1 **Explain in detail from the viewpoint of transversality theory, why the sum of the indices of a vector field with isolated zeroes on a compact orientable manifold M is independent of what vector field we choose.**

2 **Call the index sum in problem 1 the Euler characteristic $\chi(M)$. Explain why the Euler characteristic of a genus g surface (2-sphere with g handles attached) is $2-2g$. [Do this explicitly: do *not* appeal to the theorem that the Euler characteristic in the vector field sense indicated is computable from homological information. That comes next!]**

3 **Suppose that M is a triangulated compact orientable manifold, i.e., a manifold M represented as a finite simplicial complex.**

3.1 **Show that the alternating sum of the Betti numbers $b_0 - b_1 + b_2 - \cdots$ (where $b_k = \mathrm{rank}$ of the kth homology group with real coefficients) is equal to the alternating sum (number of vertices) $-$ (number of faces) $+$ (number of $2-$ simplices) $- \cdots$**

3.2 **Show that there is a vector field with the sum of its indices equal to the number described in part 3.1. [You do not need to worry about smoothness of the vector field – just describe how to build it. In part 3.1, the result should follow from some dimension counting.]**

4 **Suppose V is a smooth (C^∞) vector field on $\mathbb{R}^3$ that is nonzero at $(0,0,0)$. The vector field is said to be gradient-like at $(0,0,0)$ if there is a neighborhood of $(0,0,0)$ and a nowhere zero smooth function $\lambda(x,y,z)$ on that neighborhood such that λV is the gradient of some smooth function in some (possibly smaller) neighborhood of $(0,0,0)$.**

4.1 **Write $V=(P,Q,R)$. Show by example that there are functions P,Q,R for which V is not gradient-like in a neighborhood of $(0,0,0)$. (Hint: the orthogonal complement of V taken at each point would have to be an integrable 2-plane field)**

4.2 **Derive a general differential condition on (P,Q,R) which is necessary and sufficient for V to be gradient-like in a neighborhood of $(0,0,0)$.**

5.1 **Define carefully the "boundary map" which defines the H_n to H_{n-1} mapping that arises in the long exact sequences arising from a short exact sequence of chain complexes.**

5.2 **Prove that the kernel of the boundary map is equal to the image of the map into the H_n.**

answer

6 Compute the homology of the real projective space $\mathbb{R}P^n$ for each $n > 1$.

7.1 Define complex projective space $\mathbb{C}P^n$ $(n = 1, 2, 3, \dots)$

7.2 Show that $\mathbb{C}P^n$ is compact for all n.

7.3 Show that $\mathbb{C}P^n$ has a cell decomposition with one cell in each dimension $0, 2, 4, \dots, 2n$ and no other cells. Include a careful description of the attaching maps.

8 Suppose a compact (real) manifold M has a (finite) cell decomposition with only even dimensional cells. Is M necessarily orientable? Justify your answer.

9 Suppose that a finite group Γ acts smoothly on a compact manifold M and that the action is free, i.e. $\gamma(x) = x$ for some x in M if and only if $\gamma =$ the identity of the group Γ.

9.1 Show that M/Γ is a manifold (i.e., can be made a manifold in a natural way)

9.2 Show that $M \to M/\Gamma$ is a covering space.

9.3 If the kth de Rham cohomology of M is 0, some particular $k > 0$, then is the kth de Rham cohomology of M/Γ necessarily 0? Prove your answer.

10 Let $M = \mathbb{R}P^2 \times \mathbb{R}P^2$ where $\mathbb{R}P^2$ is a real projective 2-space). In a product manifold like that, homology elements can arise by taking in effect the product of a cycle in one factor with a cycle in the other factor. Show that in the case of this particular M, there is an element in the 3-homology with $\mathbb{Z}$ coefficients that does not arise in this way by exhibiting such an element explicitly, e.g. in terms of a cell decomposition.