1 **Show that if V is a smooth vector field on a (smooth) manifold of dimension n and if $V(p)$ is nonzero for some point p, then there is a coordinate system defined on a neighborhood of p, say $(x_1, \ldots, x_n)$, such that on a neighborhood of p, $V =$ the x_1 coordinate vector field.**

2.1 **Demonstrate the formula $L_x = di_x + i_x d$, where L is the Lie derivative and i is the interior product.**

2.2 **Use this formula to show that a vector-field X on $\mathbb{R}^3$ has a flow (locally) that preserves volume if and only if the divergence of X is everywhere 0. [Here divergence is the classical operator that takes a vector field with components f, g, h to the function $f_x + g_y + h_z$, in the usual partial derivative notation $f_x = x$-partial of f, etc.]**

3.1 **Explain some systematic reason why there is a closed 2-form on $\mathbb{R}^3 \setminus \{(0,0,0)\}$ (Euclidean 3-space with one point removed) that is not exact. You man do this by exhibiting such a form explicitly and checking that it is closed but not exact or you may argue using theorems that such a form must exist.**

3.2 **With φ such a form (as in part 3.1), discuss why, for any smooth map of S^2 to itself, such a number**

$$\left(\int_{S_2} F^*\varphi\right) / \left(\int_{S_2} \varphi\right)$$

is the degree of f. [Note that this includes explaining why the denominator integral cannot be 0.].

4 Show without using de Rham's Theorem (but you may use the Poincare Lemma without proof), that a 2-form φ on the 2-sphere S^2 that has integral 0 is exact, i.e., equal to $d\omega$ for some 1-form ω on S^2.

5 Suppose that $V : U \to S^2$ is a smooth map, considered as a vector field of unit vectors, where $U = \mathbb{R}^3$ with a finite number of points $p_1, \ldots, p_n$ removed, all these points lying strictly inside the unit sphere S^2.

Explain carefully, from basic facts about critical values and critical points and the like, why the degree of $V|_{S^2} : S^2 \to S^2$ is equal to the sum of the indices of the vector field V at the points $p_1, \ldots, p_n$.

6.1 Explain what a short exact sequence of chain complexes is.

6.2 Describe how a short exact sequence of chain complexes gives rise to a long exact sequence in homology. Include how the connecting homomorphism (where the dimension changes) arises. You do not need to prove exactness of this sequence.

7.1 Define complex projective space $\mathbb{C}P^n$, $n = 1, 2, 3, \ldots$

7.2 Compute the homology and cohomology of $\mathbb{C}P^n$, $\mathbb{Z}$ coefficients. (Any method is allowed. Cell complexes are particularly simple to use. Be sure to explain what the attaching maps are if you adopt this approach).

8.1 Find the $\mathbb{Z}$-coefficient homology of $\mathbb{R}P^2$ by any systematic method.

8.2 Explain explicitly (not using the Künneth Theorem) how a nonzero element of the 3-homology with $\mathbb{Z}$-coefficients of $\mathbb{R}P^2 \times \mathbb{R}P^2$ arises.

9.1 State the Lefschetz Fixed Point Theorem.

9.2 Show that the Lefschetz number of any map from $\mathbb{C}P^{2n}$ to itself is nonzero and hence that every map from $\mathbb{C}P^{2n}$ to itself has a fixed point. (Hint: The action of the map on cohomology with $\mathbb{Z}$ coefficients is determined by what happens to the 2nd cohomology since the whole cohomology ring is generated by the 2nd cohomology.)

10 Compute explicitly the simplicial homology, $\mathbb{Z}$ coefficients, of the surface of a tetrahedron, thus obtaining the homology of the 2-sphere.