Homework

Frederick Robinson

8 March 2010

Contents

1 Problem 11.3

1.1 Question

Redo the consumers multiperiod utility maximization problem of Section 11.6 with $u(c) = c^{\frac{1}{3}}$. (cf. 11: 1a)

1.2 Answer

As in Section 11.6 we have a consumer faced with a T-period planning horizon. where T is a finite positive integer. He has initial wealth $w \in \mathbb{R}_+$. We say that he begins period-t with a wealth w_t, and consumes c_t $(w_t \geq c_t \geq 0)$ and begins the the next period with $(w_t - c_t)(1 + r)$, where $r \geq 0$ is the interest rate.

In unlike in the example however we have that the consumption of c in any period give a utility of $u(c) = c^{\frac{1}{3}}$.

For the purposes of this problem the state space S will be the set of possible wealth levels ($\mathbb{R}_+$) and the action space A will be the set of consumption levels (again $\mathbb{R}_+$). The reward function is just $r_t(w,c) = u(c) = c^{\frac{1}{3}}$. The transition function f_t is $f_t(w,c) = (w-c)(1+r)$. Finally we have the feasible action correspondence $\Phi_t(w) = [0,w]$ for all w.

We shall let $k = (1+r)$ as in the book for ease of notation. In the last period, given a wealth level w the consumer solves:

$$\max_{c\in[0,w]} u(c).$$

But, clearly since $u(c)$ is strictly increasing the the unique solution for any given w is to consume everything. hence we conclude that the (unique) optimal strategy for one time period is just $g_T(w) = w$ for all $w \in S$ and the one period value function is given by

$$V_T(w) = w^{\frac{1}{3}}, \quad w \in S.$$

Now we consider the case of two periods. In this case we need to maximize the reward function by solving

$$\max_{c\in[0,w]} \{c^{\frac{1}{3}} + (k(w-c))^{\frac{1}{3}}\}.$$

Since this optimization problem is strictly convex we can just employ first order conditions. Thus we see we must have

$$\frac{\partial}{\partial c}\left(c^{1/3} + (k(-c+w))^{1/3}\right) = \frac{1}{3c^{2/3}} - \frac{k}{3(k(-c+w))^{2/3}} = 0$$

$$\Rightarrow c = \frac{-w-\sqrt{k}w}{-1+k} \quad \text{or} \quad c = \frac{-w+\sqrt{k}w}{-1+k}$$

however this first solution may be discarded since for w, k in the appropriate ranges, this solution corresponds to negative c. So, the corresponding value function is just

$$V(w) = \left(\frac{w}{1+\sqrt{k}}\right)^{1/3}\left(1+\sqrt{k}\right) = w^{1/3}\left(1+\sqrt{k}\right)^{2/3}$$

Now we can compute again for the three-period problem. In this case we must solve the following optimization problem.

$$\max_{c\in[0,w]} \left\{ \underbrace{\left(k^2(w-c)\right)^{1/3}}_{\text{Period 3}} + \underbrace{\left(\frac{c}{1+\sqrt{k}}\right)^{1/3}\left(1+\sqrt{k}\right)}_{\text{Periods 1 \& 2}} \right\}.$$

again applying first order conditions we see that

$$\frac{\partial}{\partial c}\left(\left(\frac{c}{1+\sqrt{k}}\right)^{1/3}\left(1+\sqrt{k}\right)+\left(k^2(w-c)\right)^{1/3}\right)$$

$$=\frac{1}{3\left(\frac{c}{1+\sqrt{k}}\right)^{2/3}}-\frac{k^2}{3\left(k^2(-c+w)\right)^{2/3}}=0$$

$$\Rightarrow c=\frac{w+\sqrt{k}w}{1+\sqrt{k}+k}$$

and

$$\begin{aligned}
V(w) &= \left(k^2(w-\frac{w+\sqrt{k}w}{1+\sqrt{k}+k})\right)^{1/3}+\left(\frac{w+\sqrt{k}w}{(1+\sqrt{k}+k)(1+\sqrt{k})}\right)^{1/3}\left(1+\sqrt{k}\right)\\
&= \left(k^2(w-\frac{w+\sqrt{k}w}{1+\sqrt{k}+k})\right)^{1/3}+\left(\frac{(w+\sqrt{k}w)(1+\sqrt{k})^2}{(1+\sqrt{k}+k)}\right)^{1/3}\\
&= \left(\frac{\left(1+\sqrt{k}\right)^3 w}{1+\sqrt{k}+k}\right)^{1/3}+\left(\frac{k^3w}{1+\sqrt{k}+k}\right)^{1/3}\\
&= \frac{\left(1+\sqrt{k}\right)w^{1/3}+kw^{1/3}}{(1+\sqrt{k}+k)^{1/3}}=\left(1+\sqrt{k}+k\right)^{2/3}w^{1/3}
\end{aligned}$$

So, it seems likely that in general we have the relations

$$V_T(w)=\left(1+\sqrt{k}+\cdots+k^{T/2}\right)^{2/3}w^{1/3}$$

and

$$g_T(w)=w\left(\frac{1+\sqrt{k}+\cdots+k^{(T-1)/2}}{1+\sqrt{k}+\cdots+k^{T/2}}\right)$$

We will use induction to prove this relation.

If we assume that the above is true for $T=n$ we can compute the case corresponding to $T=n+1$ fairly by the following optimization:

$$\max_{c\in[0,w]}\left(\left((w-c)k^{n+1}\right)^{1/3}+\left(1+\sqrt{k}+\cdots+k^{n/2}\right)^{2/3}c^{1/3}\right)$$

again, as the above is a convex optimization we can just apply first order conditions.

$$\frac{\partial}{\partial c}\left(\left((w-c)k^{n+1}\right)^{1/3}+\left(1+\sqrt{k}+\cdots+k^{n/2}\right)^{2/3}c^{1/3}\right)$$

$$=-\frac{k^{1+n}}{3\left(k^{1+n}(-c+w)\right)^{2/3}}+\left(1+\sqrt{k}+\cdots+k^{n/2}\right)^{2/3}\frac{1}{3c^{2/3}}=0$$

$$\Rightarrow -\frac{k^{1+n}}{\left(k^{1+n}(-c+w)\right)^{2/3}}+\left(\frac{1+\sqrt{k}+\cdots+k^{n/2}}{c}\right)^{2/3}=0$$

$$\Rightarrow -\frac{k^{\frac{1}{3}(1+n)}}{(w-c)^{2/3}}+\left(\frac{1+\sqrt{k}+\cdots+k^{n/2}}{c}\right)^{2/3}=0$$

$$\Rightarrow \frac{k^{\frac{1}{2}(1+n)}}{w-c}=\frac{1+\sqrt{k}+\cdots+k^{n/2}}{c}$$

$$\Rightarrow \frac{w-c}{c}=\frac{k^{\frac{1}{2}(1+n)}}{1+\sqrt{k}+\cdots+k^{n/2}}$$

$$\Rightarrow \frac{w}{c}=\frac{k^{\frac{1}{2}(1+n)}}{1+\sqrt{k}+\cdots+k^{n/2}}+\frac{1+\sqrt{k}+\cdots+k^{n/2}}{1+\sqrt{k}+\cdots+k^{n/2}}$$

$$\Rightarrow \frac{w}{c}=\frac{1+\sqrt{k}+\cdots+k^{n/2}+k^{(n+1)/2}}{1+\sqrt{k}+\cdots+k^{n/2}}$$

$$\Rightarrow c=w\left(\frac{1+\sqrt{k}+\cdots+k^{n/2}}{1+\sqrt{k}+\cdots+k^{n/2}+k^{(n+1)/2}}\right)$$

This agrees with what we expect for $g_{n+1}(w)$. Finally we substitute to check $V_{n+1}(w)$

$$\left((w-c)k^{n+1}\right)^{1/3}+\left(1+\sqrt{k}+\cdots+k^{n/2}\right)^{2/3}c^{1/3}$$

$$=\left(\left(w-w\left(\frac{1+\sqrt{k}+\cdots+k^{n/2}}{1+\sqrt{k}+\cdots+k^{n/2}+k^{(n+1)/2}}\right)\right)k^{n+1}\right)^{1/3}+$$

$$\left(1+\sqrt{k}+\cdots+k^{n/2}\right)^{2/3}w^{1/3}\left(\frac{1+\sqrt{k}+\cdots+k^{n/2}}{1+\sqrt{k}+\cdots+k^{n/2}+k^{(n+1)/2}}\right)^{1/3}$$

$$= \left(w \left(1 - \frac{1 + \sqrt{k} + \cdots + k^{n/2}}{1 + \sqrt{k} + \cdots + k^{n/2} + k^{(n+1)/2}} \right) k^{n+1} \right)^{1/3} +$$

$$w^{1/3} \frac{1 + \sqrt{k} + \cdots + k^{n/2}}{\left(1 + \sqrt{k} + \cdots + k^{n/2} + k^{(n+1)/2}\right)^{1/3}}$$

$$= \left(w \left(\frac{1 + \sqrt{k} + \cdots + k^{n/2} + k^{(n+1)/2}}{1 + \sqrt{k} + \cdots + k^{n/2} + k^{(n+1)/2}} - \frac{1 + \sqrt{k} + \cdots + k^{n/2}}{1 + \sqrt{k} + \cdots + k^{n/2} + k^{(n+1)/2}} \right) k^{n+1} \right)^{1/3} +$$

$$w^{1/3} \frac{1 + \sqrt{k} + \cdots + k^{n/2}}{\left(1 + \sqrt{k} + \cdots + k^{n/2} + k^{(n+1)/2}\right)^{1/3}}$$

$$= \left(w \frac{k^{3(n+1)/2}}{1 + \sqrt{k} + \cdots + k^{n/2} + k^{(n+1)/2}} \right)^{1/3} + w^{1/3} \frac{1 + \sqrt{k} + \cdots + k^{n/2}}{\left(1 + \sqrt{k} + \cdots + k^{n/2} + k^{(n+1)/2}\right)^{1/3}}$$

$$= w^{1/3} \left(\frac{1 + \sqrt{k} + \cdots + k^{n/2} + k^{(n+1)/2}}{\left(1 + \sqrt{k} + \cdots + k^{n/2} + k^{(n+1)/2}\right)^{1/3}} \right)$$

$$= w^{1/3} \left(1 + \sqrt{k} + \cdots + k^{n/2} + k^{(n+1)/2}\right)^{2/3}$$

which agrees with our prediction for $V_{n+1}(w)$. Thus, we are done with the inductive step and we have confirmed that

$$V_T(w) = \left(1 + \sqrt{k} + \cdots + k^{T/2}\right)^{2/3} w^{1/3}$$

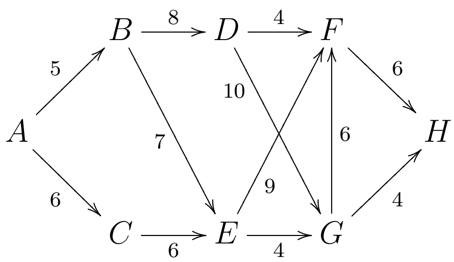
and

$$g_T(w) = w \left(\frac{1 + \sqrt{k} + \cdots + k^{(T-1)/2}}{1 + \sqrt{k} + \cdots + k^{T/2}} \right).$$

2 Problem 11.4

2.1 Question

Determine the longest path from A to H in the network in the following figure. The arrows indicate which direction an edge can be traversed.



2.2 Answer

This problem is small enough that it can be solved by brute force. Towards this we shall list all possible such paths and their lengths.

Path	Length	Sum
$A \to B \to D \to F \to H$	$5 + 8 + 4 + 6$	23
$A \to B \to D \to G \to F \to H$	$5 + 8 + 10 + 6 + 6$	35
$A \to B \to D \to G \to H$	$5 + 8 + 10 + 4$	27
$A \to B \to E \to F \to H$	$5 + 7 + 9 + 6$	27
$A \to B \to E \to G \to F \to H$	$5 + 7 + 4 + 6 + 6$	28
$A \to B \to E \to G \to H$	$5 + 7 + 4 + 4$	20
$A \to C \to E \to F \to H$	$6 + 6 + 9 + 6$	27
$A \to C \to E \to G \to F \to H$	$6 + 6 + 4 + 6 + 6$	28
$A \to C \to E \to G \to H$	$6 + 6 + 4 + 4$	20

Now we can reorder these by total length.

Path	Length	Sum
$A \to B \to D \to G \to F \to H$	$5 + 8 + 10 + 6 + 6$	35
$A \to C \to E \to G \to F \to H$	$6 + 6 + 4 + 6 + 6$	28
$A \to B \to E \to G \to F \to H$	$5 + 7 + 4 + 6 + 6$	28
$A \to C \to E \to F \to H$	$6 + 6 + 9 + 6$	27
$A \to B \to D \to G \to H$	$5 + 8 + 10 + 4$	27
$A \to B \to E \to F \to H$	$5 + 7 + 9 + 6$	27
$A \to B \to D \to F \to H$	$5 + 8 + 4 + 6$	23
$A \to C \to E \to G \to H$	$6 + 6 + 4 + 4$	20
$A \to B \to E \to G \to H$	$5 + 7 + 4 + 4$	20

In particular note that the path $A \to B \to D \to G \to F \to H$ has the maximum total length, with a length of 35.