

# 1 Give examples of Riemannian manifolds having:

## 1.1 positive scalar curvature but not positive Ricci curvature;

Let  $M = S_a^n \times S_b^m$ , and  $X, Y, U, V$  vector fields with  $X, Y$  tangent to  $S^n$ , and  $U, V$  tangent to  $S^m$ . Recall (class, or Petersen p.65) that

$$\mathfrak{R}(X \wedge V) = 0 \quad \mathfrak{R}(X \wedge Y) = aX \wedge Y \quad \mathfrak{R}(U \wedge V) = bU \wedge V$$

$$\text{Ric}(X) = (n-1)aX \quad \text{Ric}(V) = (m-1)bV \quad \text{scal} = n(n-1)a + m(m-1)b.$$

In particular if we put  $M = S^1 \times S^2$  ( $n=1, m=2$ ) we have  $\text{scal} = 2b > 0$ , but  $\text{Ric}(X) = 0$ .

## 1.2 positive Ricci curvature but not positive sectional curvature;

As above, but let  $M = S^2 \times S^2$  ( $m=n=2$ ). Now  $\text{Ric}(X) = aX > 0$ ,  $\text{Ric}(V) = bV > 0$ , and  $\text{sec}(x, v) = \mathfrak{R}(x \wedge v) = 0$ .

## 1.3 positive sectional curvature but not positive curvature operator.

**2 Show that a Riemannian manifold with parallel Ricci tensor has constant scalar curvature. In chapter 3 it will be shown that the converse is not true, and also that a metric with parallel curvature tensor doesn't have to be Einstein.**

**3 Let  $G$  be a Lie group with a bi-invariant metric. Using left-invariant fields establish the following formulas. *Hint:* First go back to the exercises to chapter 1 and take a peek at chapter 3 where some of these things are proved.**

(You can assume the following which are proved in Proposition 12 on p. 79. I suggest you read that proof.)

$$\nabla_X Y = \frac{1}{2}[X, Y]; \quad R(X, Y)Z = \frac{1}{4}[Z, [X, Y]]; \quad g(R(X, Y)Z, W) = -\frac{1}{4}(g([X, Y], [Z, W]))$$

**3.1 Show that the curvature operator is also nonnegative by showing that:**

$$g\left(\mathfrak{R}\left(\sum_{i=1}^k X_i \wedge Y_i\right), \left(\sum_{i=1}^k X_i \wedge Y_i\right)\right) = \frac{1}{4} \left| \sum_{i=1}^k [X_i, Y_i] \right|^2.$$

- 3.2 Show that  $\text{Ric}(X, X) = 0$  iff  $X$  commutes with all other left-invariant vector fields. Thus  $G$  has positive Ricci curvature if the center of  $G$  is discrete.
- 3.3 Consider the linear map  $\Lambda^2 \mathfrak{g} \rightarrow [\mathfrak{g}, \mathfrak{g}]$  that sends  $X \wedge Y$  to  $[X, Y]$ . Show that the sectional curvature is positive iff this map is an isomorphism. Conclude that this can only happen if  $n = 3$  and  $\mathfrak{g} = \mathfrak{su}(2)$ .
- 4 Consider a Riemannian metric  $(M, g)$ . Now *scale* the metric by multiplying it by a number  $\lambda^2$ . Then we get a new Riemannian manifold  $(M, \lambda^2 g)$ . Show that the new connection and  $(1, 3)$ -curvature tensor remain the same, but that  $\text{sec}$ ,  $\text{scal}$ , and  $\mathfrak{R}$  all get multiplied by  $\lambda^{-2}$ .
- 5 Recall that complex manifolds have complex tangent spaces. Thus we can multiply vectors by  $\sqrt{-1}$ . As a generalization of this we can define an *almost complex* structure. This is a  $(1, 1)$ -tensor  $J$  such that  $J^2 = -I$ . Show that the *Nijenhuis tensor*:

$$N(X, Y) = [J(X), J(Y)] - J([J(X), Y]) - J([X, J(Y)]) - [X, Y]$$

is indeed a tensor. If  $J$  comes from a complex structure then  $N = 0$ , conversely Newlander & Nirenberg have shown that  $J$  comes from a complex structure if  $N = 0$ .

A *Hermitian structure* on a Riemannian manifold  $(M, g)$  is an almost complex structure  $J$  such that

$$g(J(X), J(Y)) = g(X, Y).$$

The *Kähler form* of a Hermitian structure is

$$\omega(X, Y) = g(J(X), Y).$$

Show that  $\omega$  is a 2-form. Show that  $d\omega = 0$  iff  $\nabla J = 0$ . If the Kähler form is closed, then we call the metric a Kähler metric.

- 6 There is a strange curvature quantity we have not yet mentioned. Its definition is somewhat cumbersome and nonintuitive. First, for two symmetric  $(0, 2)$ -tensors  $h, k$  define the *Kulkarni-Nomizu product* as the  $(0, 4)$ -tensor

$$\begin{aligned} h \circ k(v_1, v_2, v_3, v_4) &= h(v_1, v_3) \cdot k(v_2, v_4) + h(v_2, v_4) \cdot k(v_1, v_3) \\ &\quad - h(v_1, v_4) \cdot k(v_2, v_3) - h(v_2, v_3) \cdot k(v_1, v_4). \end{aligned}$$

Note that  $(M, g)$  has constant curvature  $c$  iff the  $(0, 4)$ -curvature tensor satisfies  $R = c \cdot (g \circ g)$ . If we use the  $(0, 2)$  form of the Ricci tensor, then we can decompose the  $(0, 4)$ -curvature tensor as follows in dimensions  $n \geq 4$

$$R = \frac{\text{scal}}{2n(n-1)} g \circ g + \left( \text{Ric} - \frac{\text{scal}}{n} \cdot g \right) \circ g + W$$

When  $n = 3$  we have instead

$$R = \frac{\text{scal}}{12} g \circ g + \left( \text{Ric} - \frac{\text{scal}}{3} \cdot g \right) \circ g.$$

The  $(0, 4)$ -tensor  $W$  defined for  $n > 3$  is called the *Weyl tensor*.

6.1 Show that these decompositions are orthogonal, in particular:

$$|R|^2 = \left| \frac{\text{scal}}{2n(n-1)} g \circ g \right|^2 + \left| \left( \text{Ric} - \frac{\text{scal}}{n} \cdot g \right) \circ g \right|^2 + |W|^2.$$

6.2 Show that if we conformally change the metric  $\tilde{g} = f \cdot g$ , then  $\tilde{W} = f \cdot W$ .

6.3 If  $(M, g)$  has constant curvature, then  $W = 0$ .

6.4 If  $(M, g)$  is locally conformally equivalent to the Euclidean metric, i.e., locally we can always find coordinates where:

$$g = f \cdot ((dx^1)^2 + \cdots + (dx^n)^2),$$

then  $W = 0$ . The converse is also true but much harder to prove.

7 Assume that we have a Riemannian immersion of an  $n$ -manifold into  $\mathbb{R}^{n+1}$ . If  $n \geq 3$ , then show that it can't have negative curvature. If  $n = 2$  give an example where it does have negative curvature.

8 Let  $(M, g)$  be a closed Riemannian  $n$ -manifold, and suppose that there is a Riemannian embedding into  $\mathbb{R}^{n+1}$ . Show that there must be a point  $p \in M$  where the curvature operator  $\mathfrak{R} : \Lambda^2 T_p M \rightarrow \Lambda^2 T_p M$  is positive. *Hint:* Consider  $f(x) = |x|^2$  and restrict it to  $M$ , then check what happens at a maximum.