1 Let G be a compact Lie group. Show that G admits a bi-invariant metric, i.e., both right and left translations are isometries. *Hint:* Fix a left invariant metric g_L and a volume form $\omega = \sigma^1 \wedge \cdots \wedge \sigma^n$ where σ^i are left invariant 1-forms. Then define g as the average over right translations:

$$g(v,w) = \frac{1}{\int \omega} \int g_L(DR_x(v), DR_x(w))\omega.$$

Let g_L be a left invariant metric on G, i.e. $g_L(v,w) = g_L(DL_x(v), DL_x(w))$ for all $x \in G$. Suppose also $\omega = \sigma^1 \wedge \cdots \wedge \sigma^n$ for σ^i left invariant 1-forms, and define as in the hint

$$g(v,w) = \frac{1}{\int \omega} \int g_L(DR_x(v), DR_x(w))\omega.$$

First compute

$$\begin{aligned} g(DL_y(v), DL_y(w)) &= \frac{1}{\int \omega} \int g_L(DR_x(DL_y(v)), DR_x(DL_y(w)))\omega \\ &= \frac{1}{\int \omega} \int g_L(DL_y(DR_x(v)), DL_y(DR_x(w)))\omega \\ &= \frac{1}{\int \omega} \int g_L(DR_x(v), DR_x(w)) \\ &= g(v,w) \end{aligned}$$

so g left invariant.

From the right we have

$$\begin{aligned} g(DR_y(v), DR_y(w)) &= \frac{1}{\int \omega} \int g_L(DR_x(DR_y(v)), DR_x(DR_y(w)))\omega \\ &= \frac{1}{\int \omega} \int g_L(DR_{x \cdot y}(v), DR_{x \cdot y}(w))\omega \\ &= g(v,w) \end{aligned}$$

2 Consider the upper-half plane

$$\mathbb{R}^2_+ = \{(x,y) \in \mathbb{R}^2 \mid y > 0\}$$

with the hyperbolic metric

$$\frac{dx^2 + dy^2}{y^2}.$$

Show that the vertical line segment between $(0,1)$ and $(0,2)$ is the shortest path between these points.

Let $\gamma(t) = \langle \gamma_x(t), \gamma_y(t) \rangle$ be a path $\gamma : [0,a] \to \mathbb{R}^2_+$ with $\gamma(0) = (0,1)$ and $\gamma(a) = (0,2)$. Assume without loss of generality that γ is parameterized by (Euclidean) arclength. We can compute

$$|\gamma| = \int_0^a \frac{1}{\gamma_y(t)} \left[\left(\frac{\gamma_x'(t)}{1/\gamma_y(t)\left(\gamma_y'(t)^2 + \gamma_x'(t)^2\right)} \right)^2 + \left(\frac{\gamma_y'(t)}{1/\gamma_y(t)\left(\gamma_y'(t)^2 + \gamma_x'(t)^2\right)} \right)^2 \right] dt$$

since we assumed γ arclength parameterized this reduces to

$$|\gamma| = \int_0^a \gamma_y(t) dt.$$

By assumption, $\gamma_y'(t) \in [-1,1]$. In particular $\gamma_y' \leq 1 \Rightarrow \gamma_y(a-t) \geq 2-t$. Notice $a \geq 1$ since γ is arclength parameterized and the Euclidean distance between its endpoints is 1. Now compute:

$$\begin{aligned} |\gamma| &= \int_0^a \gamma_y(t)dt \\ &\geq \underbrace{\int_0^{a-1} \gamma_y(t)dt}_{(*)} + \int_1^0 (2-t)dt \end{aligned}$$

Of course the contribution of $(*)$ is strictly positive if $a > 1$ since γ_y is restricted to take only positive values. Thus, our bound is tight if and only if $a = 1$ and $\gamma_y(a-t) = 2-t$, to wit, when

$$\gamma(t) = \langle 0, 1+t \rangle$$

as desired.

3 Consider $\mathbb{R}^2_+$ with the hyperbolic metric as above. Let $v_0 = (0,1)$ be a tangent vector at the point $(0,1)$ of $\mathbb{R}^2_+$. Let $v(t)$ be the parallel transport of v_0 along the curve $x = t$, $y = 1$. Show that $v(t)$ makes an angle t with the direction of the y-axis, measured in the clockwise sense.

First note that the nonzero components of the metric and respectively inverse metric are just

$$g^{xx} = g^{yy} = y^2 \quad \text{and} \quad g_{xx} = g_{yy} = \frac{1}{y^2}.$$

We can therefore reduce the expression for Christoffel symbols

$$\Gamma_{ij}^m = \frac{1}{2}\sum_k \left(\frac{\partial}{\partial x_i} g_{jk} + \frac{\partial}{\partial x_j} g_{ki} - \frac{\partial}{\partial x_k} g_{ij}\right) g^{km} = \frac{y^2}{2}\left(\frac{\partial}{\partial x_i} g_{jm} + \frac{\partial}{\partial x_j} g_{mi} - \frac{\partial}{\partial x_m} g_{ij}\right)$$

Clearly this can only be nonzero if at least one of i, j, m is y and the others coincide. Thus compute:

$$\Gamma_{yy}^y = \Gamma_{xy}^x = \Gamma_{yx}^x = -\frac{1}{y}; \quad \Gamma_{xx}^y = \frac{1}{y}; \quad \Gamma_{ij}^m = 0 \text{ otherwise.}$$

Now let

$$v(t) = \langle \sin t, \cos t \rangle \quad x(t) = \langle t, 1 \rangle \quad \dot{x}(t) = \langle 1, 0 \rangle$$

and check

$$\frac{dv^x}{dt} + \sum_{i,j} \Gamma_{ij}^x v^j \frac{dx_i}{dt} = \cos t - \frac{1}{y}\cos t = 0 \quad \text{and} \quad \frac{dv^y}{dt} + \sum_{i,j} \Gamma_{ij}^y v^j \frac{dx_i}{dt} = -\sin t + \frac{1}{y}\sin t = 0.$$

4 For any $p \in (M,g)$ and orthonormal basis $e_1, \ldots, e_n$ for T_pM, show that there is an orthonormal frame $E_1, \ldots, E_n$ in a neighborhood of p such that $E_i = e_i$ and $(\nabla E_i)|_p = 0$. *Hint:* Fix an orthonormal frame $\overline{E}_i$ near $p \in M$ with $\overline{E}_i(p) = e_i$. If we define $E_i = \alpha_i^j \overline{E}_j$, where $[\alpha_i^j(x)] \in SO(n)$ and $\alpha_i^j(p) = \delta_i^j$, then this will yield the desired frame provided that the $D_{e_k}\alpha_i^j$ are appropriately prescribed.

5 For any point p in a Riemannian manifold (M,g), show that there exist coordinates $x^1, \ldots, x^n$ near p such that $\partial_i = e_i$ and $\nabla \partial_i = 0$ at p.

6 Let (M,g) be oriented and define the Riemannian volume form $d\mathbf{vol}$ as follows:

$$d\mathbf{vol}(v_1, \ldots, v_n) = \det(g(v_i, e_j)),$$

where $e_1, \dots, e_n$ is a positively oriented orthonormal basis for T_pM.

7 Petersen, Chapter 2, Exercise 11 on p. 58

8 Petersen, Chapter 2, Exercise 13 on p. 58