1 Let M be a connected smooth manifold. Show that for any two non-zero tangent vectors v_1 at point x_1 and v_2 at a point x_2, there is a diffeomorphism $\phi : M \to M$ such that $\phi(x_1) = x_2$ and $d\phi(v_1) = v_2$.

2 Let X and Y be submanifolds of $\mathbb{R}^n$. Prove that for almost ever $a \in \mathbb{R}^n$, the translate $X + a$ intersects Y transversely.

3 Let $M_{n\times n}(\mathbb{R}) \simeq \mathbb{R}^{n^2}$ be the space of $n \times n$ matrices with real coefficients.

3.1 Show that

$$SL_n(\mathbb{R}) = \{A \in M_{n\times n}(\mathbb{R}) \mid \det(A) = 1\}$$

is a smooth submanifold of $M_{n\times n}(\mathbb{R})$.

3.2 Identify the tangent space to $SL_n(\mathbb{R})$ at the identity matrix I_n.

3.3 Show that $SL_n(\mathbb{R})$ has trivial Euler characteristic.

4.1 Let $f_i : M \to N$, $i = 0, 1$, be two smooth maps between smooth manifolds M and N, and $f_i^* : \Omega^*(N) \to \Omega^*(M)$, $i = 0, 1$, be the induced chain maps between the respective de Rham complexes. Define the notion of chain homotopy between f_0^* and f_1^*. Here the co-boundary operators on the de Rham complexes are the exterior derivatives.

4.2 Let X be a smooth vector field on a compact smooth manifold M, and let $\phi_t : M \to M$ be the flow generated by X at time t, i.e. the solution of the differential equation $\frac{d\phi_t}{dt}(x) = X(\phi_t(x))$ with initial condition $\phi_0(x) = x$. Find an explicit chain homotopy between the chain maps ϕ_0^* and ϕ_1^*, where ϕ_i^*, $i = 0, 1$, are the induced chain maps from $\Omega^*(M)$ to itself.
(Hint: Use the formula that for any differential form ω and vector field X, the Lie derivative $\mathcal{L}_X\omega = d \circ i_X\omega + i_X \circ d\omega$. Here i_X is the contraction with respect to X.)

split

5 **Let** $\omega = dx_1 \wedge dx_2 + dx_3 \wedge dx_4 + \cdots + dx_{2n-1} \wedge dx_{2n}$ **be a 2-form on** $\mathbb{R}^{2n}$**, where** $(x_1, x_2, \ldots, x_{2n})$ **are the standard coordinates on** $\mathbb{R}^{2n}$**. Define an** S^1**-action on** $\mathbb{R}^{2n}$ **as follows: for each** $t \in S^1$**, define** $g_t : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ **by considering** $\mathbb{R}^{2n}$ **as the direct sum of** n **copies of** $\mathbb{R}^2$ **and rotating each** $\mathbb{R}^2$ **summand an angle** t**. Let** X **be the vector field on** $\mathbb{R}^{2n}$ **defined by** $X(x) = \frac{dg_t(x)}{dt}|_{t=0}$ **for any** $x \in \mathbb{R}^{2n}$**.**

5.1 **Find the Lie derivative** $\mathcal{L}_X\omega$ **and a function** f **on** $\mathbb{R}^{2n}$ **such that** $df = i_X\omega$**.**

5.2 **The** S^1**-action above induces an action on** S^{2n-1}**. Let** $\mathbb{P}^{n-1}$ **be the quotient space of** S^{2n-1} **by this** S^1**-action. Show that the quotient space** $\mathbb{P}^{n-1}$ **has a natural smooth structure and that the tangent space of** $\mathbb{P}^{n-1}$ **at any point** $\underline{\mathbf{x}}$ **can be identified with the quotient of the tangent space** T_xS^{2n-1} **by the line spanned by** $X(x)$**, for any** $x \in \underline{\mathbf{x}}$**. Here** $\underline{\mathbf{x}}$ **is the orbit of** x **under the** S^1**-action.**

5.3 **Show that** ω **descends to a well-defined 2-form on the quotient space** $\mathbb{P}^{n-1}$ **and that the 2-form so defined is closed.**

5.4 **Is the closed form in (5.3) exact?**

(Hint: For (5.3) and (5.4) use (5.1) and (5.2))

6 **Suppose that** $f : S^n \to S^n$ **is a smooth map of degree not equal to** $(-1)^{n+1}$**. Show that** f **has a fixed point.**

7.1 **Let** G **be a finitely presented group. Show that there is a topological space** X **with fundamental group** $\pi_1(X) \cong G$**.**

7.2 **Give an example of** X **in the case** $G = \mathbb{Z} * \mathbb{Z}$**, the free group on two generators.**

7.3 **How many connected, 2-sheeted covering spaces does the space** X **from (7.2) have?**

8 **Let** G **be a connected topological group. Show that** $\pi_1(G)$ **is a commutative group.**

9 **Show that if** $\mathbb{R}^m$ **and** $\mathbb{R}^n$ **are homeomorphic, them** $m = n$**.**

10 **Let** N_g **be the nonorientable surface of genus** g**, that is, the connected sum of** g **copies of** $\mathbb{RP}^3$**. Calculate the fundamental group and homology groups of** N_g**.**