Homework 5

Frederick Robinson

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Chapter 5

1 Problem 1

1.1 Question

Let f be holomorphic on a neighborhood of $\overline{D}(P,r)$. Suppose that f is not identically zero on $D(P,r)$. Prove that f has at most finitely many zeros in $D(P,r)$.

1.2 Answer

Proof. Suppose towards a contradiction that f has infinitely many zeros in $D(P,r)$. Then the set of zeros has a limit point in the closure of $D(P,r)$. So, since f is holomorphic on a neighborhood of $\overline{D}(P,r)$, and we can assume this neighborhood to be connected, $f = 0$ as desired. □

2 Problem 2

2.1 Question

Let f, g be continuous on $\overline{D}(0,1)$, holomorphic on $D(0,1)$. Assume that f has zeros at $P_1, P_2, \ldots, P_k \in D(0,1)$ and no zero in $\partial D(0,1)$. Let γ be the boundary circle of $\overline{D}(0,1)$, traversed counterclockwise. Compute

$$\frac{1}{2\pi i}\oint_\gamma \frac{f'(z)}{f(z)} \cdot g(z)dz.$$

2.2 Answer

If we evaluate this integral on a small neighborhood D_i containing only one P_i we just get

$$\int_{\partial D_i} \frac{f'(z)}{f(z)} g(z) dz = \int_{\partial D_i} \frac{d_i g(z)}{z - P_i} dz = 2\pi i d_i g(P_i)$$

where d_i is the vanishing order of f at P_i.

So, integrating around all of γ we get

$$\sum_i d_i g(P_i)$$

3 Problem 6

3.1 Question

Let $f : D(0,1) \to \mathbb{C}$ be holomorphic and nonvanishing. Prove that f has well-defined holomorphic logarithm on $D(0,1)$ by showing that the differential equation

$$\frac{\partial}{\partial z} g(z) = \frac{f'(z)}{f(z)}$$

has a suitable solution and checking that this solution g does the job.

3.2 Answer

f is nonvanishing, so f'/f is holomorphic. By Thm 1.5.3, there exists a holomorphic solution $g(z)$ in $D(0,1)$. To verify $g = \log f$, we substitute $f = e^g$ into the equation:

$$\frac{f'}{f} = \frac{g' e^g}{e^g} = g'$$

4 Problem 10a

4.1 Question

Estimate the number of zeros of $f(z) = z^8 + 5z^7 - 20$ in $D(0,6)$.

4.2 Answer

We verify that $|5z^7 - 20| \leq 5 \cdot 6^7 + 20 < 6^8 = |z^8|$. Hence, the number of zeros is just 8.

5 Problem 10d

5.1 Question

Estimate the number of zeros of $f(z) = z^{10} + 10ze^{z+1} - 9$ in $D(0, 1)$.

5.2 Answer

If $|z| = 1$, $z \neq -1$ then $|z^{10} - 9| \leq 10 < 10|e^{z+1}| = |10ze^{z+1}|$. If $z = -1$, however we just observe $|z^{10} - 9| = 8 < 10 = |10ze^{z+1}|$. Thus counting the number of zeros of $10ze^{z+1}$ we have 1.

6 Problem 11

6.1 Question

Imitate the proof of the argument principle to prove the following formula: If $f : U \to \mathbb{C}$ is holomorphic in U and invertible, $P \in U$, and if $D(P, r)$ is a sufficiently small disc about P, then

$$f^{-1}(w) = \frac{1}{2\pi i} \oint_{\partial D(P,r)} \frac{\zeta f'(\zeta)}{f(\zeta) - w} d\zeta$$

for all w in some disc $D(f(P), r_1), r_1 > 0$ sufficiently small. Derive from this formula

$$(f^{-1})'(w) = \frac{1}{2\pi i} \oint_{\partial D(P,r)} \frac{\zeta f'(\zeta)}{(f(\zeta) - w)^2} d\zeta.$$

Set $Q = f(P)$. Integrate by parts and use some algebra to obtain

$$(f^{-1})'(w) = \frac{1}{2\pi i} \oint_{\partial D(P,r)} \left(\frac{1}{f(\zeta) - Q} \right) \cdot \left(1 - \frac{w - Q}{f(\zeta) - Q} \right)^{-1} d\zeta. \tag{1}$$

Let a_k be the k^{th} coefficient of the power series expansion of f^{-1} about the point Q:

$$f^{-1}(w) = \sum_{k=0}^{\infty} a_k (w - Q)^k.$$

Then the formula (1) may be expanded and integrated term by term (prove this!) to obtain

$$\begin{aligned} na_n &= \frac{1}{2\pi i}\oint_{\partial D(P,r)} \frac{1}{|f(\zeta)-Q|^n} d\zeta \\ &= \frac{1}{(n-1)!}\left(\frac{\partial}{\partial\zeta}\right)^{n-1} \frac{(\zeta-P)^n}{[f(\zeta)-Q]^n}\Bigg|_{\zeta=P}. \end{aligned}$$

This is called *Lagrange's formula.*

6.2 Answer

By problem 6 applied to $F(z)-\omega$ we have

$$f^{-1}(w) = \frac{1}{2\pi i}\int_{\partial D(P,r)} \frac{F'(\zeta)\zeta}{F(\zeta)} d\zeta = \frac{1}{2\pi i}\int_{\partial D(P,r)} \frac{f'(\zeta)\zeta}{f(\zeta)-w} d\zeta$$

and hence

$$(f^{-1})'(w) = \frac{1}{2\pi i}\int_{\partial D(P,r)} \frac{f'(\zeta)\zeta}{(f(\zeta)-w)^2} d\zeta$$

Integrating by parts, reveals

$$(f^{-1})'(w) = \frac{1}{2\pi i}\int_{\partial D(P,r)} \zeta d\left(\frac{-1}{f(\zeta)-w}\right) = \frac{1}{2\pi i}\int_{\partial D(P,r)} \frac{1}{f(\zeta)-w} d\zeta$$

Write

$$\frac{1}{f(\zeta)-w} = \left(\frac{1}{f(\zeta)-Q}\right)\left(1-\frac{w-Q}{f(\zeta)-Q}\right)^{-1} = \sum_{k=0}^{\infty}\left(\frac{1}{f(\zeta)-Q}\right)^{k+1}(w-Q)^k$$

The radius of convergence is $\inf_\zeta |f(\zeta)-Q| > r_1$. Integrating with respect to w, we have $f^{-1}(w)=\sum_0^\infty a_k(w-Q)^k$ where

$$ka_k = \frac{1}{2\pi i}\int_{\partial D(P,r)} \frac{1}{(f(\zeta)-Q)^k} d\zeta = \frac{1}{(k-1)!}\left(\frac{\partial}{\partial\zeta}\right)^{k-1} \frac{(\zeta-P)^k}{[f(\zeta)-Q]^k}\Bigg|_{\zeta=P}$$

7 Problem 13

7.1 Question

Prove: If f is a polynomial on $\mathbb{C}$, then the zeros of f' are contained in the closed convex hull of the zeros of f. (Here the *closed convex hull* of a set S is the intersection of all closed convex sets that contain S.) [*Hint:* If the zeros of f are contained in a half plane V, then so are the zeros of f'.]

7.2 Answer

First rewrite f as a product of terms $(z - P)$ for all its roots.

$$f(x) = \prod_i (z - P_k).$$

Assume $f'(Q) = 0 \neq f(Q)$ as, if both are zero, then the zero of f' is written as a trivial linear combination of zeros of f.

$$0 = \frac{f'(Q)}{f(Q)} = \sum_i \frac{1}{Q - P_i} = \sum_i \frac{1}{Q - P_i} = \sum_i \frac{\overline{Q} - \overline{P}_i}{|Q - P_i|^2}$$

Denoting

$$x_i = \frac{1/|Q - P_i|^2}{\sum_i 1/|Q - P_i|^2}$$

we have

$$Q = \sum_i x_i P_i$$

But $0 \leq x_i \leq 1, \sum_i x_i = 1$. Thus, we have explicitly demonstrated each zero of the derivative as a convex linear combination of the zeros of the original function, and are done.

8 Problem 14

8.1 Question

Let $P_t(z)$ be a polynomial in z for each fixed value of t, $0 \leq t \leq 1$. Suppose that $P_t(z)$ is continuous in t in the sense that

$$P_t(z) = \sum_{j=0}^{N} a_j(t) z^j$$

and each $a_j(t)$ is continuous. Let $\mathcal{Z} = \{(z, t) \mid P_t(z) = 0\}$. By continuity, $\mathcal{Z}$ is closed in $\mathbb{C} \times [0, 1]$. If $P_{t_0}(z_0) = 0$ and $(\partial/\partial z) P_{t_0}(z)|_{z=z_0} \neq 0$, then show, using the argument principle, that there is an $\epsilon > 0$ such that for t suficiently near t_0 there is a unique $z \in D(z_0, \epsilon)$ with $P_t(z) = 0$. What can you say if $P_{t_0}(\cdot)$ vanishes to order k at z_0?

8.2 Answer

Let

$$N(z,r,t) = \frac{1}{2\pi i}\int_{\partial D(z,r)} \frac{p_t'(\zeta)}{p_t(\zeta)} d\zeta,$$

Then, $N(z_0, r, t_0) = 1$ for some sufficiently small ϵ. As $\partial_z P_{t_0}(z_0) \neq 0$, we have $\partial_z P_t(z_0) \neq 0$ for t suffficiently close to t_0. Hence, for such t, $N(z_0, \epsilon, t)$ is defined and equal to 1.

When the vanishing has order greater than 1, it may split into distinct roots.

9 Problem 18

9.1 Question

Let $p_t(z) = a_0(t) + a_1(t)z + \cdots + a_n(t)z^n$ be a polynomial in which the coefficients depend continuously on a parameter $t \in (-1, 1)$. Prove that if the roots of p_{t_0} are distinct (no multiple roots), for some fixed value of the parameter, then the same is true for p_t when t is sufficiently close to t_0—*provided* that the degree of p_t remains the same as the degree of p_{t_0}.

9.2 Answer

When $a_n(t)$ remains nonzero as t moves, the simpleness of roots is controlled by the discriminant $D(t) = D(p_t)$ of $p(z)$, which is a polynomial of its coefficients. As $D(t) \neq 0$ is an open condition, we are done.