Math 226A: Problem Set 4
due Monday, December 2

1. Petersen, Chapter 7, Exercise 24 on p. 233.

2. Describe all harmonic forms on the following Riemannian manifolds:

(a) The torus $T^n = S^1 \times \cdots \times S^1$ with the flat product metric;

(b) The sphere S^n with its round metric;

(c) The complex projective space $\mathbb{CP}^n$ with the Fubini-Study metric.

Feel free to use (without proof) the computation of the deRham cohomology groups of these spaces. Thus, in each case it suffices to exhibit a basis of harmonic forms of the required cardinality.

Hint: For (c), consider the powers of the Kähler form.

3. Show that the Laplacian on forms commutes with the Hodge star operator.

4. Consider the Laplacian $\Delta : \Omega^p(M) \to \Omega^p(M)$, where M is a closed, oriented Riemannian manifold.

(a) Prove that the eigenvalues of Δ are nonnegative, and have no finite accumulation points.

(b) Prove that the eigenspaces of Δ are finite dimensional.

(c) Prove that the eigenspaces corresponding to distinct eigenvalues are orthogonal.

Note: It can also be shown that Δ has infinitely many eigenvalues, and that the direct sum of all eigenspaces is dense in $\Omega^p(M)$. If you are interested, see Exercise 16 on p.254 in Warner, "Foundations of Differentiable Manifolds and Lie Groups."

5. Let M be a closed, oriented, Riemannian 4-manifold. Let us identify $H^2(M;\mathbb{R})$ with the space of harmonic 2-forms using the Hodge theorem. Note that the star operator $*$ acting on $\Omega^2(M;\mathbb{R})$ satisfies $*^2 = 1$, and therefore we have a direct sum decomposition

$$\Omega^2(M;\mathbb{R}) = \Omega^+(M) \oplus \Omega^-(M),$$

where $\Omega^\pm$ are the eigenspaces of $*$ corresponding to eigenvalues ± 1. Restricting this decomposition to harmonic forms, we obtain another direct sum decomposition

$$H^2(M;\mathbb{R}) = \mathcal{H}^+ \oplus \mathcal{H}^-,$$

where $\mathcal{H}^\pm$ are the eigenspaces of $*$ acting on $H^2(M;\mathbb{R})$, corresponding to eigenvalues ± 1.

Let $d^+ : \Omega^1(M) \to \Omega^+(M)$ be the composition of d with orthogonal projection to Ω^+, and consider the three-term complex

$$\Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d^+} \Omega^+(M).$$

Show that the cohomology groups of this complex can be naturally identified with $H^0(M;\mathbb{R})$, $H^1(M;\mathbb{R})$, and $\mathcal{H}^+(M)$, respectively.