

- 1.1 Show that the Lie group  $SL_2(\mathbb{R}) = \{A \in M_{2 \times 2}(\mathbb{R}) \mid \det(A) = 1\}$  is diffeomorphic to  $S^1 \times \mathbb{R}^2$ .
- 1.2 Show that the Lie group  $SL_2(\mathbb{C}) = \{A \in M_{2 \times 2}(\mathbb{C}) \mid \det(A) = 1\}$  is diffeomorphic to  $S^3 \times \mathbb{R}^3$ .
- 2 For  $n \geq 1$ , construct an everywhere non-vanishing smooth vector field on the odd-dimensional real projective space  $\mathbb{R}P^{2n-1}$ .
- 3 Let  $M^m \subset \mathbb{R}^n$  be a smooth submanifold of dimension  $m < n - 2$ . Show that its complement  $\mathbb{R}^n \setminus M$  is connected and simply connected.
- 4.1 Show that for any  $n \geq 1$  and  $k \in \mathbb{Z}$ , there exists a continuous map  $f : S^n \rightarrow S^n$  of degree  $k$ .
- 4.2 Let  $X$  be a compact, oriented  $n$ -dimensional manifold. Show that for any  $k \in \mathbb{Z}$ , there exists a continuous map  $f : X \rightarrow S^n$  of degree  $k$ .
- 5 Assume that  $\Delta = \{X_1, \dots, X_k\}$  is a  $k$ -dimensional distribution spanned by vector fields on an open set  $\Omega \subset M^n$  in an  $n$ -dimensional manifold. For each open subset  $V \subset \Omega$  define

$$\mathcal{Z}_V = \{u \in C^\infty(V) \mid X_1 u = 0, \dots, X_k u = 0\}$$

Show that the following two statements are equivalent:

- 5.1 The distribution  $\Delta$  is integrable.
- 5.2 For each  $x \in \Omega$  there exists an open neighborhood  $x \in V \subset \Omega$  and  $n - k$  functions  $u_1, \dots, u_{n-k} \in \mathcal{Z}_V$  such that the differentials  $du_1, \dots, du_{n-k}$  are linearly independent at each point in  $V$ .
- 6 On  $\mathbb{R}^n \setminus \{0\}$  define the  $(n - 1)$ -forms

$$\sigma = \sum_{i=1}^n (-1)^{i-1} x^i dx^1 \wedge \dots \wedge \widehat{dx^i} \wedge \dots \wedge dx^n$$

$$\omega = \frac{1}{|x|^n} \sum_{i=1}^n (-1)^{i-1} x^i dx^1 \wedge \dots \wedge \widehat{dx^i} \wedge \dots \wedge dx^n$$

- 6.1 Show that  $\omega = r^* \circ i^*(\sigma)$ , where  $i : S^{n-1} \rightarrow \mathbb{R}^n \setminus \{0\}$  is the natural inclusion of the unit sphere and  $r(x) = \frac{x}{|x|} : \mathbb{R}^n \setminus \{0\} \rightarrow S^{n-1}$  the natural retraction.
- 6.2 Show that  $\sigma$  is not a closed form.
- 6.3 Show that  $\omega$  is a closed form that is not exact.
- 7 Let  $n \geq 0$  be an integer. Let  $M$  be a compact, orientable, smooth manifold of dimension  $4n + 2$ . Show that  $\dim H^{2n+1}(M; \mathbb{R})$  is even.
- 8 Show that there is no compact three-dimensional manifold  $M$  whose boundary is the real projective space  $\mathbb{R}P^2$ .
- 9 Consider the coordinate axes in  $\mathbb{R}^n$ :

$$L_i = \{(x_1, \dots, x_n) \mid x_j = 0 \text{ for all } j \neq i\}$$

Calculate the homology groups of the complement  $\mathbb{R}^n \setminus (L_1 \cup \dots \cup L_n)$ .

- 10.1 Let  $X$  be a finite CW complex. Explain how the homology groups of  $X$  are related to the homology groups of  $X \times S^1$ .
- 10.2 For each integer  $n \geq 0$ , give an example of a compact smooth manifold of dimension  $2n + 1$  such that  $H_i(X) = \mathbb{Z}$  for all  $i = 0, \dots, 2n + 1$ .