

- 1 Let M be a closed (compact, without boundary) manifold. Show that any smooth function

$$f : M \rightarrow \mathbb{R}$$

has a critical point.

- 2.1 Show that every closed 1-form on S^n , $n > 1$, is exact.
- 2.2 Use this to show that every closed 1-form on \mathbb{RP}^n , $n > 1$, is exact.
- 3 Let M^d be a d -dimensional manifold and $\omega_1, \dots, \omega_p$ be pointwise linearly independent 1-forms. If $\theta_1, \dots, \theta_p$ are 1-forms so that

$$\sum_{i=1}^p \omega_i \wedge \theta_i = 0,$$

then there exist smooth functions f_{ij} such that

$$\theta_i = \sum_{j=1}^p f_{ij} \omega_j, \quad i = 1, \dots, p.$$

(Hint: try $p = 1$)

- 4 Let M be the set of all straight lines in \mathbb{R}^2 (not just those which pass through the origin). Show that M is a smooth manifold and identify it with a well-known manifold.

(Hint: Lines not through the origin have a unique closest point to the origin and that point determines the line uniquely. What happens at the origin?)

- 5 Let $f : M^m \rightarrow N^n$ be a smooth bijection so that $Df : T_p M \rightarrow T_{f(p)} N$ is injective for all p . Show that f is a diffeomorphism.
- 6.1 Show that if $f : S^n \rightarrow S^n$ has no fixed points then $\deg(f) = (-1)^{n+1}$.
- 6.2 Show that if X has S^{2n} as universal covering space then $\pi_1(x) = \{1\}$ or \mathbb{Z}_2 .
- 6.3 Show that if X has S^{2n+1} as universal covering space then X is orientable.
- 7.1 Outline the construction of the universal covering of a path connected locally simply connected space X .
- 7.2 Give an example of a path connected space which does not have a universal covering space.
- 8 Let X be a finite cell complex constructed inductively by gluing all p -cells onto cells of dimension $< p$. Assume no $p - 1$ and $p + 1$ cells are used to construct X . Show that

$$H_p(X, \mathbb{Z}) \simeq \mathbb{Z}^{n_p}$$

when n_p is the number of p -cells used in the construction.

9 Let $(M, \partial M)$ be a compact oriented n -manifold with a connected boundary ∂M . Show that there is no retract $r : M \rightarrow \partial M$ so that $r(x) = x$ if $x \in \partial M$. (Hint: Prove that $H_{n-1}(\partial M) \rightarrow H_{n-1}(M)$ is trivial.)

10 Let $X = T^2 \setminus \{p, q\}$, $p \neq q$ be the twice punctured 2-dimensional torus.

10.1 Compute the homology groups $H_*(X, \mathbb{Z})$.

10.2 Compute the fundamental group of X .