

- 1 Let M_n be the space of all $n \times n$ matrices with real entries and let S_n be the subset consisting of all symmetric matrices. Consider the map $F : M_n \rightarrow S_n$ defined by $F(A) = AA^t - I$, where I is the identity matrix and A^t is the transpose of A .
 - 1.1 Show that $0_{n \times n}$ (the $n \times n$ matrix with all entries 0) is a regular value of F .
 - 1.2 Deduce that $O(n)$, the set of all $n \times n$ matrices such that $A^{-1} = A^t$ is a submanifold of M_n .
 - 1.3 Find the dimension of $O(n)$ and determine the tangent space of $O(n)$ at the identity matrix as a subspace of the tangent space of M_n which is M_n itself.
- 2 Show that $T^2 \times S^n$, $n \geq 1$ is parallelizable, where S^n is the n sphere, $T^2 = S^1 \times S^1$ is the two torus, and a manifold of dimension k is said to be parallelizable if there are k vector fields V_1, \dots, V_k on it with $V_1(p), \dots, V_k(p)$ linearly independent for all points p of the manifold.
- 3 Suppose $\pi : M_1 \rightarrow M_2$ is a C^∞ map of one connected differentiable manifold to another. And suppose for each $p \in M_1$, the differential $\pi_* : T_p M_1 \rightarrow T_{\pi(p)} M_2$ is a vector space isomorphism.
 - 3.1 Show that if M_1 is connected, then π is a covering space projection.
 - 3.2 Give an example where M_2 is compact but $\pi : M_1 \rightarrow M_2$ is not a covering space (but has the π_* isomorphism property).
- 4 Let $\mathcal{F}^k(M)$ denote the differentiable (C^∞) k -forms on a manifold M . Suppose U and V are open subsets of a differentiable manifold.
 - 4.1 Explain carefully how the usual exact sequence

$$0 \rightarrow \mathcal{F}(U \cup V) \rightarrow \mathcal{F}(U) \oplus \mathcal{F}(V) \rightarrow \mathcal{F}(U \cap V) \rightarrow 0$$

arises.

- 4.2 Write down the “long exact sequence” in de Rham cohomology associated to the short exact sequence in part 4.1 and describe explicitly how the map

$$H_{deR}^k(U \cap V) \rightarrow H_{deR}^{k+1}(U \cup V)$$

arises.

- 5 Explain carefully why the following holds: if $\pi : S^N \rightarrow M$, $N > 1$ is a covering space with M orientable, then every closed k -form on M , $1 \leq k < M$ is exact. (Hint: Recall that the covering transformations in this situation form a group G with $S^N/G \simeq M$.)
- 6 Calculate the singular homology of \mathbb{R}^n , $n > 1$, with k -points removed, $k \geq 1$. (Your answer will depend on k and n).
- 7.1 Explain what is meant by adding a handle to a 2-sphere for a two dimensional orientable surface in general.
- 7.2 Show that a 2-sphere with a positive number of handles attached cannot be simply connected.
- 8.1 Define the degree $\deg f$ of a C^∞ map $f : S^2 \rightarrow S^2$ and prove that $\deg f$ as you present it is well-defined and independent of any choices you need to make in your definition.
- 8.2 Prove in detail that for each integer k (possibly negative), there is a C^∞ map $f : S^2 \rightarrow S^2$ of degree k .
- 9 Explain how Stokes Theorem for manifolds with boundary gives, as a special case, the classical divergence theorem (about $\iiint_U \operatorname{div} V d(\operatorname{vol})$, where U is a bounded open set in \mathbb{R}^3 with smooth boundary and V is a C^∞ vector field on \mathbb{R}^3).
- 10.1 Show that every map $F : S^n \rightarrow S^1 \times \cdots \times S^1$ (k copies of S^1) is null-homotopic (homotopic to a constant map).
- 10.2 Show that there is a map $F : S^1 \times \cdots \times S^1$ (n copies) $\rightarrow S^n$ such that F is not null-homotopic.
- 10.3 Show that every map $F : S^n \rightarrow S^{n_1} \times S^{n_2} \times \cdots \times S^{n_k}$, $n_1 + \cdots + n_k = n$, $n_j > 0$, $k \geq 2$, has degree 0. (You may use any definition of degree you like, and you may assume F is C^∞).