1 Explain in detail from the viewpoint of transversality theory, why the sum of the indices of a vector field with isolated zeroes on a compact orientable manifold M is independent of what vector field we choose.

2 Call the index sum in problem 1 the Euler characteristic $\chi(M)$. Explain why the Euler characteristic of a genus g surface (2-sphere with g handles attached) is $2 - 2g$. [Do this explicitly: do *not* appeal to the theorem that the Euler characteristic in the vector field sense indicated is computable from homological information. That comes next!]

3 Suppose that M is a triangulated compact orientable manifold, i.e., a manifold M represented as a finite simplicial complex.

3.1 Show that the alternating sum of the Betti numbers $b_0 - b_1 + b_2 - \cdots$ (where $b_k = \mathrm{rank}$ of the kth homology group with real coefficients) is equal to the alternating sum (number of vertices) $-$ (number of faces) $+$ (number of $2-$ simplices) $- \cdots$

3.2 Show that there is a vector field with the sum of its indices equal to the number described in part **3.1**. [You do not need to worry about smoothness of the vector field $-$ just describe how to build it. In part **3.1**, the result should follow from some dimension counting.]

4 Suppose V is a smooth (C^∞) vector field on $\mathbb{R}^3$ that is nonzero at $(0,0,0)$. The vector field is said to be gradient-like at $(0,0,0)$ if there is a neighborhood of $(0,0,0)$ and a nowhere zero smooth function $\lambda(x,y,z)$ on that neighborhood such that λV is the gradient of some smooth function in some (possibly smaller) neighborhood of $(0,0,0)$.

4.1 Write $V = (P, Q, R)$. Show by example that there are functions P, Q, R for which V is not gradient-like in a neighborhood of $(0,0,0)$. (Hint: the orthogonal complement of V taken at each point would have to be an integrable 2-plane field)

4.2 Derive a general differential condition on (P, Q, R) which is necessary and sufficient for V to be gradient-like in a neighborhood of $(0,0,0)$.

5.1 Define carefully the "boundary map" which defines the H_n to H_{n-1} mapping that arises in the long exact sequences arising from a short exact sequence of chain complexes.

5.2 Prove that the kernel of the boundary map is equal to the image of the map into the H_n.

answer

6   Compute the homology of the real projective space $\mathbb{RP}^n$ for each $n > 1$.

7.1   Define complex projective space $\mathbb{C}P^n$ $(n = 1, 2, 3, \dots)$

7.2   Show that $\mathbb{C}P^n$ is compact for all n.

7.3   Show that $\mathbb{C}P^n$ has a cell decomposition with one cell in each dimension $0, 2, 4, \dots, 2n$ and no other cells. Include a careful description of the attaching maps.

8   Suppose a compact (real) manifold M has a (finite) cell decomposition with only even dimensional cells. Is M necessarily orientable? Justify your answer.

9   Suppose that a finite group Γ acts smoothly on a compact manifold M and that the action is free, i.e. $\gamma(x) = x$ for some x in M if and only if $\gamma =$ the identity of the group Γ.

9.1   Show that M/Γ is a manifold (i.e., can be made a manifold in a natural way)

9.2   Show that $M \to M/\Gamma$ is a covering space.

9.3   If the kth de Rham cohomology of M is $\mathbf{0}$, some particular $k > 0$, then is the kth de Rham cohomology of M/Γ necessarily $\mathbf{0}$? Prove your answer.

10   Let $M = \mathbb{RP}^2 \times \mathbb{RP}^2$ where $\mathbb{RP}^2$ is a real projective 2-space). In a product manifold like that, homology elements can arise by taking in effect the product of a cycle in one factor with a cycle in the other factor. Show that in the case of this particular M, there is an element in the 3-homology with $\mathbb{Z}$ coefficients that does not arise in this way by exhibiting such an element explicitly, e.g. in terms of a cell decomposition.