# Final Exam

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5 June 2011

$D(a, r)$ is the disc of radius $r$ centered at $a$.

## 1 Problem 1

### 1.1 Question

Let

$$p_N = z^N + a_{N-1}z^{N-1} + \cdots + a_0$$

be a monic polynomial of degree $N$ and consider the function $|p_N(z)|^2$ on $\mathbb{C}$. The following is a sequence of questions about the critical points and level curves $\{z : |p_N(z)|^2 = t\}$ for the polynomial. In each case, prove that your answer is correct. Note that the leading coefficient equals 1 (i.e. $p_N$ is monic).

(a) Exactly how many local minima does $|p_N(z)|^2$ have on $\mathbb{C}$?

(b) Exactly how many local maxima does it have?

(c) Exactly how many saddle points?

(d) For each $t \in (0, \infty)$, what is the maximum number of connected components of the "level curve" $\{z : |p_N(z)|^2 = t\}$?

(e) Show that $M(r) = \sup_{|z|=r} |p_N(z)|^2$ is increasing with $r$.

(f) Show that $m(r) = r^{-2N} \sup_{|z|=r} |p_N(z)|^2$ is decreasing with $r$. [*Hint:* $z^N p_N(\frac{1}{z})$ is also a polynomial of degree $N$.]

### 1.2 Answer

Assume throughout that $p_N$ is nonconstant.

(a) The function $|p_N(z)|^2$ has as many local minima as $p_N(z)$ has distinct zeros.

*Proof.* If $a$ is a zero of $p_N$ then it is a local minimum of $|p_N|^2$, since $|p_N|^2$ has only finitely many zeros, and is nonnegative. Conversely, if $a$ is a local minimum of $|p_N|^2$ it is a local minimum of $|p_N(a)|$ and must therefore have $|p_N(a)| = |p_N(a)|^2 = 0$ by the minimum modulus principle (apply the maximum modulus principle to $1/z$). □

(b) By the maximum modulus principle (Theorem 5.4.2) there are no local maxima of $|p_N|$, and there may therefore by no local maxima of $|p_N|^2$.

(c) Denote $p_N = u + iv$, and assume $a$ is a critical point of $|p_N|^2$. Then, by definition

$$\frac{\partial}{\partial x}\left(u^2 + v^2\right) = 0 \tag{1}$$

$$2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x} = 0. \tag{2}$$

Similarly,

$$2u\frac{\partial u}{\partial y} + 2v\frac{\partial v}{\partial y} = 0. \tag{3}$$

Then, applying the Cauchy Riemann equations to (3), we have

$$2u\frac{\partial v}{\partial x} = 2v\frac{\partial u}{\partial x}.$$

Combining this, with (1) we have

$$-v^2\frac{\partial v}{\partial x} = u^2\frac{\partial v}{\partial x} \quad \text{and} \quad -u^2\frac{\partial u}{\partial x} = v^2\frac{\partial u}{\partial x}$$

or equivalently,

$$(u^2 + v^2)\frac{\partial v}{\partial x} = 0 \quad \text{and} \quad (u^2 + v^2)\frac{\partial u}{\partial x} = 0$$

However, if $u^2 + v^2 = 0$ at some point it must be a local minimum. Therefore, every saddle point has $\partial v/\partial x = \partial u/\partial x = 0$ (i.e. $\partial/\partial z = 0$).

So, saddle points are all $a$ with $\partial f/\partial z(a) = 0$ which do not have also $f(a) = 0$. This is bounded above by the number of roots of $f$ less one, as multiple roots of $f$ are also roots of $\partial f \partial z$ with multiplicity one less than their multiplicity in $f$.

(d) Denote

$$A = \{f(a) \mid \frac{\partial f}{\partial z}(a) = 0, f(a) \neq 0\}$$

where $f(a)$ appears $n$ times if $n$ is the multiplicity of the root $a$ of $\partial f/\partial z$. This is the set of saddle points with multiplicity. If we denote also

$$C(x) = |\{b \mid b \in A, b > x\}|$$

then the number of connected components of $L_t$ is just $C(t) + 1$.

*Proof.* The number of connected components of the level curve $L_t = \{z : |p_N(z)|^2 = t\}$ is the same as the number of local minima for some sufficiently small $t$ by definition of local minimum, together with the fact that all local minima have $f(m) = 0$.

As we increase $t$ the connected components grow, getting closer together until they reach a member of $A$, where they are tangent at the saddle point. If the saddle point is a multiple root, then more than one connected component is tangent at the same point corresponding to the multiplicity of the root. □

(e) *Proof.* Suppose towards a contradiction that $M$ is not increasing with $r$. Then, by continuity, there exist $r, \epsilon > 0$ such that for every $r' < r + \epsilon, M(r') \leq M(r)$. Fix such $r, \epsilon$, and fix $x$ such that $|x| = r, M(r) = f(x)$.

We must have $f'(x) = 0$ as if $f'(x) \neq 0$, then $f'(x)$ is normal to the circle, and pointing outwards (as $f(x)$ is assumed to be maximal on the circle). Thus, an arbitrarily small increase in the size of the circle will increase the value of $M$, choosing $|x'| = r + \epsilon$ as $x' = (1 + \epsilon/r)(x)$.

So, $x$ is a saddle point, or a local minimum. If it's a local minimum we have the desired contradiction, so assume that it's a saddle point. This too is contradictory however. There is a direction in which the second derivative is positive. If this is tangent to the circle, then $x$ was not a maximal choice, and if it is not, then we may move in this direction to find an $x'$ with $f(x') > f(x)$. □

(f) Per the hint observe that $z^N p_N(\frac{1}{z})$ is a polynomial of degree $N$, and so by the previous part, the following is increasing in $r$

$$\sup_{|z|=r} \left| z^N p_N\left(\frac{1}{z}\right) \right|^2 = r^{2N} \sup_{|z|=r} \left| p_N\left(\frac{1}{z}\right) \right|^2$$

Thus, substituting $z = 1/z$ we have that

$$r^{-2N} \sup_{|z|=r} |p_N(z)|^2$$

is decreasing in $r$, as desired.

# 2 Problem 2

## 2.1 Question

Let $P(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ be a monic polynomial of degree $n \geq 2$. Let $C$ be the boundary of a disc containing all of the zeros of $P(z)$. Evaluate the following integrals:

(a)

$$\int_C \frac{dz}{P(z)}.$$

(b)

$$\int_C \frac{zP'}{P} dz.$$

## 2.2 Answer

(a) We can apply the Residue theorem, to obtain

$$2\pi i \sum_{j=1}^{m} \mathrm{Res}_f(p_j)$$

since the winding number about any pole is just 1. More specifically, if we write

$$P(z) = (z-p_1)^{k_1}(z-p_2)^{k_2}\cdots(z-p_m)^{k_m}$$

that is

$$\int_C \frac{dz}{P(z)} = 2\pi i \sum_{j=1}^{m} \mathrm{Res}_f(p_j)$$
$$= 2\pi i \sum_{j=1}^{m} \frac{1}{(k_j-1)!} \left(\frac{\partial}{\partial z}\right)^{k_j-1} \left((z-p_j)^{k_j} f(z)\right)\Big|_{z=p_j}$$

(b) We can apply the Residue theorem, to obtain

$$2\pi i \sum_{j=1}^{m} \mathrm{Res}_f(p_j)$$

since the winding number about any pole is just 1. Some manipulation reveals that this is just

$$\sum_{j=1}^{m} k_j p_j.$$

# 3 Problem 3

## 3.1 Question

Does there exist a holomorphic function $f : D(0,1) \to D(0,1)$ such that $f(0) = \frac{1}{2}$ and such that $f'(0) = \frac{4}{5}$? Prove that your answer is correct.

## 3.2 Answer

No.

*Proof.* Assume that $f : D(0,1) \to D(0,1)$ is holomorphic with $f(0) = \frac{1}{2}$. By the Schwarz-Pick theorem,

$$f'(0) \leq 1 - \left|\frac{1}{2}\right|^2 = \frac{3}{4}$$

□

# 4 Problem 4

## 4.1 Question

Let $p(z;t) = z^n + a_{n-1}(t)z^{n-1} + \cdots + a_1(t)z + a_0(t)$ be a family of polynomials of degree $n$ with $a_j(t)$ continuous in the real parameter $t \in [0,1]$. Suppose that $p(z;0)$ has $k$ zeros in the disc $|z-a| < r$ and no zeros on the circle $|z-a| = r$.

(a) Show that for sufficiently small $t, p(z;t)$ has $k$ zeros in $|z-a| < r$.

(b) Show that the zeros of $p(z;t)$ are continuous in $t$.

## 4.2 Answer

(a) This is a consequence of Rouché's Theorem. If we denote $f = p(z;0), g = z;\epsilon)$, then for some $\epsilon > 0$, for each $\zeta$ in the boundary of our disk, we have

$$|f(\zeta) - g(\zeta)| < |g(\zeta)|,$$

by continuity of the $a_j$ and since there are no zeros in the boundary.

(b) We need to show that if $p(z;t_0)$ has zero at $p_0$, given $\delta > 0$, there is $\epsilon$, such that the zeros of $f(z; D(t_0,\epsilon))$ are in $D(p_0,\delta)$.

This follows directly from the first part, if we take $a = \delta$.

# 5 Problem 5

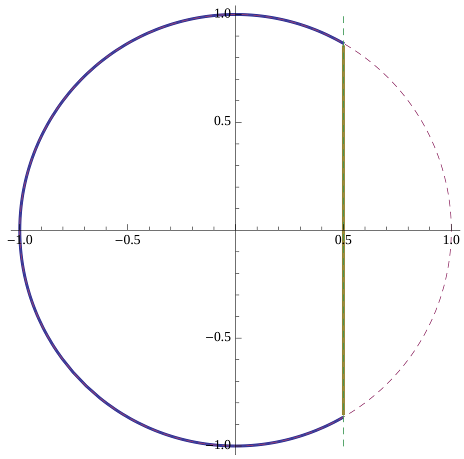
## 5.1 Question

Let $c > 0$, $a > 0 \in \mathbb{R}$. Use the residue calculus to evaluate the integrals

$$\psi(z) := \int_{\Re z = c} \frac{a^z}{z^2} dz$$

## 5.2 Answer

We will integrate around the contour pictured below, say $\gamma_r$ where $\alpha_r$ is the left portion of the circle of radius $r$ centered at 0.




$$\int_{\gamma_r} \psi(z) = \int_{\substack{\Re z = c \\ |z| \leq r}} \psi(z) + \int_{\alpha_r} \psi(z).$$

Observe that

$$\begin{aligned} \left| \int_{\alpha_r} \psi(z) \right| &\leq |\alpha_r| \cdot |\sup_{\alpha_r} \psi(z)| \\ &\leq |\alpha_r| \cdot |\frac{a^c}{r^2}| \\ &\leq r|\frac{a^c}{r^2}| \\ &= |\frac{a^c}{r}| \end{aligned}$$

where $|\alpha_r|$ denotes the length of $\alpha_r$.

Therefore, as $r \to \infty$

$$\int_{\substack{\Re z = c \\ |z| \leq r}} \psi(z) \to \int_{\gamma_r} \psi(z).$$

and by the residue theorem

$$\int_{\gamma_r} \psi(z) = 2\pi i \frac{\partial}{\partial z} a^z \bigg|_{z=0} = 2\pi \log a = \int_{\Re z = c} \psi(z).$$

# 6 Problem 6

## 6.1 Question

Find all entire holomorphic functions $f$ on $\mathbb{C}$ such that $|f(z)| \geq 1$ for all $z$.

### 6.2 Answer

Clearly, any constant value $f(z) = c$ with $|c| \geq 1$ fulfills the requirements. This is the only possibility.

*Proof.* Suppose that $f$ is a finite (nonconstant) polynomial. Then it has zeros, and so cannot have the desired property.

The only other possibility is that $f$ be nonpolynomial. However, by examining the Laurent series of $f(1/z)$ we see that this implies $f$ has an essential singularity at $\infty$, and so the function applied to some neighborhood of $\infty$ is dense in $\mathbb{C}$ a contradiction. $\square$

## 7 Problem 7

### 7.1 Question

Let $U$ be a bounded holomorphically simply connected domain. Let $a \in U$. Suppose that $f : U \to U$ is a holomorphic function such that $f(a) = a$, $|f'(a)| = 1$. Show that $f$ is 1-1 and onto.

### 7.2 Answer

*Proof.* By Schwarz's Lemma $f$ is a rotation about $a$. Thus, $f$ must be injective.

The map $f^{-1} : f(U) \to U$ (the rotation backwards by the same amount as $f$) is well defined as $f$ is injective, and an injection being itself a rotation. Hence, $f$ is bijective, as desired. $\square$

## 8 Problem 8

### 8.1 Question

Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be an entire function.

(a) State the most general Cauchy estimates for $|\frac{d^k}{dz^k} f(0)|$.

(b) Suppose that

$$|f(z)| \leq Ce^{c|z|^\rho}.$$

Show that

$$\limsup_{n\to\infty} |a_n|^{1/n} n^{\frac{1}{\rho}} < \infty. \tag{4}$$

(c) Conversely, suppose that (4) holds. Show that for all $\epsilon > 0$, $|f(z)| \leq Ce^{c|z|^{\rho+\epsilon}}$.

## 8.2 Answer

(a)
$$\left| \frac{d^k}{dz^k} f(0) \right| \leq \frac{Mk!}{r^k}$$
for $M = \sup_{z \in \overline{D}(0,r)} |f(z)|$, $r > 0$.

(b) Since
$$a_k = \frac{1}{k!} \left| \frac{d^k}{dz^k} f(0) \right|,$$
Cauchy estimates imply that
$$a_k \leq \frac{M}{r^k}.$$
Thus,
$$\begin{aligned} |a_n|^{1/n} n^{\frac{1}{\rho}} &\leq \left| \frac{M}{r^n} \right|^{1/n} n^{\frac{1}{\rho}} \\ &= \left| \frac{M^{1/n}}{r} \right| n^{\frac{1}{\rho}} \\ &\leq \left| \frac{(Ce^{c|z|^\rho})^{1/n}}{z} \right| n^{\frac{1}{\rho}} \end{aligned}$$

Choosing $|z| = n^{1/\rho}$ this is
$$|a_n|^{1/n} n^{\frac{1}{\rho}} \leq |Ce^c|$$
As this bound is not dependent on $n$, we have the desired result.

(c) Suppose
$$\limsup_{n \to \infty} |a_n|^{1/n} n^{1/\rho} = C < \infty.$$
Then, for all $\epsilon > 0$ there exists $N$ such that for all $n > N$
$$|a_n|^{1/n} n^{1/\rho} < C + \epsilon.$$

Rewriting, we have

$$|a_n| < \left( \frac{(C+\epsilon)^\rho}{n} \right)^n$$

for sufficiently large $n$. Thus,

$$|f(z)| = \sum_{i=0}^{\infty} |a_i||z|^i < K \left( \sum_{i=0}^{\infty} \left( \frac{(C+\epsilon)^\rho}{i} \right)^i |z|^i \right)$$

where the $K$ is introduced to take care of the first (finitely many) terms until the inequality holds. Now, observe

$$\begin{aligned} |f(z)| &< K \sum_{i=0}^{\infty} \left( \frac{(C+\epsilon)^\rho}{i} |z| \right)^i \\ &< K \sum_{i=0}^{\infty} \frac{1}{i!} ((C+\epsilon)^\rho |z|)^i \\ &= K e^{(C+\epsilon)^\rho |z|}. \end{aligned}$$

After a relabeling of constants, we have

$$|f(z)| \le C e^{c|z|^{\rho+\epsilon}}$$

as desired.

# 9 Problem 9

## 9.1 Question

Let $\{f_n\}$ be a uniformly bounded family of holomorphic functions in an open set $U \subset \mathbb{C}$. Suppose that there exists a subset $E \subset U$ which has an accumulation point in $U$ such that $\lim_{n\to\infty} f_n(w)$ exists for all $w \in E$.

(a) Prove that $f_n$ converges uniformly on compact subsets $K \subset U$ to a holomorphic function $f$.

(b) How is the conclusion stronger than that of Montel's theorem?

## 9.2 Answer

(a) By Montel's theorem, there is some subsequence $\{f_m\} \subseteq \{f_n\}$ which converges normally on $U$ to a limit holomorphic function $f$. Since

$\lim_{n\to\infty} f_n(w)$ exists for each $w \in E$, we know that the whole sequence converges, not just the subsequence:

$$\lim_{n\to\infty} f_n(w) = f(w).$$

However, since $E$ has a limit point, this generalizes to the entire domain $U$ (3.6.3) and

$$\lim_{n\to\infty} f_n(x) = f(x)$$

without restriction.

(b) This is stronger than Montel's theorem since it gives us a particular sequence which converges, not just convergent subsequences.

# 10 Problem 10

## 10.1 Question

Consider the following 'functional' on the space $\mathcal{P}_N$ of monic polynomials of degree $N$:

$$M(P) = \sup_{|z|=1} |P(z)|.$$

Find the polynomial $P$ which *minimizes* $M$, i.e. with the minimum value of $M(P)$. Prove that your answer is correct.

## 10.2 Answer

The desired polynomial is $f_n = z^n \in \mathcal{P}_N$.

*Proof.* Employing Cauchy Estimates on $D(0,1)$, we have

$$\left|\frac{\partial^k f}{\partial z^k}(0)\right| \leq Lk!$$

for $L = \sup_{z\in\overline{D}(0,1)} |f(z)|$. By 1(e) $\sup_{z\in|r|} |f(z)|$ is increasing in $r$. Thus, $L = M(P)$, and

$$\frac{1}{k!}\left|\frac{\partial^k f}{\partial z^k}(0)\right| \leq M(P).$$

For $f$ a monic polynomial of degree $k$, we then have

$$M(P) \geq 1.$$

This bound is tight for the polynomial $f_n = z^n \in \mathcal{P}_N$. □