Homework 4

Frederick Robinson

27 April 2011

Chapter 4

1 Problem 5

1.1 Question

Let $P = 0$. classify each of the following as having a removable singularity , a pole, or an essential singularity at P.

1. $$\frac{1}{z}$$

2. $$\sin\frac{1}{z}$$

3. $$\frac{1}{z^3} - \cos z$$

4. $$z \cdot e^{1/z} \cdot e^{-1/z^2}$$

5. $$\frac{\sin z}{z}$$

6. $$\frac{\cos z}{z}$$

7. $$\frac{\sum_{k=2}^{\infty} 2^k z^k}{z^3}$$

1.2 Answer

1. Pole. $\lim_{z\to 0} f(z) = \infty$.
2. Essential Singularity. $\lim_{z\to 0} f(z)$ does not exist.
3. Pole. $\lim_{z\to 0} f(z) = \infty$.
4. Essential Singularity. $\lim_{z\to 0} f(z)$ does not exist.
5. Removable Singularity. $\lim_{z\to 0} f(z) = 1$.
6. Pole. $\lim_{z\to 0} f(z) = \infty$.
7. Pole. $f(z) = \sum_{k=2}^{\infty} 2^k z^{k-3}$ which is a pole by our analysis of Laurent series.

2 Problem 6

2.1 Question

Prove that

$$\sum_{j=1}^{\infty} 2^{-(2^j)} \cdot z^{-j}$$

converges for $z \neq 0$ and defines a function which has an essential singularity at $P = 0$.

2.2 Answer

Proof. Let $|z| = r > 0$.

$$\sum_{j=1}^{\infty} |2^{-(2^j)} \cdot z^{-j}| = \sum_{j=1}^{\infty} 2^{-(2^j)} \cdot r^{-j}$$

Now, compute

$$\limsup_{n\to\infty} \frac{2^{-(2^{n+1})} r^{-(n+1)}}{2^{-(2^n)} r^{-n}} = \limsup_{n\to\infty} 2^{-(2^n)} r^{-1} = 0$$

□

Hence, the series is convergent for $|z| > 0$.

At $z = 0$ there are infinitely many terms of the Laurent series with nonzero coefficient, and negative exponent. Thus, there is an essential singularity here.

3 Problem 8

3.1 Question

Let $U = D(P, r) \setminus \{P\}$. Prove the following two refined versions of Riemann's Theorem

1. If f is holomorphic on U and $\lim_{z \to P}(z-P) \cdot f(z) = 0$, then f continues holomorphically across P (to all of U).
2. If f is holomorphic on U and if
$$\int_U |f(z)|^2 dxdy < \infty,$$
then f extends holomorphically across P.

[*Hint:* First show that if F is a holomorphic function on $D(Q, \epsilon)$, then

$$|F(Q)|^2 \leq \frac{1}{\pi\epsilon^2} \int_{D(Q,\epsilon)} |F(z)|^2 dxdy$$

by using the Cauchy integral formula for F^2 and writing the integral in polar coordinates centered at Q; cf. Exercise 33 in Chapter 3. In detail, observe that

$$\begin{aligned}
\int_{D(Q,\epsilon)} |F(z)|^2 dxdy &= \int_0^\epsilon \left(\int_0^{2\pi} |F^2(re^{i\theta})| d\theta \right) rdr \\
&\geq \int_0^\epsilon \left| \int_0^{2\pi} F^2(re^{i\theta}) d\theta \right| rdr \\
&= \int_0^\epsilon |2\pi F^2(Q)| rdr \\
&= \pi\epsilon^2 |F^2(Q)|.
\end{aligned}$$

Use this to show that F as in the statement of the problem has at most a simple pole or removable singularlity at P (not an essential singularity). Then show that $\int |F|^2 = +\infty$ if F has a pole.]

3.2 Answer

1. Let $f = \sum_{-\infty}^{\infty} a_j x^j$ be the Laurent series of f. The function $z - pf(z)$ has series with coefficients $b_i = a_{i-1}$. Finally, by our classification of laurent series, together with the fact that the limit at P is 0, we have that $b_i = 0$ for $i \leq 0 \Rightarrow a_i = 0$ for $i < 0$, and we have the desired result.

4 Problem 9

4.1 Question

Prove that if $f : D(P, r) \setminus \{P\} \to \mathbb{C}$ has an essential singularity at P, then for each positive integer N there is a sequence $\{z_n\} \subseteq D(P, r) \setminus \{P\}$ with $\lim_{n\to\infty} z_n = P$ and

$$|(z_n - P)^N \cdot f(z_n)| \geq N.$$

[Informally, we can say that f "blows up" faster than any positive power of $1/(z - P)$ along some sequence converging to P.]

4.2 Answer

Proof. Suppose not. Then, there exists N such that for all sequences $\{z_n\}$ converging to P,

$$|(z_n - P)^N \cdot f(z_n)| < N.$$

So, consider the function $g = f(z)(z_n - P)^N$. This is bounded on our region by assumption, and therefore has a removable singularity. However, this means that the Laurent series has no nonzero negative power terms, and thus $f(z) = g(z)(z - P)^{-k}$ has all but finitely many negative power terms zero, a contradiction, since this would mean that f has a pole. □

5 Problem 13

5.1 Question

Calculate the annulus of convergence (including any boundary points) for each of the following Laurent series:

1. $$\sum_{j=-\infty}^{\infty} 2^{-j} z^j$$

2. $$\sum_{j=0}^{\infty} 4^{-j} z^j + \sum_{j=-\infty}^{-1} 3^j z^j$$

3. $$\sum_{j=-\infty}^{\infty} z^j / j^2$$

4.
$$\sum_{j=-\infty, j\neq 0}^{\infty} z^j/j^j$$

5.
$$\sum_{j=-\infty}^{10} z^j/|j|! \quad (0! = 1)$$

6.
$$\sum_{j=-20}^{\infty} j^2 z^j$$

5.2 Answer

1.
$$\sum_{j=-\infty}^{\infty} 2^{-j} z^j = \sum_{j=-\infty}^{\infty} (\frac{z}{2})^j$$

Converges for $0 \leq |z| < 2$: that is, on $D(0,2)$.

2. The series
$$\sum_{j=0}^{\infty} 4^{-j} z^j$$
converges on $0 \leq |z| < 4$, while the series
$$\sum_{j=-\infty}^{-1} 3^j z^j$$
converges for $|z| > 1/3$. Thus, taking these together, we have convergence on $A(0, 1/3, 4)$, but nowhere on the boundary.

3. We use the root test
$$\limsup_{j\to\infty} j^{-2/j} = 1$$
so, the positive part of the series converges on $D(0,1)$. However, the negative part converges on $A(0,1,\infty)$, so if there is any convergence it must be on $|z| = 1$. There is convergence for all such, as
$$\sum_{j=-\infty}^{\infty} 1/j^2 = 2\sum_{1}^{\infty} 1/j^2$$
Hence the answer is just $A(0,1,1)$.

4. If we split the sum, considering it as

$$\sum_{j=-\infty}^{1} z^j/j^J + \sum_{j=1}^{\infty} z^j/j^J$$

it is clearly divergent, as the first piece converges, whereas the second does not.

5. We can discard the (finitely many) terms with $j > 0$. Then, find the radius of convergence in $1/z$ by the root test

$$1/\limsup_{j\to\infty} \sqrt[j]{|j|!} = 0$$

Thus, it converges nowhere.

6. We just discard negative terms and apply the root test to get

$$1/\limsup_{n\to\infty} \sqrt[j]{j^2} = 1$$

So, we have convergence on $D(0,1)$. Clearly there is no convergence for $|z| = 1$.

6 Problem 15

6.1 Question

Make the discussion of pp. 117-118 completely explicit by doing the following proofs and continuing with Exercise 16:

1. Prove that if f is holomorphic on $D(P,r)\setminus\{P\}$ and f has a pole at P, then $1/f$ has a removable singularity at P with the "filled in" value of $1/f$ at P equal to 0.

2. Let k be the order of the zero of $1/f$ at P. Prove that $(z-P)^k f$ has a removable singularity at P.

3. Conversely show that if g is holomorphic on $D(P,r)\setminus\{P\}$, if g is *not* bounded, and if there is a $m < 0 \in \mathbb{Z}$ such that $(z-P)^m g$ is bounded, then g has a pole at P. Prove that the least such m is precisely the order of the pole.

6.2 Answer

1. The function $1/f$ is holomorphic on some disk around P, except potentially at P, since f is.

   Since f has a removable singularity at P, for any $n > 0$ there exists some $\epsilon > 0$ such that $|f(x)| > n$ for all $|x - P| < \epsilon$. Hence, taking $1/f$ we get convergence to 0. Also, observe that $1/f$ is bounded in modulus (by $1/n$) on this disk of radius ϵ. Thus, by Riemann's Theorem, there is a removable singularity, and the "filled in" value is 0 since this is the limit we computed.

2. By definition, f has a pole of order k. Then $f(z - P)^k$ has a Laurent series expansion with coefficients $a_j = 0$ for all $j < 0$, and thus has a removable singularity as claimed.

3. If the assumption is satisfied then $(z - P)^m g$ has a removable singularity by the Riemann removable singularity theorem. Hence, it has a Laurent series expansion with $a_j = 0$ for all $j < 0$. Dividing by $(z - P)^m$ we have a Laurent series for g which has $a_j = 0$ for all $-\infty < j < -k$. This is the Laurent series of a function with a pole of order k or less.

   If we use the minimal k, we get precisely the order.

7 Problem 18

7.1 Question

Let $f : \mathbb{C} \to \mathbb{C}$ be a nonconstant entire function. Define $g(z) = f(1/z)$. Prove that f is a polynomial if and only if g has a pole at 0. In other words, f is transcendental (nonpolynomial) if and only if g has an essential singularity at 0.

7.2 Answer

Clearly, if f is a polynomial, g has a pole at zero. In particular,

Proof. If f is a polynomial, its modulus diverges at infinity. Since, $f(\epsilon) = f(1/\epsilon)$, and $1/\epsilon \to \infty$ as $\epsilon \to 0$, this suffices to show that g has a pole at 0. □

Conversely, if g has a pole at 0, then f is a polynomial.

Proof. Note that since f is entire and holomorphic, it has a Taylor series which converges everywhere. Moreover, our definition of g means that its Laurent series has coefficients $a_j = b_{-j}$ for b_j the Taylor series coefficients of f.

Since g has a pole, it has $a_j = 0$ for all $-\infty < j < -k$. Moreover, as it arises from a Taylor series as explained above, it has $a_j = 0$ for $j > 0$. Thus, only finitely many coefficients for the Taylor series of f are nonzero, and f is a polynomial. □

8 Problem 34a

8.1 Question

Use the calculus of residues to compute the following integral:

$$\frac{1}{2\pi i} \oint_{\partial D(0,5)} f(z) dz$$

where

$$f(z) = \frac{z}{(z+1)(z+2i)}.$$

8.2 Answer

The function f is holomorphic on $D(0,5) \setminus \{-1, -2i\}$. Thus, the integral is by the residue theorem

$$\begin{aligned} \operatorname{Res}(-1) \cdot \operatorname{Ind}(-1) + \operatorname{Res}(-2i) \cdot \operatorname{Ind}(-2i) &= \operatorname{Res}(-1) + \operatorname{Res}(-2i) \\ &= \frac{-1}{-1+2i} + \frac{-2i}{-2i+1} \\ &= 1 \end{aligned}$$

where the second equality comes from Prop 4.5.6.

9 Problem 34d

9.1 Question

Use the calculus of residues to compute the following integral:

$$\frac{1}{2\pi i} \oint_{\gamma} f(z) dz$$

where

$$f(z) = \frac{e^z}{z(z+1)(z+2)}$$

and γ is the negatively (clockwise) oriented triangle with vertices $1 \pm i$ and -3.

9.2 Answer

Our f is holomorphic on $\{\text{interior of } \gamma\} \setminus \{0, -1, -2\}$. Thus, by the residue theorem the integral is just

$$\begin{aligned}\text{Res}(0) \cdot \text{Ind}(0) + \text{Res}(-1) \cdot \text{Ind}(-1) \\ + \text{Res}(2) \cdot \text{Ind}(2) &= \text{Res}(0) + \text{Res}(-1) + \text{Res}(-2) \\ &= \frac{1}{2} - \frac{1}{e} + \frac{1}{2e^2} \\ &= \frac{e^2}{2e^2} - \frac{2e}{2e^2} + \frac{1}{2e^2}\end{aligned}$$

Where again the second equality comes from Prop 4.5.6.

10 Problem 34g

10.1 Question

Use the calculus of residues to compute the following integral:

$$\frac{1}{2\pi i} \oint_\gamma f(z) dz$$

where

$$f(z) = \frac{\sin z}{z(z+2i)^3}$$

and γ is as in Figure 4.12.

10.2 Answer

Our f is holomorphic on $\{\text{interior of } \gamma\} \setminus \{0\}$. Thus, by the residue theorem the integral is just

$$\begin{aligned}\text{Res}(0) \cdot \text{Ind}(0) &= \frac{\sin 0}{(2i)^3} \\ &= 0\end{aligned}$$

Where again the second equality comes from Prop 4.5.6.

11 Problem 47

11.1 Question

Use the calculus of residues to compute the integral

$$\int_{-\infty}^{+\infty} \frac{\cos x}{1+x^4} dx.$$

11.2 Answer

Let's choose γ_R^1 and γ_R^2 as in Figure 4.4. That is, γ_R^1 is a straight line from $-R$ to R, and γ_R^2 is the upper half of the circle centered at the origin and passing through these two points.

Integrating

$$\oint_{\gamma_R^1+\gamma_R^2} \frac{\cos x}{1+z^4}$$

we have by the residue theorem

$$\begin{aligned}
\operatorname{Res}(e^{\pi i/4}) \cdot \operatorname{Ind}(e^{\pi i/4}) + \operatorname{Res}(e^{3\pi i/4}) \cdot \operatorname{Ind}(e^{3\pi i/4}) &= \operatorname{Res}(e^{\pi i/4}) + \operatorname{Res}(e^{3\pi i/4}) \\
&= \frac{\cos e^{\pi i/4}}{1 \cdot i \cdot (1+i)} + \frac{\cos e^{3\pi i/4}}{-1 \cdot i \cdot (-1+i)} \\
&= \frac{\cos e^{\pi i/4}}{i-1} + \frac{\cos e^{3\pi i/4}}{i+1}
\end{aligned}$$

The integral over γ_R^2 is bounded above by $(\pi R)/(1+R^4)$, which goes to 0 in the limit. Therefore, the value computed above is the integral of just the portion we want.

12 Problem 52

12.1 Question

Use the calculus of residues to compute the integral

$$\int_0^{\infty} \frac{1}{p(x)} dx$$

where $p(x)$ is any polynomial with no zeros on the nonnegative real axis.

12.2 Answer

We just use contours that go from 0 to R along the real axis, and from 0 to R along a circle in the 1st quadrant.

The total integral is given by the residue theorem to be

$$2\pi i$$

Clearly the contour away from the real line has integral zero, hence this is the answer we want.