

- 1 Let  $G$  be a compact Lie group. Show that  $G$  admits a bi-invariant metric, i.e., both right and left translations are isometries. *Hint:* Fix a left invariant metric  $g_L$  and a volume form  $\omega = \sigma^1 \wedge \cdots \wedge \sigma^n$  where  $\sigma^i$  are left invariant 1-forms. Then define  $g$  as the average over right translations:

$$g(v, w) = \frac{1}{\int \omega} \int g_L(DR_x(v), DR_x(w)) \omega.$$

Let  $g_L$  be a left invariant metric on  $G$ , i.e.  $g_L(v, w) = g_L(DL_x(v), DL_x(w))$  for all  $x \in G$ . Suppose also  $\omega = \sigma^1 \wedge \cdots \wedge \sigma^n$  for  $\sigma^i$  left invariant 1-forms, and define as in the hint

$$g(v, w) = \frac{1}{\int \omega} \int g_L(DR_x(v), DR_x(w)) \omega.$$

First compute

$$\begin{aligned} g(DL_y(v), DL_y(w)) &= \frac{1}{\int \omega} \int g_L(DR_x(DL_y(v)), DR_x(DL_y(w))) \omega \\ &= \frac{1}{\int \omega} \int g_L(DL_y(DR_x(v)), DL_y(DR_x(w))) \omega \\ &= \frac{1}{\int \omega} \int g_L(DR_x(v), DR_x(w)) \omega \\ &= g(v, w) \end{aligned}$$

so  $g$  left invariant.

From the right we have

$$\begin{aligned} g(DR_y(v), DR_y(w)) &= \frac{1}{\int \omega} \int g_L(DR_x(DR_y(v)), DR_x(DR_y(w))) \omega \\ &= \frac{1}{\int \omega} \int g_L(DR_{x \cdot y}(v), DR_{x \cdot y}(w)) \omega \\ &= g(v, w) \end{aligned}$$

## 2 Consider the upper-half plane

$$\mathbb{R}_+^2 = \{(x, y) \in \mathbb{R}^2 \mid y > 0\}$$

with the hyperbolic metric

$$\frac{dx^2 + dy^2}{y^2}.$$

Show that the vertical line segment between  $(0, 1)$  and  $(0, 2)$  is the shortest path between these points.

Let  $\gamma(t) = \langle \gamma_x(t), \gamma_y(t) \rangle$  be a path  $\gamma : [0, a] \rightarrow \mathbb{R}_+^2$  with  $\gamma(0) = (0, 1)$  and  $\gamma(a) = (0, 2)$ . Assume without loss of generality that  $\gamma$  is parameterized by (Euclidean) arclength. We can compute

$$|\gamma| = \int_0^a \frac{1}{\gamma_y(t)} \sqrt{\gamma_x'(t)^2 + \gamma_y'(t)^2} dt$$

since we assumed  $\gamma$  arclength parameterized this reduces to

$$|\gamma| = \int_0^a \frac{1}{\gamma_y(t)} dt.$$

By assumption,  $\gamma'_y(t) \in [-1, 1]$ . In particular  $\gamma'_y \leq 1 \Rightarrow \gamma_y(t) \leq 1 + t$ . Notice  $a \geq 1$  since  $\gamma$  is arclength parameterized and the Euclidean distance between its endpoints is 1. Now compute:

$$\begin{aligned} |\gamma| &= \int_0^a \frac{1}{\gamma_y(t)} dt \\ &\geq \int_0^1 \frac{1}{1+t} dt + \underbrace{\int_1^a \frac{1}{\gamma_y(t)} dt}_{(*)} \end{aligned}$$

Of course the contribution of  $(*)$  is strictly positive if  $a > 1$  since  $\gamma_y$  is restricted to take only positive values. Thus, our bound is tight if and only if  $a = 1$  and  $\gamma_y(t) = 1 + t$ , to wit, when

$$\gamma(t) = \langle 0, 1 + t \rangle$$

as desired.

**3 Consider  $\mathbb{R}_+^2$  with the hyperbolic metric as above. Let  $v_0 = (0, 1)$  be a tangent vector at the point  $(0, 1)$  of  $\mathbb{R}_+^2$ . Let  $v(t)$  be the parallel transport of  $v_0$  along the curve  $x = t, y = 1$ . Show that  $v(t)$  makes an angle  $t$  with the direction of the  $y$ -axis, measured in the clockwise sense.**

First note that the nonzero components of the metric and respectively inverse metric are just

$$g^{xx} = g^{yy} = y^2 \quad \text{and} \quad g_{xx} = g_{yy} = \frac{1}{y^2}.$$

We can therefore reduce the expression for Christoffel symbols

$$\Gamma_{ij}^m = \frac{1}{2} \sum_k \left( \frac{\partial}{\partial x_i} g_{jk} + \frac{\partial}{\partial x_j} g_{ki} - \frac{\partial}{\partial x_k} g_{ij} \right) g^{km} = \frac{y^2}{2} \left( \frac{\partial}{\partial x_i} g_{jm} + \frac{\partial}{\partial x_j} g_{mi} - \frac{\partial}{\partial x_m} g_{ij} \right)$$

Clearly this can only be nonzero if at least one of  $i, j, m$  is  $y$  and the others coincide. Thus compute:

$$\Gamma_{yy}^y = \Gamma_{xy}^x = \Gamma_{yx}^x = -\frac{1}{y}; \quad \Gamma_{xx}^y = \frac{1}{y}; \quad \Gamma_{ij}^m = 0 \text{ otherwise.}$$

Now let

$$v(t) = \langle \sin t, \cos t \rangle \quad x(t) = \langle t, 1 \rangle \quad \dot{x}(t) = \langle 1, 0 \rangle$$

and check

$$\frac{dv^x}{dt} + \sum_{i,j} \Gamma_{ij}^x v^j \frac{dx_i}{dt} = \cos t - \frac{1}{y} \cos t = 0 \quad \text{and} \quad \frac{dv^y}{dt} + \sum_{i,j} \Gamma_{ij}^y v^j \frac{dx_i}{dt} = -\sin t + \frac{1}{y} \sin t = 0.$$

**4 For any  $p \in (M, g)$  and orthonormal basis  $e_1, \dots, e_n$  for  $T_p M$ , show that there is an orthonormal frame  $E_1, \dots, E_n$  in a neighborhood of  $p$  such that  $E_i = e_i$  and  $(\nabla E_i)|_p = 0$ . *Hint:* Fix an orthonormal frame  $\bar{E}_i$  near  $p \in M$  with  $\bar{E}_i(p) = e_i$ . If we define  $E_i = \alpha_i^j \bar{E}_j$ , where  $[\alpha_i^j(x)] \in SO(n)$  and  $\alpha_i^j(p) = \delta_i^j$ , then this will yield the desired frame provided that the  $D_{e_k} \alpha_i^j$  are appropriately prescribed.**

Let  $\bar{E}_i$  be an orthonormal frame on a neighborhood of  $p$ . Define  $E_i = \alpha_i^j \bar{E}_j$  and suppose that

$$\nabla_j E_l = \left( \frac{\partial E_l^i}{\partial x^j} + E_l^k \Gamma_{jk}^i \right) e_i = 0 \text{ for all } i, j, l \quad (1)$$

Computing, we get

$$\frac{\partial}{\partial x^j} E_l^i = \alpha_l^m \frac{\partial}{\partial x^j} \bar{E}_m^i + \bar{E}_m^i \frac{\partial}{\partial x^j} \alpha_l^m$$

which reduces, at  $p$  to

$$\frac{\partial}{\partial x^j} E_l^i = \frac{\partial}{\partial x^j} \bar{E}_l^i + \frac{\partial}{\partial x^j} \alpha_l^i.$$

Thus, (1) is equivalent (at  $p$ ) to

$$\begin{aligned} 0 &= \frac{\partial}{\partial x^j} \bar{E}_l^i + \frac{\partial}{\partial x^j} \alpha_l^i + \bar{E}_l^k \Gamma_{jk}^i \\ &= \frac{\partial}{\partial x^j} \bar{E}_l^i + \frac{\partial}{\partial x^j} \alpha_l^i + \Gamma_{jl}^i \\ &= \Gamma_{ij}^l + \frac{\partial}{\partial x^j} \alpha_l^i + \Gamma_{jl}^i \end{aligned}$$

Now, let  $[\alpha_l^i]$  be a family of matrices defined on a neighborhood of  $p$  by setting  $\alpha_i^j(p) = \delta_i^j$  and demanding that  $\alpha_l^i$  satisfy the family of differential equations

$$\frac{\partial}{\partial x^j} \alpha_l^i = -\Gamma_{ij}^l - \Gamma_{jl}^i$$

Since these derivatives are antisymmetric in  $i, l$  the matrices  $[\alpha_l^i]$  must belong to  $SO(n)$ . Furthermore,  $E_i = \alpha_i^j \bar{E}_j$  has the desired  $(\nabla E_i)|_p = 0$  by our previous computation.

- 5 For any point  $p$  in a Riemannian manifold  $(M, g)$ , show that there exist coordinates  $x^1, \dots, x^n$  near  $p$  such that  $\partial_i = e_i$  and  $\nabla \partial_i = 0$  at  $p$ .**
- 6 Let  $(M, g)$  be oriented and define the Riemannian volume form  $d\text{vol}$  as follows:**

$$d\text{vol}(v_1, \dots, v_n) = \det(g(v_i, e_j)),$$

**where  $e_1, \dots, e_n$  is a positively oriented orthonormal basis for  $T_p M$ .**

- 7 Petersen, Chapter 2, Exercise 11 on p. 58**
- 8 Petersen, Chapter 2, Exercise 13 on p. 58**