Homework

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1 Chapter 3 Section 3

1.1 Problem 16

1.1.1 Question

Suppose X is locally compact and Hausdorff. Given $x \in X$ and a neighbourhood U of x, find a compact neighborhood of x which is contained in U.

1.1.2 Answer

Let $x \in X$ be a point in a topological space which is both locally compact and Hausdorff. I claim that given a neighborhood U of x there is some compact neighborhood of x which is contained in U.

Proof. There exists some open strict subset of U say L. So moreover we have $\overline{O} \subset U$. Since $\overline{O}$ is closed and a subset of U. Now we can find the intersection of $\overline{O}$ and the neighborhood of x which is compact. Since both must be closed their intersection must be closed. Hence we have constructed a space which contains x, is a subset of U and is compact as desired. $\square$

1.2 Problem 19

1.2.1 Question

Let X and Y be locally compact Hausdorff spaces and let $f : X \to Y$ be an onto map. Show f extends to a map from $X \cup \{\infty\}$ onto $Y \cup \{\infty\}$ if and only if $f^{-1}(K)$ is compact for each compact subset K of Y. Deduce that if X and Y are homeomorphic space then so are their one-point compactifications. Find two spaces which are not homeomorphic but which have homeomorphic one-point compactifications.

1.2.2 Answer

We will first prove ($\Rightarrow$) that f extends to a map from $X \cup \{\infty\}$ onto $Y \cup \{\infty\}$ if $f^{-1}(K)$ is compact for each compact subset K of Y.

Proof. Let $L \subset Y \cup \{\infty\}$ be an open set. There are two cases. Either L does not contain ∞ in which case it must have an open inverse image since f is a map, or L does contain ∞. If this latter is the case it must be that L^C is compact in Y by definition of one-point compactification and therefore that $f^{-1}(L^C)$ is compact in X by assumption. Thus we have $f^{-1}(L)$ is open in $X \cup \{\infty\}$ as desired. □

Now we show ($\Leftarrow$) that if f extends to a map from $X \cup \{\infty\}$ onto $Y \cup \{\infty\}$ then $f^{-1}(K)$ is compact for each compact subset K of Y.

Proof. Suppose towards a contradiction that there exists a compact subset K of Y such that $f^{-1}(K)$ is not compact, and that f extends to a map f' from X^* to Y^*. Then $\infty \cup f'(f^{-1}(K^C))$ is not open yet $\infty \cup K^C$ is open. Contradiction. □

If X and Y are homeomorphic by a homeomorphism f then since preimages of compact sets under homeomorphisms are themselves compact it must be that the compactifications of X and Y X^* and Y^* are homeomorphic by the natural extension of f to $f' : X^* \to Y^*$.

The set $X = (-\infty, -3] \cup [3, \infty)$ has one point compactification X^* with $X^* \cong [a, b]$. The set $Y = (1, 2]$ has one point compactification Y^* so that $Y^* \cong [a, b]$. However since clearly X and Y are not homeomorphic since X is not connected though Y is.

2 Chapter 3 Section 6

2.1 Problem 40

2.1.1 Question

If A and B are path-connected subsets of a space, and if $A \cap B$ is nonempty, prove that $A \cup B$ is path-connected.

2.1.2 Answer

First I prove

Lemma 1. *Any two paths P_1 and P_2 in a topological space X which have nonempty intersection can be connected to form a new path say P which goes from the start of P_1 to the end of P_2.*

Proof. In particular say that the first point of intersection (defined as the leat x, y such that $P_1(x) = P_2(1 - y)$ is z. Then define P by taking P_1 from $0 \to x$ and P_2 from $y \to 1$. P is a path.

In particular take an open set $O \subset X \cap P$. If the open set is wholly contained in the part of P which is taken from P_1 or P_2 we are done by the continuity of P_1 and P_2. Thus, the remaining case is that there is a nonempty intersection $O \cap P_1$ and $O \cap P_2$. However this means also that $P_1(x) = P_2(1 - y) \in O$ so the preimage of O $P^{-1}(O$ is of the form $(a, x] \cup [x, b)$ for some a and b and is therefore open.

So we have proven that two paths may be connected to form a new path from the beginning of the first path to the end of the second. $\square$

Let A and B be path-connected subsets of a space X with nonempty intersection. I claim that $A \cup B$ is path connected.

Proof. Since $A \cap B$ is nonempty it in particular contains at least one point, say x. Since $x \in A$ and A is path connected there exists some path from any point a to x. Similarly since B is path connected and $x \in B$ there exsits a path from any point $b \in B$ to x.

Thus, since two paths can be connected to form a new path by the lemma there exists a path from any point $a \in A$ to any point $b \in B$. $\square$