

- 1 Assume that (M, g) has the property that all geodesics exist for a fixed time $\epsilon > 0$. Show that (M, g) is geodesically complete.**

Fix $p \in M, v \in T_p M$. By assumption there exists a geodesic $\gamma : (-\epsilon/2, \epsilon/2) \rightarrow M$ with $\gamma(0) = p, \dot{\gamma}(0) = v$. However, taking $p' = \gamma(\epsilon/2), v' = \dot{\gamma}(\epsilon/2)$ we get another geodesic $\tilde{\gamma} : (-\epsilon/2, \epsilon/2) \rightarrow M$. Since they coincide at p', v' we can extend $\gamma : (-\epsilon/2, \epsilon) \rightarrow M$. Repeating this process we produce $\gamma : \mathbb{R} \rightarrow M$.

- 2 A Riemannian manifold is said to be homogeneous if the isometry group acts transitively. Show that homogeneous manifolds are geodesically complete.**

Fix $p \in M$ and let $\epsilon > 0$ be the injectivity radius around p . Let $q \in M$. By transitivity there exists an isometry $f : M \rightarrow M$ with $f(p) = q$. However, geodesics are taken to geodesics by isometries so the injectivity radius at q is at least ϵ .

Since there is a global injectivity radius we can extend any geodesic indefinitely, as in the previous exercise.

- 3 Let $N \subset (M, g)$ be a submanifold. Let ∇^N denote the connection on N that comes from the metric induced by g . Define the second fundamental form of N in M by**

$$\text{II}(X, Y) = \nabla_X^N Y - \nabla_X Y$$

Show that $\text{II} = 0$ on N iff N is totally geodesic. (The definition of *totally geodesic* is on p. 145)

A submanifold $N \subset (M, g)$ is said to be *totally geodesic* if for each $p \in N$ a neighborhood of $0 \in T_p N$ is mapped into N via the exponential map \exp_p .

Suppose that $\text{II} = 0$ and γ a geodesic in N . Then $\nabla_{\dot{\gamma}} \dot{\gamma} = \nabla_{\dot{\gamma}}^N \dot{\gamma} = 0$

Conversely, suppose that N is totally geodesic, and recall

$$\text{II}(X, Y) = \frac{1}{2} (\text{II}(X + Y, X + Y) - \text{II}(X, X) - \text{II}(Y, Y)).$$

It therefore suffices to show that $\text{II}(v, v) = 0$ for all $v \in T_p N, p \in N$. Let $\gamma : [0, 1] \rightarrow N$ the geodesic with $\gamma(0) = p, \dot{\gamma}(0) = v$. Notice $\nabla_{\dot{\gamma}}^N \dot{\gamma} = \nabla_{\dot{\gamma}} \dot{\gamma} = 0$, so $\text{II}(v, v) = 0$

- 4 Let p be a point in a Riemannian manifold (M, g) and $\sigma \subset T_p M$ a two-dimensional subspace. For small $r > 0$, let $\Sigma_\sigma \subset M$ be the (diffeomorphic) image of $B(0, r) \cap \sigma \subset T_p M$ under the exponential map \exp_p . Show that the sectional curvature $\text{sec}(\sigma)$ at p (computed inside M) is equal to the sectional curvature (that is, Gaussian curvature) of the surface Σ_σ at p , in the induced metric.**

Clearly Σ_σ is totally geodesic. Thus, by the above exercise $\nabla_X^{\Sigma_\sigma} Y = \nabla_X Y$ and, if u, v a basis for σ

$$\begin{aligned} \text{sec}(\sigma) &= \frac{\langle R(u, v)v, u \rangle}{\langle u, u \rangle \langle v, v \rangle - \langle u, v \rangle^2} \\ &= \frac{\langle \nabla_u \nabla_v v - \nabla_v \nabla_u v - \nabla_{[u, v]} v, u \rangle}{\langle u, u \rangle \langle v, v \rangle - \langle u, v \rangle^2} \\ &= \frac{\langle \nabla_u^{\Sigma_\sigma} \nabla_v^{\Sigma_\sigma} v - \nabla_v^{\Sigma_\sigma} \nabla_u^{\Sigma_\sigma} v - \nabla_{[u, v]}^{\Sigma_\sigma} v, u \rangle}{\langle u, u \rangle \langle v, v \rangle - \langle u, v \rangle^2} \\ &= \text{sec}^{\Sigma_\sigma}(\sigma) \end{aligned}$$

- 5 Let $SO(n)$ be the Lie group of orthogonal matrices of determinant 1. Equip $SO(n)$ with a bi-invariant Riemannian metric g of volume one, as constructed in the previous homework. The tangent space $T_I SO(n)$ can be identified with the space $\mathfrak{so}(n)$ of skew-adjoint matrices. Show that the exponential map (with respect to g)

$$\exp_I : \mathfrak{so}(n) \rightarrow SO(n)$$

coincides with the usual matrix exponentiation $A \rightarrow e^A$.

Hint: Feel free to use Proposition 12 on p.79 in Petersen's book. Compare also exercise 19 in Petersen, Chapter 5, p.151.

Let $\gamma_A(t) = e^{tA}$. Clearly $\gamma_A(0) = I, \dot{\gamma}_A(0) = A, \gamma(1) = e^A$. Thus, if γ is a geodesic we have the desired result. But if we have left invariance we compute $\nabla_{\dot{\gamma}}\dot{\gamma} = \frac{1}{2}[\dot{\gamma}, \dot{\gamma}] = 0$. It remains only to check left-invariance

- 6 Let $\gamma : [0, 1] \rightarrow M$ be a geodesic. Show that $\exp_{\gamma(0)}$ has a critical point at $t\dot{\gamma}(0)$ iff there is a Jacobi field J along γ such that $J(0) = 0, \dot{J}(0) \neq 0$, and $J(1) = 0$.

Suppose that such a J exists, and let γ_τ be the one parameter family of geodesics specified by J . Because $J(0) = J(1) = 0$, we have $\gamma_\tau(0) = \gamma(0), \gamma_\tau(1) = \gamma(1)$. Because $\dot{J} \neq 0$, γ_τ is not the trivial family of geodesics. Consider then the path $\alpha(\tau) = t\dot{\gamma}_\tau(0) \subseteq T_{\gamma(0)}M$. By construction $\frac{d}{d\tau} \exp_{\gamma(0)} \alpha(\tau) = 0$.

Conversely, suppose that $\exp_{\gamma(0)}$ has a critical point at $t\dot{\gamma}(0)$. If w is the direction of zero derivative put

$$J(t) = (d\exp_{\gamma(0)})_{t\dot{\gamma}(0)}(tw)$$

- 7 Let γ be a geodesic and X a Killing field in a Riemannian manifold. Show that the restriction of X to γ is a Jacobi field. (See the definition of a **Killing field** on p.23.)

Let X a Killing field (i.e. $L_X g = 0$), and let $\{E_1, E_2, \dots, E_n\}$ be a parallel orthonormal frame along γ with $\dot{\gamma} = \lambda E_1$. Compute:

$$\begin{aligned} 0 &= (L_X g)(\dot{\gamma}, E_i) \\ &= D_X(g(\dot{\gamma}, E_i)) - g(L_X \dot{\gamma}, E_i) - g(\dot{\gamma}, L_X E_i) \\ &= -g(\nabla_X \dot{\gamma}, E_i) + g(\nabla_{\dot{\gamma}} X, E_i) - g(\dot{\gamma}, \nabla_X E_i) + g(\dot{\gamma}, \nabla_{E_i} X) \\ &= g(\nabla_{\dot{\gamma}} X, E_i) + g(\dot{\gamma}, \nabla_{E_i} X) \end{aligned}$$

so differentiating we have

$$\begin{aligned} 0 &= D_{\dot{\gamma}}(g(\nabla_{\dot{\gamma}} X, E_i)) + D_{\dot{\gamma}}(g(\dot{\gamma}, \nabla_{E_i} X)) \\ &= g(\nabla_{\dot{\gamma}} \nabla_{\dot{\gamma}} X, E_i) + g(\dot{\gamma}, \nabla_{\dot{\gamma}} \nabla_{E_i} X) \\ &= g(\nabla_{\dot{\gamma}} \nabla_{\dot{\gamma}} X, E_i) + g(\dot{\gamma}, \nabla_{\dot{\gamma}} \nabla_{E_i} X - \nabla_{E_i} \nabla_{\dot{\gamma}} X - \nabla_{[\dot{\gamma}, E_i]} X) \end{aligned}$$

since

$$0 = g(\nabla_{E_i} \dot{\gamma}, \nabla_{\dot{\gamma}} X) + g(\dot{\gamma}, \nabla_{\nabla_{E_i} \dot{\gamma}} X) = D_{E_i}(g(\dot{\gamma}, \nabla_{\dot{\gamma}} X)) - g(\nabla_{E_i} \dot{\gamma}, \nabla_{\dot{\gamma}} X) - g(\dot{\gamma}, \nabla_{\nabla_{E_i} \dot{\gamma}} X) = g(\dot{\gamma}, \nabla_{E_i} \nabla_{\dot{\gamma}} X + \nabla_{[\dot{\gamma}, E_i]} X)$$

so

$$\begin{aligned} 0 &= g(\nabla_{\dot{\gamma}} \nabla_{\dot{\gamma}} X, E_i) + g(R(\dot{\gamma}, E_i)X, \dot{\gamma}) \\ &= g(\nabla_{\dot{\gamma}} \nabla_{\dot{\gamma}} X, E_i) + g(R(X, \dot{\gamma})\dot{\gamma}, E_i) \\ &= \nabla_{\dot{\gamma}} \nabla_{\dot{\gamma}} X + R(X, \dot{\gamma})\dot{\gamma} \end{aligned}$$

8 A Riemannian manifold is said to be k -point homogeneous if for all pairs of points (p_1, \dots, p_k) and (q_1, \dots, q_k) with $d(p_i, p_j) = d(q_i, q_j)$ there is an isometry F with $F(p_i) = q_i$. When $k = 1$ we simply say that the space is homogeneous.

8.1 Show that a homogeneous space has constant scalar curvature.

Since there is an isometry taking $p \mapsto q$ for all p, q and scalar curvature is a local condition, it must be the same throughout the manifold.

8.2 Show that if $k > 2$ and (M, g) is k -point homogeneous, then M is also $(k - 1)$ -point homogeneous.

Consider the pairs $(p_1, \dots, p_{k-1}, p_{k-1})$ and $(q_1, \dots, q_{k-1}, q_{k-1})$

8.3 Show that if (M, g) is two-point homogeneous, then (M, g) is an Einstein metric.

We'll show that $\text{Ric}_p(v, v) = \text{Ric}_q(w, w)$. By symmetry of the Ricci tensor $\text{Ric}_p(v, v) = \text{Ric}_p(w, w)$ is enough to conclude that Ric is Einstein at p , so this is sufficient to show the desired result.

Let $p, q \in M$, $v \in T_p M, w \in T_q M$. There exists some $\epsilon > 0$, geodesics $\gamma, \tilde{\gamma} : (-\epsilon, \epsilon) \rightarrow M$ with $\gamma(0) = p, \dot{\gamma}(0) = v, \tilde{\gamma}(0) = q, \dot{\tilde{\gamma}}(0) = w$. The isometry taking $f(p) = q, f(\gamma(\epsilon)) = \tilde{\gamma}(\epsilon)$ must have $D_f(v) = w$.

8.4 Show that if (M, g) is three-point homogeneous, then (M, g) has constant curvature.

We argue as above, this time constructing an isometry $f(p) = q$ with $D_f(\sigma) = \tau$ for $p, q \in M$, σ, τ planes in the respective tangent bundles.

Thus, we can take $\sec_p(\sigma) = \sec_q(\tau)$ as desired.

8.5 Classify all three-point homogeneous spaces. *Hint:* The only one that isn't simply connected is the real projective space.