1.1 Show that a closed 1-form θ on $S^1 \times (-1,1)$ is dF for some function $F : S^1 \times (-1,1) \to \mathbb{R}$ if and only if $\int_{S^1} i^*\theta = 0$ where $i : S^1 \to S^1 \times (-1,1)$ is defined by $i(p) = (p,0)$ for $p \in S^1$.

1.2 Show that a 2-form ω on S^2 is $d\theta$ for some 1-form θ on S^1 if and only if $\int_{S^2} \omega = 0$.

2 Suppose that M, N are connected C^∞ manifolds of the same dimension $n \geq 1$ and $F : M \to N$ is a C^∞ map such that $dF : T_pM \to T_{F(p)}N$ is surjective for each $p \in M$.

2.1 Prove that if M is compact, then F is onto and F is a covering map.

2.2 Find an example of such an everywhere nonsingular equidimensional map where N is compact, F is onto, $F^{-1}(p)$ is finite for each $p \in N$, but F is not a covering map.
(Note: A clearly explained pictorial version of F will be acceptable; you do not have to have a "formula" for F.)

3.1 Suppose that M is a C^∞ connected manifold. Prove that, given an open subset U of M and a finite set of points $p_1, p_2, \ldots, p_k$ in M, there is a diffeomorphism $F : M \to M$ such that $f(\{p_1, p_2, \ldots, p_k\}) \subset U$. (Hint: Construct F one point at a time)

3.2 Use part (3.1) to show that if M is compact and the Euler characteristic $\chi(M) = 0$, then there is a vector field on M which vanishes nowhere. You may assume that if a vector field has isolated zeros, then the sum of the indices at the zero points equals $\chi(M)$.

4 A smooth vector field V on $\mathbb{R}^3$ is said to be "gradient-like" if, for each $p \in \mathbb{R}^3$, there is a neighborhood U_p of p and a function $\lambda_p : U_p \to \mathbb{R} \setminus \{0\}$ such that $\lambda_p V$ on U_p is the gradient of some C^∞ function on U_p. Suppose V is nowhere zero on $\mathbb{R}^3$. Then show that V is gradient-like if and only if $\operatorname{curl} V$ is perpendicular to V at each point of $\mathbb{R}^3$.

5 Suppose that M is a compact C^∞ manifold of dimension n.

5.1 Show that there is a positive integer k such that there is an immersion $F : M \to \mathbb{R}^k$.

5.2 Show that if $k > 2n$, there is a $(k-1)$-dimensional subspace H of $\mathbb{R}^k$ such that $P \circ F$ is an immersion, where $P : \mathbb{R}^k \to H$ is orthogonal projection.

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6 Let $GL_n^+(\mathbb{R})$ be the set of $n \times n$ matrices with determinant > 0. (Note: $GL_n^+(\mathbb{R})$ can be considered to be a subset of $\mathbb{R}^{n^2}$ and this subset is open.)

6.1 Prove that $SL_n^+(\mathbb{R}) = \{A \in GL_n^+(\mathbb{R}) \mid \det A = 1\}$ is a submanifold.

6.2 Identify the tangent space of $SL_n^+(\mathbb{R})$ at the identity matrix I_n.

6.3 Prove that, for every $n \times n$ matrix B, the series $I_n + B + \frac{1}{2}B^2 + \cdots + \frac{1}{n!}B^n + \cdots$ converges to some $n \times n$ matrix. Call this sum e^B.

6.4 Prove that if $e^{tB} \in SL_n^+(\mathbb{R})$ for all $t \in \mathbb{R}$, then $\operatorname{tr} B = 0$

**6.5 Prove that if $\operatorname{tr} B = 0$, then $e^B \in SL_n^+(\mathbb{R})$.
(Hint: Use one-parameter subgroups or note that it suffices to treat complex-diagonalizable B since such are dense.)**

7.1 Define complex projective space $\mathbb{C}P^n$.

7.2 Calculate the homology of $\mathbb{C}P^n$. Any systematic method such as Mayer-Vietoris or cellular homology is acceptable.

8 Let $p : E \to B$ be a covering space and $f : X \to B$ a map. Define $E^* = \{(x, e) \in X \times B : f(x) = p(e)\}$. Prove that $q : E^* \to X$ defined by $q(x, e) = x$ is a covering space.

9.1 Explain carefully and concretely what it means for two (smooth) maps of S^1 into $\mathbb{R}^2$ to be transversal.

9.2 Do the same for maps of S^1 into $\mathbb{R}^3$.

9.3 Explain what it means for transversal maps to be "generic" and prove that they are indeed generic in the cases of (9.1) and (9.2).

10 Let M be the 3-manifold with boundary obtained as the union of the two-holed torus in 3-space and the bounded component of its complement. Let X be the space obtained from M by deleting k points from the interior of M.

10.1 Calculate the fundamental group of X.

10.2 Calculate the homology of X.

11 Let P be a finite polyhedron.

11.1 Define the Euler characteristic $\chi(P)$ of P.

11.2 Prove that if P_1, P_2 are subpolyhedra of P such that $P_1 \cap P_2$ is a point and $P_1 \cup P_2 = P$, then $\chi(P) = \chi(P_1) + \chi(P_2) - 1$.

11.3 Suppose that $p : E \to P$ is an n-sheeted covering space of P, that is $p^{-1}(x)$ is n points for each $x \in P$. Prove that $\chi(E) = n\chi(P)$.

12 Let $f : T \to T = S^1 \times S^1$ be a map of the torus inducing $f_\pi : \pi_1(T) \to \pi_1(T) = \mathbb{Z} \oplus \mathbb{Z}$ and let F be a matrix representing f_π. Prove that the determinant of F equals the degree of the map f.