Homework

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1 Chapter 5 Section 2.6

1.1 Problem 7

1.1.1 Question

Apply the method of images to solve the initial-value problem

$$u_t = Ku_{xx} \qquad t > 0, x > 0$$

$$u_x(0;t) = 0 \qquad t > 0$$
$$u(x;0) = \begin{cases} 1 & 0 \leq x \leq L_1 \\ 0 & x > L_1 \end{cases}$$

Show that $u(x;t) = O(t^{-1/2})$ when $t \to \infty$

1.1.2 Answer

So we recognize that this is a Neumann boundary condition so the solution is given as in [1, Page 305] by

$$u(x;t) = \frac{1}{\sqrt{4\pi Kt}} \int_0^\infty \left(e^{\frac{-(x-\xi)^2}{4Kt}} + e^{\frac{-(x+\xi)^2}{4Kt}} \right) f(\xi) d\xi$$

and clearly we can split the domain of integration to

$$u(x;t) = \frac{1}{\sqrt{4\pi Kt}} \left(\int_0^{L_1} \left(e^{\frac{-(x-\xi)^2}{4Kt}} + e^{\frac{-(x+\xi)^2}{4Kt}} \right) f(\xi) d\xi + \int_{L_1}^\infty \left(e^{\frac{-(x-\xi)^2}{4Kt}} + e^{\frac{-(x+\xi)^2}{4Kt}} \right) f(\xi) d\xi \right)$$

and since $f(\xi)$ is uniformly zero in the second integral and one otherwise we may rewrite this as

$$\frac{1}{\sqrt{4\pi Kt}} \int_0^{L_1} \left(e^{\frac{-(x-\xi)^2}{4Kt}} + e^{\frac{-(x+\xi)^2}{4Kt}} \right) d\xi$$

So, we have found the solution to the boundary value problem as desired. Now we will look for some bound for the solution as $x \to \infty$.

But observe that each of $\frac{-(x-\xi)^2}{4Kt}$ and $\frac{-(x+\xi)^2}{4Kt}$ are negative for $t > 0$. Thus, $e^{\frac{-(x-\xi)^2}{4Kt}}$ and $e^{\frac{-(x+\xi)^2}{4Kt}}$ are less or equal to 1. So, in particular we must have that

$$\left| \frac{1}{\sqrt{4\pi Kt}} \int_0^{L_1} \left(e^{\frac{-(x-\xi)^2}{4Kt}} + e^{\frac{-(x+\xi)^2}{4Kt}} \right) d\xi \right| = |u(x;t)| \leq \frac{1}{\sqrt{4\pi Kt}} \int_0^{L_1} (2)\, d\xi = \frac{L_1}{\sqrt{\pi Kt}}$$

So, $u(x;t) = O(t^{-1/2})$ when $t \to \infty$ as desired.

1.2 Problem 11

1.2.1 Question

Consider the following initial-value problem for a heat equation with a linear source term:

$$u_t = Ku_{xx} + au \qquad t > 0, -\infty < x < \infty$$

$$u(x;0) = f(x)$$

where a is a positive constant that represents the strength of the source term, per unit of temperature.

(a) Find a Fourier representation of the solution

(b) Find an explicit representation of the solution corresponding to the Gauss-Weierstrass integral (5.2.10).

1.2.2 Answer

So we will look for solutions as Fourier series in analogy to the process used to determine the solution to such equations without a source term in 5.2.2 [1, Page 295]. So, let $U(\mu;t)$ be the Fourier transform of $u(x;t)$. So, from the definition of a Fourier transform we have

$$u(x;t) = \int_{-\infty}^{\infty} U(\mu;t)e^{i\mu x}d\mu \tag{1}$$

and

$$U(\mu;t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} u(x;t)e^{-i\mu x}dx \tag{2}$$

Now we shall assume that derivatives can be taken under the integral signs to obtain

$$u_t(x;t) = \int_{-\infty}^{\infty} U_t(\mu;t)e^{i\mu x}d\mu$$

$$u_x(x;t) = \int_{-\infty}^{\infty} U(\mu;t)i\mu e^{i\mu x}d\mu$$

$$u_{xx}(x;t) = \int_{-\infty}^{\infty} U(\mu;t)(i\mu)^2 e^{i\mu x}d\mu$$

so, in order to satisfy the given heat equation we must have

$$0 = u_t - Ku_{xx} - au = \int_{-\infty}^{\infty} \left(U_t + KU\mu^2 - aU\right)e^{i\mu x}d\mu$$

Thus, U must satisfy the ordinary differential equation

$$0 = U_t + K\mu^2 U - aU = U_t + (K\mu^2 - a)U \tag{3}$$

We may determine the initial conditions for this ODE by taking $t = 0$ in Equation 1. So, $U(\mu; 0)$ must be given by the Fourier transform of the initial condition f for the original problem, that is

$$U(\mu; 0) = F(\mu)$$

Where $F(\mu)$ is the Fourier transform of f.

So, our solution to Equation 3 is given by

$$U(\mu; t) = F(\mu)e^{(-K\mu^2+a)t}$$

Now we can just substitute this into Equation 1 in order to recover the solution we want. We have in particular

$$u(x; t) = \int_{-\infty}^{\infty} U(\mu; t)e^{i\mu x} d\mu$$

$$= \int_{-\infty}^{\infty} F(\mu)e^{i\mu x}e^{(a-K\mu^2)t} d\mu$$

Now, it remains only to compute an explicit representation of the solution corresponding to the Gauss-Weierstrass integral. We have

$$u(x; t) = \int_{-\infty}^{\infty} F(\mu)e^{i\mu x}e^{(a-K\mu^2)t} d\mu$$

But note that this is just the product of F the Fourier transform of f with $e^{(a-K\mu^2)t}$. Furthermore we know that

$$\int_{-\infty}^{\infty} e^{i\mu(x-\xi)}e^{(a-k\mu^2)t} d\mu = 2\pi \left(\frac{e^{at-\frac{(x-\xi)^2}{4Kt}}}{\sqrt{4Kt\pi}} \right)$$

and so, employing convolution properties of the Fourier tranform [1, Theorem 5.2] we get the explicit solution:

$$u(x; t) = \int_{-\infty}^{\infty} f(\xi) \left(\frac{e^{at-\frac{(x-\xi)^2}{4Kt}}}{\sqrt{4Kt\pi}} \right) d\xi$$

as desired.

1.3 Problem 15

1.3.1 Question

Solve the heat equation $u_t = Ku_{xx}$ with the initial conditions $u(x;0) = T_1$ if $x < 0$ and $u(x;0) = 0$ if $x > 0$. Show that the level curves $u(x;t) = C$ are parabolas passing through $(0,0)$ in the (x,t) plane. Plot these level curves if $K = \frac{1}{2}$, $T_1 = 100$ for the values $C = 10$, $C = 30$, $C = 50$.

1.3.2 Answer

The solution to the heat equation on an infinite rod with no source term is given [1, Page 297] by

$$u(x;t) = \int_{-\infty}^{\infty} f(\xi) \frac{e^{-(x-\xi)^2/4Kt}}{\sqrt{4\pi Kt}} d\xi$$

and so, since we have that

$$f = \begin{cases} T_1 & x < 0 \\ 0 & x > 0 \end{cases}$$

this just becomes

$$u(x;t) = \frac{T_1}{\sqrt{4\pi Kt}} \int_{-\infty}^{0} e^{-(x-\xi)^2/4Kt} d\xi$$

integrating yields

$$u(x;t) = T_1 \left(1 - \Phi\left(\frac{x}{\sqrt{2Kt}}\right)\right)$$

just as in Example 5.2.3 [1, Page 301].

Now, we examine level curves of $u(x;t)$. These take the form

$$u(x;t) = C = T_1 \left(1 - \Phi\left(\frac{x}{\sqrt{2Kt}}\right)\right)$$

for some constant C. So we must have

$$1 - \frac{C}{T_1} = \Phi\left(\frac{x}{\sqrt{2Kt}}\right)$$

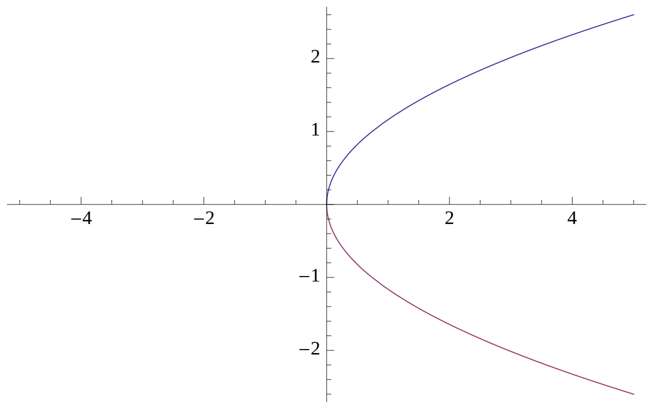

Figure 1: $K = \frac{1}{2}$, $T_1 = 100$, $C = 10$

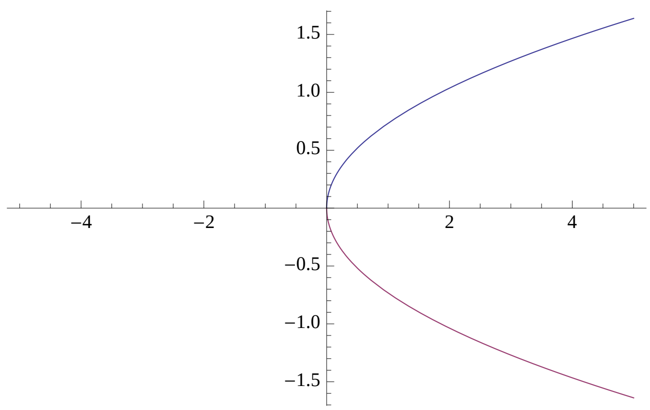

Figure 2: $K = \frac{1}{2}$, $T_1 = 100$, $C = 30$

$$\Phi^{-1}\left(1 - \frac{C}{T_1}\right) = \frac{x}{\sqrt{2Kt}}$$

but since for constant $1 - \frac{C}{T_1}$ we have that $\Phi^{-1}\left(1 - \frac{C}{T_1}\right)$ is just some constant (call it L) this is just the equation for a parabola through the origin in the (x, t) plane given by

$$2KL^2t = x^2$$

Figures 1-3 are plots of x as a function of t for fixed values of K, T_1, C.

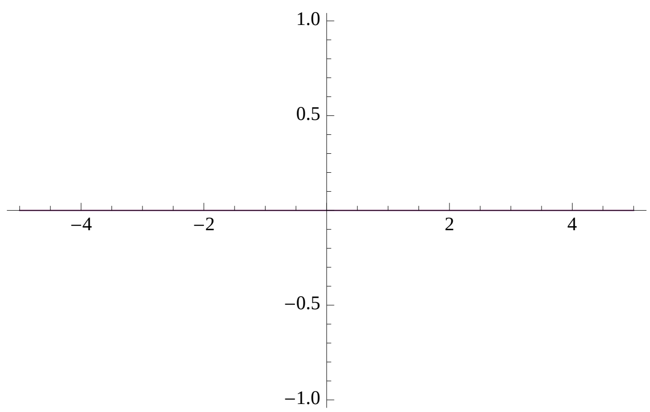

Figure 3: $K = \frac{1}{2}$, $T_1 = 100$, $C = 100$

1.4 Problem 17

1.4.1 Question

Solve the heat equation $u_t = K u_{xx}$ with the initial condititions $u(x;0) = T_1$ if $-L < x < 0$, $u(x;0) = T_2$ if $0 < x < L$, and $u(x;0) = 0$ if $|x| > L$. What is $\lim_{t\to\infty} u(x;t)$?

1.4.2 Answer

The solution to the heat equation on an infinite rod with no source term is given [1, Page 297] by

$$u(x;t) = \int_{-\infty}^{\infty} f(\xi) \frac{e^{-(x-\xi)^2/4Kt}}{\sqrt{4\pi Kt}} d\xi$$

so, splitting the integral appropriately and employing our particular initial condition we get

$$u(x;t) = T_1 \int_{-L}^{0} \frac{e^{-(x-\xi)^2/4Kt}}{\sqrt{4\pi Kt}} d\xi + T_2 \int_{0}^{L} \frac{e^{-(x-\xi)^2/4Kt}}{\sqrt{4\pi Kt}} d\xi$$

integrating we get

$$u(x;t) = T_1 \int_{-L}^{0} \frac{e^{-(x-\xi)^2/4Kt}}{\sqrt{4\pi Kt}} d\xi + T_2 \int_{0}^{L} \frac{e^{-(x-\xi)^2/4Kt}}{\sqrt{4\pi Kt}} d\xi$$

$$= T_1\left(\Phi\left(\frac{x+L}{\sqrt{2Kt}}\right)-\Phi\left(\frac{x}{\sqrt{2Kt}}\right)\right)+T_2\left(\Phi\left(\frac{x}{\sqrt{2Kt}}\right)-\Phi\left(\frac{x-L}{\sqrt{2Kt}}\right)\right)$$

In the limit when $t \to \infty$ this expression goes to

$$u(x;t) = T_1\left(\Phi\left(0\right)-\Phi\left(0\right)\right)+T_2\left(\Phi\left(0\right)-\Phi\left(0\right)\right) = 0$$

as we would expect.

2 Chapter 5 Section 2.8

2.1 Problem 1

2.1.1 Question

Use the generating function for Hermite polynomials to prove the equations $H_k'(x) = kH_{k-1}(x), k = 1, 2, \ldots$

2.1.2 Answer

Proof. The generating function for Hermite polynomials is

$$e^{tx-t^2/2} = \sum_{k=0}^{\infty}\frac{t^k}{k!}H_k(x) = H_0(x) + tH_1(x) + \frac{t^2}{2}H_2(x) + \ldots$$

So,

$$\frac{d}{dx}e^{tx-t^2/2} = te^{tx-t^2/2}$$

and

$$\frac{d}{dx}\sum_{k=0}^{\infty}\frac{t^k}{k!}H_k(x) = \sum_{k=0}^{\infty}\frac{t^k}{k!}H_k'(x)$$

thus

$$t\sum_{k=0}^{\infty}\frac{t^k}{k!}H_k(x) = \sum_{k=0}^{\infty}\frac{t^k}{k!}H_k'(x)$$

$$\Rightarrow \sum_{k=0}^{\infty}\frac{t^{k+1}}{k!}H_k(x) = \sum_{k=0}^{\infty}\frac{t^k}{k!}H_k'(x)$$

equating like terms we get

$$H_k'(x) = kH_{k-1}(x), k = 1, 2, \ldots$$

as desired. □

2.2 Problem 3

2.2.1 Question

Combine the results of the two previous exercises to prove the following differential equation satisfied by the Hermite polynomials: $H_k''(x) - xH_k' = -kH_k(x), k = 0, 1, 2, \ldots$

2.2.2 Answer

First we use the first exercise to get

$$H_k'(x) = kH_{k-1}(x)$$

$$\Rightarrow H_k'' = kH_{k-1}'$$

then, rewriting the result of the second exercise as

$$kH_{k-1} = xH_k - H_{k+1}$$

we can apply it to what we have already to yield

$$H_k'' = xH_k' - H_{k+1}'$$

finally, applying the result of the first exercise to the last term we get

$$H_k'' = xH_k' - kH_k$$

rearranging terms we get

$$H_k''(x) - xH_k' = -kH_k(x)$$

as desired.

2.3 Problem 5

2.3.1 Question

Use Exercise 3 to show that the functions $\psi_k(x) = e^{-x^2/4}H_k(x)$ satisfy the differential equation $\psi_k''(x) - (x^2/4)\psi_k(x) = -(k + \frac{1}{2})\psi_k(x)$ for $k = 1, 2, \ldots$

2.3.2 Answer

First we use the definition of $\psi(x)$ to compute some of its derivatives

$$\psi_k' = e^{-x^2/4}H_k' - \frac{1}{2}xe^{-x^2/4}H_k$$
$$\psi_k'' = e^{-x^2/4}H_k'' - xe^{-x^2/4}H_k' - \left(\frac{1}{2}e^{-x^2/4} - \frac{1}{4}e^{-x^2/4}x^2\right)H_k$$

so we see that

$$\psi_k'' - (x^2/4)\psi_k = e^{-x^2/4}H_k'' - xe^{-x^2/4}H_k' - e^{-x^2/4}\left(\frac{1}{2} - \frac{1}{4}x^2 + x^2/4\right)H_k$$
$$= e^{-x^2/4}H_k'' - xe^{-x^2/4}H_k' - \left(\frac{1}{2}e^{-x^2/4}\right)H_k$$
$$= e^{-x^2/4}H_k'' - xe^{-x^2/4}H_k' - \frac{1}{2}e^{-x^2/4}H_k$$

Employing the result from the previous exercise we get

$$= e^{-x^2/4}\left(H_k'' - xH_k' - \frac{1}{2}H_k\right)$$
$$= e^{-x^2/4}\left(-kH_k - \frac{1}{2}H_k\right)$$
$$= e^{-x^2/4}H_k\left(-k - \frac{1}{2}\right)$$
$$= -\psi_k\left(k + \frac{1}{2}\right)$$

But this is what we wanted to prove.

3 Chapter 5 Section 3

3.1 Problem 1

3.1.1 Question

Use d'Alembert's formula to solve the wave equation $y_{tt} = c^2y_{xx}$ with the initial conditions $y(x;0) = 3\sin 2x$, $y_t(x;0) = 0$.

3.1.2 Answer

D'Alembert's formula [1, Page 320] states

$$y(x;t) = \frac{1}{2}\left(f_1(x+ct) + f_1(x-ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} f_2(\xi)d\xi$$

In our case we have $f_1(x) = 3\sin 2x$ and $f_2(x) = 0$. Thus, substituting into the formula we arrive at

$$y(x;t) = \frac{3}{2}\left(\sin\left(2x+2ct\right) + \sin\left(2x-2ct\right)\right)$$

Now we may employ the trigonometric identity

$$\sin\left(\alpha+\beta\right) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

together with the oddness of $\sin$ to arrive at

$$y(x;t) = 3\sin 2x\cos 2ct$$

3.2 Problem 3

3.2.1 Question

Suppose that f_1 has two continuous derivatives and f_2 has one continuous derivative. Show that (5.3.7) is a solution of the initial-value problem (5.3.1).

3.2.2 Answer

So, in particular we must show that

$$y(x;t) = \frac{1}{2}\left(f_1(x+ct) + f_1(x-ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} f_2(\xi)d\xi$$

is a solution to the initial value problem

$$y_{tt} = c^2 y_{xx}$$

$$y(x;0) = f_1(x)$$

$$y_t(x;0) = f_2(x)$$

$$-\infty < x < \infty \quad t > 0$$

First we verify that the solution satisfies the wave equation.

$$y_{tt} = \frac{c^2}{2}\left(f_1''(x+ct) + f_1''(x-ct)\right) + \frac{c}{2}\left(f_2'(x+ct) - f_2'(x-ct)\right)$$

Moreover,

$$y_{xx} = \frac{1}{2}\left(f_1''(x+ct) + f_1''(x-ct)\right) + \frac{1}{2c}\left(f_2'(x+ct) - f_2'(x-ct)\right)$$

so we see that $y_{tt} = c^2 y_{xx}$ as desired. Now we check to make sure that our solution satisfies the initial conditions.

$$y(x;0) = \frac{1}{2}\left(f_1(x) + f_1(x)\right) + \frac{1}{2c}\int_x^x f_2(\xi)d\xi$$

$$= f_1(x)$$

as desired. Also,

$$y_t(x;t) = \frac{c}{2}\left(f_1'(x+ct) - f_1'(x-ct)\right) + \frac{1}{2}\left(f_2(x+ct) + f_2(x-ct)\right)$$

$$\Rightarrow y_t(x;0) = f_2(x)$$

as desired. Thus, we have verified that 5.3.7 is a solution for 5.3.1 as desired.

3.3 Problem 5

3.3.1 Question

Find the solution of the wave equation $y_{tt} = c^2 y_{xx}$ for $t > 0$, $x > 0$ satisfying the boundary conditions $y(0;t) = s(t)$ and the initial conditions $y(x;0) = 0$, $y_t(x;0) = g(x)$.

3.3.2 Answer

We follow the method outlined in [1, Example 5.3.1 Page 323]. So we look for y in the form $y(x;t) = f(x+ct) + h(x-ct)$. Substituting the initial conditions and boundary conditions this implies in particular that

$$y(0;t) = s(t) = f(ct) + h(-ct) \tag{4}$$

$$y(x;0) = 0 = f(x) + h(x) \tag{5}$$
$$y_t(x;0) = g(x) = cf'(x) - ch'(x) \tag{6}$$

taking the derivative and rearranging Equation 5 gives us that

$$f'(x) = -h'(x)$$

and employing this together with the Equation 6 we see that

$$g(x) = -ch'(x) - ch'(x)$$

$$g(x) = -2ch'(x)$$

and now we may take the derivative on both sides yielding

$$\int g(x)dx = -2c \int h'(x)dx$$

$$\Rightarrow G(x) = -2ch(x) + K$$

Where G is the antiderivative of g and K is an arbitrary constant.

So since by Equation 5 $f(x) = -h(x) \Rightarrow G(x) = 2cf(x) + K$ we have determined both f and h. In particular we have

$$f(x) = \frac{G(x) - K}{2c}$$

$$h(x) = \frac{K - G(x)}{2c}$$

so, substituting back into the original form of the solution we get

$$y(x;t) = \frac{G(x+ct) - K}{2c} + \frac{K - G(x-ct)}{2c}$$

$$= \frac{1}{2c}\left(G(x+ct) - G(x-ct)\right)$$

$$= \frac{1}{2c}\int_{x-ct}^{x+ct} g(\xi)d\xi$$

This is a solution to the initial value problem and holds for $x > ct$. Now we need only incorporate the boundary value defined by Equation 4 to find the solution on the region $0 < x \leq ct$. In order to do this examine Equation 4 revealing

$$s(t) - f(ct) = h(-ct)$$

and substitute $t = t - \frac{x}{c}$ to reveal that in particular

$$s(t - \frac{x}{c}) - f(ct - x) = h(x - ct)$$

substituting into the form of the solution gives us

$$y(x; t) = f(x + ct) + h(x - ct)$$

$$= f(x + ct) + s(t - \frac{x}{c}) - f(ct - x)$$

Now we merely rearrange terms and substitute our f computed above to get

$$\frac{G(x + ct) - K}{2c} - \frac{G(ct - x) - K}{2c} + s(t - \frac{x}{c})$$

$$= \frac{1}{2c} \int_{ct-x}^{x+ct} g(\xi) d\xi + s(t - \frac{x}{c})$$

for $0 < x \leq ct$.

References

[1] Mark Pinsky, *Partial Differential Equations and Boundary Value Problems with Applications*. Waveland Press, Illinois, 3rd Edition, 2003.