

- 1 Explain carefully how the classical “divergence

$$\iint_S \vec{V} \cdot \vec{n} d(\text{area}) = \iiint_V \operatorname{div} \vec{V} d(\text{volume})$$

(V a bounded volume in \mathbb{R}^3 , $S = \text{boundary of } V$) follows from Stokes’ Theorem for differential forms.

- 2 Without using deRham’s Theorem, prove:

- 2.1 every closed 1-form on S^2 is exact.
 2.2 a two-form Ω is exact on S^2 if and only if

$$\int_{S^2} \Omega = 0.$$

- 3 Show that the set of all lines in \mathbb{R}^2 has a natural structure as a differentiable manifold. What (already familiar) manifold is it?
- 4 Show that S^3 is the union of two solid tori ($S^1 \times 2\text{-disc}$) with an embedded torus ($S^1 \times S^1$) as their common boundary. (Hint: Express \mathbb{R}^3 with a solid torus removed as a union of circles and a single straight line and then add a point at infinity.)
- 5 Suppose M is a compact manifold (with empty boundary).
- 5.1 Prove that, if $f : M \rightarrow \mathbb{R}$ is a C^∞ function, then f has at least two critical points.
- 5.2 A C^∞ function on $S^1 \times S^1$ cannot have only two critical points. Prove this (e.g.) by deforming a homotopically nontrivial S^1 along the gradient flow of $f : M \rightarrow \mathbb{R}$.
- 6.1 Prove carefully that a group of homeomorphisms of S^{2n} , each of which has no fixed points (unless it is the identity map) contains at most two elements.
- 6.2 Give a counterexample for some S^{2n+1} , $n \geq 1$.
- 7 Find the homology groups with \mathbb{Z} coefficients, of $\mathbb{R}P^n$, $n = 2, 3, 4, \dots$ by some systematic rigorous method.
- 8 Find the homology and the fundamental group of $S^1 \times S^1$ with two points removed.
- 9 Prove that if a compact (empty boundary) manifold X has S^{2n+1} , $n \geq 1$, as a covering space, then X is orientable.
- 10 Suppose M is a compact manifold (empty boundary). Prove that

$$H_n(M, \mathbb{Z}) \simeq \mathbb{Z}.$$

(You may assume M is triangulated.)