\left|\frac{1}{x}\right|$$

and for fixed n this last value clearly goes to 0 as $|x| \to \infty$.

Since this is the case and each f_n is smooth the extrema of some f_n must occur at a point where the derivative is 0. In particular

$$\frac{d}{dx}\left(\frac{x}{1 + nx^2}\right) = \frac{1 - nx^2}{(1 + nx^2)^2}$$

and

$$\frac{1 - nx^2}{(1 + nx^2)^2} = 0 \Leftrightarrow x = \pm\frac{1}{\sqrt{n}}.$$

Evaluating at these points reveals

$$f_n\left(\pm\frac{1}{\sqrt{n}}\right) = \pm\frac{1}{2\sqrt{n}}.$$

Since these are the extrema of the function we have that

$$\sup |f_n(x) - f(x)| = \frac{1}{2\sqrt{n}}.$$

Since $\lim_{n\to\infty}(2\sqrt{n})^{-1} = 0$ we have that $\{f_n\}$ converges uniformly by Theorem 7.9.

□

1.3 Problem 10

1.3.1 Question

Letting (x) denote the fractional part of the real number x (see Exercise 16, Chap. 4, for the definition), consider the function

$$f(x) = \sum_{n=1}^{\infty} \frac{(nx)}{n^2} \quad (x \text{ real}).$$

Find all discontinuities of f, and show that they form a countable dense set. Show that f is nevertheless Riemann-integrable on every bounded interval.

1.3.2 Answer

It will be useful to have a minor extension of Theorem 7.11

Lemma 1. *If $f_n \to f$ uniformly on a set $E \subset X$ and x is a limit point of E such that*

$$\lim_{t\to x^+} f_n(t) = A_n$$

then

$$\lim_{t\to x^+} f(t) = \lim_{n\to\infty} A_n$$

and similarly for left handed limits.

Proof. Construct a new series of functions say g_n from f_n defined by

$$g_n(t) = \begin{cases} A_n & t < x \\ f_n(t) & \text{for } t > x \end{cases}$$

The right handed limit at x depends only on the function for values greater than x, so

$$\lim_{t\to x^+} f(t) = \lim_{t\to x^+} g(t).$$

However, since f_n converges uniformly, and g_n is the same as f_n on $t > x$ it converges uniformly for such values. Moreover it converges uniformly for values $t < x$ since A_n converges by uniform convergence of f_n.

Since we set the value of g_n to be $\lim_{t\to x^+} g(t)$ on $t < x$ we have $\lim_{t\to x^+} g(t) = \lim_{t\to x^-} g(t) = \lim_{t\to x} g(t)$. By Theorem 7.11 $\lim_{t\to x} g(t)$ exists and is $\lim_{n\to\infty} A_n$. Hence, so does $\lim_{t\to x^+} f(t)$ and it is exactly $\lim_{n\to\infty} A_n$ as claimed.

The proof for left handed limits follows similarly. □

For convenience we say

$$f_n(x) = \frac{(nx)}{n^2} \quad \text{and} \quad F_N(x) = \sum_{n=1}^{N} f_n(x)$$

We observe that $\{F_n\}$ converges uniformly since,

$$\left|\frac{(nx)}{n^2}\right| \leq \frac{1}{n^2}.$$

So, as

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

converges, $\{F_n\}$ converges uniformly by Weierstrass M-Test.

I claim that f is discontinuous precisely on $\mathbb{Q}\backslash\{0\}$.

Proof. We will prove this for $x \in \mathbb{R}^+$ only, however the proof follows almost identically for $x \in \mathbb{R}^-$ and 0.

For any $m \in \mathbb{N}$ we know that on $[m/n, (m+1)/n)$

$$f_n(x) = \frac{(nx)}{n^2} = \frac{nx - m}{n^2}.$$

(This follows directly from the definition of the fractional part of a real number.) Thus, it is clear that either

$$\lim_{t\to x^-} f_n(t) = f_n(x) = \lim_{t\to x^+} f_n(t) \quad \text{or} \quad \lim_{t\to x^-} f_n(t) = \frac{1}{n^2} \neq 0 = \lim_{t\to x^+} f_n(t)$$

with the first being the case when $x \neq m/n$ and the second occurring when $x = m/n$.

Since the sum of the limit of two functions is the limit of their sum each F_n must also have property

$$\lim_{t\to x^-} F_n(t) = F_n(x) = \lim_{t\to x^+} F_n(t) \quad \text{for } x\cdot m \notin \mathbb{N}\ m \le n$$

$$\lim_{t\to x^-} F_n(t) - \lim_{t\to x^+} F_n(t) = \sum_{\substack{x\cdot m\in\mathbb{N}\\ m\le n}} \frac{1}{m^2} \neq 0 \quad \text{for } x\cdot m \in \mathbb{N}\ m \le n \tag{1}$$

by induction.

So, as we have already proven uniform convergence, Theorem 7.11 implies that for $x \in \mathbb{R}\backslash\mathbb{Q}$

$$\lim_{n\to\infty} F_n(x) = f(x).$$

This however is just the definition of continuity at x. Moreover by Lemma 1 Fact 1 implies that for all $x \in \mathbb{Q}$

$$\lim_{t\to x^-} f(t) - \lim_{t\to x^+} f(t) \neq 0.$$

Hence, f is discontinuous precisely on $\mathbb{Q}$ as claimed. □

The function f is Riemann-integrable in every bounded interval since, each $f_n(x)$ has only finitely many discontinuities in each bounded interval and is therefore Riemann-integrable. Each member of the sequence of partial sums $\{F_n\}$ is therefore also Riemann-integrable, being a finite sum of integrable functions. So, since $F_n(x)$ converges uniformly its limit is integrable as well by Theorem 7.16.