1 **Let M_n be the space of all $n \times n$ matrices with real entries and let S_n be the subset consisting of all symmetric matrices. Consider the map $F : M_n \to S_n$ defined by $F(A) = AA^t - I$, where I is the identity matrix and A^t is the transpose of A.**

1.1 **Show that $0_{n\times n}$ (the $n \times n$ matrix with all entries 0) is a regular value of F.**

1.2 **Deduce that $O(n)$, the set of all $n \times n$ matrices such that $A^{-1} = A^t$ is a submanifold of M_n.**

1.3 **Find the dimension of $O(n)$ and determine the tangent space of $O(n)$ at the identity matrix as a subspace of the tangent space of M_n which is M_n itself.**

2 **Show that $T^2 \times S^n$, $n \geq 1$ is parallelizable, where S^n is the n sphere, $T^2 = S^1 \times S^1$ is the two torus, and a manifold of dimension k is said to be parallelizable if there are k vector fields $V_1, \ldots, V_k$ on it with $V_1(p), \ldots, V_k(p)$ linearly independent for all points p of the manifold.**

3 **Suppose $\pi : M_1 \to M_2$ is a C^∞ map of one connected differentiable manifold to another. And suppose for each $p \in M_1$, the differential $\pi_* : T_pM_1 \to T_{\pi(p)}M_2$ is a vector space isomorphism.**

3.1 **Show that if M_1 is connected, then π is a covering space projection.**

3.2 **Give an example where M_2 is compact but $\pi : M_1 \to M_2$ is not a covering space (but has the π_* isomorphism property).**

4 **Let $\mathcal{F}^k(M)$ denote the differentiable (C^∞) k-forms on a manifold M. Suppose U and V are open subsets of a differentiable manifold.**

4.1 **Explain carefully how the usual exact sequence**

$$0 \to \mathcal{F}(U \cup V) \to \mathcal{F}(U) \oplus \mathcal{F}(V) \to \mathcal{F}(U \cap V) \to 0$$

arises.

4.2 **Write down the "long exact sequence" in de Rham cohomology associated to the short exact sequence in part 4.1 and describe explicitly how the map**

$$H^k_{deR}(U \cap V) \to H^{k+1}_{deR}(U \cup V)$$

arises.

5 Explain carefully why the following holds: if $\pi : S^N \to M$, $N > 1$ is a covering space with M orientable, then every closed k-form on M, $1 \leq k < M$ is exact. (Hint: Recall that the covering transformations in this situation form a group G with $S^n/G \simeq M$.)

6 Calculate the singular homology of $\mathbb{R}^n$, $n > 1$, with k-points removed, $k \geq 1$. (Your answer will depend on k and n).

7.1 Explain what is meant by adding a handle to a 2-sphere for a two dimensional orientable surface in general.

7.2 Show that a 2-sphere with a positive number of handles attached cannot be simply connected.

8.1 Define the degree $\deg f$ of a C^∞ map $f : S^2 \to S^2$ and prove that $\deg f$ as you present it is well-defined and independent of any choices you need to make in your definition.

8.2 Prove in detail that for each integer k (possibly negative), there is a C^∞ map $f : S^2 \to S^2$ of degree k.

9 Explain how Stokes Theorem for manifolds with boundary gives, as a special case, the classical divergence theorem (about $\iiint_U \operatorname{div} V d(\mathrm{vol})$, where U is a bounded open set in $\mathbb{R}^3$ with smooth boundary and V is a C^∞ vector field on $\mathbb{R}^3$).

10.1 Show that every map $F : S^n \to S^1 \times \cdots \times S^1$ (k copies of S^1) is null-homotopic (homotopic to a constant map).

10.2 Show that there is a map $F : S^1 \times \cdots \times S^1$ (n copies) $\to S^n$ such that F is not null-homotopic.

10.3 Show that every map $F : S^n \to S^{n_1} \times S^{n_2} \times \cdots \times S^{n_k}$, $n_1 + \cdots + n_k = n$, $n_j > 0$, $k \geq 2$, has degree 0. (You may use any definition of degree you like, and you may assume F is C^∞).