

- 1** Suppose  $M$  is a compact connected 3-manifold and  $\omega$  is a nowhere zero 1-form defined on  $M$ . Suppose that the distribution  $\ker \omega$  is integrable, and  $\ker \omega = T\mathcal{F}$  for a foliation  $\mathcal{F}$ .

**1.1** Show that  $\omega \wedge d\omega = 0$

**1.2** Use a partition of unity to show that there is a 1-form  $\alpha$  such that  $d\omega = \alpha \wedge \omega$ .

**1.3** Show  $d\alpha \wedge \omega = 0$ .

**1.4** Suppose that  $\alpha'$  is some other 1-form satisfying  $d\omega = \alpha' \wedge \omega$ . Show that  $\alpha' = \alpha + g\omega$  for some function  $g$ , and that  $\alpha \wedge d\alpha = \alpha' \wedge d\alpha'$ .

**1.5** Suppose that  $\omega'$  is a nowhere zero 1-form and  $\ker \omega = \ker \omega'$ . If  $d\omega' = \gamma \wedge \omega$ , show that  $\alpha \wedge d\alpha - \gamma \wedge d\gamma$  is exact.

- 2** On the compact connected manifold  $M$ , suppose  $\alpha$  is a  $p$ -form and  $\beta$  is an  $(n - p - 1)$ -form. Suppose  $\partial M$  has two components:  $\partial_0 M$  and  $\partial_1 M$ . Let  $i_0$  and  $i_1$  be the inclusions of  $\partial_0 M$  and  $\partial_1 M$  into  $M$ . Given that  $i_0^* \alpha = 0$  and  $i_1^* \beta = 0$ , show that

$$\int_M d\alpha \wedge \beta = (-1)^{p+1} \int_M \alpha \wedge d\beta$$

- 3** Suppose  $f : S^1 \rightarrow \mathbb{R}^2$  and  $g : S^1 \rightarrow \mathbb{R}^2$  are smooth embeddings. Let

$$M = \{(a, b, \vec{v}) \in S^1 \times S^1 \times \mathbb{R}^2 : f(a) - g(b) = \vec{v}\}.$$

Show that  $M$  is a compact submanifold of  $S^1 \times S^1 \times \mathbb{R}^2$ . Let  $\pi : M \rightarrow \mathbb{R}^2$  be the projection  $\pi(a, b, \vec{v}) = \vec{v}$ . Apply Sard's Theorem to  $\pi$  and deduce that for almost every  $\vec{v} \in \mathbb{R}^2$ ,  $f(S^1)$  is transverse to  $g(S^1) + \vec{v}$ .

4 Suppose that  $f : M \rightarrow N$  is a  $C^\infty$  map,  $M$  and  $N$  are compact connected  $n$ -manifolds, and  $\text{rank}(df) = n$ . Show that  $f$  is a covering map.

5 Let  $X$  be a polyhedron,  $A$  a subpolyhedron,  $p : \tilde{X} \rightarrow X$  the universal covering space of  $X$  and  $\bar{A}$  the path component of  $p^{-1}(A)$  containing the equivalence class of the constant path at  $x_0 \in A$ .

5.1 Give an example in which  $\bar{p} : \bar{A} \rightarrow A$  (where  $\bar{p}$  is the restriction of  $p$ ) is not the universal covering space of  $A$ .

5.2 Prove that  $\bar{p} : \bar{A} \rightarrow A$  is the covering space of the kernel of  $i_* : \pi_1(A, x_0) \rightarrow \pi_1(X, x_0)$  where  $i$  is inclusion.

6 Let  $D^n$  be the unit ball in  $\mathbb{R}^n$ ,  $S^{n-1}$  its boundary, and  $0 \in \mathbb{R}^n$  the origin.

6.1 Prove that the inclusion  $i : (D^n, S^{n-1}) \rightarrow (D^n, D^n - 0)$  induces an isomorphism  $i_* : H_n(D^n, S^{n-1}) \rightarrow H_n(D^n, D^n - 0)$ .

6.2 Prove that  $i$  is not a homotopy equivalence of pairs, that is, there is no map  $g : (D^n, D^n - 0) \rightarrow (D^n, S^{n-1})$  such that  $gi$  and  $ig$  are homotopic, as maps of pairs, to identity maps.

7 Given a map  $f : X \rightarrow X$  of a polyhedron, there is an exact sequence

$$\rightarrow H_k(X) \xrightarrow{1-f_*} H_k(X) \rightarrow H_k(T_f) \rightarrow H_{k-1}(X) \rightarrow$$

where  $T_f$  is the mapping torus of  $f$ . Use the sequence to calculate the homology of the 3-manifold  $M$  obtained from  $S^2 \times I$  by identifying  $(x, 0)$  to  $(-x, 1)$  for all  $x \in S^2$ .

8 Let  $A \subseteq X \subseteq Q$  and consider

$$H_k(Q, A) \xrightarrow{i_*} H_k(Q, X) \xrightarrow{\partial} H_{k-1}(X, A)$$

where  $i$  is inclusion and  $\partial[z] = [\partial_k[z]]$  for  $\partial_k : H_k(Q, X) \rightarrow H_{k-1}(X)$  that comes from the exact sequence of  $(Q, X)$ .

8.1 Prove that  $\partial$  is well-defined.

8.2 Prove that the image of  $i_*$  equals the kernel of  $\partial$ .

9 Let  $J(X, x_0) \subseteq \pi_1(X, x_0)$  be the subgroup of cyclic classes, where a class  $\alpha$  is *cyclic* if there is a homotopy  $\{h_t : X \rightarrow X\}$  with  $h_0 = h_1 = \text{identity}$  such that  $[h_t(x_0)] = \alpha$ .

9.1 Prove that  $J(X, x_0)$  is contained in the center of  $\pi_1(X, x_0)$ .

9.2 Prove that if  $X$  is a topological group, then  $J(X, x_0) = \pi_1(X, x_0)$ .