Homework 2

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Chapter 2

1 Problem 3

1.1 Question

Let $U \subseteq \mathbb{C}$ be an open disc with center 0. Let f be holomorphic on U. If $z \in U$, then define γ_z to be the path

$$\gamma_z(t) = tz, \quad 0 \leq t \leq 1.$$

Define

$$F(z) = \oint_{\gamma_z} f(\zeta) d\zeta.$$

Prove that F is a holomorphic antiderivative for f.

1.2 Answer

Proof. We proved in Theorem 2.3.2 that the function F' acquired by integrating

$$F' = \int_0^{\Re(z)} \Re f + \int_0^{\Im(z)} \Im(f)$$

is has the desired properties. It therefore suffices to show that $F' = F$. This is just a straightforward application of the Cauchy Integral Theorem though.

Observe that

$$F' = \int_{\gamma'} f$$

where γ' is the union of the line segments from 0 to $\Re z$ and $\Re z$ to $\Im z$. If we denote by $\gamma' - \gamma$ the path acquired by traveling along γ', then backwards along γ, the Cauchy Integral Theorem gives us

$$0 = \oint_{\gamma'-\gamma} f(\zeta)d\zeta = \oint_{\gamma'} f(\zeta)d\zeta - \oint_{\gamma} f(\zeta)d\zeta = F' - F$$

which is what we wanted to show. □

2 Problem 4a

2.1 Question

Compute

$$\oint_\gamma \frac{1}{z}dz$$

where γ is the unit circle (center 0) with counterclockwise orientation.

2.2 Answer

The Cauchy integral formula with $z = 0, f(z) = 1$ is

$$1 = \frac{1}{2\pi i}\oint_\gamma \frac{1}{\zeta}d\zeta.$$

Therefore the desired integral is just $2\pi i$.

3 Problem 4c

3.1 Question

Compute

$$\oint_\gamma \frac{z}{8+z^2}dz$$

where γ is the triangle with vertices $1, i, -i$ and γ is equipped with counterclockwise orientation.

3.2 Answer

The zeroes of the denominator lie outside the triangle. The function is therefore holomorphic on the interior, and the integral evaluates to

$$\oint_\gamma \frac{z}{8+z^2}dz = 0.$$

4 Problem 5

4.1 Question

Evaluate

$$\oint_\gamma z^j dz,$$

for every integer value of j, where γ is a circle with counterclockwise orientation and whose interior contains 0.

4.2 Answer

In the case of $j = -1$ we solved this in exercise 4a. Assuming $j \neq -1$, the function is holomorphic on $\mathbb{C} \setminus \{0\}$. Hence,

$$\oint_\gamma z^j dz = \begin{cases} 2\pi i & j = -1 \\ 0 & \text{otherwise} \end{cases} .$$

5 Problem 18b

5.1 Question

Compute

$$\oint_\gamma \frac{\zeta}{(\zeta+4)(\zeta-1+i)} d\zeta$$

where γ describes the circle of radius 1 with center 0 and counterclockwise orientation.

5.2 Answer

As the function is holomorphic on the interior of the curve we just have

$$\oint_\gamma \frac{\zeta}{(\zeta+4)(\zeta-1+i)} d\zeta = 0$$

6 Problem 18d

6.1 Question

Compute

$$\oint_\gamma \zeta(\zeta+4) d\zeta$$

where γ is the circle of radius 2 and center 0 with clockwise orientation.

6.2 Answer

As $\zeta^2+4\zeta$ is a polynomial, it's holomorphic. Hence, by the Cauchy Integral Theorem

$$\oint_\gamma \zeta(\zeta+4)d\zeta = 0.$$

7 Problem 18e

7.1 Question

Compute

$$\oint_\gamma \overline{\zeta}d\zeta$$

where γ is the circle of radius 1 and center 0 with counterclockwise orientation.

7.2 Answer

We can compute the integral as

$$\begin{aligned}\oint_\gamma \overline{\zeta}d\zeta &= \oint_\gamma \Re\overline{\zeta}d\zeta + i\oint_\gamma \Im\overline{\zeta}d\zeta \\ &= \oint_\gamma \Re\zeta d\zeta - i\oint_\gamma \Im\zeta d\zeta.\end{aligned}$$

However since the function $f(z)=z$ is holomorphic, we have $0-0=0$.

8 Problem 28a

8.1 Question

Compute explicitly the integrals

$$\oint_{\partial D(8i,2)} z^3 dz,$$

$$\oint_{\partial D(6+i,3)} (\overline{z}-i)^2 dz.$$

8.2 Answer

$$\begin{aligned}\oint_{\partial D(8i,2)} z^3 dz &= \int_0^1 (8i + 2e^{2\pi it})^3 2e^{2\pi i(t+1/4)} dt \\ &= \int_0^1 -512ie^{2i\pi t} - 384e^{4i\pi t} + 96ie^{6i\pi t} + 8e^{8i\pi t} dt \\ &= 0\end{aligned}$$

For the second computation we have

$$\begin{aligned}\oint_{\partial D(6+i,3)} (\overline{z} - i)^2 dz &= \int_0^1 ((6 - i + 3e^{-2\pi it}) - i)^2 3e^{2\pi i(t+1/4)} dt \\ &= \int_0^1 (36 + 108i) + 27ie^{-2i\pi t} + (72 + 96i)e^{2i\pi t} dt \\ &= 36 + 108i\end{aligned}$$

9 Problem 43

9.1 Question

If f is a holomorphic polynomial and if

$$\oint_{\partial D(0,1)} f(z)\overline{z}^j dz = 0, \quad j = 0, 1, 2, \ldots,$$

then prove that $f \equiv 0$.

9.2 Answer