1.1 **Show that the Lie group** $SL_2(\mathbb{R}) = \{A \in M_{2\times 2}(\mathbb{R}) \mid \det(A) = 1\}$ **is diffeomorphic to** $S^1 \times \mathbb{R}^2$**.**

1.2 **Show that the Lie group** $SL_2(\mathbb{C}) = \{A \in M_{2\times 2}(\mathbb{C}) \mid \det(A) = 1\}$ **is diffeomorphic to** $S^3 \times \mathbb{R}^3$**.**

2 **For** $n \geq 1$**, construct an everywhere non-vanishing smooth vector field on the odd-dimensional real projective space** $\mathbb{R}P^{2n-1}$**.**

3 **Let** $M^m \subset \mathbb{R}^n$ **be a smooth submanifold of dimension** $m < n-2$**. Show that its complement** $\mathbb{R}^n \setminus M$ **is connected and simply connected.**

4.1 **Show that for any** $n \geq 1$ **and** $k \in \mathbb{Z}$**, there exists a continuous map** $f : S^n \to S^n$ **of degree** k**.**

4.2 **Let** X **be a compact, oriented** n**-dimensional manifold. Show that for any** $k \in \mathbb{Z}$**, there exists a continuous map** $f : X \to S^n$ **of degree** k**.**

5 **Assume that** $\Delta = \{X_1, \ldots, X_k\}$ **is a** k**-dimensional distribution spanned by vector fields on an open set** $\Omega \subset M^n$ **in an** n**-dimensional manifold. For each open subset** $V \subset \Omega$ **define**

$$\mathcal{Z}_V = \{u \in C^\infty(V) \mid X_1 u = 0, \ldots, X_k u = 0\}$$

Show that the following two statements are equivalent:

5.1 **The distribution** Δ **is integrable.**

5.2 **For each** $x \in \Omega$ **there exists an open neighborhood** $x \in V \subset \Omega$ **and** $n - k$ **functions** $u_1, \ldots, u_{n-k} \in \mathcal{Z}_V$ **such that the differentials** $du_1, \ldots, du_{n-k}$ **are linearly independent at each point in** V**.**

6 **On** $\mathbb{R}^n \setminus \{0\}$ **define the** $(n-1)$**-forms**

$$\sigma = \sum_{i=1}^n (-1)^{i-1} x^i dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n$$

$$\omega = \frac{1}{|x|^n} \sum_{i=1}^n (-1)^{i-1} x^i dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n$$

6.1 Show that $\omega = r^* \circ i^*(\sigma)$, where $i : S^{n-1} \to \mathbb{R}^n \setminus \{0\}$ is the natural inclusion of the unit sphere and $r(x) = \frac{x}{|x|} : \mathbb{R}^n \setminus \{0\} \to S^{n-1}$ the natural retraction.

6.2 Show that σ is not a closed form.

6.3 Show that ω is a closed form that is not exact.

7 Let $n \geq 0$ be an integer. Let M be a compact, orientable, smooth manifold of dimension $4n+2$. Show that $\dim H^{2n+1}(M;\mathbb{R})$ is even.

8 Show that there is no compact three-dimensional manifold M whose boundary is the real projective space $\mathbb{R}P^2$.

9 Consider the coordinate axes in $\mathbb{R}^n$:

$$L_i = \{(x_1, \ldots, x_n) \mid x_j = 0 \text{ for all } j \neq i\}$$

Calculate the homology groups of the complement $\mathbb{R}^n \setminus (L_1 \cup \cdots \cup L_n)$.

10.1 Let X be a finite CW complex. Explain how the homology groups of X are related to the homology groups of $X \times S^1$.

10.2 For each integer $n \geq 0$, give an example of a compact smooth manifold of dimension $2n+1$ such that $H_i(X) = \mathbb{Z}$ for all $i = 0, \ldots, 2n+1$.