1 In dimensions $4n$ we have that the Hodge $*: H^{2n}(M) \to H^{2n}(M)$ satisfies $** = I$. The difference in the dimensions of the eigenspaces for ± 1 is called the *signature* of M :

$$\tau(M) = \sigma(M) = \dim(\ker(* - I) - \ker(* + I)).$$

One can show that this does not depend on the metric used to define $*$, by observing that it is the index of the symmetric bilinear map

$$H^{2n}(M) \times H^{2n}(M) \to \mathbb{R},$$

$$(\omega_1, \omega_2) \to \int \omega_1 \wedge \omega_2.$$

Recall that the index of a symmetric bilinear map is the difference between positive and negative diagonal elements when it has been put into diagonal form. In dimension 4 one can show that

$$\sigma(M) = \frac{1}{12\pi^2} \int_M \left(\left| W^+ \right|^2 - \left| W^- \right|^2 \right).$$

Using the exercises from chapter 4, show that for an Einstein metric in dimension 4 we have

$$\chi(M) \geq \frac{3}{2}\sigma(M),$$

with equality holding iff the metric is Ricci flat and $W^- = 0$. Conclude that not all four manifolds admit Einstein metrics. In higher dimensions there are no known obstructions to the existence of Einstein metrics. *Hint:* consider connected sums of $\mathbb{CP}^2$ with itself k times.

Petersen, Chapter 7, Exercise 24 on p. 233.

2 Describe all harmonic forms on the following Riemannian manifolds:

2.1 The torus $T^n = S^1 \times \cdots \times S^1$ with the flat product metric;

2.2 The sphere S^n with its round metric;

2.3 The complex projective space $\mathbb{CP}^n$ with the Fubini-Study metric.

Feel free to use (without proof) the computation of the deRham cohomology groups of these spaces. Thus, in each case it suffices to exhibit a basis of harmonic forms of the required cardinality. *Hint:* For 2.3, consider the powers of the Kähler form.

3 Show that the Laplacian on forms commutes with the Hodge star operator.

4 Consider the Laplacian $\Delta : \Omega^p(M) \to \Omega^p(M)$, where M is a closed, oriented Riemannian manifold.

4.1 Prove that the eigenvalues of Δ are nonnegative, and have no finite accumulation points.

4.2 Prove that the eigenspaces of Δ are finite dimensional.

4.3 Prove that the eigenspaces corresponding to distinct eigenvalues are orthogonal.

(*Note:* It can also be shown that Δ has infinitely many eigenvalues, and that the direct sum of all eigenspaces is dense in $\Omega^p(M)$. If you are interested, see Exercise 16 on p.254 in Warner, "Foundations of Differentiable Manifolds and Lie Groups")

5 Let M be a closed, oriented, Riemannian 4-manifold. Let us identify $H^2(M;\mathbb{R})$ with the space of harmonic 2-forms using the Hodge theorem. Note that the star operator $*$ acting on $\Omega^2(M;\mathbb{R})$ satisfies $*^2 = 1$, and therefore we have a direct sum decomposition

$$\Omega^2(M;\mathbb{R}) = \Omega^+(M) \oplus \Omega^-(M),$$

where $\Omega^\pm$ are the eigenspaces of $*$ corresponding to the eigenvalues ± 1. Restricting this decomposition to harmonic forms, we obtain another direct sum decomposition

$$H^2(M;\mathbb{R}) = \mathcal{H}^+ \oplus \mathcal{H}^-,$$

where $\mathcal{H}^\pm$ are the eigenspace of $*$ acting on $H^2(M;\mathbb{R})$, corresponding to eigenvalues ± 1.

Let $d^+ : \Omega^1(M) \to \Omega^+(M)$ be the composition of d with orthogonal projection to Ω^+, and consider the three-term complex

$$\Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d^+} \Omega^+(M).$$

Show that the cohomology groups of this complex can be naturally identified with $H^0(M;\mathbb{R})$, $H^1(M;\mathbb{R})$, and $\mathcal{H}^+(M)$, respectively.