Homework 1

Frederick Robinson

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Chapter 1

1 Problem 29

1.1 Question

Compute each of the following derivatives:

1. $\dfrac{\partial}{\partial z}(x^2 - y)$
2. $\dfrac{\partial}{\partial \overline{z}}(x + y^2)$
3. $\dfrac{\partial^4}{\partial z \partial \overline{z}^3}(xy^2)$
4. $\dfrac{\partial^2}{\partial \overline{z} \partial z}(\overline{z} z^2 - z^3 \overline{z} + 7z)$

1.2 Answer

Recall that

$$\frac{\partial}{\partial z} f = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) f \quad \text{and} \quad \frac{\partial}{\partial \overline{z}} f = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) f.$$

We may therefore compute

1.
$$\frac{\partial}{\partial z}(x^2 - y) = \frac{1}{2}(2x + i)$$

2.

$$\frac{\partial}{\partial \overline{z}}(x+y^2) = \frac{1}{2}(1+2yi)$$

3.

$$\begin{aligned}
\frac{\partial^4}{\partial z \partial \overline{z}^3}(xy^2) &= \frac{\partial^3}{\partial z \partial \overline{z}^2}\frac{\partial}{\partial \overline{z}}(xy^2) \\
&= \frac{\partial^3}{\partial z \partial \overline{z}^2}\frac{1}{2}\left(y^2+2xyi\right) \\
&= \frac{\partial^2}{\partial z \partial \overline{z}}\frac{\partial}{\partial \overline{z}}\frac{1}{2}y^2+xyi \\
&= \frac{\partial^2}{\partial z \partial \overline{z}}\frac{1}{2}(-x+4yi) \\
&= \frac{\partial}{\partial z}\frac{\partial}{\partial \overline{z}}-\frac{1}{2}x+2yi \\
&= \frac{\partial}{\partial z}\frac{1}{2}\left(-\frac{5}{2}\right) \\
&= 0
\end{aligned}$$

Which should have been obvious, since our original polynomial was of degree 3.

4.

$$\begin{aligned}
\frac{\partial^2}{\partial \overline{z} \partial z}(\overline{z}z^2 - z^3\overline{z} + 7z) &= \frac{\partial}{\partial \overline{z}}2\overline{z}z - 3z^2\overline{z} + 7 \\
&= 2z - 3z^2
\end{aligned}$$

2 Problem 36

2.1 Question

Write

$$\frac{\partial}{\partial z} \quad \text{and} \quad \frac{\partial}{\partial \overline{z}}$$

in polar coordinates.

2.2 Answer

We relate polar coordinates to rectangular by

$$x = r\cos\theta \quad y = r\sin\theta \quad r = x^2 + y^2 \quad \theta = \tan^{-1} y/x.$$

Therefore compute

$$\frac{\partial r}{\partial x} = 2x = 2r\cos\theta \quad \frac{\partial r}{\partial y} = 2y = 2r\sin\theta$$

$$\frac{\partial \theta}{\partial x} = -\frac{y}{x^2+y^2} = -\frac{\sin\theta}{r} \quad \frac{\partial \theta}{\partial y} = \frac{x}{x^2+y^2} = \frac{\cos\theta}{r}.$$

Now, recall that for $f = u + iv$ we can write

$$\frac{\partial}{\partial z} f = \frac{1}{2}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \frac{i}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right).$$

Substituting

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} \quad \frac{\partial v}{\partial y} = \frac{\partial v}{\partial r}\frac{\partial r}{\partial y} \quad \frac{\partial v}{\partial x} = \frac{\partial v}{\partial \theta}\frac{\partial \theta}{\partial x} \quad \frac{\partial u}{\partial y} = \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial y}$$

we get

$$\frac{1}{2}\left(\frac{\partial u}{\partial r}2r\cos\theta + \frac{\partial v}{\partial r}2r\sin\theta\right) + \frac{i}{2}\left(\frac{\partial v}{\partial \theta}\frac{-\sin\theta}{r} - \frac{\partial u}{\partial \theta}\frac{\cos\theta}{r}\right),$$

or equivalently

$$(u\cos\theta + v\sin\theta)\left(r\frac{\partial}{\partial r} - \frac{i}{2r}\frac{\partial}{\partial \theta}\right).$$

Similarly, we can write

$$\frac{\partial}{\partial \bar{z}} f = \frac{1}{2}\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) + \frac{i}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)$$

as

$$\frac{1}{2}\left(\frac{\partial u}{\partial r}2r\cos\theta - \frac{\partial v}{\partial r}2r\sin\theta\right) + \frac{i}{2}\left(\frac{\partial v}{\partial \theta}\frac{-\sin\theta}{r} + \frac{\partial u}{\partial \theta}\frac{\cos\theta}{r}\right).$$

3 Problem 43

3.1 Question

Prove that if f is holomorphic on $U \subseteq \mathbb{C}$, then

$$\triangle(|f|^2) = 4\left|\frac{\partial f}{\partial z}\right|^2.$$

3.2 Answer

Proof. Recall that $|f|^2 = f\bar{f}$. Since f is holomorphic, $\frac{\partial}{\partial z} f = 0$, and $\frac{\partial}{\partial \bar{z}} \bar{f} = 0$. Finally, since we can define

$$\triangle f = 4\frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}} f$$

we just compute

$$\begin{aligned}\triangle(|f|^2) &= 4\frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}}|f|^2 \\ &= 4\frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}} f\bar{f} \\ &= 4\frac{\partial}{\partial z} f\frac{\partial \bar{f}}{\partial \bar{z}} \\ &= 4\frac{\partial \bar{f}}{\partial \bar{z}}\frac{\partial f}{\partial z} \\ &= 4\left|\frac{\partial f}{\partial z}\right|^2 -\end{aligned}$$

which is the desired result. □

4 Problem 44

4.1 Question

Prove that if f is holomorphic on $U \subseteq \mathbb{C}$ and f is nonvanishing, then

$$\triangle(|f|^p) = p^2|f|^{p-2}\left|\frac{\partial f}{\partial z}\right|^2, \quad \text{any } p > 0.$$

4.2 Answer

Proof. Recall that $|f|^2 = f\bar{f}$. Thus, we have in general $|f|^p = (f\bar{f})^{p/2} = f^{p/2}\bar{f}^{p/2}$. Since f is holomorphic, $\frac{\partial}{\partial z} f = 0$, and $\frac{\partial}{\partial \bar{z}} \bar{f} = 0$. Finally, since we can define

$$\triangle f = 4\frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}} f$$

we just compute

$$\begin{aligned}\triangle(|f|^p) &= 4\frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}}|f|^p\\ &= 4\frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}}f^{p/2}\bar{f}^{p/2}\\ &= 2p\frac{\partial}{\partial z}f^{p/2}\bar{f}^{p/2-1}\frac{\partial \bar{f}}{\partial \bar{z}}\\ &= p^2 f^{p/2-1}\bar{f}^{p/2-1}\frac{\partial \bar{f}}{\partial \bar{z}}\frac{\partial f}{\partial z}\\ &= p^2|f|^{p-2}\left|\frac{\partial f}{\partial z}\right|^2\end{aligned}$$

which is the desired result. □

5 Problem 45

5.1 Question

Prove that if f is harmonic and real-valued on $U \subseteq \mathbb{C}$ and if f is nonvanishing, then

$$\triangle(|f|^p) = p(p-1)|f|^{p-2}|\nabla f|^2, \quad \text{any } p \geq 1.$$

5.2 Answer

Just compute

$$\begin{aligned}\triangle(|f|^p) &= \frac{\partial}{\partial x^2}|f|^p + \frac{\partial}{\partial y^2}|f|^p\\ &= \frac{\partial}{\partial x}\left(p|f|^{p-1}\frac{\partial f}{\partial x}\right) + \frac{\partial}{\partial y^2}|f|^p\\ &= p|f|^{p-1}\frac{\partial^2 f}{\partial x^2} + \left(\frac{\partial f}{\partial x}\right)^2 p(p-1)|f|^{p-2} + \frac{\partial}{\partial y^2}|f|^p\\ &= p|f|^{p-1}\frac{\partial^2 f}{\partial x^2} + \left(\frac{\partial f}{\partial x}\right)^2 p(p-1)|f|^{p-2} + p|f|^{p-1}\frac{\partial^2 f}{\partial y^2} + \left(\frac{\partial f}{\partial y}\right)^2 p(p-1)|f|^{p-2}\\ &= \left(\frac{\partial f}{\partial x}\right)^2 p(p-1)|f|^{p-2} + \left(\frac{\partial f}{\partial y}\right)^2 p(p-1)|f|^{p-2}\\ &= \left(\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2\right) p(p-1)|f|^{p-2}\end{aligned}$$

$$= |\nabla f|^2 p(p-1)|f|^{p-2}$$

6 Problem 47

6.1 Question

Prove that if f is C^2, holomorphic, and nonvanishing, then $\log|f|$ is harmonic.

6.2 Answer

Proof. If we write $f = u + iv$ we have $\log|f| = (1/2)\log(u^2+v^2)$. Thus, $\log|f|\frac{\partial}{\partial x} = (u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial x})/(u^2+v^2)$ and

$$\log|f|\frac{\partial}{\partial^2 x}$$

$$= \frac{1}{(u^2+v^2)^2}\left(\left(u\frac{\partial^2 u}{\partial x^2} + v\frac{\partial^2 v}{\partial x^2} + \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2\right)(u^2+v^2) - 2\left(u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial x}\right)^2\right).$$

Clearly then we have

$$\log|f|\frac{\partial}{\partial^2 y}$$

$$= \frac{1}{(u^2+v^2)^2}\left(\left(u\frac{\partial^2 u}{\partial y^2} + v\frac{\partial^2 v}{\partial y^2} + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2\right)(u^2+v^2) - 2\left(u\frac{\partial u}{\partial y} + v\frac{\partial v}{\partial y}\right)^2\right).$$

If we sum these values, and use the Cauchy-Riemann equations to simplify we get

$$\triangle(\log|f|) = \frac{1}{(u^2+v^2)}\left(u\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + v\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)\right).$$

However, f is holomorphic, and therefore u and v are harmonic. Hence, as f is nonvanishing, we have 0, and $\log|f|$ is harmonic, as desired. $\square$

7 Problem 48

7.1 Question

Give an explicit description of all harmonic polynomials of second degree. Can you do the same for the third degree?

7.2 Answer

Given an arbitrary second degree polynomial

$$w_5\bar{z}^2 + w_4z^2 + w_3\bar{z}z + w_2\bar{z} + w_1z + w_0,$$

requiring that it be harmonic is equivalent to requiring that the Laplacian be zero, or

$$\triangle f = 4\frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}}f = 0.$$

Computing we have

$$\begin{aligned}
4\frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}}f = 0 &\Leftrightarrow \\
\frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}}f = 0 &\Leftrightarrow \\
\frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}}w_5\bar{z}^2 + w_4z^2 + w_3\bar{z}z + w_2\bar{z} + w_1z + w_0 = 0 &\Leftrightarrow \\
\frac{\partial}{\partial z}2w_5\bar{z} + w_3z + w_2 = 0 &\Leftrightarrow \\
w_3 = 0.
\end{aligned}$$

Therefore, a degree two polynomial is harmonic if and only if it has no terms involving both z and $\bar{z}$. A similar analysis reveals that the same condition holds true for degree three polynomials.

8 Problem 51

8.1 Question

Let (v_1, v_2) be a pair of harmonic functions on a disc $U \subseteq \mathbb{C}$. Suppose that

$$\frac{\partial v_1}{\partial y} = \frac{\partial v_2}{\partial x} \quad \text{and} \quad \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} = 0.$$

Prove that $\langle v_1, v_2\rangle$ is the gradient (i.e., the vector $\langle \partial h/\partial x, \partial h/\partial y\rangle$) of a harmonic function h.

8.2 Answer

By Corollary 1.5.2 we can find a holomorphic function F which has $\text{Re}F = v_1$, $\text{Im}F = v_2$. Then, by Theorem 1.5.3, we can find a holomorphic antiderivative of F. However, the components of this function are just the desired harmonic functions.

9 Problem 52

9.1 Question

The function $f(z) = 1/z$ is holomorphic on $U = \{z \in \mathbb{C} \mid 1 < |z| < 2\}$. Prove that f does not have a holomorphic antiderivative on U. [*Hint:* If there were an antiderivative, then its imaginary part would differ from arg z by a constant.]

9.2 Answer

Proof. Suppose towards a contradiction that such an antiderivative does exists. Suppose further that its imaginary part say, $g(z) = \arg z + \theta_0$ differs from θ, by θ_0. Now, consider this value at $\gamma(t) = 3/2e^{2\pi i t}$. This restriction must be continuous. This is a contradiction though, as

$$g(\gamma(t)) = 2\pi t + \theta_0.$$

Notice that $\gamma(0) = \gamma(1)$, but $g(\gamma(0)) = \theta_0 \neq 2\pi + \theta_0 = g(\gamma(1))$. $\square$