

- 1 Suppose $P(x, y, z)$, $Q(x, y, z)$, and $R(x, y, z)$ are C^∞ functions on \mathbb{R}^3 which vanish identically if $|x| \geq 5$, $|y| \geq 5$, or $|z| \geq 5$. Prove that the volume integral

$$\int_{-6}^6 \int_{-6}^6 \int_{-6}^6 d(Pdy \wedge dz + Qdx \wedge dz + Rdx \wedge dy) = 0$$

(Do this directly, not by quoting Stokes' Theorem: this is a special case of the proof of Stokes' Theorem!)

- 2 Suppose that $V = P(x, y, z)\frac{\partial}{\partial x} + Q(x, y, z)\frac{\partial}{\partial y} + R(x, y, z)\frac{\partial}{\partial z}$ is a C^∞ vector field on \mathbb{R}^3 with $V \neq \vec{0}$ at the origin. Find a necessary and sufficient condition for there to exist a C^∞ function $\lambda(x, y, z)$ in some neighborhood of the origin such that λV is the gradient of a C^∞ function on the neighborhood.
- 3 Let $T_t : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be the right-hand rule rotation around the positive z -axis by t degrees and $S_s : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be the right-hand rule rotation around the positive x -axis by t degrees.
- 3.1 Find the infinitesimal generators of the flows T_t and S_t , i.e., the vector fields X and Y , respectively, on \mathbb{R}^3 whose flows are $\{T_t\}$ and $\{S_t\}$.
- 3.2 Compute the commutator

$$T_{-t} \circ S_{-t} \circ T_t \circ S_t.$$

- 3.3 Compare the result of 3.2 (lowest order non-identically zero term) with the Lie bracket $[X, Y]$.
- 4 Take as given that a C^∞ 2-form ω on S^2 is of the form $d\theta$ for some C^∞ 1-form θ if and only if $\int_{S^2} \omega = 0$. Use this to show that every C^∞ 2-form Ω on $\mathbb{R}P^2$ has the form $d\Lambda$ for some C^∞ 1-form Λ . (Do not just quote DeRham's Theorem here.)

- 5.1 Suppose $F : S^1 \rightarrow \mathbb{R}^3$ is a C^∞ function such that dF is nowhere zero (on S^1). Prove that there is a two-dimensional subspace P of \mathbb{R}^3 such that $\pi_P \circ F : S^1 \rightarrow \mathbb{R}^3$ has nowhere vanishing differential, where $\pi_P =$ orthogonal projection on P .
- 5.2 Show by example a picture with explanation is all right) that there is such an F that is also 1 to 1 (injective) but is such that, for all P , $\pi_P \circ F$ fails to be injective.
- 5.3 Show that if $F : S^1 \rightarrow \mathbb{R}^4$ is C^∞ and injective then there is a three-dimensional subspace H of \mathbb{R}^4 such that $\pi_H \circ F$ is injective, where $\pi_H =$ orthogonal projection on H .
- 6.1 Suppose $F : S^n \rightarrow S^n$ is fixed-point free (i.e. for all $p \in S^n$, $p \neq F(p)$). Show that F is homotopic to the antipodal map $p \rightarrow -p$, $p \in S^n$.
- 6.2 Use part 6.1 to show that every vector field on (tangent to) S^{2n} , $n = 1, 2, 3, \dots$ vanishes somewhere on S^{2n} (i.e. has a zero).
- 7.1 Discuss carefully how to obtain the long exact sequence in homology from a short exact sequence of chain complexes. (Include definitions of the maps in the long exact sequence.)
- 7.2 If the short exact sequence is

$$0 \rightarrow C_1 \rightarrow C_2 \rightarrow C_3 \rightarrow 0,$$

prove exactness of the long exact sequence at $H_k(C_3)$ [in $\dots H_k(C_2) \rightarrow H_k(C_3) \rightarrow H_{k-1}(C_1) \dots$]

- 8.1 Suppose $F : T^2 \rightarrow T^2$ (where $T^2 = S^1 \times S^1$ is a continuous function such that $F(p) = p$ for some $p \in T^2$ and

$$F_* : \pi_1(T^2, p) \rightarrow \pi_1(T^2, p)$$

is the identity map. Is F necessarily homotopic to the identity map from T^2 to itself?

- 8.2 Is a C^∞ map $F : T^2 \rightarrow T^2$ of degree 1 necessarily homotopic to the identity map of T^2 to itself? Explain / prove your answer.
- 9.1 Discuss the (a) representation of $\mathbb{C}P^n$ as a simplicial complex.
- 9.2 Use part (a) to find the homology of $\mathbb{C}P^n$: prove carefully that your calculation is correct.
- 10.1 Let $X =$ the space obtained by attaching two discs to S^1 , the first disc being attached by $S^1 = \partial D_1 \rightarrow S^1$ being the 7 times around (counterclockwise) map, e.g., $z \mapsto z^7$, $|z| = 1$, $z \in \mathbb{C}$ and the second being attached by $S^1 = \partial D_2 \rightarrow S^1$ being the 5 times around map $z \mapsto z^5$. Find the homology of X .
- 10.2 Can X be made a C^∞ manifold? Why or why not?