1 Let M be a smooth three dimensional manifold and α is a 1-form on M s.t. $\alpha \wedge d\alpha \neq 0$ at every point of M.

1.1 Let $H = \ker \alpha \subseteq TM$. Show that H is a two-dimensional plane field of M which is not integrable.

(Hint: Use the formula $d\alpha(X,Y) = X(\alpha(Y)) - Y(\alpha(X)) - \alpha([X,Y])$ where X, Y are two arbitrary vector fields.)

1.2 Show that there exists a unique vector field V s.t.

$$(a)\ \alpha(V) = 1, \quad (b)\ \langle V \rangle \oplus H = TM, \quad (c)\ d\alpha(V,W) = 0$$

for any vector field W. Here $\langle V \rangle$ is the line field generated by V.

2 Let M be a closed smooth manifold and X be a vector field on M. Denote the flow generated by X by $\varphi_t : M \to M$, i.e., φ_t is defined by:

$$\frac{d\varphi_t}{dt}(x) = X(\varphi_t(x)) \quad \text{for any} \quad x \in M.$$

Given a function f prove that:

$$f \circ \varphi_1 - f \circ \varphi_0 = \int_0^1 \varphi_t^*(df)(X)dt.$$

3 Let M_n be the space of $n \times n$ real matrices and M_n^k be the subspace of all matrices of rank k in M_n.

3.1 Show that M_n^k is a submanifold of M_n

3.2 Find the dimension of M_n^k.

4 Let $S^2 = \{(x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$ as usual.

4.1 Show that, for each C^∞ 1-form ω on S^2 with $d\omega = 0$, there is a C^∞ function $f : S^2 \to \mathbb{R}$ such that $df = \omega$

4.2 Show that, for each 2-form Ω on S^2 such that $\Omega = d\theta$ for some 1-form θ,

$$\int_{S^2} \Omega = 0.$$

4.3 Is the converse of 4.2 true, i.e., is it true that if Ω is a 2-form on S^2 with $\int_{S^2} \Omega = 0$ then there is always a 1-form θ on S^2 such that $\Omega = d\theta$? Prove your answer.

5 Let S^2 be as in Problem 4. Consider the 2-form on $\mathbb{R}^3 \setminus \{(0,0,0)\}$

$$\sigma = (x^2 + y^2 + z^2)^{-3/2}(x\ dy \wedge dz - y\ dx \wedge dz + z\ dx \wedge dy).$$

5.1 Show that σ is closed on $\mathbb{R}^3 \setminus \{(0,0,0)\}$.

5.2 Show that the 2-form

$$\omega = x\ dy \wedge dz - y\ dx \wedge dz + z\ dx \wedge dy$$

is closed but not exact on S^2.

5.3 Find $\int_{S^2} \omega$.

5.4 Suppose M is compact, 2-dimensional, oriented embedded submanifold of $\mathbb{R}^3 \setminus \{(0,0,0)\}$. What are the possible values of $\int_M \sigma$? Prove your answer.

6.1 Define: chain complex, chain map, chain homotopy.

6.2 Prove that if $f_1, f_2 : C \to C'$ and $g_1, g_2 : C' \to C''$ are chain homotopic chain maps then $g_1 \circ g_2, g_2 \circ f_2 : C \to C''$ are also chain homotopic.

7 Let $p : \tilde{X} \to X$ be a covering space and let $f : X \to X$ be a map such that $f(x_0) = x_0$. A map $\tilde{f} : \tilde{X} \to \tilde{X}$ such that $f(\tilde{x_0}) = \tilde{x}_0$ for some $\tilde{x}_0 \in p^{-1}(x_0)$ is a *lift* of f if $p\tilde{f} = fp$.

7.1 Prove that f has a *lift* if and only if $f_*(H) \subseteq H$ where $H = p_*(\pi_1(\tilde{X}, \tilde{x}_0)) \subseteq \pi_1(X, x_0)$.

7.2 Give an example of a space X, a map $f : X \to X$ and a covering space $p = \tilde{X} \to X$ such that f has *no lifts* to $\tilde{X}$.

8 The following diagram of groups and homomorphisms is commutative and both horizontal sequences are exact. The symbol "id" denotes the identity. Prove that if $c \in C$ such that $\gamma(c) = 1$ then there exists $b \in B$ such that $\beta(b) = 1$ and $\varphi(b) = c$, and thus that $\varphi(\ker\beta) = \ker\gamma$.

$$\begin{tikzcd}
A \arrow[r, "\alpha"] \arrow[d, "\mathrm{id}"'] & B \arrow[r, "\varphi"] \arrow[d, "\beta"'] & C \arrow[r, "\delta"] \arrow[d, "\gamma"'] & D \arrow[d, "\mathrm{id}"] \\
A \arrow[r, "\alpha'"'] & B' \arrow[r, "\varphi'"'] & C' \arrow[r, "\delta'"'] & D
\end{tikzcd}$$

9 Let (X_1, A_1) and (X_2, A_2) be pairs of finite polyhedra and subpolyhedra.

9.1 Write the *relative* Mayer-Vietoris sequence for the pair $(X_1 \cup X_2, A_1 \cup A_2)$. You do not have to define the homomorphism or prove anything about it.

9.2 Use part 9.1 to prove that if X is a finite polyhedron, S^r is the r-sphere $p_0 \in S^r$ and $k > r$ then

$$H_k(X \times S^r, X \times p_0) \simeq H_{k-r}(X).$$

10 Let $p : E \to B$ be a covering space and $f : X \to B$ a map. Define

$$E^* = \{(x,e) \in X \times B \mid f(x) = p(e)\}.$$

Prove that $q = E^* \to X$ define $q(x, e) = x$ is a covering space.