1 Give examples of Riemannian manifolds having:

1.1 positive scalar curvature but not positive Ricci curvature;

Let $M = S^n_a \times S^m_b$, and X, Y, U, V vector fields with X, Y tangent to S^n, and U, V tangent to S^m. Recall (class, or Petersen p.65) that

$$\mathfrak{R}(X \wedge V) = 0 \quad \mathfrak{R}(X \wedge Y) = aX \wedge Y \quad \mathfrak{R}(U \wedge V) = bU \wedge V$$

$$\operatorname{Ric}(X) = (n-1)aX \quad \operatorname{Ric}(V) = (m-1)bV \quad \operatorname{scal} = n(n-1)a + m(m-1)b.$$

In particular if we put $M = S^1 \times S^2$ $(n = 1, m = 2)$ we have $\operatorname{scal} = 2b > 0$, but $\operatorname{Ric}(X) = 0$.

1.2 positive Ricci curvature but not positive sectional curvature;

As above, but let $M = S^2 \times S^2$ $(m = n = 2)$. Now $\operatorname{Ric}(X) = aX > 0$, $\operatorname{Ric}(V) = bV > 0$, and $\sec(x, v) = \mathfrak{R}(x \wedge v) = 0$.

1.3 positive sectional curvature but not positive curvature operator.

Recall (Petersen p. 86) that $\mathbb{CP}^n$ has eigenvalues of $\mathfrak{R}$ in $[0, 6]$, but $\sec \geq 1$.

2 Show that a Riemannian manifold with parallel Ricci tensor has constant scalar curvature. In chapter 3 it will be shown that the converse is not true, and also that a metic with parallel curvature tensor doesn't have to be Einstein.

We have

$$\nabla \operatorname{scal} = \nabla(\operatorname{tr}(\operatorname{Ric})) = \operatorname{tr}(\nabla \operatorname{Ric}) = 0.$$

Therefore, scal is constant as desired.

3 Let G be a Lie group with a bi-invariant metric. Using left-invariant fields establish the following formulas. *Hint:* First go back to the exercises to chapter 1 and take a peek at chapter 3 where some of these things are proved.

(You can assume the following which are proved in Proposition 12 on p. 79. I suggest you read that proof.)

$$\nabla_X Y = \tfrac{1}{2}[X, Y]; \quad R(X, Y)Z = \tfrac{1}{4}[Z, [X, Y]]; \quad g(R(X, Y)Z, W) = -\tfrac{1}{4}(g([X, Y], [Z, W]))$$

3.1 Show that the curvature operator is also nonnegative by showing that:

$$g\left(\mathfrak{R}\left(\sum_{i=1}^{k} X_i \wedge Y_i\right), \left(\sum_{i=1}^{k} X_i \wedge Y_i\right)\right) = \frac{1}{4}\left|\sum_{i=1}^{k} [X_i, Y_i]\right|^2.$$

Compute:

$$
\begin{aligned}
g\left(\Re\left(\sum_{i=1}^{k} X_i \wedge Y_i\right), \left(\sum_{i=1}^{k} X_i \wedge Y_i\right)\right) &= \sum_{1\le i,j\le k} R(X_i, Y_i, Y_j, X_j) \\
&= \frac{1}{4}\sum_{1\le i,j\le k} g\left([X_i, Y_i], [X_j, Y_j]\right) \\
&= \frac{1}{4}\left|\sum_{i=1}^{k}[X_i, Y_i]\right|^2
\end{aligned}
$$

3.2 Show that $\mathrm{Ric}(X, X) = 0$ iff X commutes with all other left-invariant vector fields. Thus G has positive Ricci curvature if the center of G is discrete.

Fixing a basis for T_eM, observe that X commutes with all left-invariant vector fields if and only if $[X, E_i] = 0$ for the left-invariant $E_i(e) = e_i$. However,

$$
0 = \mathrm{Ric}(X, X) = \sum_{i=1}^{n} g(R(E_i, X)X, E_i)
$$

if and only if

$$
0 = g(R(E_i, X)X, E_i) = \frac{1}{4}g([E_i, X], [E_i, X])
$$

for all i.

3.3 Consider the linear map $\Lambda^2\mathfrak{g} \overset{p}{\to} [\mathfrak{g}, \mathfrak{g}]$ that sends $X \wedge Y$ to $[X, Y]$. Show that the sectional curvature is positive iff this map is an isomorphism. Conclude that this can only happen if $n = 3$ and $\mathfrak{g} = \mathfrak{su}(2)$.

Clearly the given map is onto. It is therefore an isomorphism exactly when it is an injection, i.e. when $[X, Y] = 0 \iff X = \alpha Y$. Of course,

$$
\begin{aligned}
\sec(v, w) &= \frac{g(R(w, v)v, w)}{g(v \wedge w, v \wedge w)} \\
&= \frac{g([v, w], [v, w])}{4g(v \wedge w, v \wedge w)}
\end{aligned}
$$

which is positive if and only if $|[v, w]| > 0$ for all $v \neq \lambda w$.

Now if $\dim \mathfrak{g} = n$ we have $\dim \Lambda^2\mathfrak{g} = \frac{1}{2}n(n-1)$ and, since the derived Lie algebra is a subalgebra, $\dim[\mathfrak{g}, \mathfrak{g}] \le \dim \mathfrak{g}$. Therefore for the desired isomorphism to occur we must have

$$
\frac{1}{2}n(n-1) \le n \iff 0 \le n \le 3.
$$

4 Consider a Riemannian metric (M, g). Now *scale* the metric by multiplying it by a number λ^2. Then we get a new Riemannian manifold $(M, \lambda^2 g)$. Show that the new connection and $(1, 3)$-curvature tensor remain the same, but that sec, scal, and $\Re$ all get multiplied by λ^{-2}.

The Levi-Civita connection is given by

$$
g(\nabla_X Y, Z) = \frac{1}{2}\{X(g(Y, Z)) + Y(g(Z, X)) - Z(g(X, Y)) + g([X, Y], Z) - g([Y, Z], X) - g([X, Z], Y)\}
$$

which formula is invariant under scaling of the metric. Similarly, the $(1, 3)$-curvature tensor is defined soley in terms of the connection so ∇ remaining unchanged means that R is as well.

Denote the scaled metric g' and compute

$$g'(x \wedge y, v \wedge w) = g'(x,v)g'(y,w) - g'(x,w)g'(y,v) = \lambda^4 \left(g(x,v)g(y,w) - g(x,w)g(y,v)\right) = \lambda^4 g(x \wedge y, v \wedge w).$$

Therefore if $\mathfrak{S}$ denotes the scaled curvature operator we have

$$\begin{aligned} \lambda^4 g(\mathfrak{S}(X \wedge Y), V \wedge W) = \\ g'(\mathfrak{S}(X \wedge Y), V \wedge W) &= R'(X \wedge Y, V \wedge W) \\ &= g'(R(X,Y)Z, W) \\ &= \lambda^2 g(R(X,Y)Z, W). \end{aligned}$$

So, $\mathfrak{S}$, defined by the relation $g(\mathfrak{S}(X \wedge Y), V \wedge W) = \frac{1}{\lambda^2} g(R(X,Y)Z,W)$ is precisely $\mathfrak{R}/\lambda^2$ as desired.

Finally compute

$$\sec'(v,w) = \frac{g'(\mathfrak{S}(v \wedge w), v \wedge w)}{g'(v \wedge w, v \wedge w)} = \frac{1}{\lambda^2} \frac{g(\mathfrak{R}(v \wedge w), v \wedge w)}{g(v \wedge w, v \wedge w)} = \frac{1}{\lambda} \sec(v,w)$$

and

$$\operatorname{scal}' = 2 \cdot \operatorname{tr} \mathfrak{S} = \frac{2}{\lambda^2} \operatorname{tr} \mathfrak{R} = \frac{1}{\lambda^2} \operatorname{scal}.$$

5 Recall that complex manifolds have complex tangent spaces. Thus we can multiply vectors by $\sqrt{-1}$. As a generalization of this we can define an *almost complex* structure. This is a $(1,1)$-tensor J such that $J^2 = -I$. Show that the *Nijenhuis tensor:*

$$N(X,Y) = [J(X), J(Y)] - J([J(X), Y]) - J([X, J(Y)]) - [X,Y]$$

is indeed a tensor. If J comes from a complex structure then $N = 0$, conversely Newlander & Nirenberg have shown that J comes from a complex structure if $N = 0$.

A *Hermitian structure* on a Riemannian manifold (M,g) is an almost complex structure J such that

$$g(J(X), J(Y)) = g(X,Y).$$

The *Kähler form* of a Hermitian structure is

$$\omega(X,Y) = g(J(X), Y).$$

Show that ω is a 2-form. Show that $d\omega = 0$ iff $\nabla J = 0$. If the Kähler form is closed, then we call the metric a Kähler metric.

To confirm tensoriality we compute:

$$\begin{aligned} N(X, fY) &= [J(X), fJ(Y)] - J([J(X), fY]) - J([X, fJ(Y)]) - [X, fY] \\ &= J(X)(f)J(Y) + f[J(X), J(Y)] - J(J(X)(f)Y + f[J(X), Y]) \\ &\qquad - J(X(f)J(Y) + f[X, J(Y)]) - X(f)Y - f[X,Y] \\ &= f[J(X), J(Y)] - fJ([J(X), Y]) + f[X, J(Y)]) - f[X,Y] \\ &= f \cdot N(X,Y) \end{aligned}$$

$$\begin{aligned} N(X, A+B) &= [J(X), J(A+B)] - J([J(X), A+B]) - J([X, J(A+B)]) - [X, A+B] \\ &= N(X,A) + N(X,B) \end{aligned}$$

and similarly in the first place.

Tensoriality of ω is immediate. For antisymmetry compute:

$$\omega(Y,X) = g(J(Y), X) = g(J^2(Y), J(X)) = -g(J(X), Y)$$

6 There is a strange curvature quantity we have not yet mentioned. Its definition is somewhat cumbersome and nonintuitive. First, for two symmetric $(0,2)$-tensors h, k define the *Kulkarni-Nomizu product* as the $(0,4)$-tensor

$$h \circ k(v_1, v_2, v_3, v_4) = h(v_1, v_4) \cdot k(v_2, v_3) + h(v_2, v_3) \cdot k(v_1, v_4)$$
$$-h(v_1, v_3) \cdot k(v_2, v_4) - h(v_2, v_4) \cdot k(v_1, v_3).$$

Note that (M, g) has constant curvature c iff the $(0,4)$-curvature tensor satisfies $R = c \cdot (g \circ g)$. If we use the $(0,2)$ form of the Ricci tensor, then we can decompose the $(0,4)$-curvature tensor as follows in dimensions $n \geq 4$

$$R = \frac{\text{scal}}{2n(n-1)} g \circ g + \frac{1}{n-2} \left(\text{Ric} - \frac{\text{scal}}{n} \cdot g \right) \circ g + W$$

When $n = 3$ we have instead

$$R = \frac{\text{scal}}{12} g \circ g + \left(\text{Ric} - \frac{\text{scal}}{3} \cdot g \right) \circ g.$$

The $(0,4)$-tensor W defined for $n > 3$ is called the *Weyl tensor.*

6.1 Show that these decompositions are orthogonal, in particular:

$$|R|^2 = \left| \frac{\text{scal}}{2n(n-1)} g \circ g \right|^2 + \left| \left(\text{Ric} - \frac{\text{scal}}{n} \cdot g \right) \circ g \right|^2 + |W|^2.$$

6.2 Show that if we conformally change the metric $\tilde{g} = f \cdot g$, then $\tilde{W} = f \cdot W$.

6.3 If (M, g) has constant curvature, then $W = 0$.

6.4 If (M, g) is locally conformally equivalent to the Euclidean metric, i.e., locally we can always find coordinates where:

$$g = f \cdot \left((dx^1)^2 + \cdots + (dx^n)^2 \right),$$

then $W = 0$. The converse is also true but much harder to prove.

7 Assume that we have a Riemannian immersion of an n-manifold into $\mathbb{R}^{n+1}$. If $n \geq 3$, then show that it can't have negative curvature. If $n = 2$ give an example where it does have negative curvature.

The sectional curvature is given by $K(X_i, X_j) = k_i k_j$ where the k_i are eigenvalues of the second fundamental form. These cannot all be negative for dim > 2 for purely combinatorial reasons. Suppose k_1k_2, k_2k_3, k_1k_3 are all negative. Then exactly one of k_1, k_2 negative. Say WLOG k_1. Then as k_1k_3 negative, $k_3 > 0$. Thus $k_3k_2 > 0$ contradiction.

Consider the saddle embedded in $\mathbb{R}^3$. This has negative curvature at the saddle point.

8 Let (M, g) be a closed Riemannian n-manifold, and suppose that there is a Riemannian embedding into $\mathbb{R}^{n+1}$. Show that there must be a point $p \in M$ where the curvature operator $\mathfrak{R} : \Lambda^2 T_pM \to \Lambda^2 T_pM$ is positive. *Hint:* Consider $f(x) = |x|^2$ and restrict it to M, then check what happens at a maximum.