1 Give examples of Riemannian manifolds having:

1.1 positive scalar curvature but not positive Ricci curvature;

Let $M = S^n_a \times S^m_b$, and X, Y, U, V vector fields with X, Y tangent to S^n, and U, V tangent to S^m. Recall (class, or Petersen p.65) that

$$\mathfrak{R}(X \wedge V) = 0 \quad \mathfrak{R}(X \wedge Y) = aX \wedge Y \quad \mathfrak{R}(U \wedge V) = bU \wedge V$$

$$\mathrm{Ric}(X) = (n-1)aX \quad \mathrm{Ric}(V) = (m-1)bV \quad \mathrm{scal} = n(n-1)a + m(m-1)b.$$

In particular if we put $M = S^1 \times S^2$ $(n = 1, m = 2)$ we have $\mathrm{scal} = 2b > 0$, but $\mathrm{Ric}(X) = 0$.

1.2 positive Ricci curvature but not positive sectional curvature;

As above, but let $M = S^2 \times S^2$ $(m = n = 2)$. Now $\mathrm{Ric}(X) = aX > 0$, $\mathrm{Ric}(V) = bV > 0$, and $\sec(x, v) = \mathfrak{R}(x \wedge v) = 0$.

1.3 positive sectional curvature but not positive curvature operator.

2 Show that a Riemannian manifold with parallel Ricci tensor has constant scalar curvature. In chapter 3 it will be shown that the converse is not true, and also that a metic with parallel curvature tensor doesn't have to be Einstein.

We have

$$\nabla \mathrm{scal} = \nabla(\mathrm{tr}(\mathrm{Ric})) = \mathrm{tr}(\nabla \mathrm{Ric}) = 0.$$

Therefore, scal is constant as desired.

3 Let G be a Lie group with a bi-invariant metric. Using left-invariant fields establish the following formulas. *Hint:* First go back to the exercises to chapter 1 and take a peek at chapter 3 where some of these things are proved.

(You can assume the following which are proved in Proposition 12 on p. 79. I suggest you read that proof.)

$$\nabla_X Y = \tfrac{1}{2}[X, Y]; \quad R(X, Y)Z = \tfrac{1}{4}[Z, [X, Y]]; \quad g(R(X, Y)Z, W) = -\tfrac{1}{4}(g([X, Y], [Z, W]))$$

3.1 Show that the curvature operator is also nonnegative by showing that:

$$g\left(\mathfrak{R}\left(\sum_{i=1}^k X_i \wedge Y_i\right), \left(\sum_{i=1}^k X_i \wedge Y_i\right)\right) = \frac{1}{4}\left|\sum_{i=1}^k [X_i, Y_i]\right|^2.$$

3.2 **Show that Ric$(X,X)=0$ iff X commutes with all other left-invariant vector fields. Thus G has positive Ricci curvature if the center of G is discrete.**

3.3 **Consider the linear map $\Lambda^2\mathfrak{g}\to[\mathfrak{g},\mathfrak{g}]$ that sends $X\wedge Y$ to $[X,Y]$. Show that the sectional curvature is positive iff this map is an isomorphism. Conclude that this can only happen if $n=3$ and $\mathfrak{g}=\mathfrak{su}(2)$.**

4 **Consider a Riemannian metric (M,g). Now *scale* the metric by multiplying it by a number λ^2. Then we get a new Riemannian manifold $(M,\lambda^2 g)$. Show that the new connection and $(1,3)$-curvature tensor remain the same, but that sec, scal, and $\mathfrak{R}$ all get multiplied by λ^{-2}.**

5 **Recall that complex manifolds have complex tangent spaces. Thus we can multiply vectors by $\sqrt{-1}$. As a generalization of this we can define an *almost complex* structure. This is a $(1,1)$-tensor J such that $J^2=-I$. Show that the *Nijenhuis tensor:***

$$N(X,Y)=[J(X),J(Y)]-J([J(X),Y])-J([X,J(Y)])-[X,Y]$$

is indeed a tensor. If J comes from a complex structure then $N=0$, conversely Newlander & Nirenberg have shown that J comes from a complex structure if $N=0$.

A *Hermitian structure* on a Riemannian manifold (M,g) is an almost complex structure J such that

$$g(J(X),J(Y))=g(X,Y).$$

The *Kähler form* of a Hermitian structure is

$$\omega(X,Y)=g(J(X),Y).$$

Show that ω is a 2-form. Show that $d\omega=0$ iff $\nabla J=0$. If the Kähler form is closed, then we call the metric a Kähler metric.

6 **There is a strange curvature quantity we have not yet mentioned. Its definition is somewhat cumbersome and nonintuitive. First, for two symmetric $(0,2)$-tensors h,k define the *Kulkarni-Nomizu product* as the $(0,4)$-tensor**

$$\begin{aligned}h\circ k(v_1,v_2,v_3,v_4)&=h(v_1,v_3)\cdot k(v_2,v_4)+h(v_2,v_4)\cdot k(v_1,v_3)\\&\quad-h(v_1,v_4)\cdot k(v_2,v_3)-h(v_2,v_3)\cdot k(v_1,v_4).\end{aligned}$$

Note that (M,g) has constant curvature c iff the $(0,4)$-curvature tensor satisfies $R=c\cdot(g\circ g)$. If we use the $(0,2)$ form of the Ricci tensor, then we can decompose the $(0,4)$-curvature tensor as follows in dimensions $n\geq 4$

$$R=\frac{\mathbf{scal}}{2n(n-1)}g\circ g+\left(\mathbf{Ric}-\frac{\mathbf{scal}}{n}\cdot g\right)\circ g+W$$

When $n=3$ we have instead

$$R=\frac{\mathbf{scal}}{12}g\circ g+\left(\mathbf{Ric}-\frac{\mathbf{scal}}{3}\cdot g\right)\circ g.$$

The $(0,4)$-tensor W defined for $n > 3$ is called the *Weyl tensor.*

6.1 Show that these decompositions are orthogonal, in particular:

$$|R|^2 = \left|\frac{\textbf{scal}}{2n(n-1)} g \circ g\right|^2 + \left|\left(\textbf{Ric} - \frac{\textbf{scal}}{n} \cdot g\right) \circ g\right|^2 + |W|^2.$$

6.2 Show that if we conformally change the metric $\tilde{g} = f \cdot g$, then $\tilde{W} = f \cdot W$.

6.3 If (M, g) has constant curvature, then $W = 0$.

6.4 If (M, g) is locally conformally equivalen to the Euclidean metric, i.e., locally we can always find coordinates where:

$$g = f \cdot \left((dx^1)^2 + \cdots + (dx^n)^2\right),$$

then $W = 0$. The converse is also true but much harder to prove.

7 Assume that we have a Riemannian immersion of an n-manifold into $\mathbb{R}^{n+1}$. If $n \geq 3$, then show that it can't have negative curvature. If $n = 2$ give an example where it does have negative curvature.

8 Let (M, g) be a closed Riemannian n-manifold, and suppose that there is a Riemannian embedding into $\mathbb{R}^{n+1}$. Show that there must be a point $p \in M$ where the curvature operator $\mathfrak{R} : \Lambda^2 T_pM \to \Lambda^2 T_pM$ is positive. *Hint:* Consider $f(x) = |x|^2$ and restrict it to M, then check what happens at a maximum.