1 Let M be a connected smooth manifold. Show that for any two non-zero tangent vectors v_1 at point x_1 and v_2 at a point x_2, there is a diffeomorphism $\phi : M \to M$ such that $\phi(x_1) = x_2$ and $d\phi(v_1) = v_2$.

2 Let X and Y be submanifolds of $\mathbb{R}^n$. Prove that for almost ever $a \in \mathbb{R}^n$, the translate $X + a$ intersects Y transversely.

3 Let $M_{n\times n}(\mathbb{R}) \simeq \mathbb{R}^{n^2}$ be the space of $n \times n$ matrices with real coefficients.

3.1 Show that

$$SL(n, \mathbb{R}) = \{A \in M_{n\times n}(\mathbb{R}) \mid \det(A) = 1\}$$

is a smooth submanifold of $M_{n\times n}(\mathbb{R})$.

3.2 Identify the tangent space to $SL(n, \mathbb{R})$ at the identity matrix I_n.

3.3 Show that $SL(n, \mathbb{R})$ has trivial Euler characteristic.

4.1 Let $f_i : M \to N$, $i = 0, 1$, be two smooth maps between smooth manifolds M and N, and $f_i^* : \Omega^*(N) \to \Omega^*(M)$, $i = 0, 1$, be the induced chain maps between the respective de Rham complexes. Define the notion of chain homotopy between f_0^* and f_1^*. Here the co-boundary operators on the de Rham complexes are the exterior derivatives.

4.2 Let X be a smooth vector field on a compact smooth manifold M, and let $\phi_t : M \to M$ be the flow generated by X at time t, i.e. the solution of the differential equation $\frac{d\phi_t}{dt}(x) = X(\phi_t(x))$ with initial condition $\phi_0(x) = x$. Find an explicit chain homotopy between the chain maps ϕ_0^* and ϕ_1^*, where ϕ_i^*, $i = 0, 1$, are the induced chain maps from $\Omega^*(M)$ to itself.
(Hint: Use the formula that for any differential form ω and vector field X, the Lie derivative $\mathcal{L}_X\omega = d \circ i_X\omega + i_X \circ d\omega$. Here i_X is the contraction with respect to X.)

5 Let $\omega = dx_1 \wedge dx_2 + dx_3 \wedge dx_4 + \cdots + dx_{2n-1} \wedge dx_{2n}$ be a 2-form on $\mathbb{R}^{2n}$, where $(x_1, x_2, \ldots, x_{2n})$ are the standard coordinates on $\mathbb{R}^{2n}$. Define an S^1-action on $\mathbb{R}^{2n}$ as follows: for each $t \in S^1$, define $g_t : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ by considering $\mathbb{R}^{2n}$ as the direct sum of n copies of $\mathbb{R}^2$ and rotating each $\mathbb{R}^2$ summand an angle t. Let X be the vector field on $\mathbb{R}^{2n}$ defined by $X(x) = \frac{dg_t(x)}{dt}|_{t=0}$ for any $x \in \mathbb{R}^{2n}$.