Homework

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1 Chapter 5 Section 1

1.1 Problem 1

1.1.1 Question

Let C denote the unit circle in the plane. Suppose $f : C \to C$ is a map which is not homotopic to the identity. Prove that $f(x) = -x$ for some point x of C.

1.1.2 Answer

Proof. Let $f : C \to C$ be a map such that $f(x) \neq -x$ for any point x of C. We shall demonstrate that f is homotopic to the identity. But notice that this is just a special case of Example 2 Page 89 which states that if $f, g : X \to S^n$ are maps which never give a pair of antipodal points when evaluated at a point of X then f and g are homotopic. Since we have that $f(x) \neq -x$ for every point x of C this assumption is satisfied if we take $g(x) = x$ to be the identity function so we are done. □

1.2 Problem 5

1.2.1 Question

Let $f : X \to S^n$ be a map which is not onto. Prove that f is *null homotopic*, that is to say f is homotopic to a map which takes all of X to a single point of S^n

1.2.2 Answer

Proof. Since f is not surjective there exists some point say y of S^n such that $f(x) \neq y$ for any choice of x. Take z to be a point antipodal from y. Now take $g : X \to S^n$ defined by $g(x) = z$ to be a constant function. Now the assumptions of Example 2 are satisfied for f and g so, they are homotopic. Thus, f is null homotopic as desired. □

1.3 Problem 6

1.3.1 Question

As usual, CY denotes the cone on Y. Show that any two maps $f, g : X \to CY$ are homotopic.

1.3.2 Answer

Proof. Any map $f : X \to CY$ is homotopic to the constant map $g : X \to CY$ defined by $g(x) = (x, 1)$ (that is the 'tip' of the cone) via the straight line homotopy. It is useful to think of f as two functions $f_x : X \to Y$ and $f_i : X \to [0, 1]$. With this we express our homotopy F by

$$F(x,t) = (f_x(x), f_i(x)(1-t) + t)$$

Since homotopy is an equivalence relation this suffices to prove that any two maps $f : X \to CY$ are homotopic. □

1.4 Problem 7

1.4.1 Question

Show that a map from X to Y is null homotopic if and only if it extends to a map from the cone on X to Y.

1.4.2 Answer

Proof. If a map $f : X \to Y$ extends to a map say $F : CX \to Y$ then f is null homotopic. Recall that we defined $CX = X \times [0, 1]$ modulo the equivalence relation which associates all points of the form $(x, 1)$. So, if we define $F' : X \to Y \times [0, 1]$ by

$$F'(x,t) = F(x,t)$$

we know that F' is a map (continuous) by the continuity of F. Moreover, we know that $F'(x, 1) = c$ for some constant c since $F(x, 1) = c$ for any x (indeed all choices of x are the same in the cone). So, we have constructed a homotopy that takes f to a point, and f is null homotopic.

Conversely let $f : X \to Y$ be null homotopic. Then there exists some homotopy say $F : X \times [0, 1] \to Y$ which takes f to a point. Then, we can extend f to F' by

$$F'(x,t) = F(x,t)$$

which is necessarily well defined since, as F is a null homotopy $F(x, 1)$ gives the same result regardless of x. Moreover F' is continuous, since F is. Thus, we have extended f to CX as desired. □