1 Let G be a compact Lie group. Show that G admits a bi-invariant metric, i.e., both right and left translations are isometries. ***Hint:*** Fix a left invariant metric g_L and a volume form $\omega - \sigma^1 \wedge \cdots \wedge \sigma^n$ where σ^i are left invariant 1-forms. Then define g as the average over right translations:

$$g(v,w) = \frac{1}{\int \omega} \int g_L(DR_x(v), DR_x(w))\omega.$$

2 Consider the upper-half plane

$$\mathbb{R}^2_+ = \{(x,y) \in \mathbb{R}^2 \mid y > 0\}$$

with the hyperbolic metric

$$\frac{dx^2 + dy^2}{y^2}.$$

Show that the vertical line segment between $(0,1)$ and $(0,2)$ is the shortest path between these points.

Let $\gamma(t) = \langle \gamma_x(t), \gamma_y(t) \rangle$ be a path $\gamma : [0,a] \to \mathbb{R}^2_+$ with $\gamma(0) = (0,1)$ and $\gamma(a) = (0,2)$. Assume without loss of generality that γ is parameterized by (Euclidean) arclength. We can compute

$$|\gamma| = \int_0^a \frac{1}{\gamma_y(t)} \left[\left(\frac{\gamma_x'(t)}{1/\gamma_y(t) \left(\gamma_y'(t)^2 + \gamma_x'(t)^2\right)} \right)^2 + \left(\frac{\gamma_y'(t)}{1/\gamma_y(t) \left(\gamma_y'(t)^2 + \gamma_x'(t)^2\right)} \right)^2 \right] dt$$

since we assumed γ arclength parameterized this reduces to

$$|\gamma| = \int_0^a \gamma_y(t) dt.$$

By assumption, $\gamma_y'(t) \in [-1,1]$. In particular $\gamma_y' \leq 1 \Rightarrow \gamma_y'(a-t) \geq 2-t$. Notice $a \geq 1$ since γ is arclength parameterized and the Euclidean distance between its endpoints is 1. Now compute:

$$\begin{aligned} |\gamma| &= \int_0^a \gamma_y(t) dt \\ &\geq \underbrace{\int_0^{a-1} \gamma_y(t) dt}_{(*)} + \int_1^0 (2-t) dt \end{aligned}$$

Of course the contribution of $(*)$ is strictly positive if $a > 1$ since γ_y is restricted to take only positive values. Thus, our bound is tight if and only if $a = 1$ and $\gamma_t'(a-t) = 2-t$, to wit, when

$$\gamma(t) = \langle 0, 1+t \rangle$$

as desired.

3 **Consider $\mathbb{R}^2_+$ with the hyperbolic metric as above. Let $v_0 = (0,1)$ be a tangent vector at the point $(0,1)$ of $\mathbb{R}^2_+$. Let $v(t)$ be the parallel transport of v_0 along the curve $x = t$, $y = 1$. Show that $v(t)$ makes an angle t with the direction of the y-axis, measured in the clockwise sense.**

4 **For any $p \in (M, g)$ and orthonormal basis $e_1, \ldots, e_n$ for T_pM, show that there is an orthonormal frame $E_1, \ldots, E_n$ in a neighborhood of p such that $E_i = e_i$ and $(\nabla E_i)|_p = 0$. *Hint:* Fix an orthonormal frame $\overline{E}_i$ near $p \in M$ with $\overline{E}_i(p) = e_i$. If we define $E_i = \alpha_i^j \overline{E}_j$, where $[\alpha_i^j(x)] \in SO(n)$ and $\alpha_i^j(p) = \delta_i^j$, then this will yield the desired frame provided that the $D_{e_k}\alpha_i^j$ are appropriately prescribed.**

5 **For any point p in a Riemannian manifold (M, g), show that there exist coordinates $x^1, \ldots, x^n$ near p such that $\partial_i = e_i$ and $\nabla \partial_i = 0$ at p.**

6 **Petersen, Chapter 2, Exercise 10 (e) on p. 57**

7 **Petersen, Chapter 2, Exercise 11 on p. 58**

8 **Petersen, Chapter 2, Exercise 13 on p. 58**