

- 1 Let M be a connected smooth manifold. Show that for any two non-zero tangent vectors v_1 at point x_1 and v_2 at a point x_2 , there is a diffeomorphism $\phi : M \rightarrow M$ such that $\phi(x_1) = x_2$ and $d\phi(v_1) = v_2$.
- 2 Let X and Y be submanifolds of \mathbb{R}^n . Prove that for almost ever $a \in \mathbb{R}^n$, the translate $X + a$ intersects Y transversely.
- 3 Let $M_{n \times n}(\mathbb{R}) \simeq \mathbb{R}^{n^2}$ be the space of $n \times n$ matrices with real coefficients.

3.1 Show that

$$SL_n(\mathbb{R}) = \{A \in M_{n \times n}(\mathbb{R}) \mid \det(A) = 1\}$$

is a smooth submanifold of $M_{n \times n}(\mathbb{R})$.

3.2 Identify the tangent space to $SL_n(\mathbb{R})$ at the identity matrix I_n .

3.3 Show that $SL_n(\mathbb{R})$ has trivial Euler characteristic.

4.1 Let $f_i : M \rightarrow N$, $i = 0, 1$, be two smooth maps between smooth manifolds M and N , and $f_i^* : \Omega^*(N) \rightarrow \Omega^*(M)$, $i = 0, 1$, be the induced chain maps between the respective de Rham complexes. Define the notion of chain homotopy between f_0^* and f_1^* . Here the co-boundary operators on the de Rham complexes are the exterior derivatives.

4.2 Let X be a smooth vector field on a compact smooth manifold M , and let $\phi_t : M \rightarrow M$ be the flow generated by X at time t , i.e. the solution of the differential equation $\frac{d\phi_t}{dt}(x) = X(\phi_t(x))$ with initial condition $\phi_0(x) = x$. Find an explicit chain homotopy between the chain maps ϕ_0^* and ϕ_1^* , where ϕ_i^* , $i = 0, 1$, are the induced chain maps from $\Omega^*(M)$ to itself.

(Hint: Use the formula that for any differential form ω and vector field X , the Lie derivative $\mathcal{L}_X \omega = d \circ i_X \omega + i_X \circ d\omega$. Here i_X is the contraction with respect to X .)

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- 5 Let $\omega = dx_1 \wedge dx_2 + dx_3 \wedge dx_4 + \cdots + dx_{2n-1} \wedge dx_{2n}$ be a 2-form on \mathbb{R}^{2n} , where $(x_1, x_2, \dots, x_{2n})$ are the standard coordinates on \mathbb{R}^{2n} . Define an S^1 -action on \mathbb{R}^{2n} as follows: for each $t \in S^1$, define $g_t : \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n}$ by considering \mathbb{R}^{2n} as the direct sum of n copies of \mathbb{R}^2 and rotating each \mathbb{R}^2 summand an angle t . Let X be the vector field on \mathbb{R}^{2n} defined by $X(x) = \frac{dg_t(x)}{dt}|_{t=0}$ for any $x \in \mathbb{R}^{2n}$.
- 5.1 Find the Lie derivative $\mathcal{L}_X \omega$ and a function f on \mathbb{R}^{2n} such that $df = i_X \omega$.
- 5.2 The S^1 -action above induces an action on S^{2n-1} . Let \mathbb{P}^{n-1} be the quotient space of S^{2n-1} by this S^1 -action. Show that the quotient space \mathbb{P}^{n-1} has a natural smooth structure and that the tangent space of \mathbb{P}^{n-1} at any point \underline{x} can be identified with the quotient of the tangent space $T_x S^{2n-1}$ by the line spanned by $X(x)$, for any $x \in \underline{x}$. Here \underline{x} is the orbit of x under the S^1 -action.
- 5.3 Show that ω descends to a well-defined 2-form on the quotient space \mathbb{P}^{n-1} and that the 2-form so defined is closed.
- 5.4 Is the closed form in (5.3) exact?
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- (Hint: For (5.3) and (6) use (5.1) and (5.2))
- 7 Suppose that $f : S^n \rightarrow S^n$ is a smooth map of degree not equal to $(-1)^{n+1}$. Show that f has a fixed point.
- 8.1 Let G be a finitely presented group. Show that there is a topological space X with fundamental group $\pi_1(X) \cong G$.
- 8.2 Give an example of X in the case $G = \mathbb{Z} * \mathbb{Z}$, the free group on two generators.
- 8.3 How many connected, 2-sheeted covering spaces does the space X from (8.2) have?
- 9 Let G be a connected topological group. Show that $\pi_1(G)$ is a commutative group.
- 10 Show that if \mathbb{R}^m and \mathbb{R}^n are homeomorphic, then $m = n$.
- 11 Let N_g be the nonorientable surface of genus g , that is, the connected sum of g copies of \mathbb{RP}^2 . Calculate the fundamental group and homology groups of N_g .