1 Introduction

We are interested in the accuracy of the dimensional reduction proposed by Gao et al. In order to determine the accuracy, we define an error as the distance between a reduced fixed point and the fixed point of the reduced system. We then calculate the expected value and the fluctuation of this error analytically for a system of generalized Lotka-Volterra equations and specific random interaction matrices A.

Define errors in x and β direction:

$$\mathrm{err}_x = 1 - \frac{x(\beta_{\mathrm{eff}})}{x_{\mathrm{eff}}}\,, \tag{1}$$

$$\mathrm{err}_\beta = 1 - \frac{\beta(x_{\mathrm{eff}})}{\beta_{\mathrm{eff}}}\,. \tag{2}$$

For the GLV they coincide since the fixed point in the reduced system is

$$x(\beta_{\mathrm{eff}}) = -\frac{\alpha}{\beta_{\mathrm{eff}}}\,, \tag{3}$$

such that both errors become

$$\mathrm{err} = 1 + \frac{\alpha}{x_{\mathrm{eff}}\beta_{\mathrm{eff}}}\,. \tag{4}$$

Inserting the definitions of x_{eff} and β_{eff} from Gao et al.'s paper,

$$x_{\mathrm{eff}} = \frac{\sum_{ij} A_{ij}x_j}{\sum_{mn} A_{mn}}\,, \tag{5}$$

$$\beta_{\mathrm{eff}} = \frac{\sum_{ijk} A_{ij}A_{jk}}{\sum_{mn} A_{mn}}, \tag{6}$$

we get

$$\begin{aligned}\mathrm{err} &= 1 + \frac{\alpha\left(\sum_{mn} A_{mn}\right)\left(\sum_{mn} A_{mn}\right)}{-\alpha\left(\sum_{ijk} A_{ij}{A^{-1}}_{jk}\right)\left(\sum_{ijk} A_{ij}A_{jk}\right)} \\ &= 1 - \frac{\sum_{ijkl} A_{ij}A_{kl}}{S\sum_{ijk} A_{ij}A_{jk}} \\ &= 1 - \frac{n}{d}\,,\end{aligned} \tag{7}$$

where we defined

$$n := \sum_{ijkl} A_{ij}A_{kl}\,, \tag{8}$$

$$d := S \cdot \sum_{ijk} A_{ij}A_{jk}\,, \tag{9}$$

and the A_{ij} are the entries of the interaction matrix A.

In order to calculate the expected value and variance analytically, we make one key assumption, namely that the expected values of numerator and denominator of the error and its square can be taken independently of each other. This approximation should work well for strictly positive entries $A_{ij} > 0$ with vanishing probability density close to 0. For A_{ij} not following this constraint, the actual analytical expected value is expected to explode. Note that this behaviour may not be observed when sampling A_{ij}, because the probability associated with the denominator being zero may be very small.

Making this approximation, we get

$$\mathbb{E}[\text{err}] = \mathbb{E}\left[1 - \frac{n}{d}\right] \approx 1 - \frac{\mathbb{E}[n]}{\mathbb{E}[d]}\,, \tag{10}$$

and for the variance

$$\text{Var}(\text{err}) = \mathbb{E}\left[(\text{err} - \mathbb{E}[\text{err}])^2\right] = \mathbb{E}\left[\left(\frac{d\mathbb{E}[n] - n\mathbb{E}[d]}{d\mathbb{E}[d]}\right)^2\right] \tag{11}$$

we have

$$\text{Var}(\text{err}) \approx \frac{\mathbb{E}[d^2]\mathbb{E}[n]^2 - 2\mathbb{E}[nd]\mathbb{E}[n]\mathbb{E}[d] + \mathbb{E}[n^2]\mathbb{E}[d]^2}{\mathbb{E}[d^2]\mathbb{E}[d]^2}\,. \tag{12}$$

By automatizing the combinatorial problems associated with the sums, we calculate both values, as described in the next section.

2 Description of the algorithm

As stated above, we want assess the expected value Equation 11 and variance Equation 12 of the error. Hence, we need to calculate the terms

$$\mathbb{E}[d] = S \cdot \mathbb{E}\left[\sum_{ija} A_{ia}A_{aj}\right] \; ; \tag{13}$$

$$\mathbb{E}[n] = \mathbb{E}\left[\sum_{ijkl} A_{ij}A_{kl}\right] \; ; \tag{14}$$

$$\mathbb{E}[d^2] = S^2 \cdot \mathbb{E}\left[\sum_{ijklab} A_{ia}A_{aj}A_{kb}A_{bj}\right] \; ; \tag{15}$$

$$\mathbb{E}[n \cdot d] = S \cdot \mathbb{E}\left[\sum_{ijklmna} A_{ij}A_{kl}A_{ma}A_{an}\right] \; ; \tag{16}$$

$$\mathbb{E}[n^2] = \mathbb{E}\left[\sum_{ijklmnop} A_{ij}A_{kl}A_{mn}A_{op}\right] \tag{17}$$

where all indices are iterated over $\{1, 2, ..., S\}$. In the simplest version of interaction matrix A, the entries A_{ij} are all i.d.d. and we need to separate out pairs A_{ij}^2 as they will lead to contributions other than μ^2, where $\mu = \mathbb{E}[A_{ij}]$ (and similarly for higher order tuples). In order to do so, we devised an algorithm which is described for the simple example of $\mathbb{E}[n]$, which may even be calculated by hand quite easily. It consists of the following steps:

1. **Generation of a feasible index combination.** Such a combination like $(i = j \neq k \neq l, i \neq l)$ is encoded in the upper triangle of a 4×4 boolean matrix, which in this case would look like

$$P = \begin{bmatrix} - & T & F & F \\ - & - & F & F \\ - & - & - & F \\ - & - & - & - \end{bmatrix} . \tag{18}$$

   Here, $P_{ij} = T$ implies that the i-th index is equal to the j-th index (F for $\neq$), numbering the indices as $(i, j, k, l) \sim (0, 1, 2, 3)$. We dump non-feasible combinations such as

$$P_{\text{illegal}} = \begin{bmatrix} - & T & F & F \\ - & - & F & T \\ - & - & - & F \\ - & - & - & - \end{bmatrix} . \tag{19}$$

2. **Calculating the number of terms n_P associated with P.** In order to do so, we sort the indices into cliques containing only equal ones. The

number of cliques, called multiplicity factor m_P is directly related to the number of terms by

$$n_P = S(S-1)...(S-m_P) = \prod_{\alpha=0}^{m_P-1} (S-\alpha)\,. \tag{20}$$

3. **Project the matrix P down to a 2×2 boolean matrix R_P.** This matrix encodes whether the actual random variables $X = A_{ij}$ and $Y = A_{kl}$ are equal. We have $X = Y$ only if $i = k$ and $j = l$, which in terms of the matrices P and R_P is stated as

$$R_{01}(P) = P_{02} \wedge P_{13}\,. \tag{21}$$

4. **Calculating actual cliques $\mathcal{C}(R(P))$.** In this case, the only possible cliques are XY or X^2, trivially encoded by $R_{01}(P)$.

5. **Add contribution to overall sum.** Each clique is associated with a specific factor arising from taking the expected value. In this case, we only have

$$\mathbb{E}[XY] = \mu^2 \qquad \text{and} \qquad \mathbb{E}[X^2] = \sigma^2 + \mu^2\,. \tag{22}$$

We sum each term multiplied by the multiplicity n_P, iterating over all feasible index matrices P to get the corresponding expected value

$$\mathbb{E}[d] = S \cdot \sum_P n_P\, \mathbb{E}[\mathcal{C}(R(P))]\,. \tag{23}$$

To incorporate correlation of transposed entries, we make use of the lower triangle of R. In this example, we have $A_{ij} = X$ and $A_{kl} = A_{ji} = X^T$ only if $i = l$, $j = k$ and $(i, j) \neq (k, l)$, that is

$$R_{10}(P) = P_{03} \wedge P_{12} \wedge \neg R_{01}(P)\,. \tag{24}$$

The associated clique has the expected value

$$\mathbb{E}[XX^T] = \rho\sigma^2 + \mu^2\,. \tag{25}$$

If we finally want to allow for the diagonal elements to be drawn from a different distribution, we employ the free diagonal in a similar manner. In this example:

$$R_{00}(P) = P_{01}, \qquad R_{11}(P) = P_{23}\,, \tag{26}$$

and the associated expected values

$$\mathbb{E}[XY_d] = \mu\mu_d, \qquad \mathbb{E}[X_dY_d] = \mu_d^2, \qquad \mathbb{E}[X_d^2] = \sigma_d^2 + \mu_d^2\,. \tag{27}$$

For the second order terms, R becomes a 4×4 boolean matrix, and the full set of possible cliques becomes both larger (30 terms for both correlation and specific diagonal) and more complicated (incorporating higher moments such as $\mathbb{E}[X^3Y]$).

3 Results

The analytical expressions for expected value and variance of the error for different cases of interaction matrices A are listed in Table 1. We approximated the expressions to the highest order in the network size S, as we want to look at large networks.

We summarize these results by making the following observations:

- In all cases, the error (or its fluctuations) grows without bound if the ratio $\frac{\mu}{\sigma}$ goes to zero for a given network size S.
- The order of the fluctuations (namely $S^{-\frac{3}{2}}$) remains the same for all cases, while the order of the expected value changes. Thus,
  - for interaction matrices A without correlation ($\rho = 0$), the term dominating the error for large S are the fluctuations while the mean value is either zero (for iid entries A_{ij}) or of order S^{-2} (in case of a constant diagonal);
  - for networks with non-zero correlation, the mean becomes the dominating term of order S^{-1}.
- If the diagonal is of the same scale as S, the error may explode. This happens if $A_{ii} = -d_c$, where $d_c = (S-1)\mu$ corresponds to the value of d where the interaction matrix becomes stable and non-reactive for positive μ.

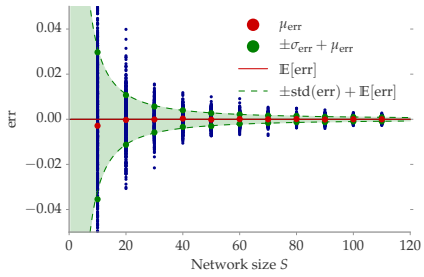
In order to test these analytical results, we sampled the interaction matrix with the corresponding statistics numerically and compared the empirical mean and standard deviation with the theoretical predictions. The results can be observed in Figure 2. In all cases, the theoretical prediction is met very well. There is a notable but small deviation for small network sizes $S = 20$, namely slight underestimation of the mean for the case of correlation, cf. Figure 1band Figure 1c.

Note that for the case of a constant diagonal close to the critical value, shown in Figure 1f, the theoretical value is not expected to give a good approximation to the empirical average, since in this case, the expected value of the denominator $\mathbb{E}[d]$ becomes zero. In this case, the approximation of taking numerator and denominator separately is not justified. Furthermore, sampling becomes difficult, as outliers may govern the empirical mean and standard deviation.

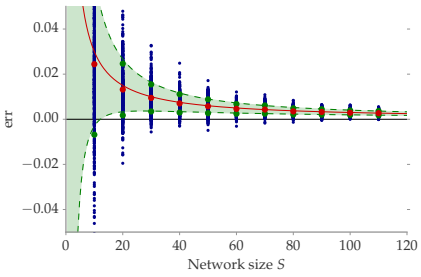
References

Table 1: Resulting analytical expressions, approximated to highest order in S.

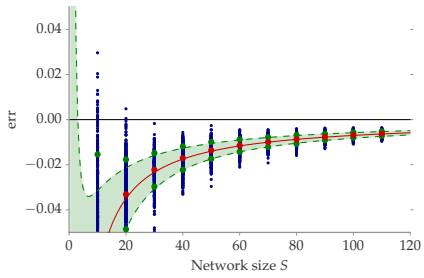
Case	$\mathbb{E}[\text{err}]$	$\text{Var}(\text{err})$
A_{ij} i.i.d.	0 (exact)	$\frac{\sigma^4}{S^3\mu^4}$
Correlation:		
$\rho = \text{corr}(A_{ij}, A_{ji}) \in [-1, 1]$	$\frac{\rho\sigma^2}{S\mu^2}$	$\frac{\sigma^4}{S^3\mu^4}\left(2\frac{\mu^2}{\sigma^2}\rho + (\rho-1)^2\right)$
Constant diagonal:		
$A_{ii} = -d$ of order 1 or $\sqrt{S}$	$\frac{\sigma^2((S-2)\rho-1)}{S^2\mu^2}$	$\frac{\sigma^4}{S^3\mu^4}\left(2\frac{\mu^2}{\sigma^2}\rho + (\rho-1)^2\right)$
for $\rho = 0$	$-\frac{\sigma^2}{S^2\mu^2}$	$\frac{\sigma^4}{S^3\mu^4}$
for $\rho \neq 0$	$\frac{\sigma^2\rho}{S\mu^2}$	$\frac{\sigma^4}{S^3\mu^4}\left(2\frac{\mu^2}{\sigma^2}\rho + (\rho-1)^2\right)$
$A_{ii} = -d$ of order S	$\frac{\sigma^2((S-2)\rho-1)}{S^2\mu^2\left(\frac{d}{d_c}-1\right)^2}$	$\frac{\sigma^4}{S^3\mu^4\left(\frac{d}{d_c}-1\right)^4}\left(2\frac{\mu^2}{\sigma^2}\rho + (\rho-1)^2\right)$
for $\rho = 0$	$-\frac{\sigma^2}{S^2\mu^2\left(\frac{d}{d_c}-1\right)^2}$	$\frac{\sigma^4}{S^3\mu^4\left(\frac{d}{d_c}-1\right)^4}$
for $\rho \neq 0$	$\frac{\sigma^2\rho}{S\mu^2\left(\frac{d}{d_c}-1\right)^2}$	$\frac{\sigma^4}{S^3\mu^4\left(\frac{d}{d_c}-1\right)^4}\left(2\frac{\mu^2}{\sigma^2}\rho + (\rho-1)^2\right)$



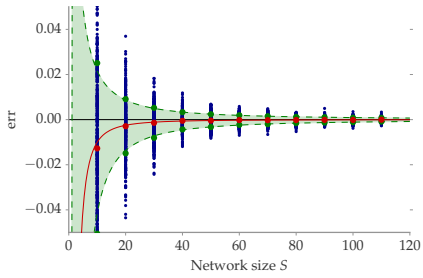
(a) All elements i.i.d.



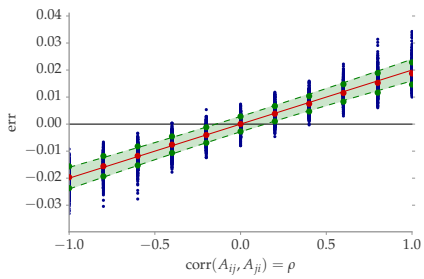
(b) Positive correlation: $\rho = 0.3$.



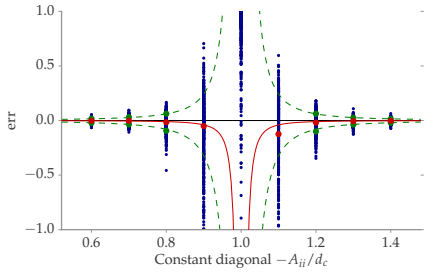
(c) Negative correlation: $\rho = -0.7$.



(d) Constant diagonal $A_{ii} = -2d_c, \rho = 0$.

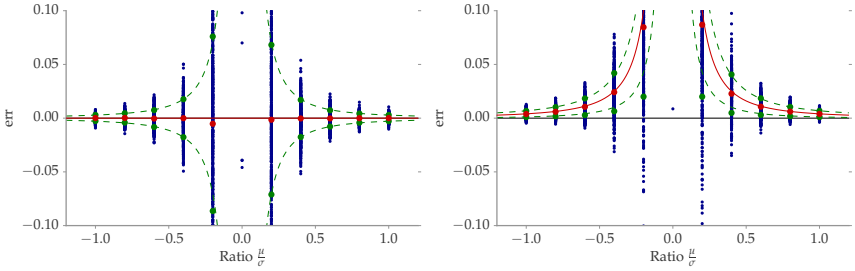


(e) Varying correlation, no specific diagonal.



(f) Varying constant diagonal, no correlation.

Figure 1: Comparison of theoretical results with numerical samples. Each blue dot corresponds to the error calculated for one interaction matrix. We sampled 500 matrices independently and calculated empirical means μ_{err} and standard deviations σ_{err} (plotted as red and green dots, respectively). The theoretical mean $\mathbb{E}[\text{err}]$ is plotted as a red line, the shaded area indicates the predicted standard deviation std(err). For all figures, $\mu = \sigma = 1$, network size for (e) and (f): $S = 50$.



(a) Varying mean μ for $\sigma = 0.5, \rho = 0$.

(b) Varying mean μ for $\sigma = 0.5, \rho = 0.2$.

Figure 2: Comparison of theoretical results with numerical samples, part 2.