Homework

Frederick Robinson

12 April 2010

1 Chapter 5 Section 2

1.1 Problem 11

1.1.1 Question

Let X be a path-connected space. When is it true that for any two points $p, q \in X$ all paths from p to q induce the same isomorphism between $\pi_1(X, p)$ and $\pi_1(X, q)$

1.1.2 Answer

This holds if and only if X is simply connected.

Proof. We have from the previous exercise that the isomorphisms induced by two paths γ and σ are the same if and only if the inner automorphism given by $\langle \sigma^{-1}\gamma \rangle$ is the identity automorphism, but this is the case if and only if $\pi_1(X)$ is abelian. □

1.2 Problem 14

1.2.1 Question

Let $\mathbb{E}^3_+$ denote those points of $\mathbb{E}^3$ which have nonnegative final coordinate. Show that the space $\mathbb{E}^3_+ - \{(x, y, z)|y = 0, 0 \leq z \leq 1\}$ has trivial fundamental group.

1.2.2 Answer

Proof. Say $X = \mathbb{E}^3_+ - \{(x, y, z)|y = 0, 0 \leq z \leq 1\}$. Fix some arbitrary point $x \in X$ and some arbitary loop $\varphi : [0, 1] \to X$ based at x. Since φ is homotopic to the constant map $\rho : [0, 1] \to X$ defined by $\rho(y) = x$ for all $y \in [0, 1]$ (via the straight line homotopy) $\pi_1(X, x)$ is the trivial group. □

2 Chapter 5 Section 3

2.1 Problem 21

2.1.1 Question

Describe the homomorphism $f_* : \pi_1(S^1, 1) \to \pi(S^1, f(1))$ induced by each of the following maps:

1. The antipodal maps $f(e^{i\theta}) = e^{i(\theta+\pi)}$, $0 \le \theta \le 2\pi$.
2. $f(e^{i\theta}) = e^{in\theta}$, $0 \le \theta \le 2\pi$, where $n \in \mathbb{Z}$
3. $f(e^{i\theta}) = \begin{cases} e^{i\theta} & 0 \le \theta \le \pi \\ e^{i(2\pi-\theta)} & \pi \le \theta \le 2\pi \end{cases}$

2.1.2 Answer

Throughout let $\varphi : \mathbb{Z} \to \mathbb{Z}$ be a homomorphism of groups.

1. $\varphi(x) = x$
2. $\varphi(x) = nx$
3. $\varphi(x) = 0$

2.2 Problem 23

2.2.1 Question

Provide a precise solution to the second part of Problem 8 as follows. Let α, β be the paths in A defined by $\alpha(s) = (s+1, 0)$ and $\beta(s) = h\alpha(s), 0 \le s \le 1$. Show that if h is homotopic to the identity relative to the two boundary circles of A then the loop $\alpha^{-1}\beta$ is homotopic rel$\{0,1\}$ to the constant loop at the point $(1,0)$. Now check that this loop represents a nontrivial element of the fundamental group of A.

2.2.2 Answer

Proof. If h is homotopic to the identity relative to the two boundary circles of A via some homotopy F then $\alpha^{-1}\beta$ is homotopic to the constant loop relative to $\{0,1\}$ via F. However, $\alpha^{-1}\beta$ is homotopic to the loop of constant radius in the annulus which we know to be nontrivial (there is a deformation retract from the annulus to the circle which takes such loops to nontrivial elements of the circle's fundamental group) $\square$