

- 1 Explain in detail from the viewpoint of transversality theory, why the sum of the indices of a vector field with isolated zeroes on a compact orientable manifold M is independent of what vector field we choose.
- 2 Call the index sum in problem 1 the Euler characteristic $\chi(M)$. Explain why the Euler characteristic of a genus g surface (2-sphere with g handles attached) is $2 - 2g$. [Do this explicitly: do *not* appeal to the theorem that the Euler characteristic in the vector field sense indicated is computable from homological information. That comes next!]
- 3 Suppose that M is a triangulated compact orientable manifold, i.e., a manifold M represented as a finite simplicial complex.
 - 3.1 Show that the alternating sum of the Betti numbers $b_0 - b_1 + b_2 - \cdots$ (where $b_k = \text{rank of the } k\text{th homology group with real coefficients}$) is equal to the alternating sum (number of vertices) $-$ (number of faces) $+ ($ number of 2-simplices) $- \cdots$
 - 3.2 Show that there is a vector field with the sum of its indices equal to the number described in part 3.1. [You do not need to worry about smoothness of the vector field – just describe how to build it. In part 3.1, the result should follow from some dimension counting.]
- 4 Suppose V is a smooth (C^∞) vector field on \mathbb{R}^3 that is nonzero at $(0, 0, 0)$. The vector field is said to be gradient-like at $(0, 0, 0)$ if there is a neighborhood of $(0, 0, 0)$ and a nowhere zero smooth function $\lambda(x, y, z)$ on that neighborhood such that λV is the gradient of some smooth function in some (possibly smaller) neighborhood of $(0, 0, 0)$.
 - 4.1 Write $V = (P, Q, R)$. Show by example that there are functions P, Q, R for which V is not gradient-like in a neighborhood of $(0, 0, 0)$. (Hint: the orthogonal complement of V taken at each point would have to be an integrable 2-plane field)
 - 4.2 Derive a general differential condition on (P, Q, R) which is necessary and sufficient for V to be gradient-like in a neighborhood of $(0, 0, 0)$.
- 5.1 Define carefully the “boundary map” which defines the H_n to H_{n-1} mapping that arises in the long exact sequences arising from a short exact sequence of chain complexes.
- 5.2 Prove that the kernel of the boundary map is equal to the image of the map into the H_n .

answer

- 6 Compute the homology of the real projective space \mathbb{RP}^n for each $n > 1$.
- 7.1 Define complex projective space \mathbb{CP}^n ($n = 1, 2, 3, \dots$)
- 7.2 Show that \mathbb{CP}^n is compact for all n .
- 7.3 Show that \mathbb{CP}^n has a cell decomposition with one cell in each dimension $0, 2, 4, \dots, 2n$ and no other cells. Include a careful description of the attaching maps.
- 8 Suppose a compact (real) manifold M has a (finite) cell decomposition with only even dimensional cells. Is M necessarily orientable? Justify your answer.
- 9 Suppose that a finite group Γ acts smoothly on a compact manifold M and that the action is free, i.e. $\gamma(x) = x$ for some x in M if and only if $\gamma =$ the identity of the group Γ .
- 9.1 Show that M/Γ is a manifold (i.e., can be made a manifold in a natural way)
- 9.2 Show that $M \rightarrow M/\Gamma$ is a covering space.
- 9.3 If the k th de Rham cohomology of M is 0, some particular $k > 0$, then is the k th de Rham cohomology of M/Γ necessarily 0? Prove your answer.
- 10 Let $M = \mathbb{RP}^2 \times \mathbb{RP}^2$ where \mathbb{RP}^2 is a real projective 2-space). In a product manifold like that, homology elements can arise by taking in effect the product of a cycle in one factor with a cycle in the other factor. Show that in the case of this particular M , there is an element in the 3-homology with \mathbb{Z} coefficients that does not arise in this way by exhibiting such an element explicitly, e.g. in terms of a cell decomposition.