1 **Let M be a closed (compact, without boundary) manifold. Show that any smooth function**

$$f : M \to \mathbb{R}$$

has a critical point.

2.1 **Show that every closed 1-form on S^n, $n > 1$, is exact.**

2.2 **Use this to show that every closed 1-form on $\mathbb{R}P^n$, $n > 1$, is exact.**

3 **Let M^d be a d-dimensional manifold and $\omega_1, \ldots, \omega_p$ be pointwise linearly independent 1-forms. If $\theta_1, \ldots, \theta_p$ are 1-forms so that**

$$\sum_{i=1}^{p} \omega_i \wedge \theta_i = 0,$$

then there exist smooth functions f_{ij} such that

$$\theta_i = \sum_{i=1}^{p} f_{ij}\omega_i, \quad i = 1, \ldots, p.$$

(Hint: try $p = 1$)

4 **Let M be the set of all straight lines in $\mathbb{R}^2$ (not just those which pass through the origin). Show that M is a smooth manifold and identify it with a well-known manifold.**

(Hint: Lines not through the origin have a unique closest point to the origin and that point determines the line uniquely. What happens at the origin?)

5 **Let $f : M^m \to N^n$ be a smooth bijection so that $Df : T_pM \to T_{f(p)}N$ is injective for all p. Show that f is a diffeomorphism.**

6.1 **Show that if $f : S^n \to S^n$ has no fixed points then $\deg(f) = (-1)^{n+1}$.**

6.2 **Show that if X has S^{2n} as universal covering space then $\pi_1(x) = \{1\}$ or $\mathbb{Z}_2$.**

6.3 **Show that if X has S^{2n+1} as universal covering space then X is orientable.**

7.1 **Outline the construction of the universal covering of a path connected locally simply connected space X.**

7.2 **Give an example of a path connected space which does not have a universal covering space.**

8 **Let X be a finite cell complex constructed inductively by gluing all p-cells onto cells of dimension $< p$. Assume no $p - 1$ and $p + 1$ cells are used to construct X. Show that**

$$H_p(X, \mathbb{Z}) \simeq \mathbb{Z}^{n_p}$$

when n_p is the number of p-cells used in the construction.

9 **Let $(M, \partial M)$ be a compact oriented n-manifold with a connected boundary ∂M. Show that there is no retract $r : M \to \partial M$ so that $r(x) = x$ if $x \in \partial M$. (Hint: Prove that $H_{n-1}(\partial M) \to H_{n-1}(M)$ is trivial.)**

10 **Let $X = T^2 \setminus \{p, q\}$, $p \neq q$ be the twice punctured 2-dimensional torus.**

10.1 **Compute the homology groups $H_*(X, \mathbb{Z})$.**

10.2 **Compute the fundamental group of X.**