1 In dimensions $4n$ we have that the Hodge $* : H^{2n}(M) \to H^{2n}(M)$ satisfies $** = I$. The difference in the dimensions of the eigenspaces for ± 1 is called the *signature* of M :

$$\tau(M) = \sigma(M) = \dim(\ker(* - I) - \ker(* + I)).$$

One can show that this does not depend on the metric used to define $*$, by observing that it is the index of the symmetric bilinear map

$$H^{2n}(M) \times H^{2n}(M) \to \mathbb{R},$$

$$(\omega_1, \omega_2) \to \int \omega_1 \wedge \omega_2.$$

Recall that the index of a symmetric bilinear map is the difference between positive and negative diagonal elements when it has been put into diagonal form. In dimension 4 one can show that

$$\sigma(M) = \frac{1}{12\pi^2} \int_M \left(\left| W^+ \right|^2 - \left| W^- \right|^2 \right).$$

Using the exercises from chapter 4, show that for an Einstein metric in dimension 4 we have

$$\chi(M) \geq \frac{3}{2}\sigma(M),$$

with equality holding iff the metric is Ricci flat and $W^- = 0$. Conclude that not all four manifolds admit Einstein metrics. In higher dimensions there are no known obstructions to the existence of Einstein metrics. *Hint:* consider connected sums of $\mathbb{CP}^2$ with itself k times.

From Petersen, Chapter 4, Exercise 9 on p. 109. we have

$$\begin{aligned}
\chi(M) &= \frac{1}{8\pi^2} \int_M \left(|W^+|^2 + |W^-|^2 + \frac{\text{scal}^2}{24} \right) \\
&= \frac{3}{2} \left[\frac{1}{12\pi^2} \int_M \left(|W^+|^2 - |W^-|^2 \right) + \frac{1}{12\pi^2} \int_M \left(\frac{\text{scal}^2}{24} + 2 \cdot |W^-|^2 \right) \right] \\
&= \frac{3}{2}\sigma(M) + \frac{1}{8\pi^2} \int_M \left(\frac{\text{scal}^2}{24} + 2 \cdot |W^-|^2 \right)
\end{aligned}$$

So clearly, $\chi(M) \geq \frac{3}{2}\sigma(M)$ with equality exactly when the metric is Ricci flat with $W^- = 0$.

Let $C_k = \overbrace{\mathbb{CP}^2 \# \cdots \# \mathbb{CP}^2}^{k \text{ times}}$ denote the connect sum of k copies of $\mathbb{CP}^2$ as suggested in hint. Since $\chi(\mathbb{CP}^2) = 3$ and $\chi(X \# Y) = \chi(X) + \chi(Y) - 2$, in general $\chi(C_k) = k + 2$.

Compute also $\sigma(\mathbb{C}_k) = k$ since by dimension constraints $\sigma(\mathbb{CP}^2) = 1$ and $\sigma(X \# Y) = \sigma(X) + \sigma(Y)$. So in this case the above inequality asserts that

$$\chi(C_k) = k + 2 \geq \frac{3}{2}k = \frac{3}{2}\sigma(C_k) \iff k \leq 4.$$

so there can be no Einstein metric on C_k for $k > 4$.

2 Describe all harmonic forms on the following Riemannian manifolds:

2.1 The torus $T^n = S^1 \times \cdots \times S^1$ with the flat product metric;

Recall,

$$H^p_{\text{dR}}(T^n) \simeq \mathbb{R}^{\binom{n}{p}}.$$

Recall also (see for instance Jost 2.1.27) that in the Euclidean metric on $\mathbb{R}^n$ the Hodge Laplacian of a p-form $\omega = \omega_{i_1 \cdots i_p} dx^{i_1} \wedge \cdots \wedge dx^{i_p}$ is given by

$$\Delta\omega = \delta d\omega + d\delta\omega = -\sum_{j=1}^{n} \frac{\partial^2 \omega_{i_1 \cdots i_p}}{(\partial x^j)^2} dx^{i_1} \wedge \cdots \wedge dx^{i_p}$$

Therefore, the harmonic p-forms on T^n are exactly those which are harmonic in each coordinate. Since T^n compact, all harmonic functions are constant.

2.2 The sphere S^n with its round metric;

Recall,

$$H^k_{\mathrm{dR}}(S^n) \simeq \begin{cases} \mathbb{R} & \text{if } k = 0, n \\ 0 & \text{otherwise} \end{cases}$$

By compactness of S^n, the harmonic 0-forms are just the constant functions. Furthermore, as Δ and $*$ commute (Exercise 3), the harmonic n-forms are just $*\omega$ for ω constant 0-forms.

2.3 The complex projective space $\mathbb{CP}^n$ with the Fubini-Study metric.

Recall,

$$H^k_{\mathrm{dR}}(\mathbb{CP}^n) \simeq \begin{cases} \mathbb{R} & k \text{ even}, \ 0 \le k \le 2n, \\ 0 & \text{otherwise} \end{cases}$$

For 0-forms we have the constant functions.

Powers of the Kähler form ω are harmonic on a Kähler manifold.

Proof. If α is a harmonic form then so is $\alpha \wedge \omega$ since this operation commutes with the Laplacian. Thus it suffices to show that the Kähler form is harmonic

Take a local frame $X_1, Y_1, \ldots, X_m, Y_m$ with $J(X_i) = Y_i$. In this frame $\omega = \sum X_i^\flat \wedge Y_i^\flat$. Furthermore

$$*\omega = \sum X_1^\flat \wedge Y_1^\flat \wedge \cdots \wedge \hat{X}_i^\flat \wedge \hat{Y}_i^\flat \wedge \cdots \wedge X_m^\flat \wedge Y_m^\flat = \frac{1}{(m-1)!}\omega^{m-1}$$

Thus ω co-closed. Since ω is a Kähler form it's automnatically closed and therefore harmonic. □

Feel free to use (without proof) the computation of the deRham cohomology groups of these spaces. Thus, in each case it suffices to exhibit a basis of harmonic forms of the required cardinality. *Hint:* For 2.3, consider the powers of the Kähler form.

3 Show that the Laplacian on forms commutes with the Hodge star operator.

First verify that we have

$$*\delta^k d^k = d^{n-k-1}\delta^{n-k-1} * \iff *\delta^{n-k-1} d^{n-k-1} = d^k \delta^k *$$

$$\begin{aligned}
*\delta^k d^k &= *(-1)^{(n-k)(k+1)} * d^{n-k-1} * d^k \\
&= (-1)^{(n-k)(k+1)} (**) d^{n-k-1} * d^k \\
&= (-1)^{(n-k)(k+1)} (-1)^{k(n-k)} d^{n-k-1} * d^k \\
&= (-1)^{(n-k)(2k+1)} d^{n-k-1} * d^k ** (-1)^{k(n-k)} \\
&= (-1)^{(n-k)(3k+1)} d^{n-k-1} * d^k ** \\
&= (-1)^{(n-k)(3k+1)} d^{n-k-1} (-1)^{(n-k)(k+1)} (-1)^{(n-(n-k-1))((n-k-1)+1)} * d^{n-(n-k-1)-1} ** \\
&= (-1)^{(n-k)(4k+2)} d^{n-k-1} \delta^{n-k-1} * \\
&= d^{n-k-1} \delta^{n-k-1} * .
\end{aligned}$$

Consequently compute:

$$\begin{aligned} *\Delta &= *d^{k-1}\delta^{k-1} + *\delta^k d^k \\ &= \delta^{n-k} d^{n-k} * + d^{n-k-1}\delta^{n-k-1} * \\ &= \Delta * \end{aligned}$$

4 Consider the Laplacian $\Delta : \Omega^p(M) \to \Omega^p(M)$, where M is a closed, oriented Riemannian manifold.

Throughout we'll use "Fact 2" from class:

Fact 1. *If $\{\alpha_n\}$ is a sequence of smooth p-forms on M such that $||\alpha_n|| \leq c$ and $||\Delta\alpha_n|| \leq c$ for all n and some constant $c > 0$ then $\{\alpha_n\}$ has a Cauchy subsequence.*

4.1 Prove that the eigenvalues of Δ are nonnegative, and have no finite accumulation points.

Suppose λ satisfies $\Delta\omega = \lambda\omega$ for some $\omega \in \Omega^p(M)$. Then $(\Delta\omega, \omega) = \lambda(\omega, \omega) \iff (\delta\omega, \delta\omega) + (d\omega, d\omega) = \lambda||\omega||^2$. Therefore, $\lambda \geq 0$.

Suppose that $\alpha < \infty$ is an accumulation point of the eigenvalues of Δ and let $\{\omega_i\}$ be an orthonormal sequence of eigenfunctions corresponding to distinct eigenvalues $\{\lambda_i\}$ converging to α. (Note: such a sequence can be constructed by 4.3). Now, by Fact 1 $\{\omega_i\}$ has a Cauchy subsequence, a contradiction.

4.2 Prove that the eigenspaces of Δ are finite dimensional.

Suppose that the eigenspace corresponding to λ is infinite dimensional. Then by definition there exists an infinite sequence $\{\omega_i\}$ of orthonormal eigenfunctions for λ. By Fact 1 this basis has a Cauchy subsequence, a contradiction.

4.3 Prove that the eigenspaces corresponding to distinct eigenvalues are orthogonal.

Let A be a self-adjoint operator, α, β eigenvectors corresponding to eigenvalues λ, μ. Then $(A\alpha, \beta) = \lambda(\alpha, \beta)$ and $(\alpha, A\beta) = \mu(\alpha, \beta)$. Since A is self-adjoint we have $\lambda(\alpha, \beta) = \mu(\alpha, \beta)$ so if $\lambda \neq \mu$ it must be that $(\alpha, \beta) = 0$ as desired.

The Laplacian is self-adjoint.

(*Note:* It can also be shown that Δ has infinitely many eigenvalues, and that the direct sum of all eigenspaces is dense in $\Omega^p(M)$. If you are interested, see Exercise 16 on p.254 in Warner, "Foundations of Differentiable Manifolds and Lie Groups")

5 Let M be a closed, oriented, Riemannian 4-manifold. Let us identify $H^2(M;\mathbb{R})$ with the space of harmonic 2-forms using the Hodge theorem. Note that the star operator $*$ acting on $\Omega^2(M;\mathbb{R})$ satisfies $*^2 = 1$, and therefore we have a direct sum decomposition

$$\Omega^2(M;\mathbb{R}) = \Omega^+(M) \oplus \Omega^-(M),$$

where $\Omega^\pm$ are the eigenspaces of $*$ corresponding to the eigenvalues ± 1. Restricting this decomposition to harmonic forms, we obtain another direct sum decomposition

$$H^2(M;\mathbb{R}) = \mathcal{H}^+ \oplus \mathcal{H}^-,$$

where $\mathcal{H}^{\pm}$ are the eigenspace of $*$ acting on $H^2(M;\mathbb{R})$, corresponding to eigenvalues ± 1.

Let $d^+ : \Omega^1(M) \to \Omega^+(M)$ be the composition of d with orthogonal projection to Ω^+, and consider the three-term complex

$$\Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d^+} \Omega^+(M).$$

Show that the cohomology groups of this complex can be naturally identified with $H^0(M;\mathbb{R})$, $H^1(M;\mathbb{R})$, and $\mathcal{H}^+(M)$, respectively.