Homework

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1 Chapter 7

1.1 Problem 16

1.1.1 Question

Suppose $\{f_n\}$ is an equicontinuous sequence of functions on a compact set K, and $\{f_n\}$ converges pointwise on K. Prove that $\{f_n\}$ converges uniformly on K.

1.1.2 Answer

Proof. We shall prove that $\{f_n\}$ converges uniformly on K.

Let $\epsilon > 0$, and pick $\delta > 0$ by equicontinuity. Let $V(x, \delta)$ be the set of all $y \in K$ with $d(x, y) < \delta$. Since there is a countable dense subset $E \subset K$ and, K is compact, there are finitely many points $x_1, \ldots, x_m$ in E such that

$$K \subset V(x_1, \delta) \cup \ldots \cup V(x_m, \delta). \tag{1}$$

Since $\{f_n(x)\}$ converges for every $x \in E$, there is an integer N such that

$$|f_l(x_s) = f_j(x_s)| < \epsilon \tag{2}$$

whenever $i \leq N, j \leq N, 1 \leq s \leq m$.

If $x \in K$ (1) shows that $x \in V(x_s, \delta)$ for some s, so that

$$f_l(x) - f_l(x_s) < \epsilon$$

for every i, If $i \geq N$ and $j \geq N$, it follows from (2) that

$$|f_l(x) - f_j(x)| \leq |f_i(x) - g_i(x_s)| + |f_i(x_s) - f_j(x_s)| + |f_j(x_s) - f_j(x)| < 3\epsilon$$

But this is just what we wanted to show.

□

1.2 Problem 18

1.2.1 Question

Let $\{f_n\}$ be a uniformly bounded sequence of functions which are Riemann-integrable on $[a, b]$, and put

$$F_n(x) = \int_a^x f_n(t)dt \quad (a \leq x \leq b).$$

Prove that there exists a subsequence $\{F_{n_k}\}$ which converges uniformly on $[a, b]$.

1.2.2 Answer

The Arzela-Ascoli Theorem states "Consider a sequence of real-valued continuous functions $\{f_n\}$ defined on a closed and bounded interval $[a, b]$ of the real line. If this sequence is uniformly bounded and equicontinuous, then there exists a subsequence $\{f_{n_k}\}$ that converges uniformly."

I will prove that the sequence of functions $\{F_n\}$ satisfies the criteria to apply this theorem, and therefore that there exists a subsequence $\{F_{n_k}\}$ which converges uniformly on $[a, b]$ as desired.

Proof. We have that each F_n is real valued, and defined on a compact subset of $\mathbb{R}$, thus it remains only to prove that each F_n is uniformly bounded, equicontinuous.

Now fix some M so that $|f_n| \leq M$ for all n. For all $x \in [a, b]$, n we have

$$|F_n(x)| \leq \int_a^x |f_n(x)| \leq (x - a)M \leq (b - a)M$$

so the F_n are all uniformly bounded.

Now observe that for any $x, y \in [a, b]$ we have

$$|F_n(y) - F_n(x)| = \left| \int_x^y f_n(t)dt \right| \leq M|y - x|,$$

and so we have equicontinuity. So we we can apply Arzela-Ascoli. □

2 Chapter 9

2.1 Problem 5

2.1.1 Question

Prove that to every $A \in L(R^n, R^1)$ corresponds a unique $y \in R^n$ such that $Ax = x \cdot y$.

Prove also that $||A|| = |y|$.

Hint: Under certain conditions, equality holds in the Schwarz inequality.

2.2 Answer

Proof. Any $A \in L(R^n, R^1)$ can be written uniquely as

$$\sum_{i=1}^{n} c_i A\hat{e}_i = \sum_{i=1}^{n} c_i d_i$$

for some fixed set of unique constants d_i (since the codomain is just R). So $\vec{d} = (d_1, \ldots, d_n)$ is they y above. □

Proof. It must be that $||A|| \geq |y|$ since in particular $A(\hat{d}) = |y|$ (where $\hat{d}$ is a unit vector parallel to d).

$$\hat{d} = \frac{d}{|d|} \Rightarrow |A(\hat{d})| = \frac{1}{|d|}|A(d)| = \frac{|d|^2}{|d|} = |d|$$

Moreover we know that $||A|| \leq |y|$ since we can write any other vector in the unit circle as a sum of the form

$$a\hat{d} + x_p$$

for some $a < 1$ and $x_p \cdot \hat{d} = 0$. However observe

$$|A(a\hat{d} + x_p)| = a|A(\hat{d})| + 0 < |y|$$

□