Homework

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1 Chapter 8 Section 3

1.1 Problem 11

1.1.1 Question

Calculate the homology groups of the following complexes: (a) three copies of the boundary of a triangle all joined together at a vertex; (b) two hollow tetrahedra glued together along an edge; (c) a complex whose polyhedron is homeomorphic to the Möbius strip; (d) a complex which triangulates the cylinder.

1.1.2 Answer

1.2 Problem 16

1.2.1 Question

Calculate the homology group of a triangulation of the sphere with k holes.

1.2.2 Answer

2 Chapter 8 Section 5

2.1 Problem 20

2.1.1 Question

Prove lemma (8.5)

If $\psi : C(L) \to C(M)$ is a second chain map then $\psi \circ \phi : C(K) \to C(M)$ is a chain map and $(\psi \circ \phi)_* = \psi_* \circ \phi_* : H_q(K) \to H_q(M)$.

2.1.2 Answer

The following diagrams commute for all q.

$$\begin{tikzcd} C_q(K) \arrow[r, "\phi_q"] \arrow[d, "\partial"'] & C_q(L) \arrow[d, "\partial"'] \\ C_{q-1}(K) \arrow[r, "\phi_{q-1}"] & C_{q-1}(L) \end{tikzcd} \qquad \begin{tikzcd} C_q(L) \arrow[r, "\psi_q"] \arrow[d, "\partial"'] & C_q(M) \arrow[d, "\partial"'] \\ C_{q-1}(L) \arrow[r, "\psi_{q-1}"] & C_{q-1}(M) \end{tikzcd}$$

Therefore, the following diagram commutes, again for all q.

$$\begin{tikzcd} C_q(K) \arrow[r, "\phi_q"] \arrow[d, "\partial"'] & C_q(L) \arrow[r, "\psi_q"] \arrow[d, "\partial"'] & C_q(M) \arrow[d, "\partial"'] \\ C_{q-1}(K) \arrow[r, "\phi_{q-1}"] & C_{q-1}(L) \arrow[r, "\psi_{q-1}"] & C_{q-1}(M) \end{tikzcd}$$

Rewriting we have what we wanted to prove

$$\begin{tikzcd} C_q(K) \arrow[r, "\psi_q \circ \phi_q"] \arrow[d, "\partial"'] & C_q(M) \arrow[d, "\partial"'] \\ C_{q-1}(K) \arrow[r, "\psi_{q-1} \circ \phi_{q-1}"] & C_{q-1}(M) \end{tikzcd}$$

Now

$$\begin{aligned}
(\psi \circ \phi)_* H_q(K) &= (\psi \circ \phi)_* \ker \partial C_q(K)/\mathrm{Im}\ \partial C_{q+1}(K) \\
&= \ker \partial(\psi \circ \phi) C_q(K)/\mathrm{Im}\ \partial(\psi \circ \phi) C_{q+1}(K) \\
&= \ker \partial\psi C_q(K)/\mathrm{Im}\ \partial\psi C_{q+1}(K) \circ \ker \partial\phi C_q(K)/\mathrm{Im}\ \partial\phi C_{q+1}(K) \\
&= (\psi_* \circ \phi_*) H_q(K)
\end{aligned}$$

2.2 Problem 21

2.2.1 Question

Check that the subdivision map $\chi : C(K) \to C(K')$ is a chain map.

2.2.2 Answer

Let σ be a typical oriented q-simplex of K. In particular say $\sigma = (v_0, v_1, \ldots, v_k, v_{k+1}, \ldots, v_q)$. Now we compute

$$\chi_q(\sigma) = \sum_{i=0}^{k} (-1)^i (v, v_0, \ldots, \hat{v_i}, \ldots, v_k, v_k + 1, \ldots, v_q)$$

and

$$\partial\chi_q(\sigma) = \sum_{j=0}^{q}\sum_{i=0}^{k}(-1)^i(v, v_0, \ldots, \hat{v_j}, \ldots, \hat{v_i}, \ldots, v_k, v_k+1, \ldots, v_q).$$

Finally we compute

$$\partial(\sigma) = \sum_{j=0}^{q}(v_0, \ldots, \hat{v_j}, \ldots, v_k, v_k+1, \ldots, v_q)$$

and

$$\partial\chi_q(\sigma) = \sum_{i=0}^{k}\sum_{j=0}^{q}(-1)^i(v, v_0, \ldots, \hat{v_j}, \ldots, \hat{v_i}, \ldots, v_k, v_k+1, \ldots, v_q).$$

So interchanging the order of summation we see that $\partial\chi_q = \chi_{q-1}\partial$ as desired.