**1 Let $M_n$ be the space of all $n \times n$ matrices with real entries and let $S_n$ be the subset consisting of all symmetric matrices. Consider the map $F : M_n \to S_n$ defined by $F(A) = AA^t - I$, where $I$ is the identity matrix and $A^t$ is the transpose of $A$.**

**1.1 Show that $0_{n\times n}$ (the $n \times n$ matrix with all entries 0) is a regular value of $F$.**

**1.2 Deduce that $O(n)$, the set of all $n \times n$ matrices such that $A^{-1} = A^t$ is a submanifold of $M_n$.**

**1.3 Find the dimension of $O(n)$ and determine the tangent space of $O(n)$ at the identity matrix as a subspace of the tangent space of $M_n$ which is $M_n$ itself.**

**2 Show that $T^2 \times S^n$, $n \geq 1$ is parallelizable, where $S^n$ is the $n$ sphere, $T^2 = S^1 \times S^1$ is the two torus, and a manifold of dimension $k$ is said to be parallelizable if there are $k$ vector fields $V_1, \ldots, V_k$ on it with $V_1(p), \ldots, V_k(p)$ linearly independent for all points $p$ of the manifold.**

**3 Suppose $\pi : M_1 \to M_2$ is a $C^\infty$ map of one connected differentiable manifold to another. And suppose for each $p \in M_1$, the differential $\pi_* : T_pM_1 \to T_{\pi(p)}M_2$ is a vector space isomorphism.**

**3.1 Show that if $M_1$ is connected, then $\pi$ is a covering space projection.**

**3.2 Give an example where $M_2$ is compact but $\pi : M_1 \to M_2$ is not a covering space (but has the $\pi_*$ isomorphism property).**

**4 Let $\mathcal{F}^k(M)$ denote the differentiable ($C^\infty$) $k$-forms on a manifold $M$. Suppose $U$ and $V$ are open subsets of a differentiable manifold.**

**4.1 Explain carefully how the usual exact sequence**

$$0 \to \mathcal{F}(U \cup V) \to \mathcal{F}(U) \oplus \mathcal{F}(V) \to \mathcal{F}(U \cap V) \to 0$$

**arises.**

**4.2 Write down the "long exact sequence" in de Rham cohomology associated to the short exact sequence in part 4.1 and describe explicitly how the map**

$$H^k_{deR}(U \cap V) \to H^{k+1}_{deR}(U \cup V)$$

**arises.**

**5 Explain carefully why the following holds: if $\pi : S^N \to M$, $N > 1$ is a covering space with $M$ orientable, then every closed $k$-form on $M$, $1 \leq k < M$ is exact. (Hint: Recall that the covering transformations in this situation form a group $G$ with $S^n/G \simeq M$.)**

**6 Calculate the singular homology of $\mathbb{R}^n$, $n > 1$, with $k$-points removed, $k \geq 1$. (Your answer will depend on $k$ and $n$).**

**7.1 Explain what is meant by adding a handle to a 2-sphere for a two dimensional orientable surface in general.**

**7.2 Show that a 2-sphere with a positive number of handles attached cannot be simply connected.**

**8.1 Define the degree $\deg f$ of a $C^\infty$ map $f : S^2 \to S^2$ and prove that $\deg f$ as you present it is well-defined and independent of any choices you need to make in your definition.**

**8.2 Prove in detail that for each integer $k$ (possibly negative), there is a $C^\infty$ map $f : S^2 \to S^2$ of degree $k$.**

**9 Explain how Stokes Theorem for manifolds with boundary gives, as a special case, the classical divergence theorem (about $\iiint_U \operatorname{div} V \, d(\mathrm{vol})$, where $U$ is a bounded open set in $\mathbb{R}^3$ with smooth boundary and $V$ is a $C^\infty$ vector field on $\mathbb{R}^3$).**

**10.1 Show that every map $F : S^n \to S^1 \times \cdots \times S^1$ ($k$ copies of $S^1$) is null-homotopic (homotopic to a constant map).**

**10.2 Show that there is a map $F : S^1 \times \cdots \times S^1$ ($n$ copies) $\to S^n$ such that $F$ is not null-homotopic.**

**10.3 Show that every map $F : S^n \to S^{n_1} \times S^{n_2} \times \cdots \times S^{n_k}$, $n_1 + \cdots + n_k = n$, $n_j > 0$, $k \geq 2$, has degree 0. (You may use any definition of degree you like, and you may assume $F$ is $C^\infty$).**