1 Let G be a compact Lie group. Show that G admits a bi-invariant metric, i.e., both right and left translations are isometries. ***Hint:*** Fix a left invariant metric g_L and a volume form $\omega = \sigma^1 \wedge \cdots \wedge \sigma^n$ where σ^i are left invariant 1-forms. Then define g as the average over right translations:

$$g(v,w) = \frac{1}{\int \omega} \int g_L(DR_x(v), DR_x(w))\omega.$$

Let g_L be a left invariant metric on G, i.e. $g_L(v,w) = g_L(DL_x(v), DL_x(w))$ for all $x \in G$. Suppose also $\omega = \sigma^1 \wedge \cdots \wedge \sigma^n$ for σ^i left invariant 1-forms, and define as in the hint

$$g(v,w) = \frac{1}{\int \omega} \int g_L(DR_x(v), DR_x(w))\omega.$$

First compute

$$\begin{aligned} g(DL_y(v), DL_y(w)) &= \frac{1}{\int \omega} \int g_L(DR_x(DL_y(v)), DR_x(DL_y(w)))\omega \\ &= \frac{1}{\int \omega} \int g_L(DL_y(DR_x(v)), DL_y(DR_x(w)))\omega \\ &= \frac{1}{\int \omega} \int g_L(DR_x(v), DR_x(w)) \\ &= g(v,w) \end{aligned}$$

so g left invariant.

From the right we have

$$\begin{aligned} g(DR_y(v), DR_y(w)) &= \frac{1}{\int \omega} \int g_L(DR_x(DR_y(v)), DR_x(DR_y(w)))\omega \\ &= \frac{1}{\int \omega} \int g_L(DR_{x\cdot y}(v), DR_{x\cdot y}(w))\omega \\ &= g(v,w) \end{aligned}$$

2 Consider the upper-half plane

$$\mathbb{R}^2_+ = \{(x,y) \in \mathbb{R}^2 \mid y > 0\}$$

with the hyperbolic metric

$$\frac{dx^2 + dy^2}{y^2}.$$

Show that the vertical line segment between $(0,1)$ and $(0,2)$ is the shortest path between these points.

Let $\gamma(t) = \langle \gamma_x(t), \gamma_y(t) \rangle$ be a path $\gamma : [0,a] \to \mathbb{R}^2_+$ with $\gamma(0) = (0,1)$ and $\gamma(a) = (0,2)$. Assume without loss of generality that γ is parameterized by (Euclidean) arclength. We can compute

$$|\gamma| = \int_0^a \frac{1}{\gamma_y(t)} \sqrt{\gamma_x'(t)^2 + \gamma_y'(t)^2}$$

since we assumed γ arclength parameterized this reduces to

$$|\gamma| = \int_0^a \frac{1}{\gamma_y(t)} dt.$$

By assumption, $\gamma_y'(t) \in [-1,1]$. In particular $\gamma_y' \leq 1 \Rightarrow \gamma_y(t) \leq 1+t$. Notice $a \geq 1$ since γ is arclength parameterized and the Euclidean distance between its endpoints is 1. Now compute:

$$\begin{aligned} |\gamma| &= \int_0^a \frac{1}{\gamma_y(t)} dt \\ &\geq \int_0^1 \frac{1}{1+t} dt + \underbrace{\int_1^a \frac{1}{\gamma_y(t)} dt}_{(*)} \end{aligned}$$

Of course the contribution of $(*)$ is strictly positive if $a > 1$ since γ_y is restricted to take only positive values. Thus, our bound is tight if and only if $a = 1$ and $\gamma_y(t) = 1+t$, to wit, when

$$\gamma(t) = \langle 0, 1+t \rangle$$

as desired.

3 Consider $\mathbb{R}^2_+$ with the hyperbolic metric as above. Let $v_0 = (0,1)$ be a tangent vector at the point $(0,1)$ of $\mathbb{R}^2_+$. Let $v(t)$ be the parallel transport of v_0 along the curve $x = t$, $y = 1$. Show that $v(t)$ makes an angle t with the direction of the y-axis, measured in the clockwise sense.

First note that the nonzero components of the metric and respectively inverse metric are just

$$g^{xx} = g^{yy} = y^2 \quad \text{and} \quad g_{xx} = g_{yy} = \frac{1}{y^2}.$$

We can therefore reduce the expression for Christoffel symbols

$$\Gamma^m_{ij} = \frac{1}{2} \sum_k \left(\frac{\partial}{\partial x_i} g_{jk} + \frac{\partial}{\partial x_j} g_{ki} - \frac{\partial}{\partial x_k} g_{ij} \right) g^{km} = \frac{y^2}{2} \left(\frac{\partial}{\partial x_i} g_{jm} + \frac{\partial}{\partial x_j} g_{mi} - \frac{\partial}{\partial x_m} g_{ij} \right)$$

Clearly this can only be nonzero if at least one of i, j, m is y and the others coincide. Thus compute:

$$\Gamma^y_{yy} = \Gamma^x_{xy} = \Gamma^x_{yx} = -\frac{1}{y}; \quad \Gamma^y_{xx} = \frac{1}{y}; \quad \Gamma^m_{ij} = 0 \text{ otherwise.}$$

Now let

$$v(t) = \langle \sin t, \cos t \rangle \quad x(t) = \langle t, 1 \rangle \quad \dot{x}(t) = \langle 1, 0 \rangle$$

and check

$$\frac{dv^x}{dt} + \sum_{i,j} \Gamma^x_{ij} v^j \frac{dx_i}{dt} = \cos t - \frac{1}{y} \cos t = 0 \quad \text{and} \quad \frac{dv^y}{dt} + \sum_{i,j} \Gamma^y_{ij} v^j \frac{dx_i}{dt} = -\sin t + \frac{1}{y} \sin t = 0.$$

4 For any $p \in (M,g)$ and orthonormal basis $e_1, \ldots, e_n$ for T_pM, show that there is an orthonormal frame $E_1, \ldots, E_n$ in a neighborhood of p such that $E_i = e_i$ and $(\nabla E_i)|_p = 0$. ***Hint:*** Fix an orthonormal frame $\overline{E}_i$ near $p \in M$ with $\overline{E}_i(p) = e_i$. If we define $E_i = \alpha_i^j \overline{E}_j$, where $[\alpha_i^j(x)] \in SO(n)$ and $\alpha_i^j(p) = \delta_i^j$, then this will yield the desired frame provided that the $D_{e_k} \alpha_i^j$ are appropriately prescribed.

Let $\overline{E}_i$ be an orthonormal frame on a neighborhood of p. Define $E_i = \alpha_i^j \overline{E}_j$ and suppose that

$$\nabla_j E_l = \left(\frac{\partial E_l^i}{\partial x^j} + E_l^k \Gamma^i_{jk} \right) e_i = 0 \text{ for all } i, j, l \tag{1}$$

Computing, we get

$$\frac{\partial}{\partial x^j} E_l^i = \alpha_l^m \frac{\partial}{\partial x^j} \overline{E}^i_m + \overline{E}^i_m \frac{\partial}{\partial x^j} \alpha_l^m$$

which reduces, at p to
$$\frac{\partial}{\partial x^j} E_l^i = \frac{\partial}{\partial x^j} \overline{E}_l^i + \frac{\partial}{\partial x^j} \alpha_l^i.$$

Thus, (1) is equivalent (at p) to
$$\begin{aligned}
0 &= \frac{\partial}{\partial x^j} \overline{E}_l^i + \frac{\partial}{\partial x^j} \alpha_l^i + \overline{E}_l^k \Gamma_{jk}^i \\
&= \frac{\partial}{\partial x^j} \overline{E}_l^i + \frac{\partial}{\partial x^j} \alpha_l^i + \Gamma_{jl}^i \\
&= \Gamma_{ij}^l + \frac{\partial}{\partial x^j} \alpha_l^i + \Gamma_{jl}^i
\end{aligned}$$

Now, let $[\alpha_l^i]$ be a family of matrices defined on a neighborhood of p by setting $\alpha_i^j(p) = \delta_i^j$ and demanding that α_l^i satisfy the family of differential equations
$$\frac{\partial}{\partial x^j} \alpha_l^i = -\Gamma_{ij}^l - \Gamma_{jl}^i$$
Since these derivatives are antisymmetric in i, l the matrices $[\alpha_l^i]$ must belong to $SO(n)$. Furthermore, $E_i = \alpha_i^j \overline{E}_j$ has the desired $(\nabla E_i)|_p = 0$ by our previous computation.

5 For any point p in a Riemannian manifold (M, g), show that there exist coordinates $x^1, \ldots, x^n$ near p such that $\partial_i = e_i$ and $\nabla \partial_i = 0$ at p.

6 Let (M, g) be oriented and define the Riemannian volume form $d$vol as follows:
$$d\mathbf{vol}(v_1, \ldots, v_n) = \det(g(v_i, e_j)),$$
where $e_1, \ldots, e_n$ is a positively oriented orthonormal basis for T_pM.

6.5 Conclude that the Laplacian has the formula
$$\Delta u = \frac{1}{\sqrt{\det(g_{ij})}} \partial_k \left(\sqrt{\det(g_{ij})} g^{kl} \partial_l u \right).$$

Using previous parts we compute
$$\begin{aligned}
\Delta u \cdot dvol(\partial_1, \ldots, \partial_n) &= (L_{g^{kl}\partial_l u \partial_k} dvol)(\partial_1, \ldots, \partial_n) \\
&= g^{kl} \partial_l u (L_{\partial_k} dvol)(\partial_1, \ldots, \partial_n) + d(g^{kl}\partial_l u)(\partial_m) dvol(\partial_1, \ldots, \partial_k, \ldots, \partial_n) \\
&= g^{kl} \partial_l u \partial_k \sqrt{\det(g_{ij})} + \partial_k (g^{kl} \partial_l u) \sqrt{\det(g_{ij})} \\
&= \partial_k \left(\sqrt{\det(g_{ij})} g^{kl} \partial_l u \right) \\
&= \frac{1}{\sqrt{\det(g_{ij})}} \partial_k \left(\sqrt{\det(g_{ij})} g^{kl} \partial_l u \right)
\end{aligned}$$

7 Let (M, g) be a oriented Riemannian manifold with volume form $d$vol as above.

7.1 If f has compact support, then
$$\int_M \Delta f \cdot d\mathbf{vol} = 0.$$

$$
\begin{aligned}
\int_M \Delta f \cdot d\mathrm{vol} &= \int_M L_{\nabla f} d\mathrm{vol} \\
&= \int_M i_{\nabla f} d(d\mathrm{vol}) + d(i_{\nabla f} d\mathrm{vol}) \\
&= 0
\end{aligned}
$$

7.2 Show that

$$\mathbf{div}(f \cdot X) = g(\nabla f, X) + f \cdot \mathbf{div} X$$

$$
\begin{aligned}
\mathrm{div}(f \cdot X) &= \mathrm{div}(f \cdot X) d\mathrm{vol}(e_1, \ldots, e_n) \\
&= (L_{f \cdot x} d\mathrm{vol})(E_1, \ldots, E_n) \\
&= f(L_X d\mathrm{vol})(E_1, \ldots, E_n) + df(E_i) d\mathrm{vol}(E_1, \ldots, X, \ldots, E_n) \\
&= f(\mathrm{div} X) d\mathrm{vol}(E_1, \ldots, E_n) + g(\nabla f, E_i) g(X, E_i) \\
&= f \cdot \mathrm{div} X + g(\nabla f, X).
\end{aligned}
$$

7.3 Show that

$$\Delta(f_1 \cdot f_2) = (\Delta f_1) \cdot f_2 + 2g(\nabla f_1, \nabla f_2) + f_1 \cdot (\Delta f_2).$$

$$
\begin{aligned}
\Delta(f_1 \cdot f_2) &= \mathrm{div}(\nabla(f_1 \cdot f_2)) \\
&= \mathrm{div}(f_1 \cdot \nabla f_2 + f_2 \cdot \nabla f_1) \\
&= f_1 \Delta f_2 + g(\nabla f_1, \nabla f_2) + f_2 \Delta f_1 + g(\nabla f_2, \nabla f_1) \\
&= f_1 \Delta f_2 + 2g(\nabla f_1, \nabla f_2) + f_2 \Delta f_1.
\end{aligned}
$$

7.4 Establish the integration by parts formula for functions with compact support:

$$\int_M f_1 \cdot \Delta f_2 \cdot d\mathbf{vol} = -\int_M g(\nabla f_1, \nabla f_2) \cdot d\mathbf{vol}.$$

$$
\begin{aligned}
\int_M f_1 \cdot \Delta f_2 \cdot d\mathrm{vol} &= \int_M f_1 \cdot \mathrm{div}(\nabla f_2) \cdot d\mathrm{vol} \\
&= \int_M (\mathrm{div}(f_1 \cdot \nabla f_2) - g(\nabla f_1, \nabla f_2)) \cdot d\mathrm{vol} \\
&= -\int_M g(\nabla f_1, \nabla f_2) \cdot d\mathrm{vol}.
\end{aligned}
$$

7.5 Conclude that if f is sub- or superharmonic (i.e., $\Delta f \geq 0$ or $\Delta f \leq 0$) then f is constant. *Hint:* first show $\Delta f = 0$; then use integration by parts on $f \cdot \Delta f$.

Let $\Delta f \geq 0$.

$$0 = \int_M \Delta f \cdot d\mathrm{vol} \geq 0,$$

so $\Delta f = 0$. Thus,

$$0 = \int_M f \cdot \Delta f \cdot d\text{vol} = -\int_M g(\nabla f, \nabla f) \cdot d\text{vol}$$

which implies $\nabla f \equiv 0$, f constant as desired.

8 Let X be a unit vector filed on (M, g) such that $\nabla_X X = 0$.

8.1 Show that x is locally the gradient of a distance function iff the orthogonal distribution is integrable.

8.2 Show that X is the gradient of a distance function in a neighborhood of $p \in M$ iff the orthogonal distribution has an integral submanifold through p. *Hint:* It might help to show that $L_X \theta_X = 0$.

8.3 Find X with the given conditions so that it is not a gradient field. *Hint:* Consider S^3