1 **Assume that (M, g) has the property that all geodesics exist for a fixed time $\epsilon > 0$. Show that (M, g) is geodesically complete.**

2 **A Riemannian manifold is said to be homogeneous if the isometry group acts transitively. Show that homogeneous manifolds are geodescially complete.**

3 **Let $N \subset (M, g)$ be a submanifold. Let ∇^N denote the connection on N that comes from teh metric induced by g. Define the second fundamental form of N in M by**

$$\mathrm{II}(X, Y) = \nabla^N_X Y - \nabla_X Y$$

Show that $\mathrm{II} = 0$ on N iff N is totally geodesic. (The definition of *totally geodesic* is on p. 145)

4 **Let p be a point in a Riemannian manifold (M, g) and $\sigma \subset T_pM$ a two-dimensional subspace. For small $r > 0$, let $\Sigma_\sigma \subset M$ be the (diffeomorphic) image of $B(0, r) \cap \sigma \subset T_pM$ under the exponential map $\exp_p$. Show that the sectional curvature $\sec(\sigma)$ at p (computed inside M) is equal to the sectional curvature (that is, Gaussian curvature) of the surface Σ_σ at p, in the induced metric.**

5 **Let $SO(n)$ be the Lie group of orthogonal matrices of determinant 1. Equip $SO(n)$ with a bi-invariant Riemannian metric g of volume one, as constructed in the previous homework. The tangent space $T_I SO(n)$ can be identified with the space $\mathfrak{so}(n)$ of skew-adjoint matrices. Show that the exponential map (with respect to g)**

$$\exp_I : \mathfrak{so}(n) \to SO(n)$$

coincides with the usual matrix exponentiation $A \to e^A$.

Hint: **Feel free to use Proposition 12 on p.79 in Petersen's book. Compare also exercise 19 in Petersen, Chapter 5, p.151.**

6 **Let $\gamma : [0,1] \to M$ be a geodesic. Show that $\exp_{\gamma(0)}$ has a critical point at $t\dot{\gamma}(0)$ iff there is a Jacobi field J along γ such that $J(0) = 0$, $\dot{J}(0) \neq 0$, and $J(t) = 0$.**

7 **Let γ be a geodesic and X a Killing field in a Riemannian manifold. Show that the restriction of X to γ is a Jacobi field. (See the definition of a *Killing field* on p.23.)**

8 **A Riemannian manifold is said to be k-*point homogeneous* if for all pairs of points $(p_1, \dots, p_k)$ and $(q_1, \dots, q_k)$ with $d(p_i, p_j) = d(q_i, q_j)$ there is an isometry F with $F(p_i) = q_i$. When $k = 1$ we simply say tat the space is homogeneous.**

8.1 **Show that a homogeneous space has constant scalar curvature.**

8.2 **Show that if $k > 2$ and (M, g) is k-point homogeneous, then M is also $(k-1)$-point homogeneous.**

8.3 **Show that if (M, g) is two-point homogeneous, then (M, g) is an Einstein metric.**

8.4 **Show that if (M, g) is three-point homogeneous, then (M, g) has constant curvature.**

8.5 **Classify all three-point homogeneous spaces. *Hint:* The only one that isn't simply connected is the real projective space.**