Homework

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1 Chapter 3 Section 5

1.1 Problem 31

1.1.1 Question

Given the set of real numbers and the finite-complement topology. What are the components of the resulting space? Answer the same question for the half-open interval topology.

1.1.2 Answer

In the finite complement topology the open sets are those which have finite complements. In this particular instance this means that the open sets are those which are of form $\mathbb{R}\backslash\{x_1, \ldots, x_n\}$ for some points $x_1, \ldots, x_n \in \mathbb{R}$.

So, there is but one connected component, namely the entire space. This follows in particular from the fact that two (nontrivial) open sets may not have empty intersection.

Proof. Assume $X \cap Y$ has empty intersection. However we know by definition of open sets under the finite complement topology that both X^C and Y^C are finite. However, this is a contradiction since X contains infinitely many points, yet, Y^C is finite. Hence, there must be some $y \in Y$ such that $y = x$ for some $x \in X$. □

I assert furthermore that the half-open interval topology on $\mathbb{R}$ has all real numbers (singeltons) as the only connected components.

Proof. Suppose towards a contradiction that there exists some connected component containing at least the two points $x \neq y \in \mathbb{R}$. Then, it must be that there are no two open sets X and Y satisfying a few properties.

Say without loss of generality that $x < y$

Now, consider the following open sets:

$$X = (x - 1, (x + y)/2)$$

$$Y = [(x + y)/2, y + 1)$$

It should be clear that these sets satisfy the requisite properties to separate x and y; In particular we see that the sets are disjoint, open, and nonempty.

Contradiction. $\square$

2 Chapter 4 Section 2

2.1 Problem 8

2.1.1 Question

Let X be a compact Hausdorff space. Show that the cone on X is homeomorphic to the one-point compactification of $X \times [0, 1)$. If A is closed in X, show that X/A is homeomorphic to the one-point compactification of $X - A$.

2.1.2 Answer

I claim that CX and the one point compactification of $X \times [0, 1)$ are homeomorphic. In particular, I will demonstrate that the following function is a homeomorphism:

$$\varphi : (X \times [0, 1])/ \equiv \to (X \times [0, 1))^*$$

(where $\equiv$ is the equivalance relation that associates each (x,1) together) with φ defined as $\varphi(x, t) = (x, t)$ for $t \neq 1$ and $\varphi(x, t) = \infty$ given $t = 1$.

We first prove that this mapping is continuous. Towards this end let Z be an open set in the codomain of φ. Then its preimage $\varphi^{-1}(Z)$ is open. There are two cases

Case 1: $\infty \notin Z$. This case is obvious, since in this restriction the mapping φ is just the identity mapping, and the topologies of the two spaces restricted to $(x, t) \neq \infty t \neq 1$ are exactly equivalent

Case 2: $\infty \in Z$. In this case we just have that $\varphi^{-1}(Z) = \varphi^{-1}(Z\backslash\{\infty\}) \cup \{(x, 1) \mid x \in X\}$. This set must be open however since we know that $Z\backslash\{\infty\}$ is open (Z^C is compact, thus closed) and $\{(x, 1) \mid x \in X\}$ is open since X is open.

Now that we have established that φ is a continuous function it remains only to show that it is bijective. It is clearly surjective since given x in the codomain there is always some x' such that $\varphi x' = x$. Take $x = x'$ if $x \neq \infty$. Otherwise just take $x' = (a, 1)$ for some a.

It is injective too since, assume $\varphi(x) = \varphi(y)$. If $\varphi(x) \neq \infty$ then we have $x = y$ since on this region φ is just the identity map. If $\varphi(x) = \varphi(y) = \infty$ then we have $x = (x_1, 1)$ $y = (x_2, 1)$ but these too are the same since the equivalence relation by which we mod out to get the cone states this explicitly.

For a closed set A we know that X/A is homeomorphic to the one point compactification of $X - A$ by the function φ which takes points in A to ∞. This is clearly bijective.

Take an open set in the codomain. The preimage of this set must be open too since, if it does not have $\infty \in U$ we are just dealing with the identity mapping. Otherwise note that there must be nonzero intersection between any open set containing ∞ and U since otherwise the complement of U would not be closed (any open set with ∞ has ∞ as a limit point.) Thus, U has open preimage even in the case that it does have ∞. The noninfinite component has open preimage as, by def of compactification its complement is compact (and therefore closed). Moreover we don't make the set nonopen by adding the preimage of ∞ since we have already shown that every open set containing ∞ is in U. Thus, it is "within" the set and can't make it nonopen.

More formally, we know that the preimage of ∞ is not the limit point of any set on the boundary.

2.2 Problem 11

2.2.1 Question

Show that the function $f : [0, 2\pi]\times[0, \pi] \to \mathbb{E}^5$ defined by $f(x, y) = (\cos x, \cos 2y, \sin 2y, \sin x \cos y, \sin x \sin y)$ induces an embedding of the Klein bottle in $\mathbb{E}^5$

2.2.2 Answer

First observe that for all y we have $f(0, y) = f(2\pi, y)$ since both evaluations yield $(1, \cos 2y, \sin 2y, 0, 0, 0)$. Moreover we see that for all x we get $f(x, 0) = f(x, 2\pi) = f(2\pi - x, 2\pi) = (\cos x, 1, 0, \sin x, \sin x, 0)$

Finally we must verify that for no pair of $(x_1, y_1), (x_2, y_2)$ that does not have $x = 0$ or $x = 2\pi$ or $y = 0$ or $y = 2\pi$ do we get $\varphi(x_1, y_1) = \varphi(x_2, y_2)$ and furthermore that the edges do not intersect except as they should.

$$(\cos x_1, \cos 2y_1, \sin 2y_1, \sin x_1 \cos y_1, \sin x_1 \sin y_1)$$

$$= (\cos x_2, \cos 2y_2, \sin 2y_2, \sin x_2 \cos y_2, \sin x_2 \sin y_2)$$

$$\Rightarrow (x = 0 \text{ or } x = 2\pi \text{ or } y = 0 \text{ or } y = 2\pi)$$

as desired.

That the edges of the square (image) do not map to intersections except as they should is obvious since for us to get $\varphi(x) = \varphi(y)$ we need in particular that the coordinates agree ($\cos x = 1$ and $\sin x = 0$), but as we checked previously this is not the case except in the corner, where it should be the case.