- 1 Give examples of Riemannian manifolds having:
- 1.1 positive scalar curvature but not positive Ricci curvature;
- 1.2 positive Ricci curvature but not positive sectional curvature;
- 1.3 positive sectional curvature but not positive curvature operator.
- 2 Show that a Riemannian manifold with parallel Ricci tensor has constant scalar curvature. In chapter 3 it will be shown that the converse is not true, and also that a metic with parallel curvature tensor doesn't have to be Einstein.
- 3 Let G be a Lie group with a bi-invariant metric. Using left-invariant fields establish the following formulas. *Hint:* First go back to the exercises to chapter 1 and take a peek at chapter 3 where some of these things are proved.

(You can assume the following which are proved in Proposition 12 on p. 79. I suggest you read that proof.)

$$\nabla_X Y = \frac{1}{2}[X,Y]. \quad R(X,Y)Z = \frac{1}{4}[Z,[X,Y]]. \quad g(R(X,Y)Z,W) = -\frac{1}{4}(g([X,Y],[Z,W]))$$

3.1 Show that the curvature operator is also nonnegative by showing that:

$$g\left(\Re\left(\sum_{i=1}^k X_i \wedge Y_i\right), \left(\sum_{i=1}^k X_i \wedge Y_i\right)\right) = \frac{1}{4} \left|\sum_{i=1}^k [X_i, Y_i]\right|^2.$$

- 3.2 Show that Ric(X, X) = 0 iff X commutes with all other left-invariant vector fields. Thus G has positive Ricci curvature if the center of G is discrete.
- 3.3 Consider the linear map $\Lambda^2 \mathfrak{g} \to [\mathfrak{g}, \mathfrak{g}]$ that sends $X \wedge Y$ to [X, Y]. Show that the sectional curvature is positive iff this map is an isomorphism. Conclude that this can only happen if n = 3 and $\mathfrak{g} = \mathfrak{su}(2)$.
- 4 Consider a Riemannian metric (M,g). Now *scale* the metric by multiplying it by a number λ^2 . Then we get a new Riemannian manifold (M,λ^2g) . Show that the new connection and (1,3)-curvature tensor remain the same, but that sec, scal, and \Re all get multiplied by λ^{-2} .
- Recall that complex manifolds have complex tangent spaces. Thus we can multiply vectors by $\sqrt{-1}$. As a generalization of this we can define an *almost complex* structure. This is a (1,1)-tensor J such that $J^2 = -I$. Show that the *Nijenhuis tensor:*

$$N(X,Y) = [J(X), J(Y)] - J([J(X), Y]) - J([X, J(Y)]) - [X, Y]$$

is indeed a tensor. If J comes from a complex structure then N=0, conversely Newlander & Nirenberg have shown that J comes from a complex structure if N=0.

A *Hermitian structure* on a Riemannian manifold (M,g) is an almost complex structure J such that

$$g(J(X), J(Y)) = g(X, Y).$$

The Kähler form of a Hermitian structure is

$$\omega(X,Y) = g(J(X),Y).$$

Show that ω is a 2-form. Show that $d\omega = 0$ iff $\nabla J = 0$. If the Kähler form is closed, then we call the metric a Kähler metric.

6 There is a strange curvature quantity we have not yet mentioned. Its definition is somewhat cumbersome and nonintuitive. First, for two symmetric (0,2)-tensors h,k define the *Kulkarni-Nomizu product* as the (0,4)-tensor

$$h \circ k(v_1, v_2, v_3, v_4) = h(v_1, v_3) \cdot k(v_2, v_4) + h(v_2, v_4) \cdot k(v_1, v_3)$$
$$-h(v_1, v_4) \cdot k(v_2, v_3) - h(v_2, v_3) \cdot k(v_1, v_4).$$

Note that (M,g) has constant curvature c iff the (0,4)-curvature tensor satisfies $R=c\cdot(g\circ g)$. If we use the (0,2) form of the Ricci tensor, then we can decompose the (0,4)-curvature tensor as follows in dimensions $n\geq 4$

$$R = \frac{\mathbf{scal}}{2n(n-1)}g \circ g + \left(\mathbf{Ric} - \frac{\mathbf{scal}}{n} \cdot g\right) \circ g + W$$

When n = 3 we have instead

$$R = \frac{\mathbf{scal}}{12}g \circ g + \left(\mathbf{Ric} - \frac{\mathbf{scal}}{3} \cdot g\right) \circ g.$$

The (0,4)-tensor W defined for n>3 is called the Weyl tensor.

6.1 Show that these decompositions are orthogonal, in particular:

$$|R|^2 = \left| \frac{\mathbf{scal}}{2n(n-1)} g \circ g \right|^2 + \left| \left(\mathbf{Ric} - \frac{\mathbf{scal}}{n} \cdot g \right) \circ g \right|^2 + |W|^2.$$

- 6.2 Show that if we conformally change the metric $\tilde{g} = f \cdot g$, then $\tilde{W} = f \cdot W$.
- **6.3** If (M, g) has constant curvature, then W = 0.
- 6.4 If (M, g) is locally conformally equivalen to the Euclidean metric, i.e., locally we can always find coordinates where:

$$g = f \cdot \left((dx^1)^2 + \dots + (dx^n)^2 \right),\,$$

then W=0. The converse is also true but much harder to prove.

- Assume that we have a Riemannian immersion of an n-manifold into \mathbb{R}^{n+1} . If $n \geq 3$, then show that it can't have negative curvature. If n = 2 give an example where it does have negative curvature.
- 8 Let (M,g) be a closed Riemannian n-manifold, and suppose that there is a Riemannian embedding into \mathbb{R}^{n+1} . Show that there must be a point $p \in M$ where the curvature operator $\mathfrak{R}: \Lambda^2 T_p M \to \Lambda^2 T_p M$ is positive. *Hint:* Consider $f(x) = |x|^2$ and restrict it to M, then check what happens at a maximum.