1 Assume that (M, g) has the property that all geodesics exist for a fixed time $\epsilon > 0$. Show that (M, g) is geodesically complete.

Fix $p \in M, v \in T_pM$. By assumption there exists a geodesic $\gamma : (-\epsilon/2, \epsilon/2) \to M$ with $\gamma(0) = p, \dot\gamma(0) = v$. However, taking $p' = \gamma(\epsilon/2), v' = \dot\gamma(\epsilon/2)$ we get another geodesic $\tilde\gamma : (-\epsilon/2, \epsilon/2) \to M$. Since they coincide at p', v' we can extend $\gamma : (-\epsilon/2, \epsilon) \to M$. Repeating this process we produce $\gamma : \mathbb{R} \to M$.

2 A Riemannian manifold is said to be homogeneous if the isometry group acts transitively. Show that homogeneous manifolds are geodesically complete.

Fix $p \in M$ and let $\epsilon > 0$ be the injectivity radius around p. Let $q \in M$. By transitivity there exists an isometry $f : M \to M$ with $f(p) = q$. However, geodesics are taken to geodesics by isometries so the injectivity radius at q is at least ϵ.

Since there is a global injectivity radius we can extend any geodesic indefinitely, as in the previous exercise.

3 Let $N \subset (M, g)$ be a submanifold. Let ∇^N denote the connection on N that comes from the metric induced by g. Define the second fundamental form of N in M by

$$\mathrm{II}(X, Y) = \nabla^N_X Y - \nabla_X Y$$

Show that $\mathrm{II} = 0$ on N iff N is totally geodesic. (The definition of *totally geodesic* is on p. 145)

A submanifold $N \subset (M, g)$ is said to be *totally geodesic* if for each $p \in N$ a neighborhood of $0 \in T_pN$ is mapped into N via the exponential map $\exp_p$.

Suppose that $\mathrm{II} = 0$ and γ a geodesic in N. Then $\nabla_{\dot\gamma}\dot\gamma = \nabla^N_{\dot\gamma}\dot\gamma = 0$

Conversely, suppose that N is totally geodesic, and recall

$$\mathrm{II}(X, Y) = \frac{1}{2}\left(\mathrm{II}(X + Y, X + Y) - \mathrm{II}(X, X) - \mathrm{II}(Y, Y)\right).$$

It therefore suffices to show that $\mathrm{II}(v, v) = 0$ for all $v \in T_pN$, $p \in N$. Let $\gamma : [0, 1] \to N$ the geodesic with $\gamma(0) = p, \dot\gamma(0) = v$. Notice $\nabla^N_{\dot\gamma}\dot\gamma = \nabla_{\dot\gamma}\dot\gamma = 0$, so $\mathrm{II}(v, v) = 0$

4 Let p be a point in a Riemannian manifold (M, g) and $\sigma \subset T_pM$ a two-dimensional subspace. For small $r > 0$, let $\Sigma_\sigma \subset M$ be the (diffeomorphic) image of $B(0, r) \cap \sigma \subset T_pM$ under the exponential map $\exp_p$. Show that the sectional curvature $\sec(\sigma)$ at p (computed inside M) is equal to the sectional curvature (that is, Gaussian curvature) of the surface Σ_σ at p, in the induced metric.

Clearly Σ_σ is totally geodesic. Thus, by the above exercise $\nabla^{\Sigma_\sigma}_X Y = \nabla_X Y$ and, if u, v a basis for σ

$$\begin{aligned}
\sec(\sigma) &= \frac{\langle R(u, v)v, u\rangle}{\langle u, u\rangle\langle v, v\rangle - \langle u, v\rangle^2} \\
&= \frac{\langle \nabla_u\nabla_v v - \nabla_v\nabla_u v - \nabla_{[u,v]}v, u\rangle}{\langle u, u\rangle\langle v, v\rangle - \langle u, v\rangle^2} \\
&= \frac{\langle \nabla^{\Sigma_\sigma}_u\nabla^{\Sigma_\sigma}_v v - \nabla^{\Sigma_\sigma}_v\nabla^{\Sigma_\sigma}_u v - \nabla^{\Sigma_\sigma}_{[u,v]}v, u\rangle}{\langle u, u\rangle\langle v, v\rangle - \langle u, v\rangle^2} \\
&= \sec^{\Sigma_\sigma}(\sigma)
\end{aligned}$$

5 Let $SO(n)$ be the Lie group of orthogonal matrices of determinant 1. Equip $SO(n)$ with a bi-invariant Riemannian metric g of volume one, as constructed in the previous homework. The tangent space $T_I SO(n)$ can be identified with the space $\mathfrak{so}(n)$ of skew-adjoint matrices. Show that the exponential map (with respect to g)

$$\exp_I : \mathfrak{so}(n) \to SO(n)$$

coincides with the usual matrix exponentiation $A \to e^A$.

Hint: **Feel free to use Proposition 12 on p.79 in Petersen's book. Compare also exercise 19 in Petersen, Chapter 5, p.151.**

Let $\gamma_A(t) = e^{tA}$. Clearly $\gamma_A(0) = I, \dot\gamma_A(0) = A, \gamma(1) = e^A$. Thus, if γ is a geodesic we have the desired result. Since we have left-invariance we can compute $\nabla_{\dot\gamma}\dot\gamma = \frac{1}{2}[\dot\gamma, \dot\gamma] = 0$.

6 Let $\gamma : [0,1] \to M$ be a geodesic. Show that $\exp_{\gamma(0)}$ has a critical point at $t\dot\gamma(0)$ iff there is a Jacobi field J along γ such that $J(0) = 0$, $\dot J(0) \neq 0$, and $J(t) = 0$.

Suppose that such a J exists, and let γ_τ be the one parameter family of geodesics specified by J. Because $J(0) = J(t) = 0$, we have $\gamma_\tau(0) = \gamma(0)$, $\gamma(t) = \gamma_\tau(t)$. Because $\dot J \neq 0$, γ_τ is not the trivial family of geodesics. Consider then the path $\alpha(\tau) = t\dot\gamma_\tau(0) \subseteq T_{\gamma(0)}M$. By construction $\frac{d}{d\tau}\exp_{\gamma(0)}\alpha(\tau) = 0$.

Conversely, suppose that $\exp_{\gamma(0)}$ has a critical point at $t\dot\gamma(0)$. If w is the direction of zero derivative put

$$J(t) = (dexp_{\gamma(0)})_{t\dot\gamma(0)}(tw)$$

7 Let γ be a geodesic and X a Killing field in a Riemannian manifold. Show that the restriction of X to γ is a Jacobi field. (See the definition of a *Killing field* on p.23.)

Let X a Killing field (i.e. $L_X g = 0$), and let $\{E_1, E_2, \dots, E_n\}$ be a parallel orthonormal frame along γ with $\dot\gamma = \lambda E_1$. Compute:

$$\begin{aligned}
0 &= (L_X g)(\dot\gamma, E_i) \\
&= D_X(g(\dot\gamma, E_i)) - g(L_X\dot\gamma, E_i) - g(\dot\gamma, L_X E_i) \\
&= -g(\nabla_X\dot\gamma, E_i) + g(\nabla_{\dot\gamma}X, E_i) - g(\dot\gamma, \nabla_X E_i) + g(\dot\gamma, \nabla_{E_i}X) \\
&= g(\nabla_{\dot\gamma}X, E_i) + g(\dot\gamma, \nabla_{E_i}X)
\end{aligned}$$

so differentiating we have

$$\begin{aligned}
0 &= D_{\dot\gamma}(g(\nabla_{\dot\gamma}X, E_i)) + D_{\dot\gamma}(g(\dot\gamma, \nabla_{E_i}X)) \\
&= g(\nabla_{\dot\gamma}\nabla_{\dot\gamma}X, E_i) + g(\dot\gamma, \nabla_{\dot\gamma}\nabla_{E_i}X) \\
&= g(\nabla_{\dot\gamma}\nabla_{\dot\gamma}X, E_i) + g(\dot\gamma, \nabla_{\dot\gamma}\nabla_{E_i}X - \nabla_{E_i}\nabla_{\dot\gamma}X - \nabla_{[\dot\gamma, E_i]}X)
\end{aligned}$$

since

$$0 = g(\nabla_{E_i}\dot\gamma, \nabla_{\dot\gamma}X) + g(\dot\gamma, \nabla_{\nabla_{E_i}\dot\gamma}X) = D_{E_i}(g(\dot\gamma, \nabla_{\dot\gamma}X)) - g(\nabla_{E_i}\dot\gamma, \nabla_{\dot\gamma}X) - g(\dot\gamma, \nabla_{\nabla_{E_i}\dot\gamma}X) = g(\dot\gamma, \nabla_{E_i}\nabla_{\dot\gamma}X + \nabla_{[\dot\gamma, E_i]}X)$$

so

$$\begin{aligned}
0 &= g(\nabla_{\dot\gamma}\nabla_{\dot\gamma}X, E_i) + g(R(\dot\gamma, E_i)X, \dot\gamma) \\
&= g(\nabla_{\dot\gamma}\nabla_{\dot\gamma}X, E_i) + g(R(X, \dot\gamma)\dot\gamma, E_i) \\
&= \nabla_{\dot\gamma}\nabla_{\dot\gamma}X + R(X, \dot\gamma)\dot\gamma
\end{aligned}$$

8 A Riemannian manifold is said to be k-*point homogeneous* if for all pairs of points $(p_1, \ldots, p_k)$ and $(q_1, \ldots, q_k)$ with $d(p_i, p_j) = d(q_i, q_j)$ there is an isometry F with $F(p_i) = q_i$. When $k = 1$ we simply say that the space is homogeneous.

8.1 Show that a homogeneous space has constant scalar curvature.

Since there is an isometry taking $p \mapsto q$ for all p, q and scalar curvature is a local condition, it must be the same throughout the manifold.

8.2 Show that if $k > 2$ and (M, g) is k-point homogeneous, then M is also $(k-1)$-point homogeneous.

Consider the pairs $(p_1, \ldots, p_{k-1}, p_{k-1})$ and $(q_1, \ldots, q_{k-1}, q_{k-1})$

8.3 Show that if (M, g) is two-point homogeneous, then (M, g) is an Einstein metric.

We'll show that $\mathrm{Ric}_p(v, v) = \mathrm{Ric}_q(w, w)$. By symmetry of the Ricci tensor $\mathrm{Ric}_p(v, v) = \mathrm{Ric}_p(w, w)$ is enough to conclude that Ric is Einstein at p, so this is sufficient to show the desired result.

Let $p, q \in M$, $v \in T_pM, w \in T_qM$. There exists some $\epsilon > 0$, geodesics $\gamma, \tilde{\gamma} : (-\epsilon, \epsilon) \to M$ with $\gamma(0) = p, \dot{\gamma}(0) = v, \tilde{\gamma}(0) = q, \dot{\tilde{\gamma}}(0) = w$. The isometry taking $f(p) = q, f(\gamma(\epsilon)) = \tilde{\gamma}(\epsilon)$ must have $D_f(v) = w$.

8.4 Show that if (M, g) is three-point homogeneous, then (M, g) has constant curvature.

We argue as above, this time constructing an isometry $f(p) = q$ with $D_f(\sigma) = \tau$ for $p, q \in M$, σ, τ planes in the respective tangent bundles.

Thus, we can take $\sec_p(\sigma) = \sec_q(\tau)$ as desired.

8.5 Classify all three-point homogeneous spaces. *Hint:* The only one that isn't simply connected is the real projective space.