A Numerical Exploration of the Local Volatility Model for Option Pricing

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Abstract

In Nobel-prize winning work, Black, Merton, and Scholes developed a model to price options. The tractability of the model revolutionized options pricing and allowed for the derivatives to become widely traded. However, the predictions of the model run counterfactual to empirically observed prices. In this paper, we consider a generalization to the Black-Scholes model, the local-volatility model. This generalization gives us more degrees of freedom to fit the prices we observe. With the additional expressiveness, however, come numerical complications. Black-Scholes requires fitting only a single number known as the *implied volatility*, but the local volatility model requires fitting an entire multi-dimensional function. We explore these complications and arrive at solutions that are assessed for numerical stability and accuracy to price out-of-sample options.

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1 Background

1.1 Options Terminology

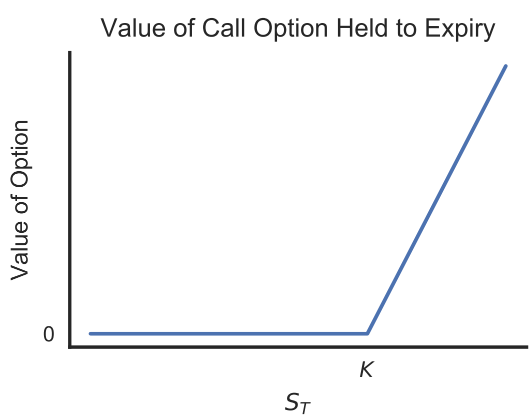
Before delving into the theoretical and numerical results, we briefly summarize options terminology. An option is a derivative on some asset, henceforth called the *underlying*—i.e. it's value is *derived* from the value of the underlying. The owner of a call/put option has the right but not the obligation to buy/sell the underlying asset at a given price at some date in the future.

The price at which the holder of the option can buy/sell is called the *strike price*, denoted K. Invoking the right to buy/sell is called *exercising* the option. The last time at which the holder can exercise is called the *expiry*, denoted as time T. The value of the underlying asset at time t is denoted as S_t.

The payoff of a call option at expiry is thus given by,

$$\max(S_T - K, 0)$$

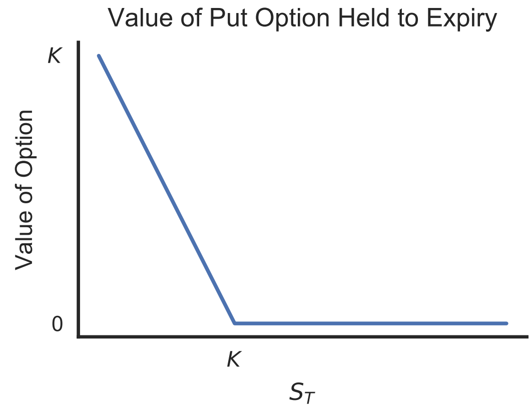
because the owner will only exercise it if the underlying is worth more than the strike price.



The payoff of a put option is simililarly,

$$\max(K - S_T, 0)$$

and the payoff curve looks like,



1.2 Risk-Neutral Pricing

The value of an option at expiry is easily observable and displayed in the figures in Section 1.1. However, before time T, the value of S_T is a random variable. Asset pricing (change of measure) theorems allow us to write the value of a call option as the discounted expectation of the payoff under what we call the risk-neutral distribution,

$$C(T, K) = e^{-rT} \int_K^\infty (S - K) \underbrace{\phi(T, S)}_{\text{risk-neutral PDF}} dS.$$

However, in order to make any progress beyond this, we need to make a further assumption about what this risk-neutral distribution looks like.

1.3 Black-Scholes Model

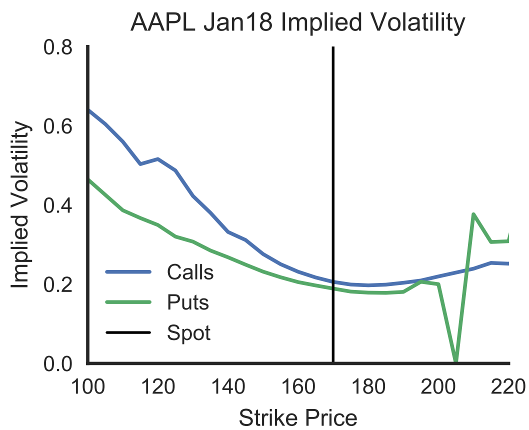
The Black-Scholes model makes one such assumption by writing how the asset diffuses over time. In particular, the Black-Scholes model treates the asset price as a geometric Brownian motion such that,

$$\frac{dS}{S} = r dt + \sigma dW$$

where W is Brownian motion and σ is a constant value called the implied volatility.

This forces the risk-neutral distribution to be log-normal and makes finding the price analytically tractable. In particular, since the only pricing input into this model that we do not observe is the implied volatility, we can quote price as a function of implied volatility (e.g. an option is said to cost $\sigma = 16\%$ if its price is consistent with the price the Black-Scholes model would predict if $\sigma = 16\%$).

Thus, Black-Scholes predicts that if σ is indeed a constant, then for any option written on the same underlying (regardless of strike price or expiry), that the implied volatility will be (approximately) constant. However, this is directly counterfactual to the options prices that we observe. For example, looking at the prices of AAPL options that expire on January 2018 retrieved on November 20$^{\text{th}}$, 2017 from Yahoo Finance, we get the following curve



While the data is noisy, particularly for deep-in-the money puts which are hardly ever traded, the graph is unmistakably not constant. Moreover, AAPL is not an exception: most graphs of implied volatility versus strike have a similar shape. These deviations from Black-Scholes also correspond to intuitive qualitative ideas. If the price of AAPL has just plunged 50%, it is palatable to think there is a lot of investor uncertainty, and that the future price of AAPL will diffuse with higher volatility than if it is just up 5%. Thus, the assumption of constant σ is suspect.

2 Local Volatility Theory

3 Pricing with Local Volatility

4 Fitting Local Volatility

5 Conclusions