Homework

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1 Chapter 5 Section 3

1.1 Problem 20

1.1.1 Question

Let $\pi : X \to Y$ be a covering map, $p \in Y$, $q \in \pi^{-1}(p)$, and $F : I \times I \to Y$ a map such that $F(0,t) = F(1,t) = p$ for $0 \leq t \leq 1$. For each t in $[0,1]$ we have a path $F_t(s) = F(s,t)$ in Y which begins at p. Let $\tilde{F}_t$ be its unique lift to a path in X which begins at q, and set $\tilde{F}(s,t) = \tilde{F}_t(s)$. Check that $\tilde{F}$ is continuous and lifts F.

1.1.2 Answer

Proof. By the homotopy lifting lemma there exists a unique lift of the homotopy F say $\tilde{F}$. By properties of homotopy $\tilde{F}$ is both continuous and a lift of F. Moreover, this $\tilde{F}$ is precisely the $\tilde{F}$ as defined above since $\tilde{F}(s,t_0)$ is a lift of a path $F(s,t_0)$ going through q and therefore unique by the path lifting lemma. □

2 Chapter 10 Section 4

2.1 Problem 18

2.1.1 Question

If $\tilde{X}$ is a covering space of X, and $\tilde{Y}$ a covering space of Y, show that $\tilde{X} \times \tilde{Y}$ is a covering space of $X \times Y$.

2.1.2 Answer

Proof. Given a point $(x,y) \in X \times Y$ there exist open sets in X, Y containing x, y such that their preimages in $\tilde{X}, \tilde{Y}$ satisfy the required properties. The product of these two open sets is open in the product space, and satisfies the required properties (by properties of product spaces). □

2.2 Problem 20

2.2.1 Question

Describe all the covering spaces of the torus, projective plane, Klein bottle, Möbius strip, and cylinder.

2.2.2 Answer

Covering spaces can be thought of as a variant of the square-with-edges-associated representation of a space in which multiple copies are attached according to the association of the edges.

1. Cylinder ($\mathbb{Z}$): Cylinder ($\mathbb{Z}$), $I \times \mathbb{R}$ (1)
2. Torus ($\mathbb{Z} \times \mathbb{Z}$): Torus ($\mathbb{Z} \times \mathbb{Z}$), $\mathbb{R}^2$ (1), $\mathbb{R} \times S^1$ ($\mathbb{Z}$)
3. Möbius Strip ($\mathbb{Z}$): Möbius Strip ($\mathbb{Z}$), $I \times \mathbb{R}$ (1), Cylinder ($\mathbb{Z}$)
4. Klein Bottle ($\mathbb{Z} * \mathbb{Z}/\left\langle aba^{-1}b\right\rangle$): Klein bottle ($\mathbb{Z} * \mathbb{Z}/\left\langle aba^{-1}b\right\rangle$), $\mathbb{R}^2$ (1), Torus ($\mathbb{Z} \times \mathbb{Z}$), $\mathbb{R} \times S^1$ ($\mathbb{Z}$), Möbius strip ($\mathbb{Z}$)
5. $\mathbb{R}P^2$ ($\mathbb{Z}/2\mathbb{Z}$): $\mathbb{R}P^2$ ($\mathbb{Z}/2\mathbb{Z}$), $\mathbb{R}^2$ (1)

2.3 Problem 21

2.3.1 Question

Find the group of covering transformations for each of the coverings of Problem 20.

2.3.2 Answer

Deck transformation are automorphisms of the collection of squares which model the covering space.

1. $\mathbb{Z}/n\mathbb{Z}$, $\mathbb{Z}$
2. $\mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$, $\mathbb{Z} \times \mathbb{Z}$, $\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$
3. $\mathbb{Z}/n\mathbb{Z}$, $\mathbb{Z}$, $\mathbb{Z}/n\mathbb{Z}$
4. $\mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$, $\mathbb{Z} \times \mathbb{Z}$, $\mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$, $\mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}$, $\mathbb{Z}$
5. 1, $\mathbb{Z}/2\mathbb{Z}$