

- 1 Let G be a compact Lie group. Show that G admits a bi-invariant metric, i.e., both right and left translations are isometries. *Hint:* Fix a left invariant metric g_L and a volume form $\omega = \sigma^1 \wedge \cdots \wedge \sigma^n$ where σ^i are left invariant 1-forms. Then define g as the average over right translations:

$$g(v, w) = \frac{1}{\int \omega} \int g_L(DR_x(v), DR_x(w)) \omega.$$

Let g_L be a left invariant metric on G , i.e. $g_L(v, w) = g_L(DL_x(v), DL_x(w))$ for all $x \in G$. Suppose also $\omega = \sigma^1 \wedge \cdots \wedge \sigma^n$ for σ^i left invariant 1-forms, and define as in the hint

$$g(v, w) = \frac{1}{\int \omega} \int g_L(DR_x(v), DR_x(w)) \omega.$$

First compute

$$\begin{aligned} g(DL_y(v), DL_y(w)) &= \frac{1}{\int \omega} \int g_L(DR_x(DL_y(v)), DR_x(DL_y(w))) \omega \\ &= \frac{1}{\int \omega} \int g_L(DL_y(DR_x(v)), DL_y(DR_x(w))) \omega \\ &= \frac{1}{\int \omega} \int g_L(DR_x(v), DR_x(w)) \omega \\ &= g(v, w) \end{aligned}$$

so g left invariant.

From the right we have

$$\begin{aligned} g(DR_y(v), DR_y(w)) &= \frac{1}{\int \omega} \int g_L(DR_x(DR_y(v)), DR_x(DR_y(w))) \omega \\ &= \frac{1}{\int \omega} \int g_L(DR_{x \cdot y}(v), DR_{x \cdot y}(w)) \omega \\ &= g(v, w) \end{aligned}$$

2 Consider the upper-half plane

$$\mathbb{R}_+^2 = \{(x, y) \in \mathbb{R}^2 \mid y > 0\}$$

with the hyperbolic metric

$$\frac{dx^2 + dy^2}{y^2}.$$

Show that the vertical line segment between $(0, 1)$ and $(0, 2)$ is the shortest path between these points.

Let $\gamma(t) = \langle \gamma_x(t), \gamma_y(t) \rangle$ be a path $\gamma : [0, a] \rightarrow \mathbb{R}_+^2$ with $\gamma(0) = (0, 1)$ and $\gamma(a) = (0, 2)$. Assume without loss of generality that γ is parameterized by (Euclidean) arclength. We can compute

$$|\gamma| = \int_0^a \frac{1}{\gamma_y(t)} \sqrt{\gamma_x'(t)^2 + \gamma_y'(t)^2} dt$$

since we assumed γ arclength parameterized this reduces to

$$|\gamma| = \int_0^a \frac{1}{\gamma_y(t)} dt.$$

By assumption, $\gamma'_y(t) \in [-1, 1]$. In particular $\gamma'_y \leq 1 \Rightarrow \gamma_y(t) \leq 1 + t$. Notice $a \geq 1$ since γ is arclength parameterized and the Euclidean distance between its endpoints is 1. Now compute:

$$\begin{aligned} |\gamma| &= \int_0^a \frac{1}{\gamma_y(t)} dt \\ &\geq \int_0^1 \frac{1}{1+t} dt + \underbrace{\int_1^a \frac{1}{\gamma_y(t)} dt}_{(*)} \end{aligned}$$

Of course the contribution of $(*)$ is strictly positive if $a > 1$ since γ_y is restricted to take only positive values. Thus, our bound is tight if and only if $a = 1$ and $\gamma_y(t) = 1 + t$, to wit, when

$$\gamma(t) = \langle 0, 1 + t \rangle$$

as desired.

3 Consider \mathbb{R}_+^2 with the hyperbolic metric as above. Let $v_0 = (0, 1)$ be a tangent vector at the point $(0, 1)$ of \mathbb{R}_+^2 . Let $v(t)$ be the parallel transport of v_0 along the curve $x = t, y = 1$. Show that $v(t)$ makes an angle t with the direction of the y -axis, measured in the clockwise sense.

First note that the nonzero components of the metric and respectively inverse metric are just

$$g^{xx} = g^{yy} = y^2 \quad \text{and} \quad g_{xx} = g_{yy} = \frac{1}{y^2}.$$

We can therefore reduce the expression for Christoffel symbols

$$\Gamma_{ij}^m = \frac{1}{2} \sum_k \left(\frac{\partial}{\partial x_i} g_{jk} + \frac{\partial}{\partial x_j} g_{ki} - \frac{\partial}{\partial x_k} g_{ij} \right) g^{km} = \frac{y^2}{2} \left(\frac{\partial}{\partial x_i} g_{jm} + \frac{\partial}{\partial x_j} g_{mi} - \frac{\partial}{\partial x_m} g_{ij} \right)$$

Clearly this can only be nonzero if at least one of i, j, m is y and the others coincide. Thus compute:

$$\Gamma_{yy}^y = \Gamma_{xy}^x = \Gamma_{yx}^x = -\frac{1}{y}; \quad \Gamma_{xx}^y = \frac{1}{y}; \quad \Gamma_{ij}^m = 0 \text{ otherwise.}$$

Now let

$$v(t) = \langle \sin t, \cos t \rangle \quad x(t) = \langle t, 1 \rangle \quad \dot{x}(t) = \langle 1, 0 \rangle$$

and check

$$\frac{dv^x}{dt} + \sum_{i,j} \Gamma_{ij}^x v^j \frac{dx_i}{dt} = \cos t - \frac{1}{y} \cos t = 0 \quad \text{and} \quad \frac{dv^y}{dt} + \sum_{i,j} \Gamma_{ij}^y v^j \frac{dx_i}{dt} = -\sin t + \frac{1}{y} \sin t = 0.$$

4 For any $p \in (M, g)$ and orthonormal basis e_1, \dots, e_n for $T_p M$, show that there is an orthonormal frame E_1, \dots, E_n in a neighborhood of p such that $E_i = e_i$ and $(\nabla E_i)|_p = 0$. *Hint:* Fix an orthonormal frame \bar{E}_i near $p \in M$ with $\bar{E}_i(p) = e_i$. If we define $E_i = \alpha_i^j \bar{E}_j$, where $[\alpha_i^j(x)] \in SO(n)$ and $\alpha_i^j(p) = \delta_i^j$, then this will yield the desired frame provided that the $D_{e_k} \alpha_i^j$ are appropriately prescribed.

Suppose as in the hint, and let $A = [\alpha_i^j]$. We would like

$$0 = D_{e_k}(A\bar{E}) = AD_{e_k}(\bar{E}) + D_{e_k}(A)\bar{E}.$$

At p this is equivalent to putting

$$0 = D_{e_k}(\bar{E}) + D_{e_k}(A) \iff D_{e_k}(A) = -D_{e_k}(\bar{E}).$$

Since $\bar{E} \subset SO(n)$, we have $-D_{e_k}(\bar{E})$ skew symmetric. Thus we can construct a family $A \subset SO(n)$ with the desired derivative.

- 5 For any point p in a Riemannian manifold (M, g) , show that there exist coordinates x^1, \dots, x^n near p such that $\partial_i = e_i$ and $\nabla \partial_i = 0$ at p .

Consider the exponential map $\exp : T_p M \rightarrow M$.

- 6 Let (M, g) be oriented and define the Riemannian volume form $d\text{vol}$ as follows:

$$d\text{vol}(v_1, \dots, v_n) = \det(g(v_i, e_j)),$$

where e_1, \dots, e_n is a positively oriented orthonormal basis for $T_p M$.

- 6.5 Conclude that the Laplacian has the formula

$$\Delta u = \frac{1}{\sqrt{\det(g_{ij})}} \partial_k \left(\sqrt{\det(g_{ij})} g^{kl} \partial_l u \right).$$

Using previous parts we compute

$$\begin{aligned} \Delta u \cdot d\text{vol}(\partial_1, \dots, \partial_n) &= (L_{g^{kl} \partial_l u \partial_k} d\text{vol})(\partial_1, \dots, \partial_n) \\ &= g^{kl} \partial_l u (L_{\partial_k} d\text{vol})(\partial_1, \dots, \partial_n) + d(g^{kl} \partial_l u)(\partial_m) d\text{vol}(\partial_1, \dots, \partial_k, \dots, \partial_n) \\ &= g^{kl} \partial_l u \partial_k \sqrt{\det(g_{ij})} + \partial_k (g^{kl} \partial_l u) \sqrt{\det(g_{ij})} \\ &= \partial_k \left(\sqrt{\det(g_{ij})} g^{kl} \partial_l u \right) \\ &= \frac{1}{\sqrt{\det(g_{ij})}} \partial_k \left(\sqrt{\det(g_{ij})} g^{kl} \partial_l u \right) \end{aligned}$$

- 7 Let (M, g) be a oriented Riemannian manifold with volume form $d\text{vol}$ as above.

- 7.1 If f has compact support, then

$$\int_M \Delta f \cdot d\text{vol} = 0.$$

$$\begin{aligned} \int_M \Delta f \cdot d\text{vol} &= \int_M L_{\nabla f} d\text{vol} \\ &= \int_M i_{\nabla f} d(d\text{vol}) + d(i_{\nabla f} d\text{vol}) \\ &= 0 \end{aligned}$$

- 7.2 Show that

$$\text{div}(f \cdot X) = g(\nabla f, X) + f \cdot \text{div} X$$

$$\begin{aligned} \text{div}(f \cdot X) &= \text{div}(f \cdot X) d\text{vol}(e_1, \dots, e_n) \\ &= (L_{f \cdot X} d\text{vol})(E_1, \dots, E_n) \\ &= f(L_X d\text{vol})(E_1, \dots, E_n) + df(E_i) d\text{vol}(E_1, \dots, X, \dots, E_n) \\ &= f(\text{div} X) d\text{vol}(E_1, \dots, E_n) + g(\nabla f, E_i) g(X, E_i) \\ &= f \cdot \text{div} X + g(\nabla f, X). \end{aligned}$$

7.3 Show that

$$\Delta(f_1 \cdot f_2) = (\Delta f_1) \cdot f_2 + 2g(\nabla f_1, \nabla f_2) + f_1 \cdot (\Delta f_2).$$

$$\begin{aligned} \Delta(f_1 \cdot f_2) &= \operatorname{div}(\nabla(f_1 \cdot f_2)) \\ &= \operatorname{div}(f_1 \cdot \nabla f_2 + f_2 \cdot \nabla f_1) \\ &= f_1 \Delta f_2 + g(\nabla f_1, \nabla f_2) + f_2 \Delta f_1 + g(\nabla f_2, \nabla f_1) \\ &= f_1 \Delta f_2 + 2g(\nabla f_1, \nabla f_2) + f_2 \Delta f_1. \end{aligned}$$

7.4 Establish the integration by parts formula for functions with compact support:

$$\int_M f_1 \cdot \Delta f_2 \cdot d\operatorname{vol} = - \int_M g(\nabla f_1, \nabla f_2) \cdot d\operatorname{vol}.$$

$$\begin{aligned} \int_M f_1 \cdot \Delta f_2 \cdot d\operatorname{vol} &= \int_M f_1 \cdot \operatorname{div}(\nabla f_2) \cdot d\operatorname{vol} \\ &= \int_M (\operatorname{div}(f_1 \cdot \nabla f_2) - g(\nabla f_1, \nabla f_2)) \cdot d\operatorname{vol} \\ &= - \int_M g(\nabla f_1, \nabla f_2) \cdot d\operatorname{vol}. \end{aligned}$$

7.5 Conclude that if f is sub- or superharmonic (i.e., $\Delta f \geq 0$ or $\Delta f \leq 0$) then f is constant. *Hint:* first show $\Delta f = 0$; then use integration by parts on $f \cdot \Delta f$.

Let $\Delta f \geq 0$.

$$0 = \int_M \Delta f \cdot d\operatorname{vol} \geq 0,$$

so $\Delta f = 0$. Thus,

$$0 = \int_M f \cdot \Delta f \cdot d\operatorname{vol} = - \int_M g(\nabla f, \nabla f) \cdot d\operatorname{vol}$$

which implies $\nabla f \equiv 0$, f constant as desired.

8 Let X be a unit vector field on (M, g) such that $\nabla_X X = 0$.

8.1 Show that x is locally the gradient of a distance function iff the orthogonal distribution is integrable.

8.2 Show that X is the gradient of a distance function in a neighborhood of $p \in M$ iff the orthogonal distribution has an integral submanifold through p . *Hint:* It might help to show that $L_X \theta_X = 0$.

8.3 Find X with the given conditions so that it is not a gradient field. *Hint:* Consider S^3