Homework

Frederick Robinson

28 May 2010

1 Problem 3

1.1 Question

For $f \in L^1[a,b]$, define

$$||f||_* = \int_a^b x^2 |f(x)| dx.$$

Show that this is a norm on $L^1[a,b]$.

1.2 Answer

We must verify the Triangle Inequality, Positive Homogeneity, and Nonnegativity. Let $f, g \in L^1[a,b]$.

Triangle Inequality

Proof.

$$||f+g||_* = \int_a^b x^2 |f(x)+g(x)| dx$$

and by properties of the integral (monotonicity, linearity) we have

$$\begin{aligned} \int_a^b x^2 |f(x)+g(x)| dx &\leq \int_a^b x^2 |f(x)| dx + \int_a^b x^2 |g(x)| dx \\ &= ||f||_* + ||g||_* \end{aligned}$$

and so

$$||f+g||_* \leq= ||f||_* + ||g||_*$$

as claimed. $\square$

Positive Homogeneity

Proof.

$$||\alpha f||_* = \int_a^b x^2 |\alpha f(x)| dx$$

so by linearity of the integral

$$\int_a^b x^2|\alpha f(x)|dx = |\alpha| \int_a^b x^2|f(x)|dx = |\alpha|||f||_*$$

and

$$||\alpha f||_* = \alpha||f||_*$$

as claimed. □

Nonnegativity

Proof.

$$||f||_* = \int_a^b x^2||f(x)||dx$$

so, since we are integrating a nonnegative function we get

$$\int_a^b x^2||f(x)||dx \geq$$

and

$$||f||_* \geq 0$$

as claimed.

Furthermore, if $f = 0$ we have

$$||f||_* = \int_a^b x^2 f(x)dx = \int_a^b 0 = 0.$$

Conversely, given $||f||_* = 0$ for some f, we have

$$||f||_* = \int_a^b x^2 f(x)dx = 0$$

and by a previous exercise (Problem 9 from homework due 7 May) this implies that $x^2 f(x) = 0$ for almost every $x \in [a, b]$. Since $x \neq 0$ for $x \neq 0$ we must have $f(x) = 0$ for almost every $x \in [a, b]$. We may therefore conclude that $||f||_* = 0$ if and only if $f(x) = 0$ almost everywhere in $[a, b]$ as desired. □

2 Problem 5

2.1 Question

Assume $m(E) < \infty$. For $f \in L^\infty(E)$, show that

$$\lim_{p\to\infty} ||f||_p = ||f||_\infty.$$

2.2 Answer

We shall assume that the function f is simple. The extension to arbitrary measurable functions f follows directly[1] from the simple function approximation theorem.

Proof. If M is the essential upper bound, then $f(x) = M$ on some set of nonzero measure say A. Thus, since

$$||f||_p = \left[\int_E |f|^p\right]^{1/p}$$

we have

$$[m(A)M^p]^{1/p} \leq ||f||_p \leq [m(E)M^p]^{1/p}$$

and so

$$M\,[m(A)]^{1/p} \leq ||f||_p \leq M\,[m(E)]^{1/p}\,.$$

Since in the limit $p \to \infty$ $x^{1/p} = 1$ for any $x > 0$ we have

$$\lim_{p\to\infty} ||f||_p = ||f||_\infty$$

as claimed. □

3 Problem 6

3.1 Question

For $1 \leq p \leq \infty$, if q is the conjugate of p, prove that for any $f \in L^p(E)$:

$$||f||_p = \max_{g\in L^q(E), ||g||_q \leq 1} \int_E fg.$$

3.2 Answer

Proof. We have from the text that the conjugate function $g = f^*$ has the desired properties i.e. $||g||_q \leq 1$ and $||f||_p = \int_E fg$. Therefore it remains only to show that there exists no function g such that $||g||_q \leq 1$ and $\int_E fg > ||f||_p$. But this is just a trivial consequence of Hölder's Inequality. □

[1] We can fix some sequence of simple functions which converge to f. Since each of these have the claimed property then their limit must as well.