1 Let $\Gamma \subset \mathbb{R}^2$ be the graph of the function $y = |x|$.

1.1 Construct a smooth function $f : \mathbb{R} \to \mathbb{R}^2$ whose image is Γ.

1.2 Can f be an immersion?

2 Let W be a smooth manifold with boundary, and $f : \partial W \to \mathbb{R}^n$ be a smooth map, for some $n \geq 1$. Show that there exists a smooth map $F : W \to \mathbb{R}^n$ such that $F|_{\partial W} = f$.

3 Let $S^n \subset \mathbb{R}^{n+1}$ be the unit sphere. Determine the values of $n \geq 0$ for which the antipodal map $S^n \to S^n$, $x \mapsto -x$ is isotopic to the identity.

4 Let $\omega_1, \ldots, \omega_k$ be 1-forms on a smooth n-dimensional manifold M. Show that $\{\omega_i\}$ are linearly independent if and only if

$$\omega_1 \wedge \omega_2 \wedge \cdots \wedge \omega_k \neq 0.$$

5 Let $M = \mathbb{R}^2/\mathbb{Z}^2$ be the two dimensional torus, L the line $3x = 7y$ in $\mathbb{R}^2$, and $S = \pi(L) \subset M$ where $\pi : \mathbb{R}^2 \to M$ is the projection map. Find a differential form on M which represents the Poincaré dual of S.

6 Let $S^n \subset \mathbb{R}^{n+1}$ be the unit sphere, equipped with the round metric g_S (the restriction of the Euclidean metric on $\mathbb{R}^{n+1}$). Consider also the hyperplane $H = \mathbb{R}^n \times \{0\} \subset \mathbb{R}^{n+1}$, equipped with the Euclidean metric g_H. Any line passing through the North Pole $p = (0, \ldots, 0, 1)$ and another point $A \in S^n$ will intersect this hyperplane in a point A'. The map

$$\Psi : S^n \setminus \{p\} \to H, \quad \Psi(A) = A'$$

is called stereographic projection. Show that Ψ is conformal, i.e. for any $x \in S^n \setminus \{p\}$, the bilinear form $(g_S)_x$ is a multiple of the bilinear form $\Psi^*((g_H)_{\Psi(x)})$.

7 Let X be the wedge sum $S^1 \vee S^1$. Give an example of an irregular covering space $X \to X$.

8 For $n \geq 2$, let X_n be the space obtained from a regular $(2n)$-gon by identifying the opposite sides with parallel orientations. For example, X_3 is

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The above description produces a cell decomposition of X_n.

8.1 Write down the associated cellular chain complex.

8.2 Show that X_n is a surface, and find its genus.

9.1 Consider the space Y obtained from $S^2 \times [0,1]$ by identifying $(x,0)$ with $(-x,0)$ and also identifying $(x,1)$ with $(-x,1)$, for all $x \in S^2$. Show that Y is homeomorphic to the connected sum $\mathbb{RP}^3 \# \mathbb{RP}^3$.

9.2 Show that $S^2 \times S^1$ is a double cover of the connected sum $\mathbb{RP}^3 \# \mathbb{RP}^3$.

10 Let X be a topological space. Define the suspension $S(X)$ to be the space obtained from $X \times [0,1]$ be contracting $X \times \{0\}$ to a point, and contracting $X \times \{1\}$ to another point. Describe the relation between the homology groups of X and $S(X)$.