Homework

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1 Problem 1

1.1 Question

Suppose that f is a function that is continuous on a closed set F of real numbers. Show that f has a continuous extension to all of $\mathbb{R}$. (**Hint**: express $\mathbb{R}\backslash F$ as a countable union of disjoint intervals and define f to be linear on the closure of these intervals).

1.2 Answer

Let the extension of f to $\mathbb{R}$ say $\bar{f}$ be as suggested above. Since $\bar{f}$ is a piecewise defined function which is continuous on each interval, and whose intervals agree on the ends $\bar{f}$ is itself continuous as claimed.

2 Problem 4

2.1 Question

State and prove an extension of Lusin's theorem to the case that E has infinite measure.

2.2 Answer

Let f be a real-valued measurable function on a set E with $m(E) = \infty$. Then for each $\epsilon > 0$, there is a continuous function g on $\mathbb{R}$ and a closed set

F containing E for which

$$f = g \text{ on } F \quad \text{and} \quad m(E \sim F) < \epsilon$$

Proof. If we fix an infinite set of compact intervals $\{I_n\}$ each separated by $\epsilon/2^n$. By the finite case of Lusin's Theorem we can find continuous functions g_n and subsets of $I_n \cap E$ which are closed, and have measure within $\epsilon/2^n$ of $I_n \cap E$ such that $g_n = f$ on each of these sets.

By the previous exercise there is a continuous function which takes on the value of g_n on each of our compact intervals. Since the measure of E less all our intervals is 2ϵ by construction, with ϵ arbitrary, we are done. □

3 Problem 7

3.1 Question

Does the Bounded Convergence Theorem hold for the Riemann integral?

3.2 Answer

No. We can construct an absolutely bounded sequence of (Riemann) integrable functions which converges to a function which is not integrable. In particular set

$$f_n = \begin{cases} 0 & x \in R_n \\ 1 & \text{otherwise} \end{cases}$$

Where we put R_n the first n rational numbers under some fixed ordering. Any f_n is integrable since it has only finitely many discontinuities. However this sequence of functions converges to

$$f_n = \begin{cases} 0 & x \in \mathbb{Q} \\ 1 & \text{otherwise} \end{cases}$$

which is not integrable.

4 Problem 9

4.1 Question

Let f be a nonnegative bounded measurable function on a set E of finite measure. Assume $\int_E f = 0$. Show that $f = 0$, a.e. on E.

4.2 Answer

Proof. Suppose towards a contradiction that there exists some subset of E say F having nonzero measure such that $f > 0$ on F. Then, there must exist a set G, $\epsilon > 0$ such that $G \subset F$ and $f > \epsilon$ on G and $m(F \backslash G) < \delta$ by Lusin.

Hence, restricted to G we get $f \leq \epsilon \cdot \mathbf{1}_G$, but this is a contradiction to monotonicity of the Lebesgue integral and our assumption that $\int_E f = 0$. $\square$