Homework 3

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20 April 2011

Chapter 3

1 Problem 1

1.1 Question

It was shown (Corollary 3.5.2) that if f_j are holomorphic on an open set $U \subseteq \mathbb{C}$ and if $f_j \to f$ uniformly on compact subsets of U, then

$$\frac{d}{dz} f_j \to \frac{d}{dz} f$$

uniformly on compact subsets of U. Give an example to show that if the word "holomorphic" is replaced by "infinitely differentiable", then the result is false.

1.2 Answer

Consider the sequence of functions

$$f_j = \frac{1}{j} \sin(jx).$$

This sequence converges uniformly to 0 on $\mathbb{C} \setminus \{0\}$, and each member of the sequence has infinitely many derivatives. However, the derivatives are $f_j' = \cos jx$, and these do not converge except for $x = n\pi$.

2 Problem 4

2.1 Question

Use Morera's theorem to give another proof of Theorem 3.5.1: If $\{f_j\}$ is a sequence of holomorphic functions on a domain U and if the sequence converges uniformly on compact subsets of U to a limit function f, then f is holomorphic on U.

2.2 Answer

Proof. Let $\gamma : [0,1] \to U$ be a closed, piecewise C^1 curve, with $\gamma(1) = \gamma(0)$. Since each $\{f_j\}$ is holomorphic

$$\oint_\gamma f_j(\zeta) d\zeta = 0.$$

Moreover, since $\{f_j\}$ converges uniformly to f,

$$\oint_\gamma f = \oint_\gamma \lim_{j\to\infty} f_j = \lim_{j\to\infty} \oint_\gamma f_j = 0.$$

Hence, by Morera's theorem, f is holomorphic. □

3 Problem 9

3.1 Question

Let $\sum_{k=0}^\infty a_k x^k$ and $\sum_{k=0}^\infty b_k x^k$ be real power series which converge for $|x| < 1$. Suppose that $\sum_{k=0}^\infty a_k x^k = \sum_{k=0}^\infty b_k x^k$ when $x = 1/2, 1/3, 1/4, \ldots$. Prove that $a_k = b_k$ for all k.

3.2 Answer

This follows from a Theorem (Rudin 8.5) which states:

Suppose two power series converge in a segement $S = (-R, R)$ and denote by E the set of all $x \in S$ where they agree. If E has a limit point in S, then the two power series are the same.

4 Problem 11b

4.1 Question

Determine the disc of convergence for the series. Then determine at which points of the boundary of the disc of convergence the series converges.

$$\sum_{k=2}^\infty k^{\log k}(z+1)^k$$

4.2 Answer

We employ the root test (Lemma 3.2.6). First compute

$$\limsup_{k\to\infty} |k^{\log k}|^{1/k} = \limsup_{k\to\infty} e^{(\log k)^2/k} = 1$$

Therefore, the disc of convergence is

$$D(-1,1)$$

To analyze the behavior on the boundary, let $z = -1 + e^{i\theta}$ be a generic point on the boundary. Now the series is

$$\sum_{k=2}^{\infty} k^{\log k} e^{ik\theta}.$$

This diverges for every choice of θ though, as $\lim_{k\to\infty} |k^{\log k} e^{ik\theta}| = \infty$.

5 Problem 11f

5.1 Question

Determine the disc of convergence for the series. Then determine at which points of the boundary of the disc of convergence the series converges.

$$\sum_{k=2}^{\infty} \frac{k}{k^2+4} z^k$$

(*Hint*: Use summation by parts.)

5.2 Answer

Let's use the root test again.

$$\limsup_{k\to\infty} \left(\frac{k}{k^2+4}\right)^{1/k} = 1.$$

Therefore, the series converges on $D(0,1)$. By a theorem involving summation by parts (Rudin 3.44) the series also converges on the boundary, except perhaps at 1. Clearly it does not converge at 1, since the sum differs by a constant from $\sum_{z=2}^{\infty} 1/z$ which diverges.

6 Problem 11g

6.1 Question

Determine the disc of convergence for the series. Then determine at which points of the boundary of the disc of convergence the series converges.

$$\sum_{k=0}^{\infty} ke^{-k}z^k$$

6.2 Answer

By the roots test, the disk of convergence is $D(0,e)$ by the root test, as

$$\limsup_{k\to\infty} k^{1/k}e^{-k/k} = \limsup_{k\to\infty} \frac{k^{1/k}}{e} = \frac{1}{e}.$$

On the boundary, we have sums of the form

$$\sum_{k=0}^{\infty} ke^{-k}(ee^{i\theta})^k = \sum_{k=0}^{\infty} k(e^{i\theta})^k$$

which has divergent modulus, and therefore does not converge for any choice of θ.

7 Problem 13

7.1 Question

Let $f:(-1,1)\to\mathbb{R}$ be C^∞. Prove that f is real analytic in some neighborhood of 0 if and only if there is a nonempty interval $(-\delta,\delta)$ and a constant $M>0$ such that $|(d/dx)^k f(x)| \le M^k \cdot k!$ for all $k\in\{1,2,\dots\}$ and all $x\in(-\delta,\delta)$

7.2 Answer

Proof. Suppose ($\Rightarrow$) that there is such a δ, M. Then, we write the absolute value of the Taylor series at a point $a\in(-\delta,\delta)$ as

$$\sum_{k=0}^{\infty}\left|\frac{1}{k!}\left(\frac{d}{dx}\right)^k f(a)(x-a)^k\right| \le \sum_{k=0}^{\infty} M^k(x-a)^k.$$

Since we can choose x to be within $1/2M$ of a, this converges on a neighborhood of a. Thus, f is analytic on $(-\delta, \delta)$ since it has an absolutely convergent (and therefore convergent) series representation on some neighborhood of any point in this interval.

Conversely ($\Leftarrow$) suppose that f is real analytic on the interval $(-\delta, \delta)$. Then there is a neighborhood of any point in the interval on which the Taylor series converges absolutely. Suppose towards a contradiction that there is no M such that

$$|(d/dx)^k f(x)| \leq M^k \cdot k! \text{ for all } k \in \{1, 2, \dots\} \text{ and all } x \in (-\delta, \delta).$$

Then, for a fixed choice of M all but finitely many choices of k, x have

$$|(d/dx)^k f(x)| > M^k \cdot k!.$$

Hence, the Taylor series

$$\sum_{k=0}^{\infty} \left| \frac{1}{k!} \left(\frac{d}{dx} \right)^k f(a) \epsilon^k \right|$$

diverges outside a neighborhood of fixed size $< \epsilon$ if we choose $M = 1/\epsilon$. This is a contradiction though, as ϵ may be chosen to be arbitrarily small. $\square$

8 Problem 23

8.1 Question

TRUE or FALSE: Let f be holomorphic on $D(0, 1)$ and assume that f^2 is a holomorphic polynomial on $D(0, 1)$. Then f is also a holomorphic polynomial on $D(0, 1)$.

8.2 Answer

FALSE

Let $f(z) = \sqrt{z + 2}$.

9 Problem 24

9.1 Question

TRUE or FALSE: Let $a_j > 0, j = 1, 2, \dots$. If $\sum a_j z^j$ is convergent on $D(0, r)$ and if $\epsilon > 0$ is sufficiently small, then $\sum (a_j + \epsilon) z^j$ is convergent on $D(0, r')$ for some $0 < r' < r$.

9.2 Answer

TRUE

Proof. Clearly

$$\sum (a_j + \epsilon) z^j = \sum (a_j z^j + \epsilon z^j) = \sum a_j z^j + \epsilon \sum z^j$$

converges on $D(0,r)$ if and only if $\sum \epsilon z^j$ does. However, this is just a geometric series, and is clear for sufficiently small z. □

10 Problem 26

10.1 Question

The functions $f_k(x) = \sin kx$ are C^∞ and bounded by 1 on the interval $[-1,1]$, yet their derivatives at 0 are unbounded.

Contrast this situation with the functions $f_k(z) = \sin kz$ on the unit disc.The Cauchy estimates provide bounds for $(\partial/\partial z) f_k(0)$. Why are these two examples not contradictory?

10.2 Answer

This is not contradictory, since the complex derivative is not completely determined by the derivative in the real direction. It has also a contribution from the derivative in the imaginary direction. In this instance the imaginary contribution must counteract the real contribution, leading to a bound on the total in general.

11 Problem 28

11.1 Question

Let $U \subseteq \mathbb{C}$ be an open set. Let $f : U \to \mathbb{C}$ be holomorphic and bounded. Let $P \in U$. Prove that

$$\left| \frac{\partial^k f}{\partial z^k}(P) \right| \leq \frac{k!}{r^k} \sup_U |f|,$$

where r is the distance of P to $\mathbb{C} \setminus U$.

11.2 Answer

Proof. By Cauchy Estimates

$$\left|\frac{\partial^k f}{\partial z^k}(P)\right| \leq \frac{k!}{r'^k}\sup_U |f|,$$

for all $0 < r' < r$ since $\sup_U |f| \geq \sup_{\overline{D}(P,r')} |f|$ for all such r'. Now just observe that by continuity of $1/x$ away from 0 the inequality must also hold in the limit $r' \to r$.

(Suppose not, then there is some $s < r$ such that

$$\frac{k!}{r^k}\sup_U |f| < \frac{k!}{s^k}\sup_U |f| < \left|\frac{\partial^k f}{\partial z^k}(P)\right|$$

a contradiction) □

12 Problem 30

12.1 Question

Let f be an entire function and $P \in \mathbb{C}$. Prove that there is a constant C, not depending on k such that

$$\left|\left(\frac{\partial}{\partial z}\right)^k f(P)\right| \leq C \cdot k!.$$

Can you improve this estimate? Is there necessarily a polynomial $p(k)$ such that

$$\left|\left(\frac{\partial}{\partial z}\right)^k f(P)\right| \leq |p(k)| \quad ?$$

12.2 Answer

For the first part, we can fix $C = \sup_{z\in\overline{D}(P,1)} |f(z)|$, then taking the Cauchy estimate, for radius 1 yields

$$\left|\frac{\partial^k f}{\partial z^k}(P)\right| \leq Ck!$$

as desired.

This estimate cannot be improved. Consider the power series

$$f(z) = \sum_{k=0}^{\infty} \frac{2^k}{k!} z^k$$

the function is analytic, by construction, however it has kth derivative 2^k, and therefore the derivatives are not bounded above by any polynomial.

13 Problem 32

13.1 Question

Suppose that f is bounded and holomorphic on $\mathbb{C} \setminus \{0\}$. Prove that f is constant. [*Hint:* Consider the function $g(z) = z^2 \cdot f(z)$ and endeavor to apply Theorem 3.4.4.]

13.2 Answer

Proof. f being holomorphic, bounded the limit $\lim_{z\to 0}$ is well defined, finite. Then, the function

$$g(z) = \begin{cases} f(z) & z \neq 0 \\ \lim_{z\to 0} f(z) & z = 0 \end{cases}$$

is holomorphic, bounded. Hence, we have the desired result by Louisville's Theorem. □

14 Problem 33

14.1 Question

1. Show that if $f : D(0, r) \to \mathbb{C}$ is holomorphic, then

$$|f(0)| \leq \frac{1}{\sqrt{\pi} r} \left(\int_{D(0,r)} |f(x, y)|^2 dxdy \right)^{1/2}.$$

[*Hint:* The function f^2 is holomorphic too. Use the Cauchy integral formula to obtain

$$\frac{1}{2\pi} \int_0^{2\pi} f^2(se^{i\theta}) d\theta = f^2(0)$$

for $0 < s < r$. Multiply both sides by a real parameter s and integrate in s from 0 to r.]

2. Let $U \subseteq \mathbb{C}$ be an open set and let k be a compact subset of U. Show that there is a constant C (depending on U and K) such that if f is holomorphic on U, then

$$\sup_K |f| \leq C \cdot \left(\int_U |f(x,y)|^2 dxdy \right)^{1/2}.$$

14.2 Answer

1. By the CIF with $z = se^{i\theta}$, $0 < s < r$ we have

$$f^2(0) = \frac{1}{2\pi i} \oint_{\partial} \frac{f^2(\zeta)}{\zeta} d\zeta = \frac{1}{2\pi} \int_0^{2\pi} f^2(se^{i\theta}) d\theta$$

Hence,

$$\int_0^r sf^2(0) ds = \int_0^r \frac{s}{2\pi} \int_0^{2\pi} f^2(se^{i\theta}) d\theta ds$$

However, as

$$\int_0^1 sf^2(0) ds = \frac{r^2}{2} f^2(0)$$

and

$$\int_0^1 \frac{s}{2\pi} \int_0^{2\pi} f^2(se)^{i\theta} d\theta ds = \frac{1}{2\pi} \oint_{D(0,r)} f^2$$

we have

$$f^2(0) = \frac{1}{\pi r^2} \oint_{D0,r)} f^2(x,y) dxdy \Rightarrow |f(0)| \leq \frac{1}{r\sqrt{\pi}} \left(\oint_{D(0,r)} |f(x,y)|^2 dxdy \right)^{1/2}$$

as desired.

2. Clearly

$$\left(\int_D |f(x,y)|^2 dxdy \right)^{1/2} \leq \left(\int_U |f(x,y)|^2 dxdy \right)^{1/2}$$

for any disk $D \subseteq U$. Now for any point $x \in K$, there is a disk of maximum radius r_x which is U. The infimum of these r_x is positive, since U is open, and K is compact. Denote $C = 1/(\sqrt{\pi} \inf_x r_x)$. This value of C gives us the claim.

15 Problem 37

15.1 Question

Let $\{p_j\}$ be holomorphic polynomials, and assume that the degree of p_j does not exceed N, all j and some fixed N. If $\{p_j\}$ converges uniformly on compact sets, prove that the limit function is a holomorphic polynomial of degree not exceeding N.

15.2 Answer

Proof. Repeated application of Corollary 3.5.2 implies that the limit f has Nth derivative 0 on compact sets. Therefore, f is a polynomial on compact sets, and by Theorem 3.5.1 everywhere. □

16 Problem 39

16.1 Question

Let $\varphi : D(0,1) \to D(0,1)$ be given by $\varphi(z) = z + a_2 z^2 + \cdots$. Define

$$\begin{aligned} \varphi_1(z) &= \varphi(z), \\ \varphi_2(z) &= \varphi \circ \varphi(z), \\ \vdots \quad & \quad \vdots \\ \varphi_j(z) &= \varphi \circ \varphi_{j-1}(z), \end{aligned}$$

and so forth. Suppose that $\{\varphi_j\}$ converges uniformly on compact sets. What can you say about φ?

16.2 Answer

If $\{\varphi_j\}$ converges uniformly it is to the identity function. Assume that f is the limit, then given $\epsilon > 0$ there is N such that $n > N \Rightarrow |\varphi_n - f| < \epsilon$ therefore, $\varphi_n(\varphi_n) = \varphi_{2n}$ has $|\varphi_{2n} - f| < \epsilon$ as well. However each φ_i is continuous and therefore, $f \circ f = f$ and clearly $f(z) = z$.

Now it's easy to see that $\varphi(z) = z$ as well. For assume not. Then, $f(z) = f(\varphi(z))$ for all z.

17 Problem 42

17.1 Question

Let f be holomorphic on a neighborhood of $\overline{D}(P,r)$. Suppose that f is not identically zero. Prove that f has at most finitely many zeros in $D(P,r)$.

17.2 Answer

Proof. Assume that there are infinitely many zeroes. Then, there is an accumulation point, and by Corollary 3.6.3, f is identically zero, a contradiction. $\square$

18 Problem *45

18.1 Question

Suppose that f is holomorphic on all of $\mathbb{C}$ and that

$$\lim_{n\to\infty}\left(\frac{\partial}{\partial z}\right)^n f(z)$$

exists, uniformly on compact sets, and that this limit is not identically zero. Then the limit function F must be a very particular kind of entire function. Can you say what kind? [*Hint:* If F is the limit function, then F is holomorphic. How is F' related to F?]

18.2 Answer

Clearly $F' = F$. Furthermore, since F is analytic, we can examine its power series. In particular

$$\sum_{k=0}^{\infty} a_k x^k = \sum_{k=0}^{\infty}(k+1)a_{k+1}x^k.$$

Therefore, $a_k = (k+1)a_{k+1}$, and the power series is

$$a_0\sum_{k=0}^{\infty}\frac{1}{k!} = a_0 e^z.$$

19 Problem *47

19.1 Question

This exercise is for those who know some functional analysis. Let $U \subseteq \mathbb{C}$ be a bounded open set. Let

$$X = \{f \in C(\overline{U}) \mid f \text{ is holomorphic on } U\}.$$

If $f \in X$, then define

$$||f|| = \sup_{\overline{U}} |f|.$$

Prove that x equipped with the norm $||\ ||$ is a Banach space. Prove that for any fixed $P \subset U$ and any $k \in \{0, 1, 2, \dots\}$ it holds that the map

$$X \ni f \mapsto \frac{\partial^k f}{\partial z^k}(P)$$

is a bounded linear functional on X.

19.2 Answer

We must show that X is a complete normed vector space. It's easy to see that the only vector with norm 0 is the constant function $f(z) = 0$, and any other function has norm > 0. Also, scalar multiplication, and the triangle inequality follow from the same properties on $\mathbb{C}$.

X is complete

Proof. Assume that $\{f_n\}$ is a sequence of functions in X such that for any $\epsilon > 0$ there exists N such that $||f_m - f_n|| < \epsilon$ for all $n, m > N$.

This implies in particular that $\{f_n\}$ is a uniformly convergent sequence of holomorphic functions. Thus, the sequence converges to another function in X, by Theorem 3.5.1. □

That the derivative at a point is a linear functional is easy to check. In particular the kth derivative of αf at p is α times the kth derivative at that point by the product rule. Furthermore, the kth derivative of the sum of two functions is the sum of the kth derivatives of the functions.

Boundedness of the functional follows from Cauchy estimates, together with the fact that members of X are bounded in absolute value.

$$\left| \frac{\partial^k f}{\partial z^k}(P) \right| \leq Mk!/r^k$$

for $M = \sup_{\overline{U}} |f|$.

20 Problem *61

20.1 Question

What can you say about the zero sets of real analytic functions on $\mathbb{R}^2$? What topological properties do they have? Can they have interior? [*Hint:* Here, by "real analytic function" of the two real variables x and y, we mean a function that can be locally represented as a convergent power series in x and y. How does this differ from the power series representation of a holomorphic function?]

20.2 Answer

The zero sets of a real analytic function on $\mathbb{R}^2$ have no interior. Suppose that they did, then in particular, there is a restriction to a function f_x on $\mathbb{R}$ which has a locally convergent power series representation by assumption, but contains an interval on which the function is zero. This is a contradiction though, because if the function is analytic, then its taylor series must converge. Since f_x is analytic, and therefore continuous, the zero set is closed. Looking at a point on the boundary it is clear that f_x doesn't have a convergent power series representation.