Homework

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10 February 2010

1 Chapter 7

1.1 Problem 5

1.1.1 Question

Let

$$f_n(x) = \begin{cases} 0 & \left(x < \dfrac{1}{n+1}\right), \\ \sin^2 \dfrac{\pi}{x} & \left(\dfrac{1}{n+1} \leq x \leq \dfrac{1}{n}\right), \\ 0 & \left(\dfrac{1}{n} < x\right). \end{cases}$$

Show that $\{f_n\}$ converges to a continuous function, but not uniformly. Use the series $\sum f_n$ to show that absolute convergence, even for all x, does not imply uniform convergence.

1.1.2 Answer

I claim that $\{f_n\}$ converges to 0.

Proof. We begin by fixing some $x \in \mathbb{R}$. If we have $x \leq 0$ we are done since for no value of $n \in \mathbb{N}$ do we have

$$\frac{1}{n+1} \leq x \leq \frac{1}{n}$$

and therefore each function in $\{f_n\}$ has $f_n(x) = 0$. For $x > 0$ we can find an N so that $n > N$ gives us $f_n(x) = 0 < \epsilon$ for any $\epsilon > 0$. In particular pick N so that the following inequality is satisfied:

$$0 < \frac{1}{N} < x.$$

We are guaranteed that such an N exists by the density of $\mathbb{Q}$ in $\mathbb{R}$ as well as the fact that decreasing the numerator of a positive rational number to 1 decreases it while keeping it greater than 0.

Since we have $\frac{1}{N} < x$ by construction, and each $n > N$ has $\frac{1}{n} < \frac{1}{N}$ we must have $\frac{1}{n} < x$ for each n, and therefore $f_n(x) = 0$ for every $n > N$, and $\{f_n\}$ converges to 0 as claimed. $\square$

Now I claim further that $\{f_n\}$ does not converge uniformly.

Proof. A sequence of functions $\{f_n\}$ converges uniformly to f if, given ϵ we can pick an N such that for all $n > N$ we have $|f_n(x) - f(x)| < \epsilon$ over all x.

Since we have shown above that $\{f_n\}$ converges pointwise to 0 if it converges uniformly it must converge to 0. However it does not converge uniformly. In particular, irrespective of n there is an x^* for which $|f_n(x) - f(x)| = 1$.

Choose

$$x^*_n = \frac{1}{n + \frac{1}{2}}.$$

For f_n we have

$$\frac{1}{n+1} < \frac{1}{n + \frac{1}{2}} < \frac{1}{n}$$

so we get

$$f_n(x*_n) = \sin^2\left(\pi n + \frac{\pi}{2}\right) = 1$$

Now, clearly for no $0 < \epsilon < 1$ may we satisfy the condition for uniform convergence. Since it is required that this condition be satisfied for all $\epsilon > 0$ $\{f_n\}$ does not converge uniformly. $\square$

The series $\sum f_n$ convergences absolutely for all x.

Proof. Each successive f_n is nonzero only on an inverval which overlaps with the previous one in only the point $1/n$. Moreover the interval on which f_n is nozero say X_n has $|x| \leq |y|$ for each $x \in X_{n+1}, y \in X_n$.

These facts together imply that if we fix some $x \in \mathbb{R}$ at most two functions in the sequence $\{f_n\}$ are nonzero at x. In particular if $x = 1/n$ for some $n \in \mathbb{N}$ then both $f_{n-1}(x) \neq 0$ and $f_n(x) \neq 0$. Otherwise we only have $f_n(x) \neq 0$ for the least $n \in \mathbb{N}$ such that $1/n < x$.

Hence, since a finite sum is convergent we see that $\sum f_n$ converges at every x.

Since the value of each f_n is always nonnegative (being everywhere 0 or a square) saying that $\sum f_n$ converges is equivalent to saying that $\sum f_n$ converges absolutely and $\sum f_n$ converges absolutely as claimed. □

The series $\sum f_n$ does not converge uniformly.

Proof. If $\{f_n\}$ does not converge uniformly neither does $\sum f_n$.

Assume towards a contradiction that $\{f_n\}$ does not converge uniformly but $\sum f_n$ does. Since $\sum f_n$ converges, the Cauchy criterion implies that there exists N such that

$$|F_n(x) - F_m(x)| \leq \epsilon$$

for any x, $n \geq N$, $m \geq N$ where F_m denotes the mth partial sum. In particular if we take $m = n + 1$ this implies that

$$|F_n(x) - F_{n+1}(x)| = |f_{n+1}(x)| \leq \epsilon$$

. However as $\epsilon > 0$ was chosen arbitrarily this is true if and only if $\{f_n\}$ converges uniformly to 0. Contradiction.

Thus, since $\{f_n\}$ does not converge uniformly neither does $\sum f_n$ as claimed. □

1.2 Problem 15

1.2.1 Question

Suppose f is a real and continuous function on $\mathbb{R}^1$, $f_n(t) = f(nt)$ for $n = 1, 2, 3, \ldots$, and $\{f_n\}$ is equicontinuous on $[0, 1]$. What conclusion can you draw about f?

1.2.2 Answer

$$f = \begin{cases} g(x) & x < 0 \\ c & x \geq 0 \end{cases}$$

where $g(0) = c$ and g is continuous on $(-\infty, 0)$. That is, I claim that $f(x)$ is constant for $x \geq 0$.

Take $f(0) = c$ for some constant c.

Now pick some $x > 0$ and $\epsilon > 0$ and set $\delta > 0$ by the equicontinuity condition (i.e., the δ such that $f(B_\delta 0) \subset B_\epsilon c$). If we choose $n > x/\delta$ we get $x/n < \delta \Rightarrow |f(x) - c| = |f_n(x/n) - c| < \epsilon$. However since we can pick ϵ arbitrarily close to 0 we get $f(x) = c$ for x. Similarly since we can pick any $x > 0$ we have $f(x) = c$ for all $x \geq 0$.

2 Chapter 8

2.1 Problem 8

2.1.1 Question

For $n = 0, 1, 2, \ldots$, and x real, prove that

$$|\sin xn| \leq n\,|\sin x|\,.$$

Note that this inequality may be false for other values of n. For instance,

$$\left|\sin \frac{1}{2}\pi\right| > \frac{1}{2}\,|\sin \pi|\,.$$

2.1.2 Answer

We proceed by induction

Proof. Base Case: For $n = 0$ we have

$$|\sin xn| = 0 \leq 0 = n\,|\sin x|\,.$$

Inductive Step: We first assume that

$$|\sin xn| \leq n\,|\sin x|\,.$$

No observe that

$$\begin{aligned} |\sin x(n+1)| &= |\sin (xn + x)| \\ &= |\sin (xn) \cos (x) \pm \sin (x) \cos (xn)| \end{aligned}$$

$$
\begin{aligned}
&\leq |\sin(xn) \pm \sin(x)| \\
&\leq |\sin(xn)| + |\sin(x)| \\
&\leq n|\sin x| + |\sin(x)| \\
&\leq (n+1)|\sin x|
\end{aligned}
$$

(Above we employ the fact that $|\cos(x)| \leq 1$) $\square$