Homework

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1 Problem 2

1.1 Question

Suppose f is a real-valued function on $\mathbb{R}$ such that $f^{-1}(c)$ is measurable for each $c \in \mathbb{R}$. Is f necessarily measurable?

1.2 Answer

No. Consider the function $f : [0, 1] \to [0, 2]$ defined by

$$f(x) = \begin{cases} x & x \notin \mathcal{V} \\ x + 1 & x \in \mathcal{V} \end{cases}$$

For $\mathcal{V}$ the Vitali set. Since this function is injective we have $f^{-1}(c)$ measurable for each $c \in \mathbb{R}$. However, taking $f^{-1}([1, 2])$ we recover the Vitali set, so the function is not measurable.

2 Problem 4

2.1 Question

Let $\{f_n\}$ be a sequence of measurable functions defined on a measurable set E. Define E_0 to be the set of points x in E at which $\{f_n(x)\}$ converges. Is the set E_0 measurable?

2.2 Answer

Yes.

Proof. By the Cauchy criterion the set of points in E at which $\{f_n(x)\}$ converges say X is precisely the set of all x such that for any $\epsilon > 0$ there exists an N which has $|f_n(x) - f_m(x)| < \epsilon$ given $m, n > N$. Moreover the function $g_{n,m} = |f_n - f_m|$ is measurable for any n, m since it is the difference of two measurable functions composed with the absolute value function which is continuous.

This established we note that we can write

$$E_0 = \bigcap_{\epsilon\in\mathbb{Q}} \bigcup_{N\in\mathbb{N}} \bigcap_{m,n>N} g_{m,n}^{-1}\left([0, \epsilon)\right)$$

so, E_0 is measurable as claimed. □

3 Problem 8

3.1 Question

(Dinis theorem) Let $\{f_n\}$ be an increasing sequence of continuous functions on $[a, b]$ that converges pointwise on $[a, b]$ to the continuous function f on $[a, b]$. Show that the convergence is actually uniform on $[a, b]$.

(Hint: let $\epsilon > 0$. For each integer $n > 0$, define $E_n = \{x \in [a, b] : f(x) - f_n(x) < \epsilon\}$. Show $\{E_n\}$ is an open cover and use compactness of $[a, b]$.)

3.2 Answer

Proof. Let $\epsilon > 0$ and define $E_n = \{x \in [a, b] : f(x) - f_n(x) < \epsilon\}$. Clearly $\{E_n\}$ covers $[a, b]$ since by definition of (pointwise) convergence, given $\epsilon > 0, x$ there exists an n such that $|f_n(x) - f(x)| < \epsilon$ and by the increasing property of this sequence $f(x) - f_n(x) \geq 0$. Furthermore each E_n is open since it is the preimage of the open set $(-1, \epsilon)$ under the continuous function $f(x) - f_n(x)$.

Now, by compactness of $[a, b]$ this open cover has a finite subcover. Moreover, since the sequence is increasing we have $E_n \subseteq E_{n+1}$. Thus, there must exist some n such that $E_n = [a, b]$. Since ϵ was arbitrary this suffices to prove uniform convergence. □

4 Problem 9

4.1 Question

Let I be an interval and let $f : I \to \mathbb{R}$ be nondecreasing. Show that f is measurable by first showing that for every integer $n > 0$, the function $x \mapsto f(x) + x/n$ is measurable.

4.2 Answer

Proof. Every $g_n = f(x) + x/n$ is strictly increasing (and thus injective). Therefore if we take $k = \inf f(x) \geq c$ then $g_n^{-1}((c, \infty)) = (g_n^{-1}(k), \infty)$ or $g_n^{-1}((c, \infty)) = [g_n^{-1}(k), \infty)$ and g_n is measurable. So, since $h_n = x/n$ is measurable $f = g_n - h_n$ is measurable, as desired. $\square$