1 Let $\mathrm{Mat}_{m\times n}(\mathbb{R})$ be the space of $m \times n$ matrices with real valued entries.

1.1 Show that the subset $S \subset \mathrm{Mat}_{m\times n}(\mathbb{R})$ of rank 1 matrices form a submanifold of dimension $m + n - 1$.

1.2 Show that the subset $T \subset \mathrm{Mat}_{m\times n}(\mathbb{R})$ of rank k matrices form a submanifold of dimension $k(m + n - k)$.

2 Let M be a smooth manifold and $\omega \in \Omega^1(M)$ a smooth 1-form.

2.1 Define the line integral

$$\int_c \omega$$

along the piecewise smooth curve $c : [0, 1] \to M$.

2.2 Show that $\omega = df$ for a smooth function $f : M \to \mathbb{R}$ if and only if $\int_c \omega = 0$ for all closed curves $c : [0, 1] \to M$, i.e., $c(0) = c(1)$.

3 Let $S_1, S_2 \subset M$ be smooth embedded submanifolds.

3.1 Define what it means for S_1, S_2 to be transversal.

3.2 Show that if $S_1, S_2 \subset M$ are transversal then $S_1 \cap S_2 \subset M$ is a smooth embedded submanifold of dimension $\dim S_1 + \dim S_2 - \dim M$.

4 Let $S \subset M$ be given as $F^{-1}(c)$ where $F = (F^1, \dots, F^k) : M \to \mathbb{R}^k$ is smooth and $c \in \mathbb{R}^k$ is a regular value for F. If $f : M \to \mathbb{R}$ is smooth, show that its restriction $f|_C$ to a submanifold $C \subset M$ has a critical point at $p \in C$ if and only if there exist constants $\lambda_1, \dots, \lambda_k$ such that

$$df_p = \sum \lambda_i dF^i_p$$

where $dg_p : T_pM \to \mathbb{R}$ denotes the differential at p of a smooth function g.

5 Let M be a smooth, orientable, compact manifold with boundary ∂M. Show that there is no (smooth) retract $r : M \to \partial M$.

6 Let $A \in Gl_{n+1}(\mathbb{C})$.

6.1 Show that A defines a smooth map $A : \mathbb{C}P^n \to \mathbb{C}P^n$.

6.2 Show that the fixed points of $A : \mathbb{C}P^n \to \mathbb{C}P^n$ correspond to eigenvectors for the original matrix.

6.3 Show that $A : \mathbb{C}P^n \to \mathbb{C}P^n$ is a Lefschetz map if the eigenvalues of A all have multiplicity 1.

6.4 Show that the Lefschetz number of $A : \mathbb{C}P^n \to \mathbb{C}P^n$ is $n + 1$. (Hint: You are allowed to use that $\mathrm{Gl}_{n+1}(\mathbb{C})$ is connected.)

7 Let $F : S^n \to S^n$ be a continuous map.

7.1 Define the degree $\deg F$ of F and show that when F is smooth

$$\deg F \int_{S^n} \omega = \int_{S^n} F^*\omega$$

for all $\omega \in \Omega^n(S^n)$.

7.2 Show that if F has no fixed points then $\deg F = (-1)^{n+1}$.

8 Let $f : S^{n-1} \to S^{n-1}$ be a continuous map and D^n the disk with $\partial D^n = S^{n-1}$.

8.1 Define the adjunction space $D^n \cup_f D^n$.

8.2 Let $\deg f = k$ and compute the homology groups $H_p(D^n \cup_f D^n, \mathbb{Z})$ for $p = 0, 1, \dots$.

8.3 Assume that f is a homeomorphism, show that $D^n \cup_f D^n$ is homeomorphic to S^n.

9 Let $F : M \to N$ be a finite covering map between closed manifolds. Either prove or find counter examples to the following questions.

9.1 Do M and N have the same fundamental groups?

9.2 Do M and N have the same de Rham cohomology groups?

9.3 When M is simply connected, do M and N have the same singular homology groups?

10 Let $A \subset X$ be a subspace of a topological space. Define the relative singular homology groups $H_p(X, A)$ and show that there is a long exact sequence

$$\cdots \to H_p(A) \to H_p(X) \to H_p(X, A) \to H_{p-1}(A) \to \cdots$$