Homework

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16 November 2009

Contents

1 Chapter 4 Section 1

1.1 Problem 9

Show that the general solution of $\nabla^2 [f(r)] = 0$ is $f(r) = A + B/r$ for arbitrary constants A, B.

We have [1, Page 237] by 4.1.5 that

$$\nabla^2 u = u_{rr} + \frac{2}{r} u_r + \frac{1}{r^2}(u_{\theta\theta} + \cot\theta u_\theta + \csc^2\theta u_{\varphi\varphi}) \tag{1}$$

but, since $u(r)$ depends only on r we have that

$$\nabla^2 [f(r)] = 0$$
$$\iff u_{rr} + \frac{2}{r}u_r + \frac{1}{r^2}(u_{\theta\theta} + \cot\theta u_\theta + \csc^2\theta u_{\varphi\varphi}) = 0$$
$$\iff f''(r) + \frac{2}{r}f'(r) = 0$$

but this is just an ordinary differential equation, whose solution we know to be

$$f(r) = A + \frac{B}{r} \tag{2}$$

So we have shown that the general solution of $\nabla^2 [f(r)] = 0$ is $f(r) = A + B/r$ for arbitrary constants A, B as desired.

1.2 Problem 11

Solve the equation $\nabla^2 [f(r)] = -1$ with the boundary conditions $f(a) = 0$ and $f(0)$ finite.

From Exercise 10 [1, Page 250] we know that the general solution for differential equations of this form is given by

$$f(r) = A + \frac{B}{r} - \frac{r^2}{6}$$

for arbitrary constants A, B.

Now we apply our boundary conditions to determine A, B. Since $f(0)$ is finite we know that $B = 0$. Thus,

$$f(a) = A + \frac{B}{a} - \frac{a^2}{6} = 0$$
$$\Rightarrow 6Aa + 6B - a^3 = 0$$
$$\Rightarrow 6A - a^2 = 0$$
$$\Rightarrow 6A = a^2$$
$$\Rightarrow A = \frac{a^2}{6}$$

Since we have determined $A = \frac{a^2}{6}$ and $B = 0$ the particular solution of the equation $\nabla^2 [f(r)] = -1$ with the boundary conditions $f(a) = 0$ and $f(0)$ finite is given by:

$$\begin{aligned} f(r) &= \frac{a^2}{6} - \frac{r^2}{6} \\ &= \frac{a^2 - r^2}{6} \end{aligned}$$

1.3 Problem 13

Solve the equation $\nabla^2 [f(r)] = -r^4$ with the boundary condition $f(a) = 0$ and $f(0)$ finite.

We employ Equation 1 to reduce the general problem to that of an ordinary differential equation as in Problem 9 (1.1).

$$\begin{aligned} &\nabla^2 [f(r)] = -r^4 \\ \iff\ & u_{rr} + \frac{2}{r}u_r + \frac{1}{r^2}(u_{\theta\theta} + \cot\theta u_\theta + \csc^2\theta u_{\varphi\varphi}) = -r^4 \\ \iff\ & f''(r) + \frac{2}{r}f'(r) = -r^4 \end{aligned}$$

but as previously stated the solution to the corresponding homogeneous ODE is given by Equation 2. To solve the current problem we need only guess a particular solution. Let's try

$$f(r) = -\frac{1}{42}r^6$$

$$\Rightarrow f''(r) + \frac{2}{r}f'(r) = -r^4$$

So this is indeed a solution to our equation. Thus the general solution to the ODE is given by a sum of the above and Equation 2.

$$f(r) = A + \frac{B}{r} - \frac{1}{42}r^6$$

Now employing our boundary conditions we see that since $f(0)$ is finite $B = 0$, and moreover

$$f(a) = 0$$

$$\Rightarrow f(a) = A - \frac{1}{42}a^6 = 0$$

$$\Rightarrow A = \frac{1}{42}a^6$$

So our final solution, including boundary conditions is given by

$$f(r) = \frac{1}{42}a^6 - \frac{1}{42}r^6$$

$$f(r) = \frac{1}{42}(a^6 - r^6)$$

1.4 Problem 15

Find the solution $u(r;t)$ of the heat equation $u_t = K\nabla^2 u$, $-\infty < t < \infty$, in the sphere $0 \leq r < a$ satisfying the boundary condition $u(a;t) = A_1 \cos\omega(t - t_0)$ where A_1, ω and t_0 are positive constants.

We note that this problem has already been solved in more generality [1, Page 239] and has solution

$$u(r;t) = \frac{a}{r}\sum_{n=-\infty}^{\infty} \alpha_n e^{i\omega_n t} \frac{e^{c_n(1+i)r} - e^{-c_n(1+i)r}}{e^{c_n(1+i)a} - e^{-c_n(1+i)a}}$$

for $\omega_n = 2\pi n/T$, $c_n = \sqrt{\pi n/KT}$

So all that remains is to specify the solution satisfying our boundary conditions by setting

$$u_0(t) = \sum_{n=-\infty}^{\infty} \alpha_n e^{t\pi int/T}$$

and computing the fourier coeffecients α_n.

It is perhaps useful to first solve this problem for $t_0 = 0$.

In this case it is easy to see that since $u_0(t) = A_1 \cos\omega t$ we have $\alpha_1 = \alpha_{-1} = \frac{1}{2}A_1$ as in Example 4.1.2[1, Page 239]. So, plugging in and taking the real part we see that the particular solution to this reduced problem is given by

$$u(r;t) = \frac{aA_1}{r} Re\ e^{i\omega t} \frac{e^{c(1+i)r} - e^{-c(1+i)r}}{e^{c(1+i)a} - e^{-c(1+i)a}}$$

for $c = \sqrt{\omega/2K}$. Again, this solution is very similar to the one derived in Example 4.1.2[1, Page 239].

However, the original problem is just a translation in time of the one which we have just solved. Thus, its solution is given by

$$u(r;t) = \frac{aA_1}{r} Re\ e^{i\omega(t-t_0)} \frac{e^{c(1+i)r} - e^{-c(1+i)r}}{e^{c(1+i)a} - e^{-c(1+i)a}}$$

for $c = \sqrt{\omega/2K}$.

1.5 Problem 17

Find the solution $u(r;t)$ of the heat equation $u_t = K\nabla^2 u + \sigma$ in the sphere $0 \leq r < a$ satisfying the boundary condition $u(a;t) = T_1$ and the initial condition $u(r;0) = T_2$. Use the five-stage method, and find the relaxation time.

Throughout we will follow the example of the book[1, Page 242].

First we will reduce this problem to a one dimensional one. Since the initial and boundary conditions are both independent of (θ, φ) we can reasonably assume that the solution is as well. Thus, we examine solutions of the form $u = u(r;t)$. Now we define a new function w by $w(r;t) = ru(r;t)$. So, $w_t = ru_t$, $w_r = ru_r + u$, $w_{rr} = ru_{rr} + 2u_r = r\nabla^2 u$. Multiplying the problem for u by r to put it in terms of w we get

$$w_t = Kw_{rr} + \sigma r$$

$$w(a;t) = aT_1$$
$$w(r;0) = rT_2$$
$$w(0;t) = 0$$

for $0 \leq r < a$.

This one-dimensional boundary vale problem for w can now be solved by the five-stage method outlined in the text[1, Page 242].

1.5.1 Stage 1

The steady-state equation is $KW_{rr} + \sigma r = 0$ with the two boundary conditions at $r = 0$, $r = a$. The general solution of this ordinary differential equation is just

$$W(r) = -\frac{\sigma r^3}{6K} + A + Br$$

for arbitrary constants A, B. The boundary condition $W(0) = 0$ requires that $A = 0$. To determine B we take

$$W(a) = -\frac{\sigma a^3}{6K} + Ba = aT_1$$

$$\Rightarrow -\frac{\sigma a^2}{6K} + B = T_1$$

$$\Rightarrow B = T_1 + \frac{\sigma a^2}{6K}$$

substituting our constants into the general solution we get

$$W(r) = -\frac{\sigma r^3}{6K} + r\left(T_1 + \frac{\sigma a^2}{6K}\right)$$

1.5.2 Stage 2

We use the steady-state solution to transform the problem. Letting $v(r;t) = w(r;t) - W(r)$, we have the equation for v:

$$v_t = Kv_{rr}$$

$$v(0;t) = 0$$

$$v(a;t) = 0$$

$$v(r;0) = \frac{\sigma r^3}{6K} - r\left(T_1 - T_2 + \frac{\sigma a^2}{6K}\right)$$

1.5.3 Stage 3

Now we determine the separated solutions to the problem for v. We write $v(r;t) = R(r)T(t)$ yielding the equations $T' + \lambda KT = 0$, $R'' + \lambda R = 0$ with boundary conditions $R(0) = 0$, $R(a) = 0$. The first equation can be solved with a constant by $T(t) = e^{-\lambda Kt}$. The equation for $R(r)$ is a Sturm-Liouville eigenvalue problem,

$$R'' + \lambda R = 0$$

$$R(0) = 0$$

$$R(a) = 0$$

which has solutions as in Example 1.6.1[1, Page 85] given by

$$\lambda_n = \left(\frac{n\pi}{a}\right)^2, \ \phi_n(x) = \sin\frac{n\pi r}{a} \ n = 1, 2, \ldots$$

Now we can write the superposition of separated solutions as

$$v(r;t) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi r}{a} e^{-\lambda_n Kt} \tag{3}$$

By Theorem 1.5[1, Page 86], the eigenfunctions must be orthogonal. Thus,

$$\int_0^a \sin\frac{m\pi r}{a} \sin\frac{n\pi r}{a} dr = 0 \ n \neq m$$

The normalization can be computed as an integral

$$\begin{aligned}\int_0^a \sin^2\frac{m\pi r}{a} dr &= \frac{1}{2}\int_0^a \left(1 - \cos\frac{2m\pi r}{a}\right) dr \\ &= \left[\frac{r}{2} - \frac{a\sin\left(\frac{2m\pi r}{a}\right)}{4m\pi}\right]_0^a \\ &= \frac{a}{2}\left(1 - \frac{\sin 2m\pi}{2m\pi}\right)\end{aligned}$$

The Fourier Coefficients A_n can be determined by setting $t = 0$ in Equation 3, multiplying by $\sin\frac{n\pi r}{a}$ and integrating; that is

$$\int_0^a \left(\frac{\sigma r^3}{6K} - r\left(T_1 - T_2 + \frac{\sigma a^2}{6K}\right)\right) \sin\frac{n\pi r}{a} dr = A_n \int_0^a \sin^2\frac{n\pi r}{a} dr$$
$$= \frac{A_n a}{2}\left(1 - \frac{\sin 2n\pi}{2n\pi}\right)$$
$$\Rightarrow \frac{1}{3Kn^4\pi^4} a^2 (3n\pi \left(Kn^2\pi^2(T_1 - T_2) + a^2\sigma\right) \cos n\pi +$$
$$\left(3Kn^2\pi^2(-T_1 + T_2) + a^2\left(-3 + n^2\pi^2\right)\sigma\right) \sin n\pi)$$

Since we are dealing only with $n \in \mathbb{N}$ we may replace $\sin n\pi$ with 0 and $\cos n\pi$ with $(-1)^n$. Making this simplification we are left with

$$\frac{1}{3Kn^4\pi^4} a^2 (3n\pi \left(Kn^2\pi^2(T_1 - T_2) + a^2\sigma\right)(-1)^n = \frac{A_n a}{2}$$

Thus, the Fourier Coefficients A_n are given by

$$A_n = \frac{2(-1)^n a(T_1 - T_2)}{n\pi} + \frac{2(-1)^n a^3\sigma}{kn^3\pi^3}$$

1.5.4 Stage 4

We have obtained the formal solution of the problem as

$$u(r;t) = U(r) + \frac{1}{r}\sum_{n=1}^{\infty} A_n \sin\frac{n\pi r}{a} e^{-\lambda_n K t}$$

with

$$U(r) = \frac{W(r)}{r} = T_1 + \frac{\sigma}{6K}(a^2 - r^2)$$
$$A_n = \frac{2(-1)^n a(T_1 - T_2)}{n\pi} + \frac{2(-1)^n a^3\sigma}{kn^3\pi^3}$$
$$\lambda_n = \left(\frac{n\pi}{a}\right)^2$$

1.5.5 Stage 5/Relaxation Time

When $t \to \infty$, the solution $u(r;t)$ tends to the steady-state solution $U(r)$. We use the method from Chapter 2 to estimate the rate of approach; thus using O from 2.2.3[1, Page 113] we get

$$\frac{1}{r}\sum_{n=1}^{\infty} A_n \sin\frac{n\pi r}{a} e^{-\lambda_n K t} = O(e^{-at})\ t \to \infty$$

Therefore $u(r;t) - U(r) = O(e^{-at})$, $t \to \infty$. Finally we compute the relaxation time by noting that

$$u(r;t) - U(r) = \frac{A_1 \sin\frac{\pi r}{a}}{r} e^{-\lambda_1 K t} + O(e^{-at})\ t \to \infty$$

If $A_1 \neq 0$ the relaxation time is given by

$$\tau = \frac{1}{\lambda_1 K} = \frac{a^2}{\pi^2 K}$$

2 Chapter 5 Section 1

In the following problems we compute the Fourier Transform of the given function $f(x)$.

2.1 Problem 1

$$f(x) = \begin{cases} 1 & : x \in (-2,2) \\ 0 & : otherwise \end{cases}$$

So we proceed employing the given definition for the Fourier Transform[1, Page 278]:

$$F(\mu) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x) e^{i\mu x} dx$$

Since the function f is zero except on the interval $x \in (-2,2)$ we may restrict the limits of our integration to this region.

Plugging in to the definition yields:

$$F(\mu) = \frac{1}{2\pi}\int_{-2}^{2} e^{i\mu x} dx$$

Now we evaluate the antiderivative to get

$$\begin{aligned} F(\mu) &= \frac{1}{2\pi}\left[-\frac{ie^{ix\mu}}{\mu}\right]_{-2}^{2} \\ &= \frac{1}{2\pi}\left(\frac{ie^{-2i\mu}}{\mu} - \frac{ie^{2i\mu}}{\mu}\right) \\ &= \frac{2}{\pi\mu}\left(\frac{ie^{-2i\mu} - ie^{2i\mu}}{4}\right) \\ &= \frac{2}{\pi\mu}\left(\frac{e^{2i\mu} - e^{-2i\mu}}{4i}\right) \end{aligned}$$

Since the numerator is just a difference of two squares so we can express this as

$$\begin{aligned} &\frac{2}{\pi\mu}\left(\frac{(e^{i\mu}+e^{-i\mu})(e^{i\mu}-e^{-i\mu})}{4i}\right) \\ &= \frac{2}{\pi\mu}\left(\frac{e^{i\mu}+e^{-i\mu}}{2}\right)\left(\frac{e^{i\mu}-e^{-i\mu}}{2i}\right) \\ &= \frac{2}{\pi\mu}\cos\mu\sin\mu \end{aligned}$$

So, the Fourier Transform of the given function is

$$F(\mu) = \frac{2}{\pi\mu}\cos\mu\sin\mu$$

2.2 Problem 3

$$f(x) = \begin{cases} e^{-3x} & : x > 0 \\ e^{2x} & : x < 0 \end{cases}$$

Again we use the definition for the Fourier Transform:

$$F(\mu) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{i\mu x} dx$$

But clearly this is equivalent to

$$F(\mu) = \frac{1}{2\pi} \left(\int_{-\infty}^{0} f(x) e^{i\mu x} dx + \int_{0}^{\infty} f(x) e^{i\mu x} dx \right) \tag{4}$$

Now we can plug in with our particular f

$$\begin{aligned} F(\mu) &= \frac{1}{2\pi} \left(\int_{-\infty}^{0} e^{2x} e^{i\mu x} dx + \int_{0}^{\infty} e^{-3x} e^{i\mu x} dx \right) \\ &= \frac{1}{2\pi} \left(\int_{-\infty}^{0} e^{2x+i\mu x} dx + \int_{0}^{\infty} e^{i\mu x - 3x} dx \right) \end{aligned}$$

Now we compute the antiderivative yielding

$$\begin{aligned} &\frac{1}{2\pi} \left(\left[-\frac{i e^{x(2+i\mu)}}{-2i+\mu} \right]_{-\infty}^{0} + \left[-\frac{i e^{ix(3i+\mu)}}{3i+\mu} \right]_{0}^{\infty} \right) \\ &= \frac{1}{2\pi} \left(\left[\frac{e^{x(2+i\mu)}}{2+i\mu} \right]_{-\infty}^{0} + \left[\frac{e^{ix(3i+\mu)}}{-3+i\mu} \right]_{0}^{\infty} \right) \end{aligned}$$

Now, since

$$\lim_{x \to -\infty} \frac{e^{x(2+i\mu)}}{2+i\mu} = 0 = \lim_{x \to \infty} \frac{e^{ix(3i+\mu)}}{-3+i\mu}$$

we evaluate at the integration limits to get

$$\begin{aligned} &\frac{1}{2\pi} \left(\frac{1}{2+i\mu} - \frac{1}{-3+i\mu} \right) \\ &= \frac{5}{2\pi \left(6 + i\mu + \mu^2\right)} \end{aligned}$$

So, the Fourier Transform of the given function is

$$F(\mu) = \frac{5}{2\pi \left(6 + i\mu + \mu^2\right)}$$

2.3 Problem 5

$$f(x) = \cos x e^{-|x|}$$

So, by Equation 4 the Fourier Transform of this function is

$$F(\mu) = \frac{1}{2\pi}\left(\int_{-\infty}^{0} f(x)e^{i\mu x}dx + \int_{0}^{\infty} f(x)e^{i\mu x}dx\right)$$

$$= \frac{1}{2\pi}\left(\int_{-\infty}^{0} \cos x e^{-|x|}e^{i\mu x}dx + \int_{0}^{\infty} \cos x e^{-|x|}e^{i\mu x}dx\right)$$

Since in each of these ranges of integration we are guaranteed $x < 0, x > 0$ respectively we may replace $|x|$ with $-x, x$ respectively yielding

$$\frac{1}{2\pi}\left(\int_{-\infty}^{0} \cos x e^{x}e^{i\mu x}dx + \int_{0}^{\infty} \cos x e^{-x}e^{i\mu x}dx\right)$$

$$= \frac{1}{2\pi}\left(\int_{-\infty}^{0} \cos x e^{i\mu x+x}dx + \int_{0}^{\infty} \cos x e^{i\mu x-x}dx\right)$$

Integrating this expression we get

$$= \frac{1}{2\pi}(\left[-\frac{ie^{x+ix\mu}}{-2-2i\mu+\mu^2}((-i+\mu)\cos x - i\sin x)\right]_{-\infty}^{0}$$

$$+\left[\frac{e^{ix(i+\mu)}}{-2+2i\mu+\mu^2}(\cos x - i\mu\cos x - \sin x)\right]_{0}^{\infty})$$

As before we observe that the $lim_{x\to\infty}$, $\lim_{x\to-\infty}$ of the respective expressions is 0. Therefore, evaluating at the limits of integration we get

$$\frac{1}{2\pi}\left(-\frac{i}{-2-2i\mu+\mu^2}((-i+\mu)) - \frac{1}{-2+2i\mu+\mu^2}(1-i\mu)\right)$$

$$= \frac{2+\mu^2}{4\pi+\pi\mu^4}$$

So we have computed the Fourier Transform of the given function

$$F(\mu) = \frac{2+\mu^2}{4\pi+\pi\mu^4}$$

2.4 Problem 7

$$f(x) = \frac{2x}{\left(1 + x^2\right)^2}$$

We observe that it is very difficult to compute the integral in the definition of the Fourier Transform for this x. However, if we realize that

$$\frac{d}{dx}\frac{-1}{(-1)^2 + x^2} = f(x)$$

and employ the relationship between the Fourier Transform of a function and that of its derivative outlined in [1, Page 280] then this problem becomes much easier.

Call

$$g(x) = \frac{-1}{(-1)^2 + x^2}$$

Since $g(x)$ is of the form $g(x) = a/(a^2 + x^2)$ its Fourier Transform is given by $G(\mu) = \frac{1}{2}e^{-a|\mu|} = \frac{1}{2}e^{|\mu|}$[1, Page 285]. Moreover we have by the aforementioned relationship $\frac{d}{dx}g(x) = f(x) \Rightarrow i\mu G(\mu) = F(\mu)$ where $F(\mu)$ is the Fourier Transform of $f(x)$.

Thus, the Fourier Transform of the given function is

$$F(\mu) = \frac{i\mu}{2}e^{-|\mu|}$$

2.5 Problem 9

$$f(x) = \cos x e^{-x^2/2}$$

Again this problem can be greatly simplified by expressing $f(x)$ in terms of other functions whose Fourier Transforms are known. First let us replace Cosine by its definition in terms of complex powers of e, that is

$$f(x) = \frac{1}{2}\left(e^{ix} + e^{-ix}\right)e^{-x^2/2}$$

Now, write

$$g(x) = e^{-x^2/2}$$

$$f(x) = \frac{1}{2}\left(e^{ix} + e^{-ix}\right) g(x)$$

$$= \frac{1}{2}e^{ix}g(x) + \frac{1}{2}e^{-ix}g(x)$$

but now, we can employ linearity, phase factor [1, Page 280] to rewrite the Fourier Transform as

$$\frac{1}{2}G(\mu+1) + \frac{1}{2}G(\mu-1)$$

where $G(\mu)$ is the Fourier Transform of $g(x)$.

Now recall that $g(x) = e^{-x^2/2}$, and observe that this is just a multiple of a special case of the general Gaussian distribution whose Fourier Transform is worked out in detail [1, Page 282] already. In particular we observe that the Fourier Transform $G(\mu)$ is given by

$$\sqrt{2\pi}G(\mu) = e^{-\mu^2/2}$$

$$\Rightarrow G(\mu) = \frac{e^{-\mu^2/2}}{\sqrt{2\pi}}$$

Thus the Fourier Transform of $f(x)$, $F(\mu)$ is given by,

$$F(\mu) = \frac{1}{2}G(\mu+1) + \frac{1}{2}G(\mu-1)$$

$$= \frac{1}{2}\frac{e^{-(\mu+1)^2/2}}{\sqrt{2\pi}} + \frac{1}{2}\frac{e^{-(\mu-1)^2/2}}{\sqrt{2\pi}}$$

$$= \frac{1}{2\sqrt{2\pi}}\left(e^{-\frac{1}{2}(\mu+1)^2} + e^{-\frac{1}{2}(\mu-1)^2}\right)$$

References

[1] Mark Pinsky, *Partial Differential Equations and Boundary Value Problems with Applications.* Waveland Press, Illinois, 3rd Edition, 2003.