Homework

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1 Chapter 5

1.1 Problem 6

1.1.1 Question

Suppose

1. f is continuous for $x \geq 0$
2. $f'(x)$ exists for $x > 0$
3. $f(0) = 0$
4. f' is monotonically increasing

Put

$$g(x) = \frac{f(x)}{x} \quad (x > 0)$$

and prove that g is monotonically increasing.

1.1.2 Answer

Proof. We will compute the derivative and show that it is always $g'(x) \geq 0$. The derivative of $g(x)$ is given by the quotient rule to be

$$g'(x) = \frac{xf'(x) - f(x)}{x^2}$$

The denominator is clearly always positive as we know it is the square of some real number $x > 0$, so in order to show that the entire derivative is always positive we need to show that the numerator is positive.

However,

$$\frac{f(x)}{x} \leq f'(x) \Leftrightarrow xf'(x) - f(x) \geq 0$$

and we know that, by the mean value theorem there is some $c \in (0, x)$ such that $\frac{f(x)}{x} = f'(x)$ since $f(0) = 0$. Since f is monotonically increasing it must be that $f'(x) \geq \frac{f(x)}{x}$.

Thus, $xf'(x) - f(x) \geq 0$ and so the derivative of g is positive, which implies that g is monotonically increasing.

□

1.2 Problem 11

1.2.1 Question

Suppose that f is defined in a neighborhood of x, and suppose that $f''(x)$ exists. Show that

$$\lim_{h \to 0} \frac{f(x+h) + f(x-h) - 2f(x)}{h^2} = f''(x).$$

Show by an example that the limit may exist even if $f''(x)$ does not.

Hint: Use Theorem 5.13.

1.2.2 Answer

Theorem 5.13 (L'Hopital's Rule)

Suppose f and g are real and differentiable in (a, b), and $g'(x) \neq 0$ for all $x \in (a, b)$ where $-\infty \leq a < b \leq +\infty$. Suppose

$$\frac{f'(x)}{g'(x)} \to A \textit{ as } x \to a.$$

If

$$f(x) \to 0 \textit{ and } g(x) \to 0 \textit{ as } x \to a,$$

or if

$$g(x) \to +\infty \textit{ as } x \to a,$$

then

$$\frac{f(x)}{g(x)} \to A \text{ as } x \to a.$$

Proof. Since the denominator of

$$\lim_{h\to 0} \frac{f(x+h)+f(x-h)-2f(x)}{h^2}$$

goes to zero as $h \to 0$ we can employ L'Hopital's rule. Using this we get

$$\lim_{h\to 0} \frac{f(x+h)+f(x-h)-2f(x)}{h^2} = \lim_{h\to 0} \frac{f'(x+h)-f'(x-h)}{2h}$$

Since the denominator is again zero we apply L'Hopital's rule again

$$= \lim_{h\to 0} \frac{f''(x+h)+f''(x-h)}{2}$$
$$= f''(x)$$

□

An example of a function that is not differentiable, but where this limit exists is

$$f(x) = \begin{cases} 0 & x \le 0 \\ x^2 & x > 0 \end{cases}$$

The second derivative of this function does not exist at zero, as

$$f'(x) = \begin{cases} 0 & x \le 0 \\ 2x & x > 0 \end{cases}$$

however we can compute the above limit at zero by

$$\lim_{h\to 0} \frac{f(x+h)+f(x-h)-2f(x)}{h^2} = \lim_{h\to 0} \frac{h^2}{h^2} = 1$$

1.3 Problem 14

1.3.1 Question

Let f be a differentiable real function defined in (a,b). Prove that f is convex if and only if f' is monotonically increasing. Assume next that $f''(x)$ exists for every $x \in (a,b)$, and prove that f is convex if and only if $f''(x) \ge 0$ for all $x \in (a,b)$.

1.3.2 Answer

A function f is said to be convex if

$$f(\lambda + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)$$

First we shall prove ($\Rightarrow$) that f' monotonically increasing implies f is convex.

Proof. Assume f' is monotonically increasing. Let $x < y < z$ for $x, y, z \in (a, b)$. The mean value theorem implies that $\exists c \in (x, y)$ such that $f'(c) = \frac{f(y)-f(x)}{y-x}$. Similarly $\exists d \in (y, z)$ such that $f'(d) = \frac{f(z)-f(y)}{z-y}$. Thus, since f' is assumed to be monotonically increasing we have

$$f'(c) \leq f'(d) \Rightarrow \frac{f(y)-f(x)}{y-x} \leq \frac{f(z)-f(y)}{z-y}$$

Because $y \in (x, z)$ we know that $y = \lambda x + (1-\lambda)z$ for some $\lambda \in (0, 1)$. Moreover, we have

$$\frac{f(y)-f(x)}{y-x} \leq \frac{f(z)-f(y)}{z-y}$$

$$\Rightarrow \frac{f(\lambda x + (1-\lambda)z) - f(x)}{\lambda x + (1-\lambda)z - x} \leq \frac{f(z) - f(\lambda x + (1-\lambda)z)}{z - \lambda x - (1-\lambda)z}$$

$$\Rightarrow \frac{f(\lambda x + z - z\lambda) - f(x)}{\lambda x + z - \lambda z - x} \leq \frac{f(z) - f(\lambda x + z - \lambda z)}{z - \lambda x - z + \lambda z}$$

$$\Rightarrow \frac{f(\lambda x + z - z\lambda) - f(x)}{\lambda x + z - \lambda z - x} \leq \frac{f(z) - f(\lambda x + z - \lambda z)}{-\lambda x + \lambda z}$$

$$\Rightarrow \lambda(z-x)\left(f(\lambda x + z - z\lambda) - f(x)\right) \leq \left(f(z) - f(\lambda x + z - \lambda z)\right)(1-\lambda)(z-x)$$

We note here that the direction of the inequality is preserved since we have $x < z$ and $\lambda > 0$ and $1 - \lambda > 0$ (*)

$$\Rightarrow \lambda\left(f(\lambda x + z - z\lambda) - f(x)\right) \leq \left(f(z) - f(\lambda x + z - \lambda z)\right)(1-\lambda)$$

$$\Rightarrow \lambda\left(f(\lambda x + z(1-\lambda)) - f(x)\right) \leq \left(f(z) - f(\lambda x + z(1-\lambda))\right)(1-\lambda)$$

$$\Rightarrow \lambda f(\lambda x + z(1-\lambda)) - \lambda f(x) \leq f(z) - f(\lambda x + z(1-\lambda)) - \lambda f(z) + \lambda f(\lambda x + z(1-\lambda))$$

reducing yields

$$-\lambda f(x) \leq f(z) - f(\lambda x + z(1-\lambda)) - \lambda f(z)$$

$$\Rightarrow -\lambda f(x) - f(z) + \lambda f(z) \leq -f(\lambda x + z(1-\lambda))$$
$$\Rightarrow \lambda f(x) + f(z) - \lambda f(z) \geq f(\lambda x + z(1-\lambda))$$
$$\Rightarrow \lambda f(x) + f(z)(1-\lambda) \geq f(\lambda x + z(1-\lambda))$$

but since the initial choice of x, y, z was arbitrary (up to ordering) this holds for any $x < z$. We must still address the case corresponding to $x > z$. This is easy though.

For let $x > z$. Then by what we have proven already

$$\lambda f(z) + f(x)(1-\lambda) \geq f(\lambda z + x(1-\lambda))$$

now let $\alpha = 1 - \lambda$ so it follows from the above that

$$(1-\alpha)f(z) + f(x)(1-1+\alpha) \geq f((1-\alpha)z + x(1-1+\alpha))$$

$$\Rightarrow (1-\alpha)f(z) + \alpha f(x) \geq f((1-\alpha)z + \alpha x)$$

Thus if f' is monotonically increasing

$$\lambda f(x) + f(z)(1-\lambda) \geq f(\lambda x + z(1-\lambda))$$

for $\forall x, y \in (a, b)$, $\lambda \in (0, 1)$ □

Now we prove ($\Leftarrow$) that f convex implies f' is monotonically increasing.

Proof. The proof of this direction follows similarly to the previous. Let's assume f is convex.

Then

$$\lambda f(x) + f(z)(1-\lambda) \geq f(\lambda x + z(1-\lambda))$$

for $x < z$ (Note: we must have $x < z$ by (*)) and $x, z \in (a, b)$, $\lambda \in (0, 1)$. Following the previous reasoning in reverse we see that

$$\frac{f(y) - f(x)}{y - x} \leq \frac{f(z) - f(y)}{z - y}$$

for $\forall y \in (x, z)$.

So in the limit $y \to x$ we have

$$f'(x) \leq \frac{f(z) - f(x)}{z - x}$$

and in the limit $y \to z$ we have

$$\frac{f(z) - f(x)}{z - x} \leq f'(z)$$

thus, combining these statements we have

$$f'(x) \leq \frac{f(z) - f(x)}{z - x} \leq f'(z)$$

so f' is monotonically increasing as desired. $\square$

Now it only remains to prove that (assuming $f''(x)$ exists for $x \in (a, b)$) f is convex if and only if $f''(x) \geq 0$ for $\forall x \in (a, b)$.

Proof. f is convex $\Leftrightarrow$ f' is monotonically increasing by the previous proof. Moreover, f' is monotonically increasing $\Leftrightarrow$ $f'' \geq 0$ (proved in class).

So we have shown that (assuming $f''(x)$ exists for $x \in (a, b)$) f is convex if and only if $f''(x) \geq 0$ for $\forall x \in (a, b)$ as desired. $\square$