## 1 Assume that $(M, g)$ has the property that all geodesics exist for a fixed time $\epsilon > 0$. Show that $(M, g)$ is geodesically complete.

Fix $p \in M, v \in T_pM$. By assumption there exists a geodesic $\gamma : (-\epsilon/2, \epsilon/2) \to M$ with $\gamma(0) = p, \dot{\gamma}(0) = v$. However, taking $p' = \gamma(\epsilon/2), v' = \dot{\gamma}(\epsilon/2)$ we get another geodesic $\tilde{\gamma} : (-\epsilon/2, \epsilon/2) \to M$. Since they coincide at $p', v'$ we can extend $\gamma : (-\epsilon/2, \epsilon) \to M$. Repeating this process we produce $\gamma : \mathbb{R} \to M$.

## 2 A Riemannian manifold is said to be homogeneous if the isometry group acts transitively. Show that homogeneous manifolds are geodesically complete.

## 3 Let $N \subset (M, g)$ be a submanifold. Let $\nabla^N$ denote the connection on $N$ that comes from the metric induced by $g$. Define the second fundamental form of $N$ in $M$ by

$$\mathrm{II}(X, Y) = \nabla^N_X Y - \nabla_X Y$$

**Show that $\mathrm{II} = 0$ on $N$ iff $N$ is totally geodesic. (The definition of *totally geodesic* is on p. 145)**

A submanifold $N \subset (M, g)$ is said to be *totally geodesic* if for each $p \in N$ a neighborhood of $0 \in T_pN$ is mapped into $N$ via the exponential map $\exp_p$.

## 4 Let $p$ be a point in a Riemannian manifold $(M, g)$ and $\sigma \subset T_pM$ a two-dimensional subspace. For small $r > 0$, let $\Sigma_\sigma \subset M$ be the (diffeomorphic) image of $B(0, r) \cap \sigma \subset T_pM$ under the exponential map $\exp_p$. Show that the sectional curvature $\sec(\sigma)$ at $p$ (computed inside $M$) is equal to the sectional curvature (that is, Gaussian curvature) of the surface $\Sigma_\sigma$ at $p$, in the induced metric.

## 5 Let $SO(n)$ be the Lie group of orthogonal matrices of determinant 1. Equip $SO(n)$ with a bi-invariant Riemannian metric $g$ of volume one, as constructed in the previous homework. The tangent space $T_ISO(n)$ can be identified with the space $\mathfrak{so}(n)$ of skew-adjoint matrices. Show that the exponential map (with respect to $g$)

$$\exp_I : \mathfrak{so}(n) \to SO(n)$$

**coincides with the usual matrix exponentiation $A \to e^A$.**

***Hint:*** **Feel free to use Proposition 12 on p.79 in Petersen's book. Compare also exercise 19 in Petersen, Chapter 5, p.151.**

**6 Let $\gamma : [0,1] \to M$ be a geodesic. Show that $\exp_{\gamma(0)}$ has a critical point at $t\dot{\gamma}(0)$ iff there is a Jacobi field $J$ along $\gamma$ such that $J(0) = 0$, $\dot{J}(0) \neq 0$, and $J(t) = 0$.**

**7 Let $\gamma$ be a geodesic and $X$ a Killing field in a Riemannian manifold. Show that the restriction of $X$ to $\gamma$ is a Jacobi field. (See the definition of a *Killing field* on p.23.)**

**8 A Riemannian manifold is said to be $k$-*point homogeneous* if for all pairs of points $(p_1, \ldots, p_k)$ and $(q_1, \ldots, q_k)$ with $d(p_i, p_j) = d(q_i, q_j)$ there is an isometry $F$ with $F(p_i) = q_i$. When $k = 1$ we simply say tat the space is homogeneous.**

**8.1 Show that a homogeneous space has constant scalar curvature.**

**8.2 Show that if $k > 2$ and $(M, g)$ is $k$-point homogeneous, then $M$ is also $(k-1)$-point homogeneous.**

**8.3 Show that if $(M, g)$ is two-point homogeneous, then $(M, g)$ is an Einstein metric.**

**8.4 Show that if $(M, g)$ is three-point homogeneous, then $(M, g)$ has constant curvature.**

**8.5 Classify all three-point homogeneous spaces. *Hint:* The only one that isn't simply connected is the real projective space.**