Homework

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1 Chapter 2 Section 1

1.1 Problem 4

1.1.1 Question

Find all the limit points of the following subsets of the real line:

1. $\{(1/m) + (1/n) \mid m, n = 1, 2, \ldots\}$
2. $\{(1/n) \sin n \mid n = 1, 2, \ldots\}$

1.1.2 Answer

In each of the following we show that the indicated points are limit points in open balls. Because such open balls are a basis for the Euclidian Topology, it follows that the indicated points are limit points for the topology

That is, if a point x has the property that every open ball containing it also includes members of a set $E \backslash x$, then every open set containing x must also contain members of $E \backslash x$ as every open set may be written as a union of open balls.

1. The limit points of this set are precisely the members of X for $E = \{1/n \mid n \in \mathbb{R}\} \cup \{0\}$

   Call the set in question $X = \{(1/m) + (1/n) \mid m, n = 1, 2, \ldots\}$

   First we will show

   Lemma 1. 0 *is a limit point of the set* $Y = \{1/n \mid n \in \mathbb{R}\}$

   Proof. Pick some $\epsilon > 0$, then by the density of $\mathbb{Q}$ in $\mathbb{R}$ there is some $p/q \in \mathbb{Q}$ such that $p/q < \epsilon$.

   However, since $1/q < p/q < \epsilon$ and $1/q \in Y$ there is some member of Y in $B_\epsilon 0$. Since ϵ was chosen arbitrarily it is clear that 0 is a limit point of Y. ☐

Now we will demonstrate that each member of E is indeed a limit point of X.

Proof. Fixing some $1/n$, we can write a subset of X as $1/n + \{1/m \mid m \in \mathbb{N}\}$, but by Lemma 1 we know that a limit point of this is $1/n$. So all points of the form $1/n$ $n \in \mathbb{N}$ are limit points of X.

Moreover, fixing $m = n \in \mathbb{N}$ we observe that there is a subset of X given by as $2 \cdot \{1/m \mid m \in \mathbb{N}\}$, and so by Lemma 1, 0 is a limit point of X. $\square$

It remains to show that each $x \in\notin E$ for is not a limit point of X

Proof. Let $x \notin E$. Now, for each $1/n$ $n \in \mathbb{N}$ there is some other $1/m$ $m \in \mathbb{N}$ for which $d(1/m + 1/n, x)$ is minimized.

For all but finitely many $1/n$ this is just whichever $1/m$ minimizes $d(1/m, x)$. For, all but finitely many $1/n$ have $1/n \in B_\epsilon 0$ given $\epsilon > 0$ and given $1/l, 1/k$ with $d(1/l, x) < d(1/k, x)$ there is some $\epsilon > 0$ with $d(1/l + \delta, x) < d(1/k + \delta, x)$ for all $\delta < \epsilon$.

This established, we suppose towards a contradiction that $X \cap B_\epsilon x$ is nonempty for all $\epsilon > 0$. In particular each $B_\epsilon x$ must contain a point of the form $1/n + 1/m$ for some $n \in \mathbb{N}$ and m chosen so as to minimize this distance.

However, there is but a finite set of such m as we have previously demonstrated. Thus, each $B_\epsilon x$ must contain a point of the form $1/n + 1/m$ for some $n \in \mathbb{N}$ and m one of a finite set of points. Yet, the limit points of a finite union consist of the union of the limit points, and we know by Lemma 1 that the limit points of $\{a + 1/m \mid m \in \mathbb{N}\}$ is just a. So, this is a contradiction as $x \notin E$ is not of the form $1/m$ $m \in \mathbb{N}$. $\square$

2. The only limit point of this set is 0.

   First we prove that 0 is a limit point.

   Proof. Since we know that $\sin x \in [-1, 1]$ for any choice of x it must be that $(1/n) \sin n \in [-1/n, 1/n]$.

   Pick some $\epsilon > 0$, then the neighborhood of 0 given by $B_\epsilon 0$ must contain a point in our set.

   In particular find some $p/q \in \mathbb{Q}$ such that $p/q < \epsilon$ (this is always possible by density of $\mathbb{Q}$ in $\mathbb{R}$). Then clearly $1/q < p/q < \epsilon$ but as we showed earlier $(1/q) \sin q \leq 1/q$ and $(1/q) \sin q$ is in our set.

   So, we have demonstrated a member of our set in $B_\epsilon 0$ and 0 is a limit point of the set, as desired. $\square$

   Now we need to show that no other $x \in \mathbb{R}$ is a limit point of our set.

Proof. All but finitely many points of our set are within some arbitrarily small neighborhood of 0. That is, given $\epsilon > 0$ all but finitely many points are in $B_\epsilon 0$.

Our proof above, that 0 is a limit point may be extended to show this. As we observed above $(1/n)\sin n \in [-1/n, 1/n]$. Hence, because for each $m < n \in \mathbb{N}$ we have $(1/n)\sin n < 1/m$ and we have at least one member of the set in any $B_\epsilon 0$ we must have all but finitely many members of the set in such a ball. Each successive member of the set must also be in the ball.

So, now that we have established that all but finitely many points are in $B_\epsilon 0$ it is clear that our set has no other limit points. For, given $x \neq 0$ we may find some neighborhood of 0 not containing x. Since this neighborhood contains all but finitely many points of the set, any neighborhood of x which does not intersect the chosen neighborhood of 0 contains only finitely many elements. Then, there must be an element of least distance. A ball of smaller radius than this least distance contains only x of all the elements in our set.

More concretely, we can take $\epsilon = x/2$. $B_\epsilon 0$ contains all but finitely many elements from the set. Moreover, by the triangle inequality $B_\epsilon x$ contains none of the elements in $B_\epsilon 0$ and therefore contains at most finitely many elements.

So, there must be an element y in our set with minimum $d(y, x)$. Therefore $B_{y/2}x$ contains no elements of the set, and $x \neq 0$ is not a limit point. $\square$

1.2 Problem 7

1.2.1 Question

Suppose Y is a subspace of X. Show that a subset of Y is closed in Y if it is the intersection of Y with a closed set in X. If A is a subset of Y, show that we get the same answer whether we take the closure of A in Y, or intersect Y with the closure of A in X.

1.2.2 Answer

Let Y be a subspace of X. We claim that a subset of Y is closed in Y if it is the intersection of Y with a closed set in X.

Proof. Let C be a closed set in X. If $C^C \cap Y = \emptyset$ then $C \cap Y = Y$ and we are done.

Otherwise, by definition X^C is open in X. Hence, $X^C \cap Y$ is open in Y, and $Y \backslash (X^C \cap Y)^C = Y \backslash (X \cup Y^C) = Y \backslash X$ is closed in Y.

So we have shown that if a set is closed in X, then its intersection with Y is also closed as desired. $\square$

Now we claim that if A is a subset of Y then $\overline{A|_Y} = Y \cap \overline{A}$

Proof. If $x \in X$ is a member of the closure of A in X. Then, by definition, some $a \in A$ is in O for every closed set O containing x. So, $Y \cap \overline{A}$ consists of all those points $y \in Y$ for which some $a \in A$ is in O for every closed set O containing y.

Thus, each $y \in Y \cap \overline{A}$ is also in $\overline{A|_Y}$ as each closed set in Y is the intersection of a closed set in X and Y by the first part, and $A \subset Y$.

Now for the reverse inclusion let $y \in \overline{A|_Y}$. Then every closed set in Y containing A also contains y. However if a set is closed in Y it is the intersection of Y with a set which is closed in X. Thus, every closed set in X which intersects Y and contains A must also contain y. All sets in X which contain A intersect Y though since $A \subset Y$. Hence, every closed set in X containing A also contains y, and y is a member of the closure of A in Y and $A \cap \overline{A}$.

□

1.3 Problem 8

1.3.1 Question

Let Y be a subspace of X . Given $A \subseteq Y$, write $\mathring{A}_Y$ for the interior of A in Y, and $\mathring{A}_x$ for the interior or A in X. Prove that $\mathring{A}_X \subseteq \mathring{A}_Y$, and give an example to show the two may not be equal.

1.3.2 Answer

$\mathring{A}_X \subseteq \mathring{A}_Y$

Proof. Let $a \in \mathring{A}_X$. Then, there is at least one open set $E \subset X$ contained wholly in A with $x \in E$.

Open sets in Y may be written as the intersection of open sets in X and the entire set Y. Because $A \subset Y$ there is at least one open set in Y contained wholly in A and containing x, namely $E \cap Y$. □

Let $X = \mathbb{R}$ and $Y = \{y \mid |y| \leq 1\}$. The interior of Y in Y is just Y since open sets are their own interiors, and topological spaces are open in themselves. However, the interior of Y in X is $\{x \mid |x| < 1\}$.