1 Assume that (M,g) has the property that all geodesics exist for a fixed time $\epsilon > 0$. Show that (M,g) is geodesically complete.

Fix $p \in M, v \in T_pM$. By assumption there exists a geodesic $\gamma : (-\epsilon/2, \epsilon/2) \to M$ with $\gamma(0) = p, \dot{\gamma}(0) = v$. However, taking $p' = \gamma(\epsilon/2), v' = \dot{\gamma}(\epsilon/2)$ we get another geodesic $\tilde{\gamma} : (-\epsilon/2, \epsilon/2) \to M$. Since they coincide at p', v' we can extend $\gamma : (-\epsilon/2, \epsilon) \to M$. Repeating this process we produce $\gamma : \mathbb{R} \to M$.

2 A Riemannian manifold is said to be homogeneous if the isometry group acts transitively. Show that homogeneous manifolds are geodesically complete.

Fix $p \in M$ and let $\epsilon > 0$ be the injectivity radius around p. Let $q \in M$. By transitivity there exists an isometry $f : M \to M$ with $f(p) = q$. However, geodesics are taken to geodesics by isometries so the injectivity radius at q is at least ϵ.

Since there is a global injectivity radius we can extend any geodesic indefinitely, as in the previous exercise.

3 Let $N \subset (M,g)$ be a submanifold. Let ∇^N denote the connection on N that comes from the metric induced by g. Define the second fundamental form of N in M by

$$\mathrm{II}(X,Y) = \nabla^N_X Y - \nabla_X Y$$

Show that $\mathrm{II} = 0$ on N iff N is totally geodesic. (The definition of *totally geodesic* is on p. 145)

A submanifold $N \subset (M,g)$ is said to be *totally geodesic* if for each $p \in N$ a neighborhood of $0 \in T_pN$ is mapped into N via the exponential map $\exp_p$.

4 Let p be a point in a Riemannian manifold (M,g) and $\sigma \subset T_pM$ a two-dimensional subspace. For small $r > 0$, let $\Sigma_\sigma \subset M$ be the (diffeomorphic) image of $B(0,r) \cap \sigma \subset T_pM$ under the exponential map $\exp_p$. Show that the sectional curvature $\sec(\sigma)$ at p (computed inside M) is equal to the sectional curvature (that is, Gaussian curvature) of the surface Σ_σ at p, in the induced metric.

Clearly Σ_σ is totally geodesic. Thus, by the above exercise $\nabla^{\Sigma_\sigma}_X Y = \nabla_X Y$ and, if u, v a basis for σ

$$\begin{aligned}
\sec(\sigma) &= \frac{\langle R(u,v)v, u\rangle}{\langle u,u\rangle\langle v,v\rangle - \langle u,v\rangle^2} \\
&= \frac{\langle \nabla_u \nabla_v v - \nabla_v \nabla_u v - \nabla_{[u,v]} v, u\rangle}{\langle u,u\rangle\langle v,v\rangle - \langle u,v\rangle^2} \\
&= \frac{\langle \nabla^{\Sigma_\sigma}_u \nabla^{\Sigma_\sigma}_v v - \nabla^{\Sigma_\sigma}_v \nabla^{\Sigma_\sigma}_u v - \nabla^{\Sigma_\sigma}_{[u,v]} v, u\rangle}{\langle u,u\rangle\langle v,v\rangle - \langle u,v\rangle^2} \\
&= \sec^{\Sigma_\sigma}(\sigma)
\end{aligned}$$

5 Let $SO(n)$ be the Lie group of orthogonal matrices of determinant 1. Equip $SO(n)$ with a bi-invariant Riemannian metric g of volume one, as constructed in the previous homework. The tangent space $T_I SO(n)$ can be identified with the space $\mathfrak{so}(n)$ of skew-adjoint matrices. Show that the exponential map (with respect to g)

$$\exp_I : \mathfrak{so}(n) \to SO(n)$$

coincides with the usual matrix exponentiation $A \to e^A$.

Hint: **Feel free to use Proposition 12 on p.79 in Petersen's book. Compare also exercise 19 in Petersen, Chapter 5, p.151.**

Let $\gamma_A(t) = e^{tA}$. Clearly $\gamma_A(0) = I, \dot{\gamma}_A(0) = A, \gamma(1) = e^A$. Thus, if γ is a geodesic we have the desired result. But compute $\nabla_{\dot{\gamma}}\dot{\gamma} = \frac{1}{2}[\dot{\gamma}, \dot{\gamma}] = 0$.

6 Let $\gamma : [0,1] \to M$ be a geodesic. Show that $\exp_{\gamma(0)}$ has a critical point at $t\dot{\gamma}(0)$ iff there is a Jacobi field J along γ such that $J(0) = 0$, $\dot{J}(0) \neq 0$, and $J(t) = 0$.

Suppose that such a J exists, and let γ_τ be the one parameter family of geodesics specified by J. Because $J(0) = J(t) = 0$, we have $\gamma_\tau(0) = \gamma(0)$, $\gamma(t) = \gamma_\tau(t)$. Because $\dot{J} \neq 0$, γ_τ is not the trivial family of geodesics. Consider then the path $\alpha(\tau) = t\dot{\gamma_\tau}(0) \subseteq T_{\gamma(0)}M$. By construction $\frac{d}{d\tau}\exp_{\gamma(0)}\alpha(\tau) = 0$.

Conversely, suppose that $\exp_{\gamma(0)}$ has a critical point at $t\dot{\gamma}(0)$. If w is the direction of zero derivative put

$$J(t) = (dexp_{\gamma(0)})_{t\dot{\gamma}(0)}(tw)$$

7 Let γ be a geodesic and X a Killing field in a Riemannian manifold. Show that the restriction of X to γ is a Jacobi field. (See the definition of a *Killing field* on p.23.)

Let X a Killing field (i.e. $L_X g = 0$), and let $\{E_1, E_2, \dots, E_n\}$ be a parallel orthonormal frame along γ with $\dot{\gamma} = \lambda E_1$. Compute:

$$\begin{aligned}
0 &= (L_X g)(\dot{\gamma}, E_i) \\
&= D_X(g(\dot{\gamma}, E_i)) - g(L_X\dot{\gamma}, E_i) - g(\dot{\gamma}, L_X E_i) \\
&= -g(\nabla_X\dot{\gamma}, E_i) + g(\nabla_{\dot{\gamma}}X, E_i) - g(\dot{\gamma}, \nabla_X E_i) + g(\dot{\gamma}, \nabla_{E_i}X) \\
&= g(\nabla_{\dot{\gamma}}X, E_i) + g(\dot{\gamma}, \nabla_{E_i}X)
\end{aligned}$$

so differentiating we have

$$\begin{aligned}
0 &= D_{\dot{\gamma}}(g(\nabla_{\dot{\gamma}}X, E_i)) + D_{\dot{\gamma}}(g(\dot{\gamma}, \nabla_{E_i}X)) \\
&= g(\nabla_{\dot{\gamma}}\nabla_{\dot{\gamma}}X, E_i) + g(\dot{\gamma}, \nabla_{\dot{\gamma}}\nabla_{E_i}X) \\
&= g(\nabla_{\dot{\gamma}}\nabla_{\dot{\gamma}}X, E_i) + g(\dot{\gamma}, \nabla_{\dot{\gamma}}\nabla_{E_i}X - \nabla_{E_i}\nabla_{\dot{\gamma}}X - \nabla_{[\dot{\gamma},E_i]}X)
\end{aligned}$$

since

$$0 = g(\nabla_{E_i}\dot{\gamma}, \nabla_{\dot{\gamma}}X) + g(\dot{\gamma}, \nabla_{\nabla_{E_i}\dot{\gamma}}X) = D_{E_i}(g(\dot{\gamma}, \nabla_{\dot{\gamma}}X)) - g(\nabla_{E_i}\dot{\gamma}, \nabla_{\dot{\gamma}}X) - g(\dot{\gamma}, \nabla_{\nabla_{E_i}\dot{\gamma}}X) = g(\dot{\gamma}, \nabla_{E_i}\nabla_{\dot{\gamma}}X + \nabla_{[\dot{\gamma},E_i]}X)$$

so

$$\begin{aligned}
0 &= g(\nabla_{\dot{\gamma}}\nabla_{\dot{\gamma}}X, E_i) + g(R(\dot{\gamma}, E_i)X, \dot{\gamma}) \\
&= g(\nabla_{\dot{\gamma}}\nabla_{\dot{\gamma}}X, E_i) + g(R(X, \dot{\gamma})\dot{\gamma}, E_i) \\
&= \nabla_{\dot{\gamma}}\nabla_{\dot{\gamma}}X + R(X, \dot{\gamma})\dot{\gamma}
\end{aligned}$$

8 A Riemannian manifold is said to be k-*point homogeneous* if for all pairs of points $(p_1, \ldots, p_k)$ and $(q_1, \ldots, q_k)$ with $d(p_i, p_j) = d(q_i, q_j)$ there is an isometry F with $F(p_i) = q_i$. When $k = 1$ we simply say that the space is homogeneous.

8.1 Show that a homogeneous space has constant scalar curvature.

8.2 Show that if $k > 2$ and (M, g) is k-point homogeneous, then M is also $(k-1)$-point homogeneous.

8.3 Show that if (M, g) is two-point homogeneous, then (M, g) is an Einstein metric.

8.4 Show that if (M, g) is three-point homogeneous, then (M, g) has constant curvature.

8.5 Classify all three-point homogeneous spaces. *Hint:* The only one that isn't simply connected is the real projective space.