# Homework

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# 1 Chapter 3 Section 4

## 1.1 Problem 23

### 1.1.1 Question

Prove that $[0,1) \times [0,1)$ is homeomorphic to $[0,1] \times [0,1)$.

### 1.1.2 Answer

We define a mapping by

$$f(x,y) = \begin{cases} (.5x, 2y-1) & y \geq .5x + .5 \\ (x-y+.5, x) & y < .5x + .5 \text{ and } y \geq x \\ (x+y-.5, y) & y < x \text{ and } y \geq 2x-1 \\ (-.5y+1, 2x-1) & y < 2x-1 \end{cases}$$

Since this is a composition of affine linear mappings which agree on the edges it is continuous with continuous inverse. Moreover it is easy to see that it is bijective.

## 1.2 Problem 25

### 1.2.1 Question

Show that the diagonal map $\Delta : X \to X \times X$ defined by $\Delta(x) = (x,x)$ is indeed a map, and check that $X$ is Hausdorff if and only if $\Delta(X)$ is closed in $X \times X$.

### 1.2.2 Answer

We check that $\Delta$ is indeed a map first. Towards this end let $Y$ be an open set in the image of $X$ ($\Delta(X)$). Since the projection map is open as has been proven previously (Theorem 3.12), and the projection map just generates the preimage of an given set in image of $\Delta$ it must be that $\Delta$ is continuous as desired.

*Proof.* Suppose that $\Delta(X)$ is closed in $X \times X$. We wish to show that $X$ is Hausdorff. Since $\Delta(X)$ is closed in $X \times X$ it must be that every point not of the form $(x, x) \in (X, X)$ is not a limit point of $\Delta(X)$ thus, in particular each such point can be separated from every point of form $(x, x)$ by open sets and in particular basis sets (as each open set is the union of basis sets). This however implies that $X$ is Hausdorff. Since, if any $(x, x)$ and $(x, y)$ can be separated by sets of the form $A \times B$ and $C \times D$ then any $x, y \in X$ can be separated by $B$ and $D$. □

*Proof.* Suppose that $X$ is Hausdorff. We wish to show that $\Delta(X)$ is closed in $X \times X$. Since $X$ is Hausdorff any points $x, y \in X$ can be separated by open sets say $A$ and $B$. So, any points in the product space say $(x, y)$ and $(x, x)$ can be separated by (open) basis set as $A \times B$ and $A \times A$.

Thus, $\Delta(X)$ is closed since it has no limit points not contained in itself. □

# 2 Chapter 3 Section 5

## 2.1 Problem 30

### 2.1.1 Question

Let $X$ be the set of all points in the plane which have at least one rational coordinate. Show that $X$, with the induced topology, is a connected space.

### 2.1.2 Answer

It is clear that each $x \in X$ is path connected to the origin. If $x = (x_1, x_2)$ with $x_1 \in \mathbb{Q}$ we have in particular

$$\varphi(z) = \begin{cases} (x_1 - 2z(x_1), x_2) & z \leq .5 \\ (0, x_2 - 2(z - .5)x_2) & z > .5 \end{cases}$$

Similarly if $x_2 \in \mathbb{Q}$. This is a linear (and hence continuous) map from $[0, 1] \to X$ which satisfies the requirements for a path.

Therefore, each $x \in X$ is path connected to each other $x' \in X$ (if we have a path from $x$ to the origin and one from $x'$ to the origin we can just reverse teh latter, scale each by $(1/2)$ and adjoin them to get a path from $x$ to $x'$). Since path connectedness implies connectedness we are done.