

- 1 Let  $G$  be a compact Lie group. Show that  $G$  admits a bi-invariant metric, i.e., both right and left translations are isometries. *Hint:* Fix a left invariant metric  $g_L$  and a volume form  $\omega = \sigma^1 \wedge \cdots \wedge \sigma^n$  where  $\sigma^i$  are left invariant 1-forms. Then define  $g$  as the average over right translations:

$$g(v, w) = \frac{1}{\int \omega} \int g_L(DR_x(v), DR_x(w)) \omega.$$

- 2 Consider the upper-half plane

$$\mathbb{R}_+^2 = \{(x, y) \in \mathbb{R}^2 \mid y > 0\}$$

with the hyperbolic metric

$$\frac{dx^2 + dy^2}{y^2}.$$

Show that the vertical line segment between  $(0, 1)$  and  $(0, 2)$  is the shortest path between these points.

Let  $\gamma(t) = \langle \gamma_x(t), \gamma_y(t) \rangle$  be a path  $\gamma : [0, a] \rightarrow \mathbb{R}_+^2$  with  $\gamma(0) = (0, 1)$  and  $\gamma(a) = (0, 2)$ . Assume without loss of generality that  $\gamma$  is parameterized by (Euclidean) arclength. We can compute

$$|\gamma| = \int_0^a \frac{1}{\gamma_y(t)} \left[ \left( \frac{\gamma'_x(t)}{1/\gamma_y(t) (\gamma'_y(t)^2 + \gamma'_x(t)^2)} \right)^2 + \left( \frac{\gamma'_y(t)}{1/\gamma_y(t) (\gamma'_y(t)^2 + \gamma'_x(t)^2)} \right)^2 \right] dt$$

since we assumed  $\gamma$  arclength parameterized this reduces to

$$|\gamma| = \int_0^a \gamma_y(t) dt.$$

By assumption,  $\gamma'_y(t) \in [-1, 1]$ . In particular  $\gamma'_y \leq 1 \Rightarrow \gamma_y(a-t) \geq 2-t$ . Notice  $a \geq 1$  since  $\gamma$  is arclength parameterized and the Euclidean distance between its endpoints is 1. Now compute:

$$\begin{aligned} |\gamma| &= \int_0^a \gamma_y(t) dt \\ &\geq \underbrace{\int_0^{a-1} \gamma_y(t) dt}_{(*)} + \int_1^0 (2-t) dt \end{aligned}$$

Of course the contribution of  $(*)$  is strictly positive if  $a > 1$  since  $\gamma_y$  is restricted to take only positive values. Thus, our bound is tight if and only if  $a = 1$  and  $\gamma_y(a-t) = 2-t$ , to wit, when

$$\gamma(t) = \langle 0, 1+t \rangle$$

as desired.

- 3 Consider  $\mathbb{R}_+^2$  with the hyperbolic metric as above. Let  $v_0 = (0, 1)$  be a tangent vector at the point  $(0, 1)$  of  $\mathbb{R}_+^2$ . Let  $v(t)$  be the parallel transport of  $v_0$  along the curve  $x = t, y = 1$ . Show that  $v(t)$  makes an angle  $t$  with the direction of the  $y$ -axis, measured in the clockwise sense.

First note that the nonzero components of the metric (resp. inverse metric) are just

$$g^{xx} = g^{yy} = y^2 \quad \text{and} \quad g_{xx} = g_{yy} = \frac{1}{y^2}.$$

We can therefore reduce the expression for Christoffel symbols

$$\Gamma_{ij}^m = \frac{1}{2} \sum_k \left( \frac{\partial}{\partial x_i} g_{jk} + \frac{\partial}{\partial x_j} g_{ki} - \frac{\partial}{\partial x_k} g_{ij} \right) g^{km} = \frac{y^2}{2} \left( \frac{\partial}{\partial x_i} g_{jm} + \frac{\partial}{\partial x_j} g_{mi} - \frac{\partial}{\partial x_m} g_{ij} \right)$$

Clearly this can only be nonzero if at least one of  $i, j, m$  is  $y$  and the others coincide. Thus compute:

$$\Gamma_{yy}^y = \Gamma_{xy}^x = \Gamma_{yx}^x = -\frac{1}{y}; \quad \Gamma_{xx}^y = \frac{1}{y}; \quad \Gamma_{ij}^m = 0 \text{ otherwise.}$$

Now let

$$v(t) = \langle \sin t, \cos t \rangle \quad x(t) = \langle t, 1 \rangle \quad \dot{x}(t) = \langle 1, 0 \rangle$$

and check

$$\frac{dv^x}{dt} + \sum_{i,j} \Gamma_{ij}^x v^j \frac{dx_i}{dt} = \cos t - \frac{1}{y} \cos t = 0 \quad \text{and} \quad \frac{dv^y}{dt} + \sum_{i,j} \Gamma_{ij}^y v^j \frac{dx_i}{dt} = -\sin t + \frac{1}{y} \sin t.$$

- 4 For any  $p \in (M, g)$  and orthonormal basis  $e_1, \dots, e_n$  for  $T_p M$ , show that there is an orthonormal frame  $E_1, \dots, E_n$  in a neighborhood of  $p$  such that  $E_i = e_i$  and  $(\nabla E_i)|_p = 0$ . **Hint:** Fix an orthonormal frame  $\bar{E}_i$  near  $p \in M$  with  $\bar{E}_i(p) = e_i$ . If we define  $E_i = \alpha_i^j \bar{E}_j$ , where  $[\alpha_i^j(x)] \in SO(n)$  and  $\alpha_i^j(p) = \delta_i^j$ , then this will yield the desired frame provided that the  $D_{e_k} \alpha_i^j$  are appropriately prescribed.
- 5 For any point  $p$  in a Riemannian manifold  $(M, g)$ , show that there exist coordinates  $x^1, \dots, x^n$  near  $p$  such that  $\partial_i = e_i$  and  $\nabla \partial_i = 0$  at  $p$ .
- 6 Petersen, Chapter 2, Exercise 10 (e) on p. 57
- 7 Petersen, Chapter 2, Exercise 11 on p. 58
- 8 Petersen, Chapter 2, Exercise 13 on p. 58