

- 1 Explain carefully how the classical “divergence

$$\iint_S \vec{V} \cdot \vec{n} d(\text{area}) = \iiint_V \operatorname{div} \vec{V} d(\text{volume})$$

( $V$  a bounded volume in  $\mathbb{R}^3$ ,  $S = \text{boundary of } V$ ) follows from Stokes’ Theorem for differential forms.

- 2 Without using deRham’s Theorem, prove:

- 2.1 every closed 1-form on  $S^2$  is exact.  
 2.2 a two-form  $\Omega$  is exact on  $S^2$  if and only if

$$\int_{S^2} \Omega = 0.$$

- 3 Show that the set of all lines in  $\mathbb{R}^2$  has a natural structure as a differentiable manifold. What (already familiar) manifold is it?
- 4 Show that  $S^3$  is the union of two solid tori ( $S^1 \times 2\text{-disc}$ ) with an embedded torus ( $S^1 \times S^1$ ) as their common boundary. (Hint: Express  $\mathbb{R}^3$  with a solid torus removed as a union of circles and a single straight line and then add a point at infinity.)
- 5 Suppose  $M$  is a compact manifold (with empty boundary).
- 5.1 Prove that, if  $f : M \rightarrow \mathbb{R}$  is a  $C^\infty$  function, then  $f$  has at least two critical points.
- 5.2 A  $C^\infty$  function on  $S^1 \times S^1$  cannot have only two critical points. Prove this (e.g.) by deforming a homotopically nontrivial  $S^1$  along the gradient flow of  $f : M \rightarrow \mathbb{R}$ .
- 6.1 Prove carefully that a group of homeomorphisms of  $S^{2n}$ , each of which has no fixed points (unless it is the identity map) contains at most two elements.
- 6.2 Give a counterexample for some  $S^{2n+1}$ ,  $n \geq 1$ .
- 7 Find the homology groups with  $\mathbb{Z}$  coefficients, of  $\mathbb{R}P^n$ ,  $n = 2, 3, 4, \dots$  by some systematic rigorous method.
- 8 Find the homology and the fundamental group of  $S^1 \times S^1$  with two points removed.
- 9 Prove that if a compact (empty boundary) manifold  $X$  has  $S^{2n+1}$ ,  $n \geq 1$ , as a covering space, then  $X$  is orientable.
- 10 Suppose  $M$  is a compact manifold (empty boundary). Prove that

$$H_n(M, \mathbb{Z}) \simeq \mathbb{Z}.$$

(You may assume  $M$  is triangulated.)