3 Is Quantum Theory Complete?

In quantum theory, every measurement entails a choice. If we choose to measure one observable (physical variable), we choose *not* to measure some other observable. In confronting us with these choices quantum theory is consistent, but does quantum theory provide a complete account of all that we observe? Is quantum theory perhaps incomplete?

The title of a paper by Einstein, Podolsky and Rosen [1] (EPR) asks this question. So does the title of a reply by Bohr [2]. Einstein, Podolsky and Rosen considered the possibility of a complete theory. Such a theory, they claimed, must represent *both* the position and the momentum of a particle at any time, with arbitrary accuracy. Quantum theory does not; hence, quantum theory is incomplete.

The next section presents the EPR claim that quantum theory is incomplete. Einstein, Podolsky and Rosen believed a complete theory to be possible, although they did not show how to complete the theory. *Is* there such a complete theory? This question has a remarkable answer – one which Einstein, Podolsky, Rosen and Bohr did not anticipate.

3.1 The Einstein–Podolsky–Rosen Paradox

Einstein, Podolsky and Rosen used quantum mechanics in their paper. But the mathematical formalism is not necessary. Here we present the EPR paradox without the formalism, just as Bohr did in his reply to the EPR paper.

Suppose two particles that once interacted are now apart and no longer interact. Since they do not interact, a measurement on one particle does not affect the other particle. But since they interacted in the past, a measurement on one particle may be an indirect measurement on the other particle. For example, we can measure the position of a particle indirectly. In Fig. 3.1, a pair of massive particles approaches a board with two slots in it, a distance L apart. Suppose that the particles pass through the board through different slots, at the same time. (It may be unlikely that they pass through the two slots, but we can try many times until two particles do pass through the two slots simultaneously.) In this way we establish that the vertical distance between two particles, as they pass through the board, is L. Now we measure the height of one of the particles right after they pass through the board. We indirectly measure the height of the other particle, too, without affecting it.

We can combine the position measurement with a momentum measurement. Consider measurements of position and momentum along the vertical axis (x-axis) in Fig. 3.2. Particles arrive at the board in pairs, as in Fig. 3.1; let x_1 and p_1 denote the position and momentum components of one particle, and x_2 and p_2 the position and momentum components of the

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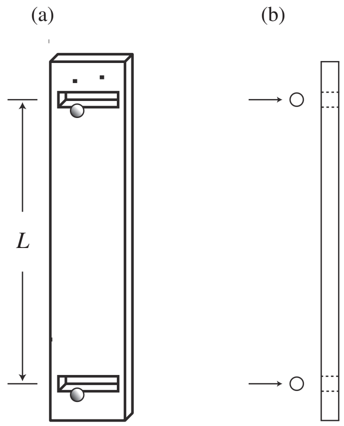

Figure 3.1: (a) An experiment to measure the height of one particle relative to another. (b) Side view of the experiment.

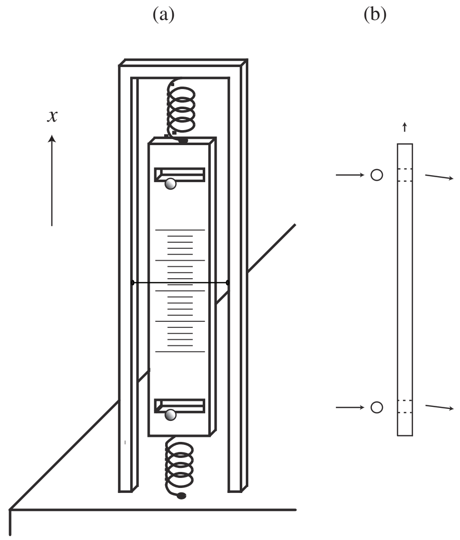

Figure 3.2: (a) An experiment to measure $x_2 - x_1$, the relative height of two particles, and $p_1 + p_2$, their total momentum along the vertical axis, at the same time. (b) Side view of the experiment. Initially, $p_1 = 0 = p_2$; while p_1 and p_2 may change as the particles pass through the slots, the vertical recoil of the board measures the change in $p_1 + p_2$.

other. We arrange that initially, $p_1 = p_2 = 0$; the initial momenta are horizontal. Also, before and after each pair arrives at the board, we accurately measure the momentum component of the board itself. Since the total momentum of the particles and the board remains constant during the passage, this experiment yields the total momentum $p_1 + p_2$ of the particles after they pass through the board (even if they knock against the slots as they pass through). It also yields the vertical distance $x_2 - x_1$ between the particles as they pass through the board (if they pass through the two slots). Right after they pass through the board, we can indirectly measure either the position or momentum of one particle in a pair without affecting it. For example, from a direct measurement of x_1 we can infer x_2 and from a direct measurement of p_1 we can infer p_2.

The measurements in this thought experiment do not contradict the uncertainty relations.[1] We measure the momentum of the board, but we never have to measure the position of the board. We prepare pairs of particles for which p_1 and p_2 vanish initially, but we assume nothing about x_1 and x_2 initially. Bohr showed, however, that the EPR experiment confronts us with a choice between complementary measurements. We can measure x_1 and infer x_2 from the result. However, measuring x_1 may affect p_1. After measuring x_1, we cannot measure p_1 and infer p_2 from the result. Alternatively, we can measure p_1 and infer p_2 from the result. However, measuring p_1 may affect x_1; after measuring p_1, we cannot measure x_1 and infer x_2 from the result. We could also measure p_2 indirectly and x_2 directly. But the x_2 measurement affects p_2 just as it does when we measure p_2 and x_2 directly. Thus Δx_2 and Δp_2 satisfy the same uncertainty relation whether we measure x_2 or p_2 directly or indirectly.

So once again, Bohr showed that quantum theory is consistent. But is quantum theory complete? Einstein, Podolsky and Rosen did not claim that quantum theory is, or is not, consistent; they claimed that quantum theory is not complete. Let us consider the EPR claim.

We return to the thought experiment. Suppose we prepare a pair of particles with $x_2 - x_1 = L$. If the board does not recoil up or down, then we also have $p_1 + p_2 = 0$. Next we measure p_2 and either x_1 or p_1. By assumption, the particles do not interact, so the result of the p_2 measurement cannot depend on whether we measure x_1 or p_1. What is p_2? If we measure p_1 and obtain $p_1 = p$, we also obtain $p_2 = -p$ because $p_1 + p_2 = 0$. If we measure x_1 we cannot measure p_1, and quantum theory does not predict the result of the p_2 measurement. But the result of the p_2 measurement cannot depend on whether we measure x_1 or p_1, by assumption. Hence we must still obtain $p_2 = -p$, i.e. some predetermined result independent of what we measure on the other particle (and which particle we measure first). Since quantum theory does not predict this result, quantum theory is incomplete. This is the EPR claim.

The EPR claim is a paradox for quantum uncertainty. Let two observers, Alice and Bob, prepare a pair of particles with $x_2 - x_1 = L$ and $p_1 + p_2 = 0$ where L is very large. Next Alice, located near one particle, measures either x_1 or p_1; Bob, located near the other particle, measures p_2. They choose independently what to measure and their measurements are spacelike separated. Later they may meet and compare their results, but the experiment is over once they complete their measurements. In some inertial reference frame, Alice measures first. She measures either the position or the momentum of her particle. If Alice measures momentum, Bob's uncertainty about the momentum of his particle (before he measures it) is subjective, according to Alice. If Alice measures position, Bob's uncertainty about the momentum of his particle (before he measures it) is objective, according to the uncertainty relations. But in another frame, Bob measures first, and we cannot distinguish subjective from objective uncertainty.

Does the EPR claim have implications for experiment? We might well wonder. Pauli, writing to Born in 1954, expressed doubt: "As O. Stern said recently, one should no more rack one's brain about the problem of whether something one cannot know anything about exists all the same, than about the ancient question of how many angels are able to sit on the point of a needle. But it seems to me that Einstein's questions are ultimately always of this kind" [3].

[1] Formally, we have $[x_2 - x_1, p_1 + p_2] = 0$, so the uncertainty relations do not constrain simultaneous measurements of $x_2 - x_1$ and $p_1 + p_2$. (See Prob. 3.10.)

Yet Pauli was wrong; the EPR claim is testable! Ten years later, Bell discovered that the EPR claim implicitly contradicts predictions of quantum mechanics.

3.2 Polarized Photons

If we replace the position and momentum measurements in the EPR thought experiment with spin or polarization measurements, it becomes a practical experiment. Bohm reformulated the EPR paradox for such an experiment [4]. At one stroke he confronted the EPR claim with experiment and with quantum theory. In this section we discuss polarization in classical and quantum theory; in Sect. 3.4 we present Bohm's formulation and apply it – as Bell did – to show that the EPR claim contradicts predictions of quantum mechanics.

Figure 3.3 shows two polarizers – disks of polaroid or another polarizing material – with a common axis of rotation. Light from a lamp or the sun shines on the first polarizer, through a pinhole on the axis; some of this light passes through the polarizers. What happens if we rotate one polarizer with respect to the other? Experiment shows a change in the intensity of light passing through the polarizers. Each polarizer has a *pass axis*, and the second polarizer passes a fraction $\cos^2(\theta-\theta')$ of the light that passes the first polarizer, where $\theta-\theta'$ is the angle between their pass axes. What accounts for this effect? Let us first consider the classical, and then the quantum, account.

In classical theory, light is an electromagnetic wave. Like any electromagnetic wave, light is either monochromatic or a superposition of monochromatic waves; a monochromatic wave has a propagation vector $\mathbf{k}$ (which fixes the angular frequency $\omega = c|\mathbf{k}|$ of the wave) and a polarization (which fixes the direction of the electric field $\mathbf{E}(\mathbf{x},t)$). Since there are two independent directions for $\mathbf{E}$ – which must be perpendicular to $\mathbf{k}$ – there are two independent polarizations. Let unit vectors $\epsilon_\mathbf{1}$ and $\epsilon_\mathbf{2}$, with $\epsilon_\mathbf{1}\cdot\epsilon_\mathbf{2} = \epsilon_\mathbf{1}\cdot\mathbf{k} = \epsilon_\mathbf{2}\cdot\mathbf{k} = 0$, represent the two (linear) polarizations. Then the electric field of a monochromatic wave is a superposition

$$\mathbf{E}(\mathbf{x},t) = E_1\epsilon_\mathbf{1}\cos(\mathbf{k}\cdot\mathbf{x}-\omega t+\phi_1) + E_2\epsilon_\mathbf{2}\cos(\mathbf{k}\cdot\mathbf{x}-\omega t+\phi_2) \tag{3.1}$$

with real phases ϕ_1, ϕ_2 and coefficients E_1, E_2. Without loss of generality we can take $|\phi_2-\phi_1| \le \pi/2$.

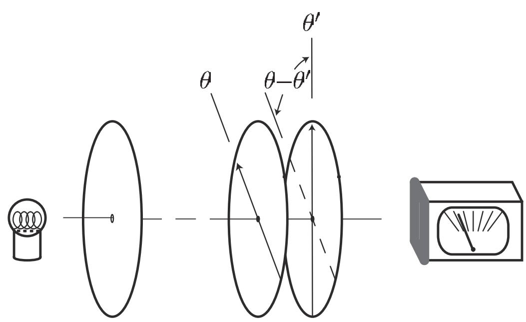

Figure 3.3: Two polarizers with an angle $\theta-\theta'$ between their pass axes.

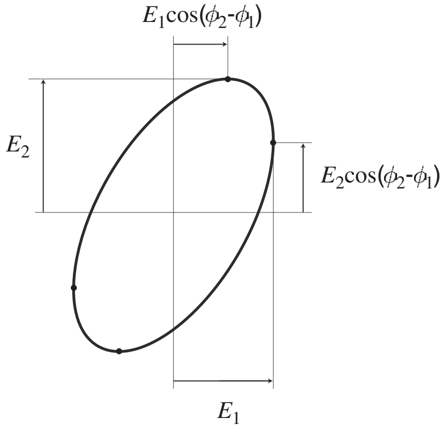

Figure 3.4: The electric field $\mathbf{E}(\mathbf{x}, t)$ of Eqs. (3.1–2) traces out an ellipse at each point $\mathbf{x}$.

If $\phi_1 = \phi_2$, the polarization is linear. Otherwise, the polarization is elliptic. A special case of elliptic polarization is circular polarization: $|E_1| = |E_2|$ and $\phi_1 - \phi_2 = \pm\pi/2$. (See Fig. 3.4.) Now a polarizer passes only the component of $\mathbf{E}$ that is parallel to its pass axis; hence light that passes through a polarizer is linearly polarized. Let us represent the polarization of the light passing through the first polarizer as a unit vector $\mathbf{P} = P_1\epsilon_\mathbf{1} + P_2\epsilon_\mathbf{2} = \cos\theta\,\epsilon_\mathbf{1} + \sin\theta\,\epsilon_\mathbf{2}$ parallel[2] to $\mathbf{E}$. Of this light, only the component parallel to $\mathbf{P}' = P_1'\epsilon_\mathbf{1} + P_2'\epsilon_\mathbf{2} = \cos\theta'\epsilon_\mathbf{1} + \sin\theta'\epsilon_\mathbf{2}$ passes through the second polarizer. So the second polarizer reduces the amplitude of $\mathbf{E}$ by $\mathbf{P}\cdot\mathbf{P}' = P_1P_1' + P_2P_2' = \cos(\theta - \theta')$ and reduces the intensity (which is proportional to the amplitude squared) by $\cos^2(\theta - \theta')$. The experiment in Fig. 3.3 shows the effect of linear polarization.

Now what happens if we reduce the incident light intensity? Classical theory predicts, and experiment confirms, that the same fraction of light, $\cos^2(\theta - \theta')$, passes the second polarizer, whatever the incident light intensity. But what if we reduce the incident light intensity until the energy of the light passing through the polarizers is roughly $\hbar\omega = h\nu$? According to quantum theory, the incident light consists of photons of energy $\hbar\omega$. (See Eq. (2.3).) Hence only one photon passes through a polarizer at a time. This photon either does or does not pass the second polarizer. It cannot split into a part that passes the polarizer and a part that does not, because its angular frequency ω, and therefore its energy $\hbar\omega$, cannot change as it passes through the polarizer. So $\cos^2(\theta - \theta')$ must be the *probability* that a photon arriving at the second polarizer will pass through it.

We can generalize this experiment by using birefringent panes [5]. It is convenient to express $\mathbf{E}(\mathbf{x}, t)$ as the real part of a complex wave [6]:

$$\mathbf{E}(\mathbf{x}, t) = \Re\left[\,(E_1e^{i\phi_1}\epsilon_\mathbf{1} + E_2e^{i\phi_2}\epsilon_\mathbf{2})e^{i\mathbf{k}\cdot\mathbf{x} - i\omega t}\,\right]\ . \tag{3.2}$$

Because electric fields add as vectors, the polarizations $E_1e^{i\phi_1}\epsilon_\mathbf{1}$ and $E_2e^{i\phi_2}\epsilon_\mathbf{2}$ add as vectors, even when they are complex. Equation (3.2) allows us to represent the polarization of an

[2] If necessary, we can change the sign of $\mathbf{E}$ by adding π to ϕ_1 and to ϕ_2.

electromagnetic wave as a vector $|\mathbf{P}\rangle$ in a two-dimensional complex vector space:

$$|\mathbf{P}\rangle = \frac{E_1}{E} e^{i\phi_1} |\epsilon_\mathbf{1}\rangle + \frac{E_2}{E} e^{i\phi_2} |\epsilon_\mathbf{2}\rangle \ ,$$

where $E = (E_1^2 + E_2^2)^{1/2}$. We let $|\mathbf{P}\rangle$, $|\epsilon_\mathbf{1}\rangle$, $|\epsilon_\mathbf{2}\rangle$ denote vectors in the complex vector space of polarizations, while $\mathbf{P}$, $\epsilon_\mathbf{1}$, $\epsilon_\mathbf{2}$, denote vectors in the real vector space of directions perpendicular to $\mathbf{k}$. Now a birefringent pane has an *optic axis*, and it shifts the relative phase between polarization components (i.e. between the component parallel to the optic axis and the component perpendicular to it) by an amount that depends on the thickness of the pane. By placing birefringent panes on either side of a polarizer, we can make it select (pass) monochromatic light with arbitrary polarization. We have a selector for arbitrary polarizations, not just linear polarizations. Figure 3.5 shows two such selectors. Suppose the first selector selects light of polarization $|\mathbf{P}\rangle$ and the second selector selects light of polarization $|\mathbf{P}'\rangle$. What fraction of light passing the first selector also passes the second selector? Prob. 3.2 shows that the fraction is

$$|\langle \mathbf{P}|\mathbf{P}'\rangle|^2 \ , \tag{3.3}$$

where the scalar product $\langle \mathbf{P}|\mathbf{P}'\rangle$ between two vectors $|\mathbf{P}\rangle = P_1|\epsilon_\mathbf{1}\rangle + P_2|\epsilon_\mathbf{2}\rangle$ and $|\mathbf{P}'\rangle = P_1'|\epsilon_\mathbf{1}\rangle + P_2'|\epsilon_\mathbf{2}\rangle$ generalizes $\mathbf{P}\cdot\mathbf{P}'$:

$$\langle \mathbf{P}|\mathbf{P}'\rangle = P_1^* P_1' + P_2^* P_2' \ .$$

(The asterisk denotes complex conjugation.) For example, if $|\mathbf{P}\rangle = |\epsilon_\mathbf{2}\rangle$ and $|\mathbf{P}'\rangle = (|\epsilon_\mathbf{1}\rangle + i|\epsilon_\mathbf{2}\rangle)/\sqrt{2}$, then $\langle \mathbf{P}|\mathbf{P}'\rangle = i/\sqrt{2}$ and the fraction is 1/2. If $|\mathbf{P}\rangle = |\mathbf{P}'\rangle$, the photon will certainly pass through the selector. If $|\mathbf{P}\rangle$ is orthogonal to $|\mathbf{P}'\rangle$, i.e. $\langle \mathbf{P}|\mathbf{P}'\rangle = 0$, the photon will certainly not pass. In general, $|\mathbf{P}\rangle$ is a complex superposition of $|\mathbf{P}'\rangle$ and the orthogonal polarization, and a fraction $|\langle \mathbf{P}|\mathbf{P}'\rangle|^2$ of the photons that pass through the first selector also pass through the second selector. When the polarizations $|\mathbf{P}\rangle$ and $|\mathbf{P}'\rangle$ are linear, Eq. (3.3) reduces to $\cos^2(\theta - \theta')$.

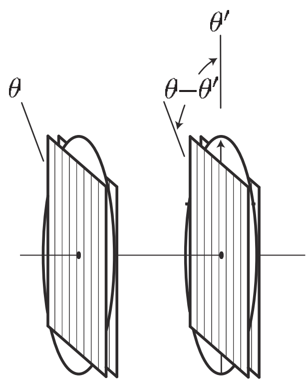

Figure 3.5: Each polarizer, sandwiched between birefringent panes, selects a particular elliptical polarization. Assume the two birefringent panes have optic axes parallel to $\epsilon_\mathbf{1}$. The first birefringent pane has just the right thickness to convert a particular elliptical polarization to the linear polarization that this polarizer passes; the second birefringent pane then converts the linear polarization back to the original elliptical polarization by restoring the relative phase $\phi_2 - \phi_1$.

3.3 Quantum States and Observables

Photon polarization illustrates a correspondence between quantum states and measurements. The polarization $|P\rangle$ is an example of a quantum state in a two-dimensional complex vector space – Hilbert space. The Hilbert space is two-dimensional, because if a photon arrives at a polarizer, there are two possible outcomes: the photon either does or does not pass through the polarizer. A measurement may have more than two possible outcomes. Section 3.4 considers measurements on *pairs* of photons. Each photon in a pair arrives at a polarizer and either does or does not pass through. There are four mutually exclusive answers to the question, "Do the two photons pass through the polarizers?" Thus polarization states of two photons are vectors in a four-dimensional Hilbert space; a basis for this space is $|\epsilon_1\rangle \otimes |\epsilon_1\rangle$, $|\epsilon_2\rangle \otimes |\epsilon_1\rangle$, $|\epsilon_1\rangle \otimes |\epsilon_2\rangle$ and $|\epsilon_2\rangle \otimes |\epsilon_2\rangle$, where the first state in each product represents one photon and the second state represents the other. The Hilbert space of a quantum system has dimension N if a complete measurement on the system has N possible outcomes.

This correspondence between quantum states and measurements follows the principle of complementarity. There are only two basis vectors for the polarization state of a photon, because an experiment can select only one (and reject only one) polarization at a time. There are infinitely many basis vectors for the position of a particle, because an experiment that selects one position rejects infinitely many other positions. (See Prob. 3.9.) But there are no additional basis vectors for the momentum of the particle, because the principle of complementarity allows us to measure *either* the position *or* the momentum of a particle accurately. There *are* basis vectors for momentum, but they are superpositions of the basis vectors for position (and vice versa). Incompatible measurements on a system correspond to incompatible bases for the Hilbert space of the system.

By measuring an observable A on a system, we choose a basis of states. Let us denote the basis states $|i\rangle$, where $i = 1, \ldots, N$ for an N-dimensional Hilbert space, and take $\langle i|j\rangle = \delta_{ij}$ – the basis is orthonormal. (See Prob. 3.5.) The correspondence with A is that if the system is in a state $|i\rangle$, a measurement of A yields an outcome a_i with certainty. [3] All other possible states $|\psi\rangle$ of the system are linear combinations of the $|i\rangle$ with complex coefficients c_i,

$$|\psi\rangle = \sum_i c_i|i\rangle \ ,$$

with the constraint $\sum_i c_i^* c_i = 1$, just as any polarization vector is a linear combination of $|\epsilon_1\rangle$ and $|\epsilon_2\rangle$ with complex coefficients satisfying the same constraint. The scalar product of two states $|\psi\rangle$ and $|\psi'\rangle$, denoted $\langle\psi|\psi'\rangle$, is the generalization of $\langle \mathbf{P}|\mathbf{P}'\rangle$ to Hilbert spaces of more than two dimensions. That is, if $|\psi'\rangle$ is

$$|\psi'\rangle = \sum_i c_i'|i\rangle \ ,$$

then the scalar (or inner) product of $|\psi\rangle$ and $|\psi'\rangle$ is

$$\langle\psi|\psi'\rangle = \sum_i c_i^* c_i' \ .$$

[3] If A is degenerate, i.e. if $a_i = a_j$ for some $i \neq j$, we also measure a nondegenerate observable on $|i\rangle$ and $|j\rangle$ for a complete measurement.

The square of the absolute value of $\langle\psi|\psi'\rangle$ is the probability that a measurement on a system in the state $|\psi\rangle$ will show it to be in the state $|\psi'\rangle$ (or vice versa), just as $|\langle\mathbf{P}|\mathbf{P}'\rangle|^2$ represents the probability that a measurement on a photon with polarization $|\mathbf{P}\rangle$ will show it to have polarization $|\mathbf{P}'\rangle$ (or vice versa).

Now consider an experiment in which we prepare a system in a state $|\psi\rangle$ and measure A. With probability $|\langle\psi|i\rangle|^2$ the measurement leaves the system in a state $|i\rangle$ and the measured value of A is a_i. (The constraint $\sum_i c_i^* c_i = 1$ insures that the probabilities $|\langle\psi|i\rangle|^2$ sum to 1.) If we repeat this experiment (the preparation and the measurement) many times, the average measured value of A will be

$$\sum_i a_i|\langle\psi|i\rangle|^2 \ ,$$

which we can write $\langle\psi|A|\psi\rangle$ by defining a linear Hermitian operator A:

$$A = \sum_i a_i|i\rangle\langle i| \ . \tag{3.4}$$

All observables of quantum theory are linear Hermitian operators. (See Probs. 3.3–4.) We call $\langle\psi|A|\psi\rangle$ the expectation value of A in the state $|\psi\rangle$, letting A denote both the physical variable and its operator. Applying A to the state $|j\rangle$ we find

$$A|j\rangle = a_j|j\rangle;$$

$|j\rangle$ is an eigenstate (or eigenvector) of A with eigenvalue a_j.

3.4 Bell's Inequality

A pion at rest decays into two photons. The photons fly off in opposite directions, with equal and opposite momenta. What is their polarization state? It must be a linear combination of the four basis vectors $|\epsilon_\mathbf{1}\rangle\otimes|\epsilon_\mathbf{1}\rangle$, $|\epsilon_\mathbf{1}\rangle\otimes|\epsilon_\mathbf{2}\rangle$, $|\epsilon_\mathbf{2}\rangle\otimes|\epsilon_\mathbf{1}\rangle$ and $|\epsilon_\mathbf{2}\rangle\otimes|\epsilon_\mathbf{2}\rangle$, but which linear combination? The same question arises for the annihilation of a positron and an electron into two photons in the decay of positronium, and for the emission of photon pairs in an atomic cascade. Experiment shows that the polarization state of photons from pion or positronium decay is

$$|\Psi_1\rangle = \frac{1}{\sqrt{2}}\left[|\epsilon_\mathbf{1}\rangle\otimes|\epsilon_\mathbf{2}\rangle - |\epsilon_\mathbf{2}\rangle\otimes|\epsilon_\mathbf{1}\rangle\right] \tag{3.5}$$

and not one of the states $|\Psi_2\rangle$, $|\Psi_3\rangle$ and $|\Psi_4\rangle$:

$$\begin{aligned}
|\Psi_2\rangle &= \frac{1}{\sqrt{2}}\left[|\epsilon_\mathbf{1}\rangle\otimes|\epsilon_\mathbf{2}\rangle + |\epsilon_\mathbf{2}\rangle\otimes|\epsilon_\mathbf{1}\rangle\right] \ ,\\
|\Psi_3\rangle &= \frac{1}{\sqrt{2}}\left[|\epsilon_\mathbf{1}\rangle\otimes|\epsilon_\mathbf{1}\rangle + |\epsilon_\mathbf{2}\rangle\otimes|\epsilon_\mathbf{2}\rangle\right] \ ,\\
|\Psi_4\rangle &= \frac{1}{\sqrt{2}}\left[|\epsilon_\mathbf{1}\rangle\otimes|\epsilon_\mathbf{1}\rangle - |\epsilon_\mathbf{2}\rangle\otimes|\epsilon_\mathbf{2}\rangle\right] \ .
\end{aligned} \tag{3.6}$$

We choose $|\Psi_1\rangle$, $|\Psi_2\rangle$, $|\Psi_3\rangle$ and $|\Psi_4\rangle$ as an orthonormal basis (rather than $|\epsilon_1\rangle \otimes |\epsilon_1\rangle$, $|\epsilon_1\rangle \otimes |\epsilon_2\rangle$, $|\epsilon_2\rangle \otimes |\epsilon_1\rangle$ and $|\epsilon_2\rangle \otimes |\epsilon_2\rangle$) for two reasons. First, $|\Psi_1\rangle$ and $|\Psi_3\rangle$, but not $|\Psi_2\rangle$ or $|\Psi_4\rangle$, are invariant under rotations around the axis of symmetry – the axis along which the photons propagate. (See Prob. 3.7.) Therefore, the polarization state of the photons must be either $|\Psi_1\rangle$ or $|\Psi_3\rangle$. Second, the parity of the state $|\Psi_1\rangle$ is odd, while the parity of the states $|\Psi_2\rangle$, $|\Psi_3\rangle$ and $|\Psi_4\rangle$ is even. (See Prob. 3.8.) Pions and positronium have odd parity and decay by emitting photons in the polarization state $|\Psi_1\rangle$.

The scalar product of $|\Psi_1\rangle$ with $|\epsilon_1\rangle \otimes |\epsilon_1\rangle$ or $|\epsilon_2\rangle \otimes |\epsilon_2\rangle$ vanishes; hence in the state $|\Psi_1\rangle$, the photons have opposite polarizations. In the state $|\Psi_3\rangle$, which could arise from an atomic cascade, the photons have the *same* polarization, because the scalar product of $|\Psi_3\rangle$ with $|\epsilon_1\rangle \otimes |\epsilon_2\rangle$ or $|\epsilon_2\rangle \otimes |\epsilon_1\rangle$ vanishes. These correlations hold even if we rotate the polarization axes from ϵ_1, ϵ_2 to $\epsilon_1{}'$, $\epsilon_2{}'$,

$$\begin{aligned} |\epsilon_1{}'\rangle &= |\epsilon_1\rangle \cos\phi - |\epsilon_2\rangle \sin\phi \ , \\ |\epsilon_2{}'\rangle &= |\epsilon_1\rangle \sin\phi + |\epsilon_2\rangle \cos\phi \ , \end{aligned} \tag{3.7}$$

for $|\Psi_1\rangle$ looks just the same in the new basis:

$$|\Psi_1\rangle = \frac{1}{\sqrt{2}} \left[|\epsilon_1{}'\rangle \otimes |\epsilon_2{}'\rangle - |\epsilon_2{}'\rangle \otimes |\epsilon_1{}'\rangle \right] \ . \tag{3.8}$$

So does $|\Psi_3\rangle$:

$$|\Psi_3\rangle = \frac{1}{\sqrt{2}} \left[|\epsilon_1{}'\rangle \otimes |\epsilon_1{}'\rangle + |\epsilon_2{}'\rangle \otimes |\epsilon_2{}'\rangle \right] \ . \tag{3.9}$$

Equations (3.8–9) are just what rotational symmetry demands. An experiment in 1949 first checked and confirmed the anticorrelated polarizations of photons pairs from pion or positronium decay [7]. The experiment ruled out a model, suggested by Einstein, in which the state $|\Psi_1\rangle$ decays spontaneously to $|\epsilon_1\rangle \otimes |\epsilon_2\rangle$ or $|\epsilon_2\rangle \otimes |\epsilon_1\rangle$ (with equal probability) as the photons fly off [8]. Later experiments checked and confirmed the correlated polarizations of photon pairs emitted in an atomic cascade. How does the EPR claim apply to these experiments?

Let Alice and Bob help us with the measurements again. Pairs of photons fly off from an atomic cascade, with one photon in each pair arriving at Alice's polarizer and the other arriving at Bob's polarizer. After each arrival, Alice and Bob reset the pass axes of their polarizers. They may reset the pass axes at random, from a finite list of settings, or according to a common plan. Either way, they find that when the pass axes of their polarizers are parallel, both photons in each pair pass the polarizers, or both do not; when the pass axes are at right angles, only one photon in each pair passes a polarizer. Suppose that Bob sets the pass axis of his polarizer to ϵ_1. If Alice sets the pass axis of her polarizer to ϵ_1, and a photon passes through, the photon arriving at Bob's polarizer also passes through. If Alice does not set the pass axis of her polarizer to either ϵ_1 or ϵ_2, quantum theory does not predict whether the photon that arrives at Bob's polarizer passes through. What would Einstein, Podolsky and Rosen say? The result of Bob's measurement cannot depend on what Alice measures. Hence the photon arriving at Bob's polarizer must pass through anyway. Quantum theory is incomplete, because quantum theory fails to predict that the photon must pass through.

Indeed, EPR would claim that quantum theory is hopelessly incomplete. The conclusion that holds when the pass axis of Bob's polarizer is $\epsilon_{\mathbf{1}}$ holds for any setting of his polarizer. Similarly, it holds for any setting of Alice's polarizer. Hence a complete theory must predict the result of *any* polarization measurement by either Alice or Bob. But the photons cannot anticipate what Alice and Bob will measure, so the EPR claim implies that every pair of photons arrives at Alice's and Bob's respective laboratories with a complete – i.e. infinite – list of results, one for each measurement that Alice and Bob might make. Still, the EPR claim seems as untestable as Bishop Berkeley's claim that a tree does not fall if no one sees it fall.

In 1964, however, Bell published a remarkable paper that showed, in effect, how to test whether unseen trees fall [9]. He showed that the EPR claim implies an inequality that some quantum correlations do not satisfy. Bell's paper appeared in the first volume of a journal that folded almost immediately. Moreover his test, involving electron spin, was not practical at the time. But five years later, Clauser, Horne, Shimony, and Holt [10] (CHSH) generalized Bell's inequality; the CHSH inequality, applied to photon polarization, allows a practical test of the EPR claim. Let us derive the CHSH inequality.

We return to Alice and Bob and their measurements. According to EPR, every pair of photons represents a complete list of answers (results) for each pair of questions (observables) that Alice and Bob might ask (measure). We refer to such lists as local plans; let λ denote a local plan. Each local plan must list results for the observables that Alice and Bob may measure, and the set of observables is infinite. But let us consider just two possible observables, A and A', that Alice may measure, and two observables, B and B', that Bob may measure. Let $\rho(\lambda)$ denote the relative probability that a photon pair carries a local plan λ. We normalize $\rho(\lambda)$:

$$\int d\lambda \rho(\lambda) = 1 ,$$

where the integration is over all λ. Given a local plan λ, let $P(A; a; \lambda)$ be the probability that a measurement A yields the result a. Similarly, let $P(A, B; a, b; \lambda)$ be the probability that measurements A and B (on two photons) yield results a and b, respectively. The plan λ is local, hence $P(A, B; a, b; \lambda)$ factorizes:

$$P(A, B; a, b; \lambda) = P(A; a; \lambda)P(B; b; \lambda) . \tag{3.10}$$

Now let $P(A, B; a, b)$ be the probability that measurements of A and B on a photon pair yield a and b, respectively. It is the average of $P(A, B; a, b; \lambda)$ weighted by $\rho(\lambda)$, i.e.

$$P(A, B; a, b) = \int d\lambda \rho(\lambda)P(A, B; a, b; \lambda) .$$

We define the correlation between measurements A and B to be

$$C(A, B) = \sum_{ij} a_i b_j P(A, B; a_i, b_j) , \tag{3.11}$$

where the a_i and b_j are possible results of measurements A and B, respectively. If A and B are photon polarization or electron spin measurements, each has two possible results; but in general, we let A and B have any finite number of possible results. Without loss of generality, we assume $-1 \le a_i, b_j \le 1$. We will prove that a combination S_{CHSH} of correlations,

$$S_{CHSH}(A, A'; B, B') = C(A, B) + C(A', B) + C(A, B') - C(A', B') \ ,$$

is bounded above and below:

$$-2 \le S_{CHSH}(A, A'; B, B') \le 2 \ . \tag{3.12}$$

To prove Eq. (3.12), we fix λ and look at the sum of products

$$\sum_{ij} a_i P(A; a_i; \lambda) \left[b_j P(B; b_j; \lambda) + b'_j P(B'; b'_j; \lambda) \right] \\ + \sum_{ij} a'_i P(A'; a'_i; \lambda) \left[b_j P(B; b_j; \lambda) - b'_j P(B'; b'_j; \lambda) \right] \ . \tag{3.13}$$

The absolute values of $\sum_i a_i P(A; a_i; \lambda)$ and $\sum_j b_j P(B; b_j; \lambda)$ etc. are bounded by 1. Hence each line in Eq. (13.3) is bounded in absolute magnitude by 2. The sum of the two lines is also bounded by 2, because (for example) if $\sum_j b_j P(B; b_j; \lambda) + \sum_j b'_j P(B'; b'_j; \lambda)$ has magnitude 2, then $\sum_j b_j P(B; b_j; \lambda) - \sum_j b'_j P(B'; b'_j; \lambda)$ vanishes, and vice versa. (See Prob. 3.12.) Thus

$$-2 \le \sum_{ij} [a_i b_j P(A; a_i; \lambda) P(B; b_j; \lambda) + a'_i b_j P(A'; a'_i; \lambda) P(B; b_j; \lambda) \\ + a_i b'_j P(A; a_i; \lambda) P(B'; b'_j; \lambda) - a'_i b'_j P(A'; a'_i; \lambda) P(B'; b'_j; \lambda)] \le 2 \ .$$

Multiplying by $\rho(\lambda)$ and integrating over λ, we obtain the CHSH inequality, Eq. (3.12):

$$-2 \le C(A, B) + C(A', B) + C(A, B') - C(A', B') \le 2 \ .$$

The CHSH inequality follows from the very assumption that local results exist, whether or not anyone measures them – that a tree falls whether or not anyone sees it fall. This reasonable ontological assumption underlies the EPR claim.

But some quantum correlations violate the CHSH inequality. To demonstrate a violation, we define the quantum correlation $C_Q(A, B)$ by replacing $P(A, B; a_i, b_j)$ in Eq. (3.11) with the quantum probability $P_Q(A, B; a_i, b_j)$:

$$C_Q(A, B) = \sum_{ij} a_i b_j P_Q(A, B; a_i, b_j) \ . \tag{3.14}$$

Alice and Bob measure linear polarizations in the plane of $\epsilon_\mathbf{1}$ and $\epsilon_\mathbf{2}$, with an angle θ_{AB} between the pass axes of A and B; let the values 1 and -1 correspond to a photon passing or not passing, respectively. We can obtain $C_Q(A, B)$ as follows. Suppose Bob's measurement yields 1, i.e. his photon passes his polarizer. The photons in each pair are correlated, hence Alice's photon is polarized parallel to Bob's. Then from Sect. 3.2 we infer that the probability that Alice's photon passes *her* polarizer is $\cos^2 \theta_{AB}$. By rotational symmetry, we have $P_Q(A, B, 1, 1) = (\cos^2 \theta_{AB})/2 = P_Q(A, B, -1, -1)$ and $P_Q(A, B, -1, 1) = (\sin^2 \theta_{AB})/2 = P_Q(A, B, 1, -1)$. Then $C_Q(A, B) = \cos^2 \theta_{AB} - \sin^2 \theta_{AB} = \cos 2\theta_{AB}$. In

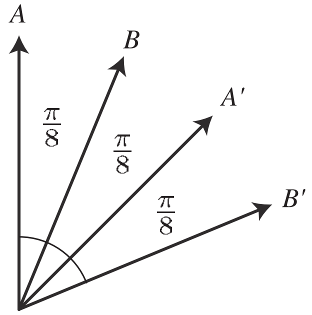

Figure 3.6: Polarizer pass axes in a test of the CHSH inequality, Eq. (3.12).

particular, let A, B, A' and B' correspond to polarizers with their pass axes in a common plane and an angle $\pi/8$ between the pass axes of A and B, B and A', and A' and B'. (See Fig. 3.6.) We have

$$C_Q(A,B) = C_Q(A,B') = C_Q(A',B) = \frac{\sqrt{2}}{2} = -C_Q(A',B')$$

and the sum

$$C_Q(A,B) + C_Q(A',B) + C_Q(A,B') - C_Q(A',B') = 2\sqrt{2}$$

violates the CHSH inequality, as we set out to demonstrate. Quantum correlations are nonlocal; they cannot arise from local plans.

An experimental test of the CHSH inequality, by Aspect, Dalibard, and Roger [11], measured correlations along the polarization axes of Fig. 3.6. The source of the photon pairs was an atomic cascade. In this experiment, the measurement settings switched pseudorandomly between A and A' and between B and B' in a time short compared to the time of flight of the photons. The correlations in this experiment were consistent with quantum correlations and violated the CHSH inequality by five standard deviations. In a more recent experiment by Wiehs et al. [12] the switching between these measurement settings was truly random and spacelike separated; the measured correlations violated the CSHS inequality by 30 standard deviations.

There are many extensions of Bell's inequality. Greenberg, Horne, and Zeilinger [13] (GHZ) found a remarkable extension involving *three* particles. (See Prob. 3.13.) The EPR paradox arises generically for any *entangled* state – any state of macroscopically separated systems that is not a product of states of each system. Any entangled state yields quantum correlations that violate a generalization of Bell's inequality [14].

3.5 Paradox and Beyond

The EPR claim assumes that Bob and Alice measure independent physical variables. Einstein, Podolsky and Rosen never anticipated that this reasonable assumption would prove inconsistent with experiment, that we cannot quite isolate systems in an entangled state from each other. But we cannot. As Bell put it, "The reasonable thing just doesn't work" [15].

Ironically, the claim that quantum theory is incomplete may well be correct, though not in the EPR sense. Quantum theory does not explain how we go from probability to observation, from possibility to actuality, as a complete theory would. There *is* such a complete theory, due to Bohm [16]. (See Prob. 9.3.) Bohm's theory is equivalent to quantum mechanics in its experimental predictions, but contains additional "hidden" variables that account for the results of every measurement. In keeping with Bell's inequality, these hidden variables are not local; so Einstein, Podolsky and Rosen would not have accepted them. But Bohm's theory is a complete quantum theory. Section 18.3 proposes final boundary conditions on the universe as a way to complete quantum theory. The hidden variables of this complete theory are nonlocal in *time*.

Problems

3.1 Consider the EPR paradox of Sect. 3.1. Does a measurement of x_1 affect a subsequent measurement of p_2? A measurement of x_1 affects the momentum component P of the measuring device according to the uncertainty relation $\Delta x_1 \Delta P \geq h$. Let m_2 and M denote the masses of particle 2 and of the measuring device, respectively. Let v_2 and $V = P/M$ denote the velocity components of particle 2 and of the measuring device, relative to a reference frame that does not accelerate during the measurement. Relative to the measuring device, the momentum of particle 2 is $p_2 = m_2(v_2 - V)$. Show that the uncertainty Δp_2 can be arbitrarily small, whatever Δx_1.

*3.2 (a) Show that a selector for polarization $|\mathbf{P}\rangle$ does not transmit any light of polarization $|\mathbf{P}'\rangle$ if $\langle\mathbf{P}|\mathbf{P}'\rangle = 0$. (See Sect. 3.2 and Fig. 3.5.)
(b) Show that Eq. (3.2) gives the fraction of light of polarization $|\mathbf{P}\rangle$ that passes a selector for polarization $|\mathbf{P}'\rangle$, and vice versa.

3.3 Let $A = |\psi_2\rangle\langle\psi_1|$ denote an operator such that $A|\phi\rangle = (\langle\psi_1|\phi\rangle)\,|\psi_2\rangle$. Show that A is a linear operator, i.e. that

$$A(c|\phi\rangle + c'|\phi'\rangle) = cA|\phi\rangle + c'A|\phi'\rangle \ .$$

Show that aA is a linear operator, if a is any complex number, and that any sum of linear operators is a linear operator.

3.4 The adjoint $A^\dagger$ of a linear operator A has the property that for any $|\phi\rangle$ and $|\phi'\rangle$,

$$\langle\phi|A^\dagger|\phi'\rangle = (\langle\phi'|A|\phi\rangle)^* \ .$$

(a) Show that if A is self-adjoint (or Hermitian), i.e. if $A = A^\dagger$, then the eigenvalues a_i of A are real.
(b) Show that if an operator U is unitary, i.e. $U^\dagger U = 1$, then its eigenvalues have absolute value 1.
(c) Unitary operators transform among incompatible bases. Show that if a set of vectors $|i\rangle$ is orthonormal, then so is the set of vectors $U|i\rangle$.

3.5 Let A be self-adjoint, with $A|\phi\rangle = a|\phi\rangle$ and $A|\phi'\rangle = a'|\phi\rangle$. Show that if $a \neq a'$, then $\langle\phi|\phi'\rangle = 0$.

3.6 An experiment to measure the spin component of an electron (along any axis) always yields $\hbar/2$ or $-\hbar/2$, corresponding to two basis vectors for the Hilbert space of electron spin. Let orthonormal vectors $|\uparrow\rangle$ and $|\downarrow\rangle$ represent states with z-component of spin equal to $\hbar/2$ and $-\hbar/2$, respectively. Any other spin state is a superposition of $|\uparrow\rangle$ and $|\downarrow\rangle$. Let $|u_1\rangle$ and $|u_2\rangle$ be two electron spin states with $\langle u_1|u_2\rangle = 0$; $|u_1\rangle$ and $|u_2\rangle$ are linear combinations of $|\uparrow\rangle$ and $|\downarrow\rangle$. Show that the entangled state

$$\frac{1}{\sqrt{2}}\left(|u_1\rangle \otimes |u_2\rangle - |u_2\rangle \otimes |u_1\rangle\right)$$

of two electrons equals the state

$$\frac{1}{\sqrt{2}}\left(|\uparrow\rangle \otimes |\downarrow\rangle - |\downarrow\rangle \otimes |\uparrow\rangle\right) \tag{3.15}$$

up to an overall phase.

3.7 (a) Show that a ϕ rotation about the z-axis, Eq. (3.7), leaves $|\Psi_1\rangle$ and $|\Psi_3\rangle$ of Eqs. (3.5–6) invariant.
(b) Show that the quantum correlation for two photons in the state $|\Psi_1\rangle$ of Eq. (3.5) is $C_Q(A, B) = -\cos 2\theta_{AB}$, where θ_{AB} is the angle between the pass axes of linear polarizers in measurements A and B.

*3.8 (a) Show that the state $|\Psi_1\rangle$ in Eq. (3.5) is odd under parity and that the states $|\Psi_2\rangle$, $|\Psi_3\rangle$ and $|\Psi_4\rangle$ in Eq. (3.6) are even under parity. (Parity reverses the direction of the propagation vector $\mathbf{k}$ of each photon, as well as the polarization direction.)
(b) Show that Eq. (3.15) (in Prob. 3.6) is odd under parity. (Parity reverses the propagation direction of each electron, but not the spin.)

*3.9 A quantum wave function $\psi(\mathbf{x})$ is a state in an infinite-dimensional Hilbert space. We denote it $|\psi\rangle$. The scalar product of two quantum wave functions $\psi_1(\mathbf{x})$, $\psi_2(\mathbf{x})$ is $\langle\psi_1|\psi_2\rangle \equiv \int_{-\infty}^{\infty} \psi_1^*(\mathbf{x})\psi_2(\mathbf{x})d\mathbf{x}$. A basis for the Hilbert space may be discrete or continuous. The basis states $|\psi_i\rangle$ of a discrete basis satisfy $\langle\psi_i|\psi_j\rangle = \delta_{ij}$, where $i, j = 1, 2, \ldots$. The basis states of a continuous basis satisfy an analogous condition, but with a δ-function in place of δ_{ij}. For example, $\delta(x - x_0)$ represents the state of a particle, in one space dimension, with position $x = x_0$. We denote it $|x_0\rangle$. We think of $\delta(x - x_0)$ as vanishing everywhere outside an arbitrarily small neighborhood of x_0. Formally, we define $\delta(x - x_0)$ via the integral

$$\int_a^b f(x)\delta(x - x_0)dx\ ,$$

which equals $f(x_0)$ if $a < x_0 < b$ and zero otherwise; $f(x)$ is an arbitrary differentiable function.

(a) A common representation of $\delta(x - x_0)$ is

$$\delta(x - x_0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ik(x-x_0)} dk \ ;$$

that is,

$$\int_a^b f(x)\delta(x - x_0) dx = \frac{1}{2\pi} \lim_{M\to\infty} \int_a^b \int_{-M}^{M} f(x) e^{ik(x-x_0)} dk dx \ .$$

Show that the double integral yields $f(x_0)$ if $a < x_0 < b$.
(b) Show that $x|x_0\rangle = x_0|x_0\rangle$ and that $\langle x_i|x_j\rangle = \delta(x_i - x_j)$. In three space dimensions, $\delta(\mathbf{x} - \mathbf{x}_i) \equiv \delta(x - x_i)\delta(y - y_i)\delta(z - z_i)$ represents the state of a particle with position $\mathbf{x} = \mathbf{x}_i$, and we denote it $|\mathbf{x}_i\rangle$. Show that $\langle \mathbf{x}_i|\mathbf{x}_j\rangle = \delta(\mathbf{x}_i - \mathbf{x}_j)$. These conditions are continuous analogues of the discrete condition $\langle \psi_i|\psi_j\rangle = \delta_{ij}$. In keeping with Sect. 3.3, we have $|\langle \psi|\mathbf{x}_0\rangle|^2 = |\psi(\mathbf{x}_0)|^2$ as the probability density, in the state $|\psi\rangle$, to find the particle at a point $\mathbf{x}_0$. (It is the probability *density*, not the probability, because the state $|\mathbf{x}_0\rangle$ is not normalized, i.e. $\langle \mathbf{x}_0|\mathbf{x}_0\rangle \neq 1$.)
(c) The quantum wave $(2\pi\hbar)^{-1/2} e^{ip_0x/\hbar}$ represents the state of a particle, in one space dimension, with momentum $p = p_0$. (See Eq. (2.4).) We denote it $|p_0\rangle$. Show that $\langle p_i|p_j\rangle = \delta(p_i - p_j)$. (d) The analogue of Eq. (3.4) for the Hermitian operator p is

$$p = \int_{-\infty}^{\infty} |p_0\rangle\langle p_0|p_0 dp_0 \ .$$

Show that $p\psi(x) = -i\hbar d\psi(x)/dx$ where $\psi(x)$ is any function with a Fourier transform. Thus $-i\hbar d/dx$ represents the observable p.

3.10 For a given state $|\Psi\rangle$ and Hermitian operators A, B, let $\langle A\rangle$ denote the expectation value $\langle\Psi|A|\Psi\rangle$ and ΔA denote the uncertainty $\Delta A = \left(\langle A^2\rangle - \langle A\rangle^2\right)^{1/2}$.
(a) Prove that

$$A|\Psi\rangle = \langle A\rangle|\Psi\rangle + \Delta A|\Psi_\perp^A\rangle \ ,$$

where $|\Psi_\perp^A\rangle$ is some state orthogonal to $|\Psi\rangle$.
(b) Use this result to prove the general uncertainty relation, $\Delta A \Delta B \geq \frac{1}{2}\left|\langle [A, B]\rangle\right|$, where $[A, B] = AB - BA$ is the commutator of A and B.
(c) Derive $\Delta x \Delta p \geq \hbar/2$. (See Prob. 3.9(d).)

*3.11 Consider two macroscopically separated systems, S_A and S_B, prepared in an arbitrary state $|\Psi\rangle$. Let A be an observable on S_A and B be an observable on S_B. Show that the probability of any result of a measurement of A on S_A is independent of which observable B is measured on S_B. (Hint: Compute the probability of the joint result a_i for A and b_j for B and obtain the probability of the result a_i for A from it by summing over j.)

3.12 Rewrite Eq. (3.13) by defining

$$S^{(A)} = \sum_i a_i P(A; a_i; \lambda) \quad , \quad S^{(A')} = \sum_i a'_i P(A'; a'_i; \lambda) \; ,$$
$$S^{(B)} = \sum_j b_j P(B; b_j; \lambda) \quad , \quad S^{(B')} = \sum_j b'_j P(B'; b'_j; \lambda) \; ,$$

and

$$S = S^{(A)}[S^{(B)} + S^{(B')}] + S^{(A')}[S^{(B)} - S^{(B')}] \; .$$

Note that $-1 \leq S^{(A)}, S^{(A')}, S^{(B)}, S^{(B')} \leq 1$.
(a) Prove that if $S^{(A)}$, $S^{(A')}$, $S^{(B)}$ and $S^{(B')}$ all have absolute value 1, then $S^2 = 4$.
(b) Suppose that S is a maximum for $|S^{(A)}| < 1$. Prove $S^2 \leq 4$ using the stationary condition $\partial K/\partial S^{(A)} = 0$. Prove $|S| \leq 2$ in general.

*3.13 The GHZ state $\frac{1}{\sqrt{2}}(|\uparrow\rangle_1 \otimes |\uparrow\rangle_2 \otimes |\uparrow\rangle_3 - |\downarrow\rangle_1 \otimes |\downarrow\rangle_2 \otimes |\downarrow\rangle_3)$ is a simultaneous eigenstate of $\sigma_y^{(1)}\sigma_y^{(2)}\sigma_x^{(3)}$, $\sigma_y^{(1)}\sigma_x^{(2)}\sigma_y^{(3)}$, $\sigma_x^{(1)}\sigma_y^{(2)}\sigma_y^{(3)}$ and $\sigma_x^{(1)}\sigma_x^{(2)}\sigma_x^{(3)}$, where the Pauli spin matrices $\sigma_x^{(j)}, \sigma_y^{(j)}$ act on the j-th state in each product and

$$\sigma_x^{(j)}|\uparrow\rangle_j = |\downarrow\rangle_j \; , \quad \sigma_x^{(j)}|\downarrow\rangle_j = |\uparrow\rangle_j \; ,$$
$$\sigma_y^{(j)}|\uparrow\rangle_j = i|\downarrow\rangle_j \; , \quad \sigma_y^{(j)}|\downarrow\rangle_j = -i|\uparrow\rangle_j \; .$$

What is the eigenvalue of each operator on the GHZ state? If the operators $\sigma_x^{(1)}$, $\sigma_x^{(2)}$, $\sigma_x^{(3)}$ and $\sigma_y^{(1)}$, $\sigma_y^{(2)}$, $\sigma_y^{(3)}$ all have simultaneous (hidden) values, what assignment of -1 or 1 to each is consistent with the quantum results?

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