1.1 Let $M = SL_2(\mathbb{R}) = \{A \in M_{2\times 2}(\mathbb{R}) \mid \det A = 1\}$. Show that M is a submanifold of $M_{2\times 2}(\mathbb{R})$ (the space of two-by-two matrices). Given $A \in M$, regard $T_A M$ as a subspace of $M_{2\times 2}(\mathbb{R})$. Consider three vector fields H, X, Y on M defined by

$$H(A) = A \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, X(A) = A \cdot \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, Y(A) = A \cdot \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} \in T_A M.$$

Find the flows of H, X and Y.

1.2 Show that $[H, X] = 2X$.

2 State the general Stokes Theorem, and explain how the classical version

$$\iint_S (\nabla \times \vec{v}) \cdot \vec{n} dA = \iint_{\partial S} \vec{v} \cdot d\vec{r}$$

follows. Here S is a compact surface in $\mathbb{R}^3$ with normal vector $\vec{n}$ and boundary ∂S, and $\vec{r}$ is the position vector.

3 Describe diffeomorphisms between $SO(3)$, $\mathbb{RP}^3$ and $UT(S^2)$, the unit tangent bundle of S^2. You need not check that the maps are smooth. ($SO(3)$ is the special orthogonal group and $UT(S^2)$ is the set of tangent vectors of length one.)

4 Let X be the space of symmetric n-by-n real matrices and let X_k be the subspace of matrices of rank k in X. Show that X_k is a submanifold and find its dimension.

5 Suppose that $f : M \to N$ is C^∞, M and N are compact connected n-manifolds, and $\operatorname{rank}(df) = n$. Show that f is a covering map.

6 Consider the exact sequence of abelian groups and homomorphisms

$$0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0.$$

Prove that if there is a homomorphism $\gamma : B \to A$ such that $\gamma\alpha : A \to A$ is the identity, then B is isomorphic to $A \oplus C$.

7 Prove that the n-sphere S^n admits a continuous field of nonzero tangent vectors if and only if n is odd.

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