1 Give examples of Riemannian manifolds having:

1.1 positive scalar curvature but not positive Ricci curvature;

1.2 positive Ricci curvature but not positive sectional curvature;

1.3 positive sectional curvature but not positive curvature operator.

2 Show that a Riemannian manifold with parallel Ricci tensor has constant scalar curvature. In chapter 3 it will be shown that the converse is not true, and also that a metic with parallel curvature tensor doesn't have to be Einstein.

3 Let G be a Lie group with a bi-invariant metric. Using left-invariant fields establish the following formuls. *Hint:* First go back to the exercises to chapter 1 and take a peek at chapter 3 where some of these things are proved.

(You can assume the following which are proved in Proposition 12 on p. 79. I suggest you read that proof.)

$$\nabla_X Y = \frac{1}{2}[X, Y]. \quad R(X, Y)Z = \frac{1}{4}[Z, [X, Y]]. \quad g(R(X, Y)Z, W) = -\frac{1}{4}(g([X, Y], [Z, W]))$$

3.1 Show that the curvature operator is also nonnegative by showing that:

$$g\left(\mathfrak{R}\left(\sum_{i=1}^{k} X_i \wedge Y_i\right), \left(\sum_{i=1}^{k} X_i \wedge Y_i\right)\right) = \frac{1}{4}\left|\sum_{i=1}^{k}[X_i, Y_i]\right|^2.$$

3.2 Show that $\mathrm{Ric}(X, X) = 0$ iff X commutes with all other left-invariant vector fields. Thus G has positive Ricci curvature if the center of G is discrete.

3.3 Consider the linear map $\Lambda^2\mathfrak{g} \to [\mathfrak{g}, \mathfrak{g}]$ that sends $X \wedge Y$ to $[X, Y]$. Show that the sectional curvature is positive iff this map is an isomorphism. Conclude that this can only happen if $n = 3$ and $\mathfrak{g} = \mathfrak{su}(2)$.

4 Consider a Riemannian metric (M, g). Now *scale* the metric by multiplying it by a number λ^2. Then we get a new Riemannian manifold $(M, \lambda^2 g)$. Show that the new connection and $(1, 3)$-curvature tensor remain the same, but that sec, scal, and $\mathfrak{R}$ all get multiplied by λ^{-2}.

5 Recall that complex manifolds have complex tangent spaces. Thus we can multiply vectors by $\sqrt{-1}$. As a generalization of this we can define an *almost complex* structure. This is a $(1, 1)$-tensor J such that $J^2 = -I$. Show that the *Nijenhuis tensor:*

$$N(X, Y) = [J(X), J(Y)] - J([J(X), Y]) - J([X, J(Y)]) - [X, Y]$$

is indeed a tensor. If J comes from a complex structure then $N = 0$, conversely Newlander & Nirenberg have shown that J comes from a complex structure if $N = 0$.

A *Hermitian structure* on a Riemannian manifold (M, g) is an almost complex structure J such that

$$g(J(X), J(Y)) = g(X, Y).$$

The *Kähler form* of a Hermitian structure is

$$\omega(X, Y) = g(J(X), Y).$$

Show that ω is a 2-form. Show that $d\omega = 0$ iff $\nabla J = 0$. If the Kähler form is closed, then we call the metric a Kähler metric.

6 Petersen, Chapter 3, Exercise 11 parts (a), (b), (c), (d) on p. 92.

7 Assume that we have a Riemannian immersion of an n-manifold into $\mathbb{R}^{n+1}$. If $n \geq 3$, then show that it can't have negative curvature. If $n = 2$ give an example where it does have negative curvature.

8 Let (M, g) be a closed Riemannian n-manifold, and suppose that there is a Riemannian embedding into $\mathbb{R}^{n+1}$. Show that there must be a point $p \in M$ where the curvature operator $\mathfrak{R} : \Lambda^2 T + pM \to \Lambda^2 T_p M$ is positive. *Hint:* Consider $f(x) = |x|^2$ and restrict it to M, then check what happens at a maximum.