1 Give examples of Riemannian manifolds having:

1.1 positive scalar curvature but not positive Ricci curvature;

1.2 positive Ricci curvature but not positive sectional curvature;

1.3 positive sectional curvature but not positive curvature operator.

2 Show that a Riemannian manifold with parallel Ricci tensor has constant scalar curvature. In chapter 3 it will be shown that the converse is not true, and also that a metic with parallel curvature tensor doesn't have to be Einstein.

3 Let G be a Lie group with a bi-invariant metric. Using left-invariant fields establish the following formulas. *Hint:* First go back to the exercises to chapter 1 and take a peek at chapter 3 where some of these things are proved.

(You can assume the following which are proved in Proposition 12 on p. 79. I suggest you read that proof.)

$$\nabla_X Y = \tfrac{1}{2}[X,Y]; \quad R(X,Y)Z = \tfrac{1}{4}[Z,[X,Y]]; \quad g(R(X,Y)Z,W) = -\tfrac{1}{4}(g([X,Y],[Z,W]))$$

3.1 Show that the curvature operator is also nonnegative by showing that:

$$g\left(\mathfrak{R}\left(\sum_{i=1}^{k} X_i \wedge Y_i\right), \left(\sum_{i=1}^{k} X_i \wedge Y_i\right)\right) = \frac{1}{4}\left|\sum_{i=1}^{k}[X_i, Y_i]\right|^2.$$

3.2 Show that $\mathrm{Ric}(X,X) = 0$ iff X commutes with all other left-invariant vector fields. Thus G has positive Ricci curvature if the center of G is discrete.

3.3 Consider the linear map $\Lambda^2\mathfrak{g} \to [\mathfrak{g},\mathfrak{g}]$ that sends $X \wedge Y$ to $[X,Y]$. Show that the sectional curvature is positive iff this map is an isomorphism. Conclude that this can only happen if $n = 3$ and $\mathfrak{g} = \mathfrak{su}(2)$.

4 Consider a Riemannian metric (M,g). Now *scale* the metric by multiplying it by a number λ^2. Then we get a new Riemannian manifold $(M,\lambda^2 g)$. Show that the new connection and $(1,3)$-curvature tensor remain the same, but that sec, scal, and $\mathfrak{R}$ all get multiplied by λ^{-2}.

5 Recall that complex manifolds have complex tangent spaces. Thus we can multiply vectors by $\sqrt{-1}$. As a generalization of this we can define an *almost complex* structure. This is a $(1,1)$-tensor J such that $J^2 = -I$. Show that the *Nijenhuis tensor:*

$$N(X,Y) = [J(X),J(Y)] - J([J(X),Y]) - J([X,J(Y)]) - [X,Y]$$

is indeed a tensor. If J comes from a complex structure then $N = 0$, conversely Newlander & Nirenberg have shown that J comes from a complex structure if $N = 0$.

A *Hermitian structure* on a Riemannian manifold (M, g) is an almost complex structure J such that

$$g(J(X), J(Y)) = g(X, Y).$$

The *Kähler form* of a Hermitian structure is

$$\omega(X, Y) = g(J(X), Y).$$

Show that ω is a 2-form. Show that $d\omega = 0$ iff $\nabla J = 0$. If the Kähler form is closed, then we call the metric a Kähler metric.

6 There is a strange curvature quantity we have not yet mentioned. Its definition is somewhat cumbersome and nonintuitive. First, for two symmetric $(0, 2)$-tensors h, k define the *Kulkarni-Nomizu product* as the $(0, 4)$-tensor

$$h \circ k(v_1, v_2, v_3, v_4) = h(v_1, v_3) \cdot k(v_2, v_4) + h(v_2, v_4) \cdot k(v_1, v_3)$$
$$- h(v_1, v_4) \cdot k(v_2, v_3) - h(v_2, v_3) \cdot k(v_1, v_4).$$

Note that (M, g) has constant curvature c iff the $(0, 4)$-curvature tensor satisfies $R = c \cdot (g \circ g)$. If we use the $(0, 2)$ form of the Ricci tensor, then we can decompose the $(0, 4)$-curvature tensor as follows in dimensions $n \geq 4$

$$R = \frac{\textbf{scal}}{2n(n-1)} g \circ g + \left(\textbf{Ric} - \frac{\textbf{scal}}{n} \cdot g \right) \circ g + W$$

When $n = 3$ we have instead

$$R = \frac{\textbf{scal}}{12} g \circ g + \left(\textbf{Ric} - \frac{\textbf{scal}}{3} \cdot g \right) \circ g.$$

The $(0, 4)$-tensor W defined for $n > 3$ is called the *Weyl tensor.*

6.1 Show that these decompositions are orthogonal, in particular:

$$|R|^2 = \left| \frac{\textbf{scal}}{2n(n-1)} g \circ g \right|^2 + \left| \left(\textbf{Ric} - \frac{\textbf{scal}}{n} \cdot g \right) \circ g \right|^2 + |W|^2.$$

6.2 Show that if we conformally change the metric $\tilde{g} = f \cdot g$, then $\tilde{W} = f \cdot W$.

6.3 If (M, g) has constant curvature, then $W = 0$.

6.4 If (M, g) is locally conformally equivalen to the Euclidean metric, i.e., locally we can always find coordinates where:

$$g = f \cdot \left((dx^1)^2 + \cdots + (dx^n)^2 \right),$$

then $W = 0$. The converse is also true but much harder to prove.

7 Assume that we have a Riemannian immersion of an n-manifold into $\mathbb{R}^{n+1}$. If $n \geq 3$, then show that it can't have negative curvature. If $n = 2$ give an example where it does have negative curvature.

8 Let (M, g) be a closed Riemannian n-manifold, and suppose that there is a Riemannian embedding into $\mathbb{R}^{n+1}$. Show that there must be a point $p \in M$ where the curvature operator $\mathfrak{R} : \Lambda^2 T_p M \to \Lambda^2 T_p M$ is positive. *Hint:* Consider $f(x) = |x|^2$ and restrict it to M, then check what happens at a maximum.