

- 1 Show that if V is a smooth vector field on a (smooth) manifold of dimension n and if $V(p)$ is nonzero for some point p , then there is a coordinate system defined on a neighborhood of p , say (x_1, \dots, x_n) , such that on a neighborhood of p , $V =$ the x_1 coordinate vector field.
 - 2.1 Demonstrate the formula $L_x = di_x + i_x d$, where L is the Lie derivative and i is the interior product.
 - 2.2 Use this formula to show that a vector-field X on \mathbb{R}^3 has a flow (locally) that preserves volume if and only if the divergence of X is everywhere 0. [Here divergence is the classical operator that takes a vector field with components f, g, h to the function $f_x + g_y + h_z$, in the usual partial derivative notation $f_x = x$ -partial of f , etc.]
 - 3.1 Explain some systematic reason why there is a closed 2-form on $\mathbb{R}^3 \setminus \{(0, 0, 0)\}$ (Euclidean 3-space with one point removed) that is not exact. You may do this by exhibiting such a form explicitly and checking that it is closed but not exact or you may argue using theorems that such a form must exist.
 - 3.2 With φ such a form (as in part 3.1), discuss why, for any smooth map of S^2 to itself, such a number

$$\left(\int_{S^2} F^* \varphi \right) / \left(\int_{S^2} \varphi \right)$$

is the degree of f . (Note: this includes explaining why the denominator integral cannot be 0.).

- 4 Show without using de Rham's Theorem (but you may use the Poincare Lemma without proof), that a 2-form φ on the 2-sphere S^2 that has integral 0 is exact, i.e., equal to $d\omega$ for some 1-form ω on S^2 .
- 5 Suppose that $V : U \rightarrow S^2$ is a smooth map, considered as a vector field of unit vectors, where $U = \mathbb{R}^3$ with a finite number of points p_1, \dots, p_n removed, all these points lying strictly inside the unit sphere S^2 .

Explain carefully, from basic facts about critical values and critical points and the like, why the degree of $V|_{S^2} : S^2 \rightarrow S^2$ is equal to the sum of the indices of the vector field V at the points p_1, \dots, p_n .

- 6.1 Explain what a short exact sequence of chain complexes is.
- 6.2 Describe how a short exact sequence of chain complexes gives rise to a long exact sequence in homology. Include how the connecting homomorphism (where the dimension changes) arises. You do not need to prove exactness of this sequence.
- 7.1 Define complex projective space $\mathbb{C}P^n$, $n = 1, 2, 3, \dots$
- 7.2 Compute the homology and cohomology of $\mathbb{C}P^n$, \mathbb{Z} coefficients. (Any method is allowed. Cell complexes are particularly simple to use. Be sure to explain what the attaching maps are if you adopt this approach).
- 8.1 Find the \mathbb{Z} -coefficient homology of $\mathbb{R}P^2$ by any systematic method.
- 8.2 Explain explicitly (not using the Künneth Theorem) how a nonzero element of the 3-homology with \mathbb{Z} -coefficients of $\mathbb{R}P^2 \times \mathbb{R}P^2$ arises.
- 9.1 State the Lefschetz Fixed Point Theorem.
- 9.2 Show that the Lefschetz number of any map from $\mathbb{C}P^{2n}$ to itself is nonzero and hence that every map from $\mathbb{C}P^{2n}$ to itself has a fixed point. (Hint: The action of the map on cohomology with \mathbb{Z} coefficients is determined by what happens to the 2nd cohomology since the whole cohomology ring is generated by the 2nd cohomology.)
- 10 Compute explicitly the simplicial homology, \mathbb{Z} coefficients, of the surface of a tetrahedron, thus obtaining the homology of the 2-sphere.