Frederick Robinson Winter 2003

1 Let M be a smooth compact manifold of dimension n. Show that there is no immersion of M into  $\mathbb{R}^n$ .

The *n*-dimensional torus  $T^n$  is defined to be  $\mathbb{R}^n/\mathbb{Z}^n$ , i.e. for any x and y in  $\mathbb{R}^n$ ,  $x \sim y$  iff  $x - y \in \mathbb{Z}^n$ . Let  $\alpha$  and  $\beta$  be two such functions on  $\mathbb{R}^n$  such that (i)  $\alpha(x) = \alpha(y)$  and  $\beta(x) = \beta(y)$  iff  $x - y \in \mathbb{Z}^n$  and (ii)  $\alpha/\beta$  is an irrational constant. Then

$$v = \alpha(x) \frac{\partial}{\partial x^1} + \beta(x) \frac{\partial}{\partial x^2}$$

is a vector field on  $\mathbb{R}^n$  descending to  $T^n$ , where

$$\left\{ \frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \dots, \frac{\partial}{\partial x^n} \right\}$$

are coordinate vector fields. Find all functions f on  $T^n$  such that vf = 0.

- 3 Let M and N be smooth compact connected manifolds and  $f: M \to N$  be a smooth map such that, for any point  $m \in M$ ,  $\operatorname{rank}(df_m) = \dim(N)$ . Show that (i) for any  $n \in N$ ,  $f^{-1}(n)$  is a submanifold of M and (ii) for any  $n_1$  and  $n_2$  in N, the submanifolds  $f^{-1}(n_1)$  and  $f^{-1}(n_2)$  of M are diffeomorphic to each other.
- 4 Let

$$\tilde{\theta} = \frac{1}{2} \left\{ (x^2 dx^1 - x^1 dx^2) + (x^4 dx^3 - x^3 dx^4) + \dots + (x^{2n} dx^{2n-1} - x^{2n-1} dx^{2n}) \right\}$$

be a 1-form on  $\mathbb{R}^{2n}$  and  $\theta$  be its restriction to the unit sphere

$$S^{2n-1} = \left\{ x = (x^1, \dots, x^{2n}) \mid (x^1)^2 + \dots + (x^{2n})^2 = 1 \right\}.$$

The kernel K of  $\theta$  is a distribution on  $S^{2n-1}$ :

$$K = \{ v \mid v \in TS^{2n-1}, \theta(v) = 0 \}.$$

Decide whether or not K is integrable.

5 Let  $T^{2m}=\mathbb{R}^{2n}/\mathbb{Z}^{2n}$  be a torus of dimension 2n. Consider the 2-form

$$\omega = dx^1 \wedge dx^{n+1} + dx^2 \wedge dx^{n+2} + \dots + dx^n \wedge dx^{2n}$$

defined on  $\mathbb{R}^{2n}$  descending to  $\mathbb{T}^{2n}$ .

- 5.1 Show that  $\omega$  is closed by not exact on  $T^{2n}$ .
- 5.2 Let  $i: T^n \to T^{2n}$  be the subtorus defined by the equation

$$x^{n+1} = x^{n+2} = \dots = x^{2n} = 0.$$

What is  $i^*\omega$ ?

5.3 Let  $\Sigma = S^2 \setminus \{\bigcup_{i=1}^m D_i\}$ , where  $D_i$ , i = 1, ..., m are m open discs in  $S^2$  which disjoint closures. Show that

$$\int_{\Sigma} f_1^* \omega = \int_{\Sigma} f_2^* \omega$$

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- if  $f_1, f_2: (\Sigma, \partial \Sigma) \to (T^{2n}, T^n)$  are homotopic to each other, where  $\partial \Sigma$  is the boundary of  $\Sigma$ .
- 6 Let X be a path connected space and let  $x_0, x_1 \in X$ . Prove carefully that  $\pi_1(X, x_0)$  is isomorphic to  $\pi_1(X, x_1)$ .
- 7.1 Define what is meant by a "chain homotopy" P between chain maps  $f_{\#}, g_{\#}: C \to D$  and prove that chain homotopic chain maps induce the same homomorphism of homology.
- 7.2 Let X and Y be spaces and let  $F: X \times I \to Y$  be a homotopy between maps f and g. Define a chain homotopy P between the induced chain maps  $f_\#, g_\#: C(X) \to C(Y)$  of singular chains.
- 7.3 Verify that P satisfies the definition of a chain homotopy *only* for the restriction of P to  $C_1(X)$ .
- 8 Let X be a locally contractible space and H a subgroup of  $\pi_1(X, x_0)$ . Describe carefully how to construct a topological space  $X_H$  and a map  $p: X_H \to X$  such that  $p_*(\pi_1(X_H, \tilde{x}_0)) = H$  and show that it has the required property of  $p_*$ . (Note: Although  $X_H$  will be a covering space, you don't have to verify this unless you want to use some general properties of covering spaces.)
- 9 Prove that the real even-dimensional projective spaces  $\mathbb{R}P^{2n}$  have the fixed point property, that is, for every map  $f: \mathbb{R}P^{2n} \to \mathbb{R}P^{2n}$  there is a solution to f(x) = x. (Hint: Consider the maps on the covering space  $S^{2n}$ .)
- Use the Mayer-Vietoris sequence to calculate the homology of  $S^1 \times S^2$ . You may assume the homology calculations for  $S^1$ ,  $S^1 \times S^1$  and  $S^2$ .